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# GLOBAL WARFIELD GROUPS

BY

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ABSTRACT. A global Warfield group is a summand of a simply presented abelian group. The theory of global Warfield groups encompasses both the theory of totally projective $p$-groups, which includes the classical Ulm-Zippin theory of countable $p$-groups, and the theory of completely decomposable torsion-free groups. This paper develops the central results of the theory including existence and uniqueness theorems. In addition it is shown that every decomposition basis of a global Warfield group has a nice subordinate with simply presented torsion cokernel, and that every global Warfield group is a direct sum of a group of countable torsion-free rank and a simply presented group.

**1. Introduction.** The theory of Warfield groups has its roots in the classical theories of countable $p$-groups and completely decomposable torsion-free groups. These two theories were first unified by the notion of a *simply presented* group, a concept introduced for $p$-groups in **[CRH]**, extended to $p$-local groups in **[WAR2]** and to arbitrary groups in **[WAR3]**. Warfield groups were originally defined as summands of simply presented groups. In this paper, as in **[HRW3]** for the local case, we define them to be extensions of direct sums of valuated cyclics by simply presented torsion groups. The point of this paper is to present the global theory. The main results are analogues of the classical theorems of Ulm and Zippin–that is, a theorem stating that two Warfield groups are isomorphic if and only if their invariants are the same, and a theorem constructing a Warfield group with any prescribed admissible set of invariants.

The local theory, in which the groups are modules over the integers localized at a prime $p$, was developed in **[KMA]**, **[ROT1]**, **[RYN]**, **[MEG2]**, **[WAR1]**, **[WAR5]**, **[WAR7]**, **[STA1]**, **[STA2]**, **[RIC]**, **[HRW2]** and **[HRW3]**. We refer the reader to **[HRW3]** for a survey.

The first developments in the global area dealt almost exclusively with groups of torsion-free rank one, and took place in **[ROT2]**, **[MEG1]**, **[WAL]**, **[WAR3]** and **[HUN]**. In **[WAR3]**, Warfield studied global simply presented groups. These groups are direct sums of groups of torsion-free rank one but are not closed under taking summands, even in the local case. The paper also contained an example **[WAR3**, Theorem 6] of a direct sum of two rank one simply presented groups which could be decomposed in two ways so that no two of the four factors involved contained isomorphic infinite cyclic valuated subgroups. This example was the first difficulty

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of a strictly global nature. With this difficulty in mind, Warfield proved an isomorphism theorem for global simply presented groups with isomorphic decomposition bases [**WAR3**, Theorem 5].

A special class of global Warfield groups appeared in [**WAR6**]. Called *balanced projectives*, these groups are simply presented (and therefore direct sums of groups of torsion-free rank one). However, their special properties ensure closure under taking summands, and exclude examples of the kind mentioned above. Thus Warfield was able to provide a complete theory in this case.

In [**STA3**] and [**AHW**] the difficulties presented by Warfield's example were resolved, removing the restrictions on Warfield's original isomorphism theorem for simply presented groups. In [**STA3**, Theorem 10] the isomorphism theorem was proved for a larger class of groups which the complete theory now reveals coincides with the class of Warfield groups.

Attention then turned to the major open problem of whether a summand of a group with a decomposition basis has a decomposition basis. This was first proved for finite rank summands by Arnold, Hunter and Walker [**WAR8**, Problem 11], then in general in [**STA6**] and in [**AHR**]. Another difficulty remained–that of showing that a summand of a Warfield group has a decomposition basis generating a nice subgroup with simply presented cokernel. The first proof that such decomposition bases could be found, even locally, appeared in the proof of the local isomorphism theorem [**WAR1**, Theorem 4]. Unfortunately, the proof used [**WAR1**, Lemma 6] which is false (Example 7.1). The development of the local theory in [**HRW3**] avoids this difficulty by passing to the category *Walk* at critical moments. However, it turns out that an appropriate substitute for [**WAR1**, Lemma 6] is necessary for a straightforward global treatment. This is our Corollary 9.4.

