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SYMMETRY PROPERTIES OF THE ZERO SETS OF NIL-THETA FUNCTIONS

BY

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ABSTRACT. Let N denote the three dimensional Heisenberg group, and let Γ be a discrete two-generator subgroup of N such that N/Γ is compact. Then we may decompose $L^2(N/\Gamma)$ into primary summands with respect to the right regular representation R of N on $L^2(N/\Gamma)$ as follows: $L^2(N/\Gamma) = \oplus \sum_{m \in \mathbf{Z}} H_m(\Gamma)$. It can be shown that for $m \neq 0$, $H_m(\Gamma)$ is a multiplicity space for the representation R of multiplicity $|m|$. The distinguished subspace theory of L. Auslander and J. Brezin singles out a finite number of the decompositions of $H_m(\Gamma)$, $m \neq 0$, which are in some ways nicer than the others. They define algebraically an integer valued function, called the index, on the set Ω_m of irreducible closed R-invariant subspaces of $H_m(\Gamma)$ such that the distinguished subspaces have index one.

In this paper, we give an analytic-geometric interpretation of the index. Every space in Ω_m contains a unique (up to constant multiple) special function, called a nil-theta function, that arises as a solution of a certain differential operator on N/Γ. These nil-theta functions have been shown to be closely related to the classical theta functions. Since the classical theta functions are determined (up to constant multiple) by their zero sets, it is natural to attempt to classify the spaces in Ω_m using various properties of the zero sets of the nil-theta functions lying in these spaces. We define the index of a nil-theta function in $H_m(\Gamma)$ using the symmetry properties of its zero set. Our main theorem asserts that the algebraic index of a space in Ω_m equals the index of the unique nil-theta function lying in that space. We have thus an analytic-geometric characterization of the index.

We then use these results to give a complete description of the zero sets of those nil-theta functions of a fixed index. We also investigate the behavior of the index under the multiplication of nil-theta functions; i.e. we discuss how the index of the nil-theta function FG relates to the indices of the nil-theta functions F and G.

Let N denote the three dimensional *Heisenberg group*, that is the connected, simply connected, nilpotent Lie group whose underlying manifold is $\mathbf{R}^3$, and whose group multiplication is given by the rule

$$(x, y, t)(a, b, c) = \big(x + a, y + b, t + c + \tfrac{1}{2}(ya - xb)\big),$$

where (x, y, t) and (a, b, c) are elements of N. It is easily verified that the center, $Z(N)$, of N equals the commutator subgroup $[N, N]$ of N, and consists of all the elements of N that are of the form $(0, 0, t)$ for $t \in \mathbf{R}$.

Let F be a complex valued function on N, and let $g \in N$. We define the left and right translates of F by g to be the functions

$$(l(g)F)(h) = F(g^{-1}h) \quad \text{and} \quad (R(g)F)(h) = F(hg),$$

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$h \in N$, respectively. A family $\mathfrak{F}$ of functions on N will be called R-invariant if $R(g)\mathfrak{F} = \mathfrak{F}$ for all $g \in N$.

Let Γ be a discrete subgroup of N such that the space N/Γ of *right* Γ-cosets is compact. N/Γ has a unique (up to constant multiple) Haar measure which we fix and omit from all notation. Any function F on N/Γ can also clearly be viewed as a function on N which is constant on Γ-cosets, that is, for which $l(\gamma)F = F$ for all $\gamma \in \Gamma$. Therefore, for every $g \in N$, the map $R(g)$ induces an operator on $L^2(N/\Gamma)$, namely:

$$(R(g)F)(\Gamma h) = F(\Gamma hg), \qquad h \in N.$$

The map R: $g \mapsto R(g)$ is clearly a unitary representation of N on the space $L^2(N/\Gamma)$ known as the right regular representation.

Let $\beta(\Gamma)$ be the positive real number such that $(0, 0, \beta(\Gamma))$ generates the discrete subgroup $\Gamma \cap Z(N)$ of $Z(N)$. For simplicity of notation, where no confusion will arise, we will denote $\beta(\Gamma)$ simply by β. Let $H_m(\Gamma)$ be the closed subspace of $L^2(N/\Gamma)$ spanned by those continuous functions F satisfying $F(x, y, t + c) = \exp(2\pi imc/\beta)F(x, y, t)$. It is well known that we can decompose $L^2(N/\Gamma)$ into an orthogonal direct sum of R-invariant subspaces:

$$L^2(N/\Gamma) = \oplus \sum_{m \in \mathbf{Z}} H_m(\Gamma).$$

It follows from the Stone-von Neumann theorem (see [7, p. 71]) that the restriction of the representation R to the space $H_m(\Gamma)$ is a multiple of a single irreducible unitary representation $U(m)$ of N, as long as $m \neq 0$. Moreover, for $m \neq n$, the representations $U(m)$ and $U(n)$ are not unitarily equivalent. It follows then that two irreducible closed R-invariant subspaces W_1 and W_2 of $L^2(N/\Gamma)$ are the underlying spaces for unitarily equivalent representations of N, if and only if both W_1 and W_2 are subspaces of the same $H_m(\Gamma)$.

A discrete cocompact subgroup of N can always be generated by three suitably chosen elements, and in fact two will suffice whenever the subgroup's center and commutator subgroup coincide. Moreover, it is easily seen that every discrete cocompact subgroup Γ of N contains a two-generator subgroup Γ' such that $\beta(\Gamma) = d\beta(\Gamma')$ and $H_{md}(\Gamma) = H_m(\Gamma')$ where d is the order of Γ'/Γ. It suffices therefore for our purposes to consider only the two-generator discrete cocompact subgroups of N, and Γ will henceforth denote such a subgroup, which we will call simply a two generator subgroup of N.

Let Π: $N \to N/Z(N) = \mathbf{R}^2$ be the natural projection defined by $\Pi(x, y, t) = (x, y)$. Making the usual identification of $\mathbf{R}^2$ with $\mathbf{C}$, we will also view Π as a map from N to $\mathbf{C}$. The space $H_0(\Gamma)$ can then be identified with $L^2(\mathbf{R}^2/\Pi(\Gamma))$ and is thus essentially $L^2(T^2)$, where T^2 denotes the 2-torus. Therefore, $H_0(\Gamma)$ can be decomposed in one and only one way into an orthogonal direct sum of irreducible (in fact, one dimensional) closed R-invariant subspaces.

For $m \neq 0$, it is well known that R restricted to $H_m(\Gamma)$ is precisely the $|m|$-fold multiple of the irreducible representation $U(m)$. There is not, however, a unique decomposition of $H_m(\Gamma)$ into irreducibles. In fact, there are an infinite number of such decompositions, none of which seem at first glance to be any better than the rest. In [3], L. Auslander and J. Brezin define an integer valued function, which they call the *index*, on the set of irreducible closed R-invariant subspaces of $H_m(\Gamma)$, $m \neq 0$. They, and in later works, Auslander and Tolimieri [2], [4], [11] show that those subspaces with index one, which the latter pair name *distinguished subspaces*, have certain special properties.

The definition of the index given in [3] is algebraic in nature. That paper ends, however, with a necessary and sufficient condition for an irreducible subspace W to be a distinguished subspace–that it must contain a function satisfying two special properties. This function was moreover closely related to a classical theta function.

It has been known for a long time (see [4], [11], [2]) that the functions on N/Γ, named *nil-theta functions*, that arise as solutions of the differential operator $V_i f = 0$,

$$V_i = \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) - \frac{i}{2}(x + iy)\frac{\partial}{\partial t}$$

are in obvious bijective correspondence with the classical theta functions of some "type" with respect to the lattice $L = \Pi(\Gamma)$, as defined in [10] and [7]. Furthermore, it is known (see [2]) that every irreducible closed R-invariant subspace W of $H_m(\Gamma)$, for $m > 0$, contains a unique (up to constant multiple) nil-theta function, and conversely, every nil-theta function in $H_m(\Gamma)$, for $m > 0$, lies in a unique irreducible closed R-invariant subspace of $H_m(\Gamma)$. Since the classical theta functions are completely determined by their zero sets, it is natural to attempt to classify the irreducible subspaces of $H_m(\Gamma)$, for $m > 0$, according to the properties of the zero sets of the nil-theta functions lying in these subspaces.

In this paper, we intend to use some simple properties of the classical theta functions to give another characterization of the indexing on the irreducible subspaces of $H_m(\Gamma)$, for $m > 0$, which is analytic-geometric rather than algebraic in nature. We show that the index of the subspace W of $H_m(\Gamma)$, $m > 0$, reflects certain symmetry properties of the zero set of the unique nil-theta function lying in W.

More precisely, the index of W as defined in [3] is given by $\operatorname{ind}(W) = m/|\Delta(W)|$ where $\Delta(W)$, the *left stabilizer* of W, is a finite abelian group, of order $|\Delta(W)|$, dividing m, which arises from representation theoretic considerations. We will define this group precisely in §2. Let θ be the unique nil-theta function lying in W. We will see that the zero set of θ is completely determined by a set S containing a certain m zeroes of θ. We will define the *symmetry group* $\Delta(\theta)$ of θ to be the finite group of symmetries of the set S, and then define the index of the function θ as

$$\operatorname{ind}(\theta) = m/|\Delta(\theta)|.$$

Our main result is that $\Delta(W) = \Delta(\theta)$, and thus $\operatorname{ind}(W) = \operatorname{ind}(\theta)$. Thus, the algebraic notion of the index of W corresponds to the index of the nil-theta function θ lying in W, which reflects analytic-geometric properties. In particular, we see that the index of the infinite dimensional space W is completely determined by the symmetry properties of the zero set of a single special function lying in W.

This characterization yields, moreover, another nice feature of the distinguished subspaces. The larger the index of the subspace, the less symmetric the zero set of the nil-theta function lying in the subspace. Thus, the distinguished subspaces are those whose nil-theta functions have zero sets which are as symmetric as possible.

These results furthermore enable us to characterize all nil-theta functions lying in subspaces with a fixed index k and left stabilizer Δ, by describing their zero sets. We show that the zero set of such a function can be partitioned into subsets of order k in such a way that the sum of the zeroes in each of these subsets is a completely determined rational expression in the coordinates of the generators of Δ. This yields still another special property of the distinguished subspaces–each subset in the partition described above contains a single element. Thus, we know the precise location of each of the zeroes of the nil-theta function lying in the distinguished subspace. In particular, we know that these zeroes are at points rationally related to the generators of Δ. Hence, if W is of index one, its stabilizer and index determine the precise location and rationality of the individual zeroes of the nil-theta function lying in W, while if the index of W is k, $k > 1$, the stabilizer and index of W determine only the location and rationality of sums of k zeroes of the corresponding nil-theta function. Thus, in some sense, the larger the index of the subspace, the less we know about the zeroes of the nil-theta function lying in that subspace. These results are presented in §2.

In §1, after the presenting the background information that we will need in this paper, we give a complete characterization of the zero sets of those nil-theta functions lying in $H_m(\Gamma)$, or in more classical language, we characterize the zero sets of all theta functions of a fixed type with respect to the lattice $L = \Pi(\Gamma)$. We then use this result to provide a refinement of the classical result, due to Jacobi, that every elliptic function can be expressed as a quotient of theta functions.

More precisely, let $f(z)$ be an elliptic function with periods $2\omega_1$ and $2\omega_2$, and let L be the lattice in $\mathbf{C}$ spanned by $2\omega_1$ and $2\omega_2$. It is well known that the set of zeroes (poles) of f is of the form $S + L$, where S is a finite set of zeroes (poles) of f. Such a set S is called a fundamental set of zeroes (poles) of f. It is also well known that all fundamental sets of zeroes of f have the same order as all fundamental sets of poles of f.

Let $\{z_1, z_2, \ldots, z_m\}$ and $\{w_1, w_2, \ldots, w_m\}$ be fundamental sets of zeroes and poles of f, respectively, chosen so that $\Sigma_{i=1}^m (z_i - w_i) = 0$. The classical result due to Jacobi (see for example **[10]**) is that one can write

$$f(z) = C \prod_{i=1}^m \frac{\theta\begin{bmatrix}1\\0\end{bmatrix}\left(\dfrac{z - z_i}{2\omega_1}, \dfrac{\omega_2}{\omega_1}\right)}{\theta\begin{bmatrix}1\\0\end{bmatrix}\left(\dfrac{z - w_i}{2\omega_1}, \dfrac{\omega_2}{\omega_1}\right)} = C \prod_{i=1}^m \frac{\theta_i(z)}{\chi_i(z)}$$

where C is a constant, and $\theta[{a \atop b}](z, \tau)$ denotes the classical Jacobi theta function of characteristic $[{a \atop b}]$ and period τ in the variable z, as defined in **[9]**.

The functions θ_i and χ_i, $i = 1, 2, \ldots, m$, are all theta functions with respect to the lattice L of various "types" in the sense of Swinnerton-Dyer (see **[10]**). We will, in §1, recall the isomorphism defined in **[1]** between the classical theta functions of a certain type with respect to the lattice L, and the nil-theta functions invariant with respect to a two-generator subgroup Γ of N with $\Pi(\Gamma) = L$. Different "types" of classical theta functions correspond to invariance with respect to different two generator subgroups Γ, all with $\Pi(\Gamma) = L$. The classical theta functions θ_i and χ_i are both of the same type; however for $i \neq j$, θ_i and θ_j need not be of the same type. In the language of nonabelian harmonic analysis, their nil-theta analogues do not all lie on a single nil-manifold N/Γ.

Tolimieri in **[11]** refined this result. He proved (in the more general setting of theta functions of several variables) that every elliptic function f with period lattice L can be written as $f = F/G$ where F and G are nil-theta functions lying in $H_n(\Gamma)$ for *one* fixed two generator subgroup Γ with $\Pi(\Gamma) = L$, and some positive integer n which may be arbitrarily large. The number n is easily seen to represent the order of the fundamental set of zeroes of the classical theta functions corresponding to the nil-theta functions F and G. Tolimieri's proof provides no bound for the integer n, and shows no relationship between n and the number of zeroes of the elliptic function f. In §1, we will show that one more than the number of zeroes of f is the best possible bound for n.

Let $\Theta_m(\Gamma)$ denote the zeroes of the operator V_i that lie in the space $H_m(\Gamma)$, and let Θ denote the (algebraic) direct sum of the $\Theta_m(\Gamma)$'s:

$$\Theta = \bigoplus \sum_{m \neq 0} \Theta_m(\Gamma).$$

The elements of Θ can be shown by standard Sobolov estimates to be infinitely differentiable. It thus follows from Leibniz's rule for differentiation of products that θ is a graded algebra over **C**. This algebra is actually a finitely generated integral domain, which one proves by an easy, but tedious computation–see **[2]**, for example.

It is natural to ask whether this algebra structure is in any way related to the indexing structure; more presicely, if $F \in \Theta_m(\Gamma)$ and $G \in \Theta_n(\Gamma)$, how does the index of the nil-theta FG relate to the indices of F and G? Some relationship is indicated, since for example, if s is any positive integer, it is easily verified that $\operatorname{ind}(F^s) = s \operatorname{ind}(F)$. This suggests the possibility that the indices might add as the functions multiply. The actual behavior of the index under multiplication of nil-theta functions is much more complicated than this. We present here the results in the special case that both of the functions being multiplied are of index one, as these can be described fairly easily. The results in the more general cases are quite messy, and we refer the interested reader to the author's thesis for further details.

1. The zeroes of nil-theta functions. In this section we summarize some of the background information that we will need about the Heisenberg group, nil-theta functions and their zero sets, before proving our first result.

1. *The automorphisms of N.* A detailed discussion of the automorphism group of N is given in **[11]**. We mention here, without proof, several facts that we will need.

Let ${}^tS$ denote the transpose of the matrix S, and let J denote the linear transformation $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ on $\mathbf{R}^2$ that arises from the complex structure of **C**. Every automorphism α of N can be described by a 3×3 matrix of the form

$$\alpha = \begin{bmatrix} & S & & 0 \\ & & & 0 \\ a_1 & & a_2 & c \end{bmatrix}$$

where a_1, a_2, and c are real numbers, and S is a real 2×2 matrix satisfying the condition ${}^tSJS = cJ$. Moreover, α is an inner automorphism of N if and only if the matrix S is the two dimensional identity matrix.

Let Γ be a two-generator subgroup of N, and let α be an automorphism of N. Then $\alpha^{-1}(\Gamma)$ is also a two-generator subgroup of N and $H_m(\Gamma) \circ \alpha = H_m(\alpha^{-1}(\Gamma))$.

2. *Nil-theta functions*. Let $L(N)$ denote the Lie algebra of N. Viewing $L(N)$ as the space of left invariant vector fields on N, we have the following basis for this space:

$$X(x, y, t) = \partial/\partial x + \tfrac{1}{2}y\partial/\partial t|(x, y, t),$$

$$Y(x, y, t) = \partial/\partial y - \tfrac{1}{2}x\partial/\partial t|(x, y, t),$$

$$T(x, y, t) = \partial/\partial t|(x, y, t),$$

for $(x, y, t) \in N$. Let α be an automorphism of N. Then α induces an automorphism of the complexification, $L_{\mathbf{C}}(N)$, of $L(N)$, whose matrix with respect to the basis X, Y, T, is the same as the matrix of α. In particular, we consider the automorphism J of $L_{\mathbf{C}}(N)$ whose matrix is

$$J = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Since J^4 is the identity matrix, J determines a direct sum decomposition of $L_{\mathbf{C}}(N)$: $L_{\mathbf{C}}(N) = V_i + V_{-i} + Z_{\mathbf{C}}$ where $V_{\pm i}$ denotes the eigenvalue $\pm i$ subspace of $L_{\mathbf{C}}(N)$ with respect to J, and $Z_{\mathbf{C}}$ denotes the center of $L_{\mathbf{C}}(N)$. V_i can be shown (see **[11]**) to have the basis vector, which we also denote by V_i

$$V_i = X + iY = \frac{\partial}{\partial \bar{z}} - \frac{i}{2} z \frac{\partial}{\partial t}$$

where $z = x + iy$ and $\partial/\partial\bar{z} = \partial/\partial x + i\partial/\partial y$.

We define the space Θ of *nil-theta functions* of N to be the space of all infinitely differentiable functions f on N that satisfy the differential equation $V_i f = 0$.

Let Γ be a two-generator subgroup of N. Because V_i is a left invariant operator, it follows that it is a well-defined operator on $C^\infty(N/\Gamma)$. Thus we can define $\Theta(\Gamma) = C^\infty(N/\Gamma) \cap \Theta$. Let p_m denote the orthogonal projection of $L^2(N/\Gamma)$ onto $H_m(\Gamma)$, $m \neq 0$. Then $p_m(\Theta(\Gamma)) \subseteq \Theta(\Gamma)$. Therefore, we can define $\Theta_m(\Gamma) = \Theta \cap H_m(\Gamma)$ and we have the following L^2 decomposition:

$$\Theta(\Gamma) = \oplus \sum_{m \in Z} \Theta_m(\Gamma).$$

We now recall the relationship between the nil-theta functions and the classical theta functions. The proofs that we omit can be found in **[2]** or **[11]**.