There are three different types of theorem concerning the construction of Warfield groups–existence, crude embedding and fine embedding. An existence theorem tells when groups with prescribed isomorphism invariants exist. The classical theorem of Zippin is an example of an existence theorem. Warfield gave existence theorems for local balanced projectives ($KT$-modules) in [**WAR5**, Theorem 4.1]; existence theorems for abritrary local Warfield groups were given in [**HRW2**].

The notions of crude and fine embedding theorems are modeled on the notions of crude and fine existence theorems [**RYN**]. A crude embedding theorem is [**RWA**, Theorem 1] which shows how to embed any valuated group as a nice subgroup of a group in such a way that the cokernel is simply presented and torsion.

A fine embedding theorem tells when a valuated group can be embedded in a specified type of group with prescribed relative invariants. For Warfield groups such a theorem tells when a direct sum of infinite cyclic valuated groups can be embedded nicely in a group with specified relative Ulm invariants and simply presented torsion cokernel. In general, fine embedding theorems are more difficult to obtain than existence theorems. In [**HRW2**] a fine embedding theorem is proved for countable local Warfield groups, but no fine embedding theorem is proved for the uncountable case, although a general existence theorem is proved. A general fine embedding theorem in the local case would yield the global existence theorem, and the global fine embedding theorem, directly. Our development requires theorems of all three types on the way to obtaining the existence theorem for Warfield groups.

We now outline the present paper. §3 introduces the global invariants, using a generalization to arbitrary groups and valuated groups of the type of an element of a torsion-free group, together with the local invariants of **[STA1]** and **[HRW3]**. Warfield groups are defined in §4, along with a class of valuated torsion groups which we call *Fuchsian*. The Fuchsian valuated groups are those having a nice composition series and, by [**RWA**, Theorem 13], are precisely the valuated $p$-groups with homological dimension 1 in the category of local valuated groups. In this section it is shown that the global Warfield invariants of the previous section are the same for two Warfield groups precisely when those groups contain isomorphic direct sums of cyclic valuated groups with torsion cokernel. In the terminology of Warfield **[WAR8]**, they have the same invariants if and only if they are $H$-isomorphic.

General globalizing techniques are given in §5. These techniques are necessary, but not sufficient, to obtain the global theory from the local. They come under the scope of the term 'Hasse principle' as used by Warfield.

§6 contains the isomorphism theorem, §7 the counterexample to Warfield's lemma, and §8 a proof that finite subsets of decomposition bases are nice–a fact tacitly assumed in [**HRW3**, Lemma 30].

In §9 we show that any decomposition basis of a Warfield group has a subordinate which generates a nice subgroup with simply presented torsion cokernel. A consequence of the main result of this section, Theorem 9.3, is that if an arbitrary group contains a decomposition basis which generates a nice subgroup, then any decomposition basis has a subordinate with this property. The results of this section are then used in §10 to prove that a summand of a Warfield group is Warfield. It is here that the global Azumaya theorems of **[AHR]** are applied.

§11 provides a generalization of the local fine embedding theorem [**HRW2**, Theorem 12], which removes the condition that the ordinals involved be countable. In §12 we give conditions for the embedding of a direct sum of valuated cyclics in a simply presented group and for the embedding of a countable direct sum of valuated cyclics in a Warfield group. Using these, we then obtain a full global existence theorem. At the same time, we show that each Warfield group is the direct sum of a group of countable torsion-free rank and a simply presented group. We also show that each Warfield group is a summand of a simply presented group with torsion complement, so that our definition does indeed coincide with that of Warfield.

**2. Preliminaries.** Throughout, the word *group* will mean an abelian group. The letter $p$ will be reserved for a prime, $m$ and $n$ for integers–usually nonnegative. The letter $\omega$ will stand both for the first infinite ordinal and the first infinite cardinal. The symbol $A_p$ will denote the group $A$ localized at the prime $p$. We assume familiarity with the category of valuated groups and the associated notation introduced in **[RWA]**.

The *p-value sequence* $V_p a$ of an element $a$ of a valuated group is the sequence $v_p a, v_p pa, \ldots$. A *value sequence* $\alpha$ is a sequence $\alpha_0 < \alpha_1 < \alpha_2 < \ldots$ of ordinals and symbols $\infty$. By $p^n\alpha$ we mean the value sequence $\alpha_n, \alpha_{n+1}, \ldots$. An ordinal $\alpha$ is identified with the value sequence $\alpha, \alpha + 1, \alpha + 2, \ldots$.