LEMMA 1.1. *Let $F \in C^{\infty}(N)$ be such that $V_i F = 0$ and such that*

$$F(x, y, t) = \exp(2\pi iat)F(x, y, 0), \qquad a \in R. \qquad (*)$$

Then the function

$$(M(a)F(z)) = \exp(-2\pi iat)\exp(-\pi iaxy)\exp(\pi ay^2)F(x, y, t)$$

is an entire function of the variable $z = x + iy$.

Conversely, if $H(z)$ is an entire function of $z = x + iy$, then

$$\big(M(a)^{-1}H\big)(x, y, t) = \exp(2\pi iat)\exp(\pi iaxy)\exp(-\pi ay^2)H(x + iy)$$

is a solution of V_i satisfying $(*)$.

Let Γ have generators γ_1 and γ_2, and let Π: $N \to \mathbf{R}^2 = \mathbf{C}$ be the natural projection. Then $L = \Pi(\Gamma)$ is a lattice in $\mathbf{R}^2$ generated by the two linearly independent vectors $\Pi(\gamma_1)$ and $\Pi(\gamma_2)$. The following two corollaries of Lemma 1.1 are proven in [2].

COROLLARY 1.2. *Let Γ be a two-generator subgroup of N. Then $\Theta_m(\Gamma)$ is trivial for $m < 0$, and consists of the constant functions for $m = 0$.*

COROLLARY 1.3. *Let F and G belong to $\Theta_m(\Gamma)$, for $m > 0$. Then F/G is a meromorphic function on the torus $\mathbf{C}/L$ for $L = \Pi(\Gamma)$.*

We will have more to say about the converse to Corollary 1.3 later on in this section.

Let $F \in \Theta_m(\Gamma)$, $m > 0$. For simplicity of notation, we will denote $\beta(\Gamma)$ by β and $M(m/\beta)$ by M_m. It is shown in [11] that $G = M_m F$ is an entire theta function of a certain "type" with respect to the lattice $L = \Pi(\Gamma)$, as defined in [10] and [7]: i.e., there exists an inhomogeneous linear function $\varphi(z, l)$ of z such that

$$G(z, l) = \exp(\varphi(z, l))G(z) \quad \text{for all } l \in L.$$

Thus, the Γ-invariance of the function F implies that the entire function G satisfies certain functional equations. For our purpose the following special case, the proof of which can be found in [2], will suffice.

Let $G = \operatorname{grp}\{g_1, g_2\}$ denote that g_1 and g_2 generate the group G.

LEMMA 1.4. *Let $\tau = \alpha + i\beta$ be an element of the complex upper half-plane, and let $a, b \in R$. We define the lifted subgroup*

$$\Gamma(\tau; a, b) = \operatorname{grp}\{(1, 0, b\beta), (\alpha, \beta, -a\beta)\}.$$

Let $F \in \Theta_m(\Gamma(\tau; a, b))$, $m > 0$, and let $H(z) = (M_m F)(x, y, t)$, $z = x + iy$. Then $H(z)$ satisfies the following functional equations:

(a) $H(z + 1) = \exp(2\pi imb)H(z)$.

(b) $H(z + \tau) = \exp(-2\pi ima)\exp(-\pi im(2z + \tau))H(z)$.

Conversely, if $H(z)$ is analytic and satisfies (a) *and* (b), *then $M_m^{-1}H \in \Theta_m(\Gamma(\tau; a, b))$.*

We see then that M_m is an isomorphism between the space $\Theta_m(\Gamma(\tau; a, b))$ and the m-dimensional complex vector space of classical Jacobi theta functions of period τ

and characteristic $[{}^{2mb}_{2ma}]$ in the variable $z = x + iy$, as discussed in [9]. Adopting the notation of [9], we denote the latter space by $\Theta_m[{}^{2mb}_{2ma}](z, \tau)$.

Let $\Gamma(\tau) = \Gamma(\tau; 0, 0) = \mathrm{grp}\{(1, 0, 0), (\alpha, \beta, 0)\}$. Then for $n = (a + b\alpha, b\beta, ab\beta/2)$, we have that $\Gamma(\tau; a, b) = n^{-1}\Gamma(\tau)n$. Now $\Theta_m(\Gamma(\tau)) = \Theta_m[{}^0_0](z, \tau)$, so that the automorphism, i_n, of N defined by $i_n(g) = n^{-1}gn$ results in a shift in the characteristic of the corresponding theta function.

3. *The zero sets of nil-theta functions.* It is well known that the zeroes of a classical theta function with respect to a lattice L are isolated and determine the function up to constant multiple. Moreover, the zero set of such a function is invariant under translations by elements of L. It will suffice then for our purpose to consider those zeroes that lie in the *fundamental parallelogram*, D, of the lattice L–that is the open parallelogram spanned by the two generators of L, together with the two sides of the parallelogram that contain zero.

Suppose that $F \in \Theta_m(\Gamma)$, $m > 0$, for some two-generator subgroup Γ of N and let $L = \Pi(\Gamma)$ be the corresponding lattice in $\mathbf{C}$. Let $\tilde{S}$ and S denote the zero sets of the functions F and $M_m F$ respectively. It follows easily from the fact that the multipler M_m never vanishes that

$$\tilde{S} = \{(x, y, t) | x + iy \in S \text{ and all } t \in R\};$$

i.e., $\tilde{S} = S \cdot Z(N)$.

We see thus that the zero set of the nil-theta function $F \in \Theta_m(\Gamma)$, $m > 0$, is completely determined by the subset of the complex plane consisting of the zeroes of the analytic function $M_m F$ that are contained in the fundamental parallelogram D of the lattice $L = \Pi(\Gamma)$. Moreover, it is not difficult to see, using a computation similar to one on p. 79 of [9], that $M_m F$ has m zeroes in D.

We are now ready to state and prove our first theorem. The expression w is congruent to u modulo L will mean that $w - u \in L$.

THEOREM 1.5. *Let Γ be a subgroup of N with generators (a_1, a_2, a_3) and (b_1, b_2, b_3), and let D be the fundamental parallelogram of the lattice $L = \Pi(\Gamma)$. Let $S = \{z_1, z_2, \ldots, z_m\}$ be a subset of D. Then there exists a function $F \in \Theta_m(\Gamma)$ such that S is the set of zeroes in D of the function $M_m F$ if and only if $\sum_{i=1}^m z_i$ is congruent modulo L to*

$$\begin{cases} mv_0, & m \text{ even}, \\ mv_0 + \frac{1}{2}(a_1 + b_1) + (i/2)(a_2 + b_2), & m \text{ odd}, \end{cases}$$

where $v_0 = (a_3 b_1 - a_1 b_3)/(b_2 a_1 - a_2 b_1) + i(a_3 b_2 - a_2 b_3)/(b_2 a_1 - a_2 b_1)$.

PROOF. We consider several cases.

Case I. Suppose that $\Gamma = \Gamma(\tau) = \mathrm{grp}\{(1, 0, 0), (\alpha, \beta, 0)\}$, for $\tau = \alpha + i\beta$, $\beta > 0$. Let $L(\tau) = \Pi(\Gamma(\tau))$, let $D(\tau)$ be the fundamental parallelogram of $L(\tau)$ and let $S = \{z_1, z_2, \ldots, z_m\} \subseteq D(\tau)$. We write $z_i = r_i + s_i\tau$ where $0 \leqslant r_i, s_i < 1$, $i = 1, 2, \ldots, m$. It is a well-known fact (see [9, p. 81]) that the classical Jacobi theta function,

$$\theta_m\left[{}^{\sigma}_{\sigma'}\right](z, \tau),$$

of period τ and characteristic $[^{\sigma}_{\sigma'}]$ ($\sigma, \sigma' \in R$), defined by

$$\theta[^{\sigma}_{\sigma'}](z, \tau) = \sum_{n \in \mathbf{Z}} \exp \pi i \left\{ \tau\left(n + \frac{\sigma}{2}\right)^2 + 2\left(n + \frac{\sigma}{2}\right)\left(z + \frac{\sigma'}{2}\right) \right\}$$

has a unique zero in $D(\tau)$ at the point z_0 which is congruent modulo $L(\tau)$ to $(1/2 + \tau/2) + (\sigma'/2 - \sigma\tau/2)$. It follows then that, up to constant multiple, the unique classical theta function with zero set S in $D(\tau)$ is the function

$$H(z) = \prod_{i=1}^{m} \theta\begin{bmatrix} 2\left(\frac{1}{2} - s_i\right) \\ -2\left(\frac{1}{2} - r_i\right) \end{bmatrix}(z, \tau) \in \Theta_m \begin{bmatrix} 2\sum_{i=1}^{m}\left(\frac{1}{2} - s_i\right) \\ -2\sum_{i=1}^{m}\left(\frac{1}{2} - r_i\right) \end{bmatrix}(z, \tau).$$

Let $F = M_m^{-1}H$. By Lemma 1.4 we have that $F \in \Theta_m(\Gamma(\tau; r, s))$, where $r = -(1/m)\sum_{i=1}^{m}(\frac{1}{2} - r_i)$, and $s = (1/m)\sum_{i=1}^{m}(\frac{1}{2} - s_i)$. F will then be the required nil-theta function if and only if $F \in \Theta_m(\Gamma(\tau)) \cap \Theta_m(\Gamma(\tau; r, s))$. In that case we must have that $L(\gamma)F = F$ for all γ in the group generated by $\Gamma(\tau)$ and $\Gamma(\tau; r, s)$. Thus, for any $g \in N$

$$\begin{aligned} F(g) &= F\left[(1, 0, s\beta)g\right] = F\left[(1, 0, 0)(0, 0, s\beta)g\right] \\ &= F\left[(0, 0, s\beta)g\right] = \exp(2\pi i m s)F(g), \end{aligned}$$

and similarly $F(g) = F[(\alpha, \beta, -r\beta)g] = \exp(-2\pi i m r)F(g)$. We must have then that both mr and ms are integers, and this is clearly equivalent to the condition that $\sum_{i=1}^{m} z_i = \sum_{i=1}^{m}(r_i + s_i\tau)$ is congruent modulo $L(\tau)$ to

$$\begin{cases} 0, & m \text{ even}, \\ \frac{1}{2} + \tau/2, & m \text{ odd}, \end{cases}$$

and the theorem is proven in this special case.

Case II. Suppose that $\Gamma = \Gamma(\tau; a, b)$, where $\tau = \alpha + i\beta$, $\beta > 0$, $a, b \in R$, and let $L(\tau) = \Pi(\Gamma)$ and $D(\tau)$ be as above. By a remark in the beginning of this section, we have that $H_m(\Gamma(\tau; a, b)) = H_m(\Gamma(\tau)) \circ i_n$ where i_n denotes the inner automorphism by $n = (a + b\alpha, b\beta, ab\beta/2)$. Thus,

$$H_m(\Gamma(\tau; a, b)) = H_m(\Gamma(\tau)) \circ i_n = l(n)R(n)H_m(\Gamma(\tau)) = l(n)H_m(\Gamma(\tau)),$$

where the last equality follows from the R-invariance of $H_m(\Gamma(\tau))$. It is shown in **[1]** that, for any $g \in N$, $l(g)\Theta = \Theta$. It follows then that $\Theta_m(\Gamma(\tau; a, b)) = l(n)\Theta_m(\Gamma(\tau))$.

We clearly have then that F is a function in $\Theta_m(\Gamma(\tau; a, b))$ such that the zero set in $D(\tau)$ of M_mF is $S = \{z_1, z_2, \ldots, z_m\}$ if and only if $l(n^{-1})F$ is a function in $\Theta_m(\Gamma(\tau))$ such that the zero set in $D(\tau)$ of $(l(n^{-1})F)$ is the set $\{-(a + b\tau) + z_1, -(a + b\tau) + z_2, \ldots, -(a + b\tau) + z_m\}$ which is true, using our previous case, precisely when $-m(a + b\tau) + \sum_{i=1}^{m} z_i$ is congruent modulo $L(\tau)$ to

$$\begin{cases} 0, & m \text{ even}, \\ \frac{1}{2} + \tau/2, & m \text{ odd}, \end{cases}$$

or equivalently, if and only if $\Sigma_{i=1}^{m} z_i$ is congruent modulo $L(\tau)$ to

$$\begin{cases} m(a+b\tau), & m \text{ even}, \\ m(a+b\tau)+\frac{1}{2}+\tau/2, & m \text{ odd}. \end{cases}$$

This proves the theorem in this case.

Case III. Suppose that Γ is an arbitrary subgroup of N with generators (a_1, a_2, a_3) and (b_1, b_2, b_3), and let D be the fundamental parallelogram of the lattice $L = \Pi(\Gamma)$. A simple computation shows that the automorphism

$$\zeta = \begin{bmatrix} u & v & 0 \\ -v & u & 0 \\ 0 & 0 & u^2+v^2 \end{bmatrix},$$

with $u = a_1/(a_1^2 + a_2^2)$ and $v = a_2/(a_1^2 + a_2^2)$, is such that $\zeta(\Gamma) = \Gamma(\tau; a, b)$, for $\tau = \alpha + i\beta$, $\alpha = (a_1b_1 + a_2b_2)/(a_1^2 + a_2^2)$, $\beta = (b_2a_1 - a_2b_1)/(a_1^2 + a_2^2)$, $a = -b_3/(b_2a_1 - a_2b_1)$, and $b = a_3/(b_2a_1 - a_2b_1)$. Thus, $H_m(\Gamma) = H_m(\Gamma(\tau; a, b)) \circ \zeta$. Moreover, a simple application of the chain rule shows that $\Theta \circ \zeta = \Theta$. Thus,

$$\Theta_m(\Gamma) = \Theta_m(\Gamma(\tau; a, b)) \circ \zeta.$$

Let $S = \{z_1, z_2, \dots, z_m\} \subseteq D$. Then $F \in \Theta_m(\Gamma)$ is such that S is the zero set in D of the analytic function M_mF if and only if the nil-theta function $F \circ \zeta^{-1} \in \Theta_m(\Gamma(\tau; a, b))$ is such that $M_m(F \circ \zeta^{-1})$ has the zero set $\{\zeta(z_1), \zeta(z_2), \dots, \zeta(z_m)\}$ in $D(\tau)$. By our previous case, this happens if and only if $\Sigma_{i=1}^{m} \zeta(z_i) = \zeta(\Sigma_{i=1}^{m} z_i)$ is congruent modulo $L(\tau)$ to

$$\begin{cases} m(a+b\tau), & m \text{ even}, \\ m(a+b\tau)+\frac{1}{2}+\tau/2, & m \text{ odd}, \end{cases}$$

or equivalently, if and only if $\Sigma_{i=1}^{m} z_i$ is congruent modulo L to

$$\begin{cases} \zeta^{-1}(m(a+b\tau)) = mv_0, & m \text{ even}, \\ \zeta^{-1}(m(a+b\tau)) + \zeta^{-1}\left(\frac{1}{2}+\frac{\tau}{2}\right) = mv_0 + \frac{1}{2}(a_1+b_1) + \frac{i}{2}(a_2+b_2), & m \text{ odd}, \end{cases}$$

and the theorem is proven.

We will end this section with a corollary of Theorem 1.5. We have already seen in Corollary 1.3 that if Γ is a lifted subgroup of N and $F, G \in \Theta_m(\Gamma)$, for $m > 0$, then F/G is a meromorphic function on the torus $\mathbf{C}/L$, where $L = \Pi(\Gamma)$. The converse of this result is true as well. It is classically known that if f is a meromorphic function on a torus $\mathbf{C}/L$, or equivalently, if f is an elliptic function with respect to a lattice L, then we can write f as a quotient of theta functions; i.e., the field $\mathcal{F}(\mathbf{C}/L)$ of meromorphic functions on $\mathbf{C}/L$ can be written as follows:

$$\mathcal{F}(\mathbf{C}/L) = \sum_{\Pi(\Gamma)=L} Q(\Theta_m(\Gamma)),$$

where we sum over all two-generator subgroups Γ such that $\Pi(\Gamma) = L$, and where Q denotes the quotients of elements of $\Theta_m(\Gamma)$. That is, various "types" of theta functions may arise. In **[11]** Tolimieri uses the methods of nonabelian harmonic analysis to refine this result. He proves that we can, in fact, locate all of the nil-theta functions necessary to form $\mathfrak{F}(\mathbf{C}/L)$ on a *single* nil-manifold N/Γ, or equivalently, by using only theta functions of a single type. Tolimieri proves there that if Γ is any *one* fixed subgroup of N such that $\Pi(\Gamma) = L$, then $\mathfrak{F}(\mathbf{C}/L) = \sum_{m>0} Q(\Theta_m(\Gamma))$; i.e., any $f \in \mathfrak{F}(\mathbf{C}/L)$ can be written in the form

$$f = \sum_{i \in I} (F_i/G_i), \qquad f_i,\, G_i \in \Theta_i(\Gamma),$$

where I is a finite set of positive integers. Equivalently (taking a common denominator in this expression for f), we have that $f = F/G$, for $F, G \in \Theta_m(\Gamma)$, for some positive integer m. We note that m may be arbitrarily large. We now present another proof of Tolimieri's result, which in addition yields a bound for the positive integer m.

COROLLARY 1.6. *Let L be a lattice in the complex plane* (*with two linearly independent generators*), *and let $f \in \mathfrak{F}(\mathbf{C}/L)$. Then for any fixed two-generator subgroup $\Gamma = \operatorname{grp}\{(a_1, a_2, a_3), (b_1, b_2, b_3)\}$ of N such that $\Pi(\Gamma) = L$, we may choose a positive integer m and functions $F, G \in \Theta_m(\Gamma)$ such that $f = F/G$. Moreover, if n is the number of zeroes* (*or poles*) *of f in the fundamental parallelogram D of L, then $n \leqslant m \leqslant n + 1$.*

PROOF. Let $S_1 = \{z_1, z_2, \ldots, z_n\}$ and $S_2 = \{w_1, w_2, \ldots, w_n\}$ denote the sets of zeroes and poles, respectively, of f in D. It is a well-known fact that $\sum_{i=1}^n z_i$ is congruent modulo L to $\sum_{i=1}^n w_i$. Let z' denote the unique element in D congruent modulo L to $\sum_{i=1}^n z_i$.

If z' is congruent modulo L to

$$\begin{cases} nv_0, & n \text{ even}, \\ nv_0 + \frac{1}{2}(a_1 + b_1) + (i/2)(a_2 + b_2), & n \text{ odd}, \end{cases}$$

where v_0 is as in Theorem 1.5, then we let F and G be the functions (unique up to constant multiple) in $\Theta_m(\Gamma)$, as in Theorem 1.5, with zero sets S_1 and S_2 in D, respectively. Clearly, we may choose these constants so that $f = F/G$.

If z' is not congruent modulo L to the above elements, we let μ be the unique element of D such that $z' + \mu$ is congruent modulo L to

$$\begin{cases} (n+1)v_0, & n \text{ odd}, \\ (n+1)v_0 + \frac{1}{2}(a_1 + b_1) + (i/2)(a_2 + b_2), & n \text{ even}. \end{cases}$$

Let $S_1' = \{z_1, z_2, \ldots, z_n, \mu\}$ and $S_2' = \{w_1, w_2, \ldots, w_n, \mu\}$. By Theorem 1.5, we may choose functions $F, G \in \Theta_{n+1}(\Gamma)$ with zero sets S_1' and S_2', respectively, and we can write $f = F/G$ as desired.