For any value sequence $\alpha$ and $p$-local group $A$, let $A(\alpha) = \{a \in A\colon v_p p^n a > \alpha_n$ for all $n\}$. Let $A(\alpha^*)$ be generated by $\{a \in A(\alpha)\colon v_p p^n a \neq \alpha_n$ for infinitely many $n\}$. Note that if $\infty$ is in $\alpha$, then $A(\alpha^*) = 0$.

The subgroup generated by a subset $X$ of a valuated group will be written $\langle X \rangle$. A subgroup $N$ of a valuated group $A$ is *nice* if every coset of $N_p$ in $A_p$ has an element of maximal value. Note that in the setting of valuated groups, niceness is transitive.

**3. Ulm and Warfield invariants.** The local invariants we require have been introduced and discussed elsewhere, so in this section only the definition and relevant references will be given. Let $A$ be a $p$-local valuated group and $\alpha$ an ordinal. The $\alpha$th *Ulm invariant of $A$* is the rank of the $p$-bounded group

$$F_A(\alpha) = \frac{\{a \in A(\alpha)\colon pa \in A(\alpha + 2)\}}{A(\alpha + 1)}$$

and is denoted $f_A(\alpha)$. The $\infty$th Ulm invariant $f_A(\infty)$ is the rank of the $p$-bounded group $F_A(\infty) = \{a \in A(\infty)\colon pa = 0\}$. A valuated embedding $A \to B$ induces an embedding $\phi\colon F_A(\alpha) \to F_B(\alpha)$. The rank of the cokernel of $\phi$ is called the $\alpha$th *Ulm invariant of $B$ relative to $A$*. The $\alpha$th *derived Ulm invariant of $A$* is the rank of

$$\frac{A(\alpha)}{\bigcap_{\theta<\alpha} (A(\alpha + 1) + A(\alpha) \cap pA(\theta))}$$

and is denoted $g_A(\alpha)$. For a full account of these invariants, see **[HRW2]** and **[HRW3]**. The Ulm invariants of a nonlocal valuated group are the Ulm invariants $f_A(p, \alpha)$ of its localizations.

For each $p$-local valuated group $A$ and value sequence $\alpha$ we get a sequence

$$\frac{A(\alpha)}{A(\alpha^*)} \to \frac{A(p\alpha)}{A(p\alpha^*)} \to \frac{A(p^2\alpha)}{A(p^2\alpha^*)} \to \cdots$$

of groups, where the maps are induced from multiplication by $p$. The *Warfield invariant* $w_A(\alpha)$ is defined to be the rank of the direct limit $W_A(\alpha)$ of this sequence. This construction of the Warfield invariants is due to Stanton **[STA1]**. Note that $W_A(\alpha)$ is $p$-bounded if $\infty$ is not in $\alpha$, and is $A(\infty)$ modulo its torsion subgroup if $\infty$ is in $\alpha$. A valuated embedding $A \to B$ induces an embedding $\phi\colon W_A(\alpha) \to W_B(\alpha)$. The rank of the cokernel of $\phi$ is called the *Warfield invariant of $B$ relative to $A$ at $\alpha$* and is denoted by $w_{B,A}(\alpha)$. This rank is taken to be the torsion-free rank if $\infty$ is in $\alpha$.

Let $A$ and $B$ be (global) valuated groups. We say that $A$ is *quasi-homomorphic* to $B$ if there is a map from a subgroup of finite index in $A$ onto a subgroup of finite index in $B$. This notion partially pre-orders the class of valuated groups. The equivalence classes of mutually quasi-homomorphic cyclic valuated groups are called *types*, and we say that a cyclic valuated group $A$ *has type* $\tau$ if $A \in \tau$. The induced partial order on the set of types will be indicated by $<$ and $\leqslant$ . All torsion cyclics have the same type $\tau_0$, and $\tau \leqslant \tau_0$ for any type $\tau$.