2. The index and symmetry. In this section we present our main results relating the indexing on the irreducible R-invariant subspaces of $H_m(\Gamma)$, for m a positive integer, and the symmetry properties of the zero sets of the nil-theta functions lying in these subspaces.

Let $\Gamma = \operatorname{grp}\{\gamma_1, \gamma_2\}$, and let $L = \Pi(\Gamma)$ be the corresponding lattice in $\mathbf{R}^2$. We first recall the definition, given in [3], of the indexing on the set Ω_m of irreducible closed R-invariant subspaces of $H_m(\Gamma)$, m a positive integer.

Let

$$\Lambda[m] = \mathrm{grp}\{m^{-1}\gamma_1, m^{-1}\gamma_2\}, \qquad \Gamma[m] = \mathrm{grp}\{\gamma_1, \gamma_2, \beta/m\},$$

and $G_m(\Gamma) = \Lambda[m]/\Gamma[m]$–the space of right cosets. Let $Z(G_m(\Gamma))$ denote the center of $G_m(\Gamma)$. Then $G_m(\Gamma)/Z(G_m(\Gamma)) = (1/m)L/L$ is a discrete group of order m^2.

Since $\Lambda[m]$ normalizes $\Gamma[m]$, the map from $\Lambda[m]$ to operators on $H_m(\Gamma)$, $L\colon \lambda \mapsto L(\lambda)$, defines a representation of $\Lambda[m]$. Since $\Gamma[m]$ is contained in the kernel of this map, we think of L as a representation of $G_m(\Gamma)$ on $H_m(\Gamma)$.

DEFINITION 2.1. Let $\mathfrak{H}$ be an irreducible closed R-invariant subspace of $H_m(\Gamma)$, $m > 0$. The *left stabilizer* of $\mathfrak{H}$ is the group

$$\tilde{\Delta}_\Gamma(\mathfrak{H}) = \{\lambda \in G_m(\Gamma) | L(\lambda)\mathfrak{H} = \mathfrak{H}\}.$$

The following facts are proven in **[1]** or **[3]**:

Let m be a positive integer.

(1) $\tilde{\Delta}_\Gamma(\mathfrak{H})$ is an abelian subgroup of $G_m(\Gamma)$ containing $Z(G_m(\Gamma))$.

(2) There exists a character $\tilde{\chi}$ of $\tilde{\Delta}_\Gamma(\mathfrak{H})$ such that $L(\lambda)F = \tilde{\chi}(\lambda)F$, for all $F \in \mathfrak{H}$ and $\lambda \in \tilde{\Delta}_\Gamma(\mathfrak{H})$, and such that $\tilde{\chi}$ restricted to $Z(G_m(\Gamma))$ is the character $\eta\colon \Gamma[m](0, 0, a\beta/m^2) \mapsto \exp((2\pi i a\beta)/m)$.

(3) If $\tilde{\Delta}$ is any abelian subgroup of $G_m(\Gamma)$, then the order of $\tilde{\Delta}$ divides m^2, and $\tilde{\Delta}$ is maximal abelian if and only if its order equals m^2.

(4) If $\tilde{\Delta}$ is any abelian subgroup of $G_m(\Gamma)$ that contains $Z(G_m(\Gamma))$, and $\tilde{\chi}$ is any character of $\tilde{\Delta}$ restricting to η on $Z(G_m(\Gamma))$, set

$$H(m, \tilde{\chi}) = \{F \in H_m(\Gamma) | L(\lambda)F = \tilde{\chi}(\lambda)F \text{ for all } \lambda \in \tilde{\Delta}\}.$$

Then

(a) $H(m, \tilde{\chi})$ is R-invariant.

(b) $H(m, \tilde{\chi})$ is of multiplicity $m^2/|\tilde{\Delta}|$, where $|G|$ denotes the order of the group G.

(c) $H_m(\Gamma) = \bigoplus \Sigma_{\tilde{\chi}} H(m, \tilde{\chi})$, the sum being over all characters $\tilde{\chi}$ of $\tilde{\Delta}$ restricting to η on $Z(G_m(\Gamma))$.

We see thus that $H(m, \tilde{\chi})$ is irreducible precisely when $\tilde{\Delta}$ is maximal abelian. In this case, $\tilde{\Delta}$ is the left stabilizer of $H(m, \tilde{\chi})$. If $\tilde{\Delta}$ is not maximal abelian, then any irreducible subspace $\mathfrak{H}$ of $H_m(\Gamma)$ having $\tilde{\Delta}$ as its left stabilizer will be properly contained in $H(m, \tilde{\chi})$ for some character $\tilde{\chi}$ on $\tilde{\Delta}$.

DEFINITION 2.2. Let $\mathfrak{H}$ be an irreducible closed R-invariant subspace of $H_m(\Gamma)$, $m > 0$. The *index of* $\mathfrak{H}$ is defined as follows:

$$\mathrm{ind}_\Gamma(\mathfrak{H}) = m^2/|\Delta_\Gamma(\mathfrak{H})|.$$

DEFINITION 2.3. $\mathfrak{H}$ is said to be a distinguished subspace of $H_m(\Gamma)$ if $\mathrm{ind}_\Gamma(\mathfrak{H}) = 1$.

We see that $\mathfrak{H}$ is distinguished precisely when $\tilde{\Delta}_\Gamma(\mathfrak{H})$ is a maximal abelian subgroup of $G_m(\Gamma)$, in which case the decomposition in fact (4) above yields a decomposition of $H_m(\Gamma)$ into distinguished subspaces.

The following characterization of the distinguished subspaces given in **[2]** will be useful to us:

LEMMA 2.4. *$\mathfrak{H}$ is a distinguished subspace of $H_m(\Gamma)$ if and only if $\mathfrak{H} = H_1(\Gamma_1)$, for some two-generator subgroup Γ_1 of N.*

The condition $H_1(\Gamma_1) \subseteq H_m(\Gamma)$ implies, moreover (see **[11]**), that $\Gamma \subseteq \Gamma_1$ and $\beta(\Gamma) = m\beta(\Gamma_1)$.

The following lemma will be useful to us.

LEMMA 2.5. *Let* Γ *and* Γ' *be two-generator subgroups of* N *and suppose that* $\mathfrak{I} \subseteq H_m(\Gamma) \subseteq H_n(\Gamma')$ *is an irreducible closed* R*-invariant space,* $m, n \in \mathbf{Z}$. *Then,* $\operatorname{ind}_\Gamma(\mathfrak{I}) = \operatorname{ind}_{\Gamma'}(\mathfrak{I})$, *so that we may denote this number simply by* $\operatorname{ind}(\mathfrak{I})$.

PROOF. It is proven in **[2]** that $H_m(\Gamma) \subseteq H_n(\Gamma')$ precisely when $n = sm$, $\Gamma \supseteq \Gamma'$, and $|\Pi(\Gamma)/\Pi(\Gamma')| = s$, for some positive integer s. It is not difficult to see that $\tilde{\Delta}_{\Gamma'}(\mathfrak{I}) = \{\mu + \tilde{\Delta}_\Gamma(\mathfrak{I}) | \mu \in \Pi(\Gamma)/\Pi(\Gamma')\}$, so that

$$\operatorname{ind}_{\Gamma'}(\mathfrak{I}) = \frac{|sm|}{|\tilde{\Delta}_{\Gamma'}(\mathfrak{I})|} = \frac{s|m|}{s|\tilde{\Delta}_\Gamma(\mathfrak{I})|} = \frac{|m|}{|\tilde{\Delta}_\Gamma(\mathfrak{I})|} = \operatorname{ind}_\Gamma(\mathfrak{I}).$$

It will be convenient for us in what follows to work modulo $Z(G_m(\Gamma))$. We therefore define $\Delta_\Gamma(\mathfrak{I}) = \tilde{\Delta}_\Gamma(\mathfrak{I})/Z(G_m(\Gamma))$. The following facts are immediate.

(5) $\Delta_\Gamma(\mathfrak{I})$ is a subgroup of $(1/m)L/L$ of order dividing m.

(6) Let χ denote the character on $\Delta_\Gamma(\mathfrak{I})$ such that $\tilde{\chi} = \chi\eta$, i.e., such that $\tilde{\chi}(\Gamma[m](a, b, c)) = \chi(\Gamma[m](a, b, 0))\eta(\Gamma[m](0, 0, c))$. Then

$$\Delta_\Gamma(\mathfrak{I}) = \left\{ \mu \in \frac{1}{m} L/L \,|\, L(\mu)F = \chi(\mu)F \text{ for all } F \in \mathfrak{I} \right\}.$$

(7) $\operatorname{ind}_\Gamma(\mathfrak{I}) = m/|\Delta_\Gamma(\mathfrak{I})|$.

We are now ready to discuss the main results of this section. It is proven in **[1]** that every irreducible closed R-invariant subspace of $H_m(\Gamma)$, $m > 0$, contains a unique (up to constant multiple) element of Θ. Conversely, if $\theta \in \Theta_m(\Gamma)$, $m > 0$, then θ lies in a unique irreducible subspace of $H_m(\Gamma)$ which we denote by $\mathfrak{I}(\theta)$. In fact, it is not difficult to show that $\mathfrak{I}(\theta)$ is the closure (in the topology arising from the L^2 norm) of the linear span of $\{R(g)\theta | g \in N\}$.

Suppose then that $\theta \in \Theta_m(\Gamma)$, $m > 0$. Let D be the fundamental parallelogram of the lattice $L = \Pi(\Gamma)$, and let S be the set of zeroes of the analytic function $\theta' = M_m\theta$ that are contained in D.

DEFINITION 2.6. The *symmetry group* of θ is the group

$$\Delta_\Gamma(\theta) = \left\{ \mu \in \frac{1}{m} L/L \,|\, -\mu + S = S \right\}.$$

THEOREM 2.7. *Let* $\theta \in \Theta_m(\Gamma)$, $m > 0$, *and let* S *be the set of zeroes of* $\theta' = M_m\theta$ *in* D. *Then* $\Delta_\Gamma(\theta) = \Delta_\Gamma(\mathfrak{I}(\theta))$, *i.e., the symmetry group of the nil-theta function* θ *is precisely the left stabilizer of the unique irreducible subspace of* $H_m(\Gamma)$ *containing* θ.

Before we prove this theorem, let us mention that what it says is that the index of any irreducible R-invariant subspace $\mathfrak{I}$ of $H_m(\Gamma)$ is completely determined by the symmetry properties of the zero set of a single special function lying in this space. Moreover, while the index of $\mathfrak{I}$, as defined in [3], is a purely algebraic object, Theorem 2.7 yields an analytic-geometric characterization of the index. This theorem also yields another way in which the distinguished subspaces of $H_m(\Gamma)$ are nicer than those of higher index–the zero set of a nil-theta function lying in such a subspace is as symmetric as possible. In fact, we can clearly write the zero set S of such a nil-theta function as follows:

$$S = \{z_1 + \gamma | \gamma \in \Delta_\Gamma(\theta)\} = z_1 + \Delta_\Gamma(\theta)$$

for some $z_1 \in D$.

DEFINITION 2.8. Let $\theta \in \Theta_m(\Gamma)$, $m > 0$. We define the *index of* θ:

$$\operatorname{ind}(\theta) = m/|\Delta_\Gamma(\theta)|.$$

Theorem 2.7 says, then, that $\operatorname{ind}(\theta) = \operatorname{ind}(\mathfrak{J}(\theta))$.

PROOF OF THEOREM 2.7. Let $\tilde{\Delta}_\Gamma(\theta) = \{\lambda \in G_m(\Gamma) | \lambda^{-1}\tilde{S} = \tilde{S}\}$, where $\tilde{S} = S \cdot Z(N)$ is the zero set of θ.

(a) We first show that $\Delta_\Gamma(\mathfrak{J}(\theta)) \subseteq \Delta_\Gamma(\theta)$.

Let $\lambda \in \tilde{\Delta}_\Gamma(\mathfrak{J}(\theta))$. Then, for all $g \in N$, $L(\lambda)\theta(g) = \theta(\lambda^{-1}g) = \tilde{\chi}(\lambda)\theta(g)$, for some character $\tilde{\chi}$ on $\tilde{\Delta}_\Gamma(\mathfrak{J}(\theta))$, so that $g \in \tilde{S}$ if and only if $\lambda^{-1}g \in \tilde{S}$. Thus, $\lambda \in \tilde{\Delta}_\Gamma(\theta)$ and $\tilde{\Delta}_\Gamma(\mathfrak{J}(\theta)) \subseteq \tilde{\Delta}_\Gamma(\theta)$. Moreover,

$$\Delta_\Gamma(\mathfrak{J}(\theta)) = \tilde{\Delta}_\Gamma(\mathfrak{J}(\theta))/Z(G_m(\Gamma)) \subseteq \tilde{\Delta}_\Gamma(\theta)/Z(G_m(\Gamma)) = \Delta_\Gamma(\theta).$$

(b) We show, conversely, that $\Delta_\Gamma(\theta) \subseteq \Delta_\Gamma(\mathfrak{J}(\theta))$.

Let $\lambda \in \tilde{\Delta}_\Gamma(\theta)$. Then $M_m\theta$ and $M_m(l(\lambda)\theta)$ are entire theta functions of the same type with respect to the lattice $L = \Pi(\Gamma)$, with the same zeroes. Therefore, $M_m(l(\lambda)\theta)$ is a scalar multiple of $M_m\theta$, and clearly $l(\lambda)\theta \in \mathfrak{J}(\theta)$. It follows easily that left multiplication by λ preserves the least right-invariant space containing θ; that is, $l(\lambda)\mathfrak{J}(\theta) = \mathfrak{J}(\theta)$ and thus $\lambda \in \tilde{\Delta}_\Gamma(\mathfrak{J}(\Theta))$ and the theorem is proven.

We have seen, then, that the stabilizer and index of an irreducible subspace $\mathfrak{J}$ of $H_m(\Gamma)$, $m > 0$, measure the symmetry properties of the zero set of the unique nil-theta function θ lying in $\mathfrak{J}$. We would now like to discuss to what extent the index and stabilizer of $\mathfrak{J}$ determine the actual locations of the zeroes of θ.

The following lemma is a simple consequence of the Fundamental Theorem of Abelian Groups.

LEMMA 2.9. *Let L be a lattice in $\mathbf{R}^2$ with two linearly independent generators γ_1 and γ_2, let m be a positive integer, and let k be a positive divisor of m. Suppose that Δ is a subgroup of $(1/m)L/L$ of order $k^{-1}m$. Then Δ has generators β_1 and β_2 such that*

(a) $\beta_1 = m^{-1}k_{11}\gamma_1$, $\beta_2 = m^{-1}k_{21}\gamma_1 + m^{-1}k_{22}\gamma_2$, *where* $k_{ij} \in \{0, 1, 2, \dots, m-1\}$, $i, j = 1, 2$.

(b) $k_{11}k_{22} = km$.

(c) *k divides k_{ij} for all i, j.*

(d) *k_{ii} divides m, $i = 1, 2$.*

Suppose now that $\Gamma = \operatorname{grp}\{(a_1, a_2, a_3), (b_1, b_2, b_3)\}$ is a subgroup of N, and let $\theta \in \Theta_m(\Gamma)$, $m > 0$, be of index k. Then $\Delta_\Gamma(\theta)$ is a subgroup of $(1/m)L/L$ of order m/k, and therefore has generators β_1 and β_2 as in Lemma 2.9. Let $L = \Pi(\Gamma)$, let L' be the lattice in $\mathbf{R}^2$ spanned by β_1 and β_2, and let D and D' be the fundamental parallelograms of L and L', respectively. Then the zero set $S = \{z_1, z_2, \dots, z_m\}$ of the entire theta function $\theta' = M_m\theta$, contained in D, can be written in the form

$$S = \bigcup_{\delta \in L'/L} \{\delta + S'\}$$

where $S' = \{z_1, z_2, \ldots, z_k\}$ is the set of those zeroes of θ' that lie in D'. Thus, to describe the set S, it suffices to describe the smaller set S'.

Let w_0 be the unique element in D congruent modulo L to

$$\begin{cases} mv_0, & m \text{ even}, \\ mv_0 + \frac{1}{2}(a_1 + b_1) + (i/2)(a_2 + b_2), & m \text{ odd}, \end{cases}$$

where v_0 is as in Theorem 1.5.

By Theorem 1.6, there exists a fixed element $l \in L$ such that $\sum_{i=1}^{m} z_i = w_0 + l$.

The following theorem shows to what extent the stabilizer of θ determines the location of the zeroes of θ.

THEOREM 2.10. *Under the conditions described above, $\sum_{i+1}^{k} z_i$ is congruent modulo L' to*

$$m^{-1}kl + kv_0 + \begin{cases} 0, & k \text{ even}, \\ (\beta_1 + \beta_2)/2, & k \text{ odd}, k_{11} \text{ and } k_{22} \text{ even}, \\ \beta_2/2, & k \text{ and } k_{22} \text{ odd}, k_{11} \text{ even}, \\ \beta_1/2, & k \text{ and } k_{11} \text{ odd}, k_{22} \text{ even}, \\ (2m)^{-1}(k_{22}\beta_1 + k_{11}\beta_2) - (2m)^{-1}k_{21}\beta_1, & k, k_{11} \text{ and } k_{22} \text{ odd}. \end{cases}$$

PROOF. As mentioned above, we may write $S = \bigcup_{\delta \in \Delta_\Gamma(\theta)} \{\delta + S'\}$ where $S = \{z_1, z_2, \ldots, z_m\}$ and $S' = \{z_1, z_2, \ldots, z_k\}$ are the zero sets of the function $\theta' = M_m, \theta$ in D and D', respectively. Now $\Delta_\Gamma(\theta) = \{a\beta_1 + b\beta_2 | a \in \{0, 1, \ldots, k^{-1}k_{22} - 1\}, b \in \{0, 1, \ldots, k^{-1}k_{11} - 1\}$, so that summing all the zeroes in S, we have that

$$\sum_{i=1}^{m} z_i = k^{-1}m \sum_{i=1}^{k} z_i - (2)^{-1}m(\beta_1 + \beta_2) + (2)^{-1}m\left(k^{-1}k_{22}\beta_1 + k^{-1}k_{11}\beta_2\right),$$

or equivalently

$$\sum_{i=1}^{k} z_i = m^{-1}kw_0 + m^{-1}kl + (2)^{-1}k(\beta_1 + \beta_2) - (2)^{-1}(k_{22}\beta_1 + k_{11}\beta_2).$$

Reducing modulo L' yields the desired results.

COROLLARY 2.11. *Under the hypotheses of Theorem 2.10, if moreover θ is of index one, then we may write the zero set S of θ' inside D as follows*:

$$S = \bigcup_{\delta \in \Delta_\Gamma(\theta)} \{\delta + z_1\} = \Delta_\Gamma(\theta) + z_1 = L_1/L + z_1,$$

where z_1 is the unique zero of this function inside D'. Moreover, z_1 is congruent modulo L' to

$$m^{-1}l + v_0 + \begin{cases} (\beta_1 + \beta_2)/2, & k_{11} \text{ and } k_{22} \text{ even}, \\ \beta_2/2, & k_{11} \text{ even}, k_{22} \text{ odd}, \\ \beta_1/2, & k_{11} \text{ odd}, k_{22} \text{ even}, \\ (2m)^{-1}(k_{22}\beta_1 + k_{11}\beta_2) - (2m)^{-1}k_{21}\beta_1, & k_{11} \text{ and } k_{22} \text{ odd}, \end{cases}$$

where l and v_0 are as in the previous theorem.