If $G$ is any valuated group, and $x \in G$, then the *type* of $x$ is defined to be the type of the cyclic subgroup generated by $x$. Note that if $mx = ny$ for $m, n \neq 0$ then $x$ and $y$ have the same type. If $G$ is a torsion-free group, then this notion of type agrees with the usual one [**FUC**, p. 109]. We let $G(\tau)$ denote $\{x \in G\colon \operatorname{type}(x) \geqslant \tau\}$, and $G(\tau^*)$ denote the subgroup generated by $\{x \in G\colon \operatorname{type}(x) > \tau\}$.

The (*global*) *Warfield invariant* $w_G(\tau, p, \alpha)$ of a valuated group $G$ is defined, for each type $\tau$, prime $p$ and value sequence $\alpha$, to be the $\alpha$th Warfield invariant of $G(\tau)_p$ relative to $G(\tau^*)_p$. Note that if $G$ is $p$-local, then $w_G(\alpha) = w_G(\tau, p, \alpha)$ if $\tau$ has an element whose generator has $p$-value sequence $\alpha$, and is 0 otherwise.

THEOREM 3.1. *If $H$ is a valuated subgroup of $G$ and $G/H$ is torsion, then $G$ and $H$ have the same Warfield invariants.*

PROOF. Consider the commutative diagram

$$\begin{array}{ccccc} H(\tau)_p & \to & G(\tau)_p & \to & G(\tau)_p/H(\tau)_p \\ \uparrow & & \uparrow & & \uparrow \\ H(\tau^*)_p & \to & G(\tau^*)_p & \to & G(\tau^*)_p/H(\tau^*)_p \end{array}$$

Since $G/H$ is torsion, so are $G(\tau)_p/H(\tau)_p$ and $G(\tau^*)_p/H(\tau^*)_p$. By [**HRW3**, Lemma 7], the local version of this theorem, $G$ and $H$ have the same Warfield invariants. $\square$

**4. Warfield groups.** A group $A$ is *simply presented* if $A$ is generated by a set $X$ and relations on $X$ of the form $nx = y$ and $nx = 0$. The set $X$ breaks up into equivalence classes if we define $x \sim y$ to mean $nx = my$ for some nonzero $n$ and $m$. This yields a decomposition of $A$ into a direct sum of simply presented groups of torsion-free rank at most one. A group $A$ is simply presented and torsion precisely when it has a *nice composition series* [**FUC**, p. 82], that is, a well-ordered ascending chain of nice subgroups $N_\alpha$ such that:

(1) $N_0 = 0$;
(2) $\bigcup N_\alpha = A$;
(3) $N_{\alpha+1}/N_\alpha$ is cyclic of prime order;
(4) $N_\alpha = \bigcup_{\theta<\alpha} N_\theta$ if $\alpha$ is a limit.

A valuated group is called *Fuchsian* if it has a nice composition series. We remark that not every Fuchsian valuated group is simply presented in the sense of [**HRW1**]. The next lemma is a useful property of Fuchsian valuated groups which follows directly from the definition.

LEMMA 4.1. *Let $A$ be a nice valuated subgroup of $B$. Then $B$ is Fuchsian if both $A$ and $B/A$ are Fuchsian.* $\square$

Let $A$ be a valuated group and $C$ a valuated subgroup of $A$. If $C$ is a direct sum of infinite valuated cyclics and $A/C$ is torsion, then a valuated basis for $C$ is called a *decomposition basis* for $A$. If in addition $A$ is a group, $C$ is nice in $A$ and $A/C$ is simply presented, then $A$ is called a *Warfield group*. If $A$ is a simply presented group, then taking one generator from each summand of torsion-free rank one yields a decomposition basis for $A$ that generates a nice subgroup with simply presented cokernel. Thus every simply presented group is Warfield. We will later show that every summand of a simply presented group is Warfield (Theorem 10.1) and, conversely, that every Warfield group is a summand of a simply presented group (Corollary 12.4).

If $A$ has a decomposition basis $X$, then by Theorem 3.1 the Warfield invariants of $A$ are the same as those of $\langle X \rangle$. The Warfield invariants of $\langle X \rangle$ are given by

$$w_{\langle X \rangle}(\tau, p, \alpha) = \operatorname{card}\{x \in X\colon \langle x \rangle \in \tau \text{ and } V_p p^m x = p^n \alpha \text{ for some } m \text{ and } n\}.$$

If $X$ and $Y$ are decomposition bases, we say that $X$ is *subordinate* to $Y$ if every element of $X$ is a multiple of an element of $Y$. If $H = \langle Y \rangle$, then we say $\langle X \rangle$ is a subordinate of $H$.