Let us now interpret these results. Let Γ be a two-generator subgroup of N, and $\Pi(\Gamma) = L$. Suppose that Δ is a subgroup of $(1/m)L/L$ of order m/k. We would like to characterize all of the subspaces $\mathfrak{J} \in \Omega_m$ that have Δ as their stabilizer by describing the zero sets of the nil-theta function θ lying in such a subspace. What we have shown is that if $S = \{z_1, z_2, \ldots, z_m\}$ is the set of zeroes of $M_m\theta$ in D, then we can partition the set S as follows: $S = S_1 \cup S_2 \cup \cdots \cup S_{m/k}$, where

(a) $|S_i| = k$, $i = 1, 2, \ldots, m/k$,

(b) $S_i = \delta_i + S_1$, $\delta_i \in \Delta$, $i = 1, 2, \ldots, m/k$,

(c) $S_1 = \{z_1, z_2, \ldots, z_k\}$ is the zero set of $M_m\theta$ in the fundamental parallelogram D' of the lattice L' spanned by Δ.

Moreover, it is not possible to further partition all of the S_i, $i = 1, 2, \ldots, m/k$, into sets with these same properties.

Summing all the zeroes in S_i yields $\Sigma_{z \in S_i} z = \Sigma_{j=1}^{k} z_j + k\delta_i$ or equivalently,

$$\lambda_i = k^{-1} \sum_{z \in S_i} z = k^{-1} \sum_{j=1}^{k} z_j + \delta_i, \qquad i = 1, 2, \ldots, m/k,$$

where λ_i is the center of gravity of the zeroes of $M_m\Theta$ lying in the translate $\delta_i + D'$ of D', $i = 1, 2, \ldots, m/k$. Moreover, the set $\{\lambda_1, \lambda_2, \ldots, \lambda_{m/k}\}$ forms a group isomorphic to Δ. By Theorem 2.10, we know exactly what these λ_i's are, up to the addition of the element $m^{-1}kl \in (k/m)L$. We now discuss this element.

In Theorem 2.10, because we have reduced $\Sigma_{i=1}^{k} z_i$ modulo L', the distinct possibilities for $(k/m)l$ correspond to the elements of the group $\hat{\Delta} = (k/m)L/L'$, which is clearly a group of order m/k. Thus, there are m/k possibilities for the sum of the k zeroes of $M_m\theta$ inside D'.

We now recall the decomposition of $H_m(\Gamma)$ described in fact (4) of this section: $H_m(\Gamma) = \bigoplus \Sigma_{\tilde{\chi}} H(m, \tilde{\chi})$, where χ ranges over a certain finite group of characters of Δ. It is not difficult to show (see [2]) that this character group is, in fact, isomorphic to our group $\hat{\Delta}$. Thus each one of the distinct possibilities for $\Sigma_{j=1}^{k} z_i$ in D' corresponds to a different one of the subspaces $H(m, \tilde{\chi})$ in the above decomposition of $H_m(\Gamma)$. It is clear that the irreducible subspace $\mathfrak{J}(\theta)$ must be contained in one of the subspaces $H(m, \tilde{\chi})$. We have then that to each possible value of $\nu_\mu = \Sigma_{i=1}^{k} z_i$ in D', there corresponds a unique character χ_μ of Δ such that $\mathfrak{J}(\theta) \subseteq H(m, \tilde{\chi}_\mu)$, $\mu = 1, 2, \ldots, m/k$.

Suppose that $\Sigma_{i=1}^{k} z_i = \nu_\mu$. Applying Theorem 1.5 to the set $\{z_1, z_2, \ldots, z_k\}$, there exists a two-generator subgroup Γ_μ with $\Pi(\Gamma_\mu) = L'$, such that $\theta \in \Theta_k(\Gamma_\mu)$. Moreover, there does not exist an integer $j < k$ such that $\theta \in \Theta_j(\Gamma'_\mu)$ for any two-generator subgroup Γ'_μ of N. Thus, if $\mathfrak{J}$ is a subspace of $H_m(\Gamma)$ of index k, then the unique nil-theta function θ in $\mathfrak{J}$ is such that $\theta \in \Theta_k(\Gamma_\mu)$ for some two-generator subgroup Γ_μ of N, and k is the smallest integer for which this is true.

Let us note now the difference between the distinguished nil-theta functions (i.e., those nil-theta functions lying in a distinguished subspace) and those of higher index. When we are given the stabilizer of the distinguished nil-theta function, we know precisely where each of its zeroes lies, up to translation by an element of $\hat{\Delta}$. In particular, these zeroes clearly occur at points whose coordinates are rational expressions in the coordinates of the generators of the subgroup Γ. For a nil-theta function θ of index $k > 1$, however, we cannot locate the zeroes precisely, nor can we even insure that they are rationally related to the coordinates of the generators of Γ. We can only locate the m/k possible values of the centers of gravity λ_i of the sets S_i, $i = 1, 2, \ldots, m/k$, occurring in the partition of the zero set S of θ described above. These λ_i are rational, but the individual zeroes need not be. Thus the larger the index of θ is, the larger the sets S_i are, and the less information we have, in some sense, about the individual zeroes of θ.

In summary, what we have seen in this section is that the stabilizer and hence also the index of an irreducible R-invariant subspace $\mathcal{J}$ of $H_m(\Gamma)$, $m > 0$, depends entirely on the symmetry properties of a single special function in that subspace. Moreover, given a subgroup Δ of $(1/m)L/L$, where $L = \Pi(\Gamma)$, we have completely described the zero sets of those nil-theta functions θ having Δ as their symmetry groups. If $|\Delta| = m/k$, then sums of k zeroes of Δ can be precisely described in terms of the generators of Γ. For the distinguished nil-theta function, the locations of the individual zeroes are thus determined.

We have found, then, two more ways in which the distinguished subspaces of $H_m(\Gamma)$, $m > 0$, are particularly nice: Firstly, the zero set of the nil-theta function lying in this subspace is as symmetrical as possible, while the zero sets of nil-theta functions of higher index are necessarily less symmetrical. Secondly, the zeroes of the distinguished nil-theta functions are located at points whose coordinates are rationally related to the coordinates of the generators of Γ, while the zeroes of the nil-theta functions of higher index need not have this property.

3. Multiplication theory. In this section, for convenience of notation, we restrict ourselves to the subgroup $\Gamma_0 = \operatorname{grp}\{(1, 0, 0), (0, 1, 0)\}$ of N. Let $L = \Pi(\Gamma_0)$, and let D be the fundamental parallelogram of L. For $F \in \Theta_m(\Gamma_0)$, we set $\Delta(F) = \Delta_\Gamma(F)$.

It is not difficult to see that if $F \in \Theta_m(\Gamma_0)$ and $G \in \Theta_n(\Gamma_0)$, then $FG \in \Theta_{m+n}(\Gamma_0)$. Let $\mathcal{Q}$ denote the algebra

$$\mathcal{Q} = \bigoplus \sum_{m>0} \Theta_m(\Gamma_0).$$

The isomorphisms

$$M_m\colon \Theta_m(\Gamma_0) \to \Theta_m\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right](z, i), \qquad m > 0,$$

extend to an algebra isomorphism

$$M\colon \mathcal{Q} \to \bigoplus \sum_{m>0} \Theta_m\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right](z, i)$$

and hence $\mathcal{Q}$ has no zero divisors. The precise structure of $\mathcal{Q}$ is discussed in **[5]**.

We would like to consider the behavior of the index under multiplication of nil-theta functions; i.e., how does $\operatorname{ind}(FG)$ relate to $\operatorname{ind}(F)$ and $\operatorname{ind}(G)$. The simplest case will be discussed in Theorem 3.2.

Let F be any complex valued function and s any positive integer. By F^s we will mean the complex valued function defined by $F^s(g) = (F(g))^s$. For $F \in \Theta_m(\Gamma_0)$ we let $F' = M_m F$.

LEMMA 3.1. *Let $F \in \Theta_m(\Gamma_0)$ and $G \in \Theta_n(\Gamma_0)$ be of index j and k respectively, where m and n are positive integers. Then for any positive integer s*

$$\Delta((FG)^s) = \Delta(FG).$$

PROOF. Let S denote the set of zeroes of the function $M_{m+n}FG$ that are contained in D. Then clearly the zero set of the function $M_{s(m+n)}(FG)^s$ in D is S^s. It follows easily that

$$\begin{aligned}\Delta(FG) &= \{\mu | -\mu + S = S\} \cap \frac{1}{m+n} L \\ &= \{\mu | -\mu + S^s = S^s\} \cap \frac{1}{s(m+n)} L = \Delta((FG)^s)\end{aligned}$$

and the lemma is proved.

THEOREM 3.2. *Let $F \in \Theta_m(\Gamma_0)$ have index j. Then for any positive integer s*

$$\operatorname{ind}(F^s) = sj.$$

PROOF.

$$\operatorname{ind}(F^s) = \frac{sm}{|\Delta(F^s)|} = \frac{sm}{|\Delta(F)|} = \frac{sm}{m/j} = sj.$$

We see thus the nice behavior of the index when nil-theta functions are raised to a power. We now consider the general multiplication theory.

Let $F \in \Theta_m(\Gamma_0)$ and $G \in \Theta_n(\Gamma_0)$, $m, n > 0$. It follows easily from Theorem 2.7 that $\Delta(F) \cap \Delta(G) \subseteq \Delta(FG)$. Moreover, if F and G are of index j and k, respectively, we may write

$$\operatorname{ind}(FG) = (p/w)j + (q/w)k,$$

where $p = |\Delta(F)/(\Delta(F) \cap \Delta(G))|$, $q = |\Delta(G)/(\Delta(F) \cap \Delta(G))|$, and $w = |\Delta(FG)/(\Delta(F) \cap \Delta(G))|$.

Let us now see what can be said about the integers p, q, and w. Let L_1 and L_2 denote the lattices in $\mathbf{R}^2$ spanned by the generators of $\Delta(F)$ and $\Delta(G)$, respectively. Choosing generators as in Lemma 2.9, we have that

$$L_1 = \operatorname{grp}\{(m^{-1}j_{11}, 0), (m^{-1}j_{21}, m^{-1}j_{22})\}, \qquad L_2 = \operatorname{grp}\{(n^{-1}k_{11}, 0), (n^{-1}k_{21}, n^{-1}k_{22})\}$$

where $j_{11}j_{22} = jm$, $k_{11}k_{22} = kn$, j divides j_{st} and k divides k_{st}, for all s, t. A rather lengthy but straightforward computation (see [6]) shows that $L_1 \cap L_2$ has generators of the form $((d_2)^{-1}, 0)$, $(*, (d_1)^{-1}A)$ where d_i is the greatest common divisor of the integers $j^{-1}j_{ii}$ and $k^{-1}k_{ii}$, $i = 1, 2$, and A is the smallest positive divisor of the integer d_1d_2, less than or equal to d_1, such that d_1d_2/A divides $Q = (j_{22}k_{21} - k_{22}j_{21})/(kj)$. Thus $|\Delta(F) \cap \Delta(G)| = (d_1d_2)/A$, $p = (mA)/(jd_1d_2)$ and $q = (nA)/(kd_1d_2)$. Thus, the integers p and q are computed easily, and are determined by the generators of $\Delta(F)$ and $\Delta(G)$.

Consider now the integer $w = |\Delta(FG)/(\Delta(F) \cap \Delta(G))|$. Now w is the order of the set

$$W = \left\{ \mu \in \frac{1}{m+n} L/(L_1 \cap L_2) | -\mu + (S_1 \cup S_2) = (S_1 \cup S_2) \right\}$$

where S_1 and S_2 are the zero sets in D of the functions $F' = M_m F$ and $G' = M_n G$, respectively. Thus w measures the interaction between the zero sets of F and G. The integer w is the number of symmetries of the set $S_1 \cup S_2$ that are not symmetries of both S_1 and S_2. This is clearly a function not merely of the symmetry groups of the functions F and G, but of the actual position of the zeroes of F' and G' relative to each other.

Although our remarks in §2 show that the symmetry group of a nil-theta function F whose index is j determines sums of j zeroes of F', this group does not determine the location of the individual zeroes of F'. Thus, the integer w cannot be computed merely by looking at the symmetry groups of F and G. We will now discuss to what extent it is possible for us to compute w. Our next theorem shows that in the special case that both F and G are of index one, then w is quite easily computed.

THEOREM 3.3. *Suppose that $F \in \Theta_m(\Gamma_0)$ and $G \in \Theta_n(\Gamma_0)$ are both of index one. Let L_1 and L_2 denote the lattices in $\mathbf{R}^2$ spanned by the generators of $\Delta(F)$ and $\Delta(G)$ respectively, and let D_i be the fundamental parallelogram of the lattice L_i, $i = 1, 2$. Then*

$$w = \begin{cases} 2, & \text{if } \Delta(F) = \Delta(G) \text{ and } z_0 - w_0 \in \frac{1}{2} L_1, \\ 1, & \text{otherwise}, \end{cases}$$

where z_0 is the unique zero of $F' = M(m)F$ in D_1 and w_0 is the unique zero of $G' = M(n)G$ in D_2.

PROOF. Let S_1 and S_2 denote the zeroes in D of F' and G', respectively. It follows easily from Corollary 2.11 that $S_1 = \{z_0 + \gamma | \gamma \in \Delta(F)\}$ and $S_2 = \{w_0 + \beta | \beta \in \Delta(G)\}$. Let $\mu \in (1/(m+n))L$ be such that $-\mu + (S_1 \cup S_2) = (S_1 \cup S_2)$. It follows easily that either $\mu \in \Delta(F) \cap \Delta(G)$ or else $-\mu + S_1 = S_2$ and $-\mu + S_2 = S_1$. Therefore, $W = \{0\} \cup \{\mu \in (1/(m+n))L | -\mu + S_1 = S_2 \text{ and } -\mu + S_2 = S_1\}$.

We consider two cases:

Case I. Suppose that $m \neq n$. Then clearly there can exist no μ such that $-\mu + S_1 = S_2$ and hence $W = \{0\}$ and $w = 1$.

Case II. Suppose that $m = n$. Let $\mu \in (1/(m+n))L$ be such that $-\mu + S_1 = S_2$. Then, there exists a $\beta \in \Delta(G)$ such that $-\mu + z_0 = w_0 + \beta$. Let $\gamma \in \Delta(F)$. Then $-\mu + z_0 + \gamma = w_0 + \beta + \gamma$. But this must belong to S_2, which implies that $\gamma \in \Delta(G)$. Thus $\Delta(F) \subseteq \Delta(G)$. Equivalently, $-\mu + S_2 = S_1$ implies that $\Delta(G) \subseteq \Delta(F)$. Hence, in order for W to contain a nonzero element, it is necessary that $\Delta(F) = \Delta(G)$.

We assume then that $\Delta(F) = \Delta(G)$. It follows then from Corollary 2.11 that there exists an element μ of the group $\hat{\Delta}(F) = (1/m)L/L_1$ such that $w_0 = z_0 - \mu$, which implies that $-\mu + S_1 = S_2$. In order for this element μ to belong to W, it is necessary and sufficient that $-\mu + S_2 = S_1$ as well. This will be true if and only if $2\mu \in \Delta(F) \cap \Delta(G) = \Delta(F)$, or equivalently if $\mu \in \frac{1}{2} L_1$. In this case, $W = \{0\}$ and $w = 2$. Otherwise, $W = \{0\}$ and $w = 1$.

We illustrate this theorem with two examples.

EXAMPLE 1. Let $F \in \Theta_4(\Gamma_0)$ and $G \in \Theta_4(\Gamma_0)$ be such that the zero sets of the functions $F' = M(4)F$ and $G' = M(4)G$ in D are the sets

$$S_1 = \left\{\left(\tfrac{1}{8}, \tfrac{1}{2}\right), \left(\tfrac{3}{8}, \tfrac{1}{2}\right), \left(\tfrac{5}{8}, \tfrac{1}{2}\right), \left(\tfrac{7}{8}, \tfrac{1}{2}\right)\right\} \quad \text{and} \quad S_2 = \left\{\left(\tfrac{1}{8}, 0\right), \left(\tfrac{3}{8}, 0\right), \left(\tfrac{5}{8}, 0\right), \left(\tfrac{7}{8}, 0\right)\right\},$$

respectively. Then $\Delta(F) = \Delta(G) = \{(0, 0), (\frac{1}{4}, 0), (\frac{2}{4}, 0), (\frac{3}{4}, 0)\}$, and $\operatorname{ind}(F) = \operatorname{ind}(G) = 1$. Now $z_0 = (\frac{1}{8}, \frac{1}{2})$ and $w_0 = (\frac{1}{8}, 0)$, so that $z_0 - w_0 = (0, \frac{1}{2}) \in \frac{1}{2}L_1$, $W = \{(0, 0), (0, \frac{1}{2})\}$, and $w = 2$. In this case $\operatorname{ind}(FG) = 1$.

EXAMPLE 2. Let $F \in \Theta_4(\Gamma_0)$ and $G \in \Theta_4(\Gamma_0)$ be such that the zero sets in D of the functions F' and G' are the sets

$$S_1 = \left\{\left(\tfrac{1}{8}, \tfrac{1}{2}\right), \left(\tfrac{3}{8}, \tfrac{1}{2}\right), \left(\tfrac{5}{8}, \tfrac{1}{2}\right), \left(\tfrac{7}{8}, \tfrac{1}{2}\right)\right\} \quad \text{and} \quad S_2 = \left\{\left(\tfrac{1}{8}, \tfrac{1}{4}\right), \left(\tfrac{3}{8}, \tfrac{1}{4}\right), \left(\tfrac{5}{8}, \tfrac{1}{4}\right), \left(\tfrac{7}{8}, \tfrac{1}{4}\right)\right\},$$

respectively. As above, $\Delta(F) = \Delta(G) = \{(0, 0), (\frac{1}{4}, 0), (\frac{2}{4}, 0), (\frac{3}{4}, 0)\}$ and $\operatorname{ind}(F) = \operatorname{ind}(G) = 1$. In this case however, $z_0 = (\frac{1}{8}, \frac{1}{2})$ and $w_0 = (\frac{1}{8}, \frac{1}{4})$ so that $z_0 - w_0 = (0, \frac{1}{4}) \notin \frac{1}{2}L_1$ so that $W = \{(0, 0)\}$, $w = 1$ and $\operatorname{ind}(FG) = 2$.

We see then that in the special case that both F and G are of index one, the multiplication theory is quite nice. The index of the function FG is seen to depend not only on $\operatorname{ind}(F)$ and $\operatorname{ind}(G)$, but on the generators of the symmetry groups of F and G, and on the location of the zeroes z_0 and w_0.

In the case that only one of the functions F and G is of index one, it is possible to describe the set W, but this is quite complicated, so we omit it and refer the reader to the author's thesis [6] for details.

If neither of the two functions is of index one, then we have not been able to find a simple way to compute w.

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