The remarks above show that, in particular, each subordinate of a direct sum of valuated cyclics has the same Warfield invariants. A further useful observation is that each such subordinate is nice and has Fuchsian cokernel.

Throughout our development we will be confronted with the problem of transforming a given decomposition basis to another with more desirable properties. It is perhaps surprising that we can always do so by passing to a subordinate. The following result, which is at the heart of our proof of the isomorphism theorem (Theorem 6.1), is an example of this process.

THEOREM 4.2. *Two direct sums of cyclic valuated groups have the same Warfield invariants if and only if they have isomorphic subordinates.*

PROOF. The 'if' follows from Theorem 3.1. Let $\{x_i\}_{i\in I}$ and $\{y_j\}_{j\in J}$ be generators of the two direct sums of cyclics. We may assume that the $x$'s and the $y$'s are all of the same type, so by [**AHW**, Theorem 5], it suffices to construct subordinates that are locally isomorphic at each prime.

For each prime $p$ and value sequence $\alpha$ define $I(p, \alpha) = \{i \in I\colon V_p(p^n x_i) = p^m \alpha$ for some $m$ and $n\}$. Define $J(p, \alpha)$ similarly. Then $\operatorname{card} I(p, \alpha) = \operatorname{card} J(p, \alpha)$ since the Warfield invariants of the two valuated groups are the same. Thus, for each $p$, we have a 1-1 correspondence between $I$ and $J$ that carries $I(p, \alpha)$ to $J(p, \alpha)$.

By the usual argument we may assume that $I$ and $J$ are countable; hence initial subsets of the positive integers. We shall construct positive integers $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ such that for all primes $p$ and value sequences $\alpha$, the two sets $\{i \in I\colon V_p(a_i x_i) = \alpha\}$ and $\{j \in J\colon V_p(b_j y_j) = \alpha\}$ have the same cardinality. Hence the indicated subordinates are isomorphic by [**AHW**, Theorem 5].

We first construct a set $S$ of triples $(i, p, j)$ with the following properties:

(1) For fixed $p$, the set of all $(i, p, j)$ in $S$ is a 1-1 correspondence of $I$ with $J$ taking $I(p, \alpha)$ to $J(p, \alpha)$ for all value sequences $\alpha$.

(2) For each $i$ the triple $(i, p, i) \in S$ for all but finitely many primes $p$.

In view of (1), $S$ will be a subset of

$$T = \{(i, p, j)\colon \text{if } i \in I(p, \alpha) \text{ then } j \in J(p, \alpha)\}.$$

We form $S$ as the union of a chain $\varnothing = S_0 \subset S_1 \subset \dots$ with each $S_n$ the union of $S_{n-1}$ and the set

$$A_n = \{(n, p, n) \in T\colon (n, p, j) \notin S_{n-1} \text{ and } (i, p, n) \notin S_{n-1}, \text{ any } i, j\}$$

together with additional elements of $T$ such that, for each prime $p$, there is exactly one $j \in J$ so that $(n, p, j) \in S_n$ and exactly one $i \in I$ so that $(i, p, n) \in S$. That these additional elements can be found at each stage is a consequence of card $I(p, \alpha) =$ card $J(p, \alpha)$ and the fact that, for each prime $p$, the set of all pairs $(i, j)$ with $(i, p, j) \in S_{n-1}$ is finite. It is easy to check that $S$ satisfies (1) and (2).

Let $h_p a$ denote the $p$-height of the integer $a$. The $a_i$'s and $b_j$'s are now chosen so that if $(i, p, j) \in S$ then $h_p a_i = n$ and $h_p b_j = m$ where $m$ and $n$ are the smallest integers with $V_p p^n x_i = V_p p^m y_j$. Since $(i, p, i) \in S$ for almost all $p$ and the $x$'s and the $y$'s all have the same type, the $a_i$'s and $b_j$'s are well defined. □

Theorem 4.2, together with Theorem 3.1, provides an interpretation of the Warfield invariants for groups having a decomposition basis–namely, two such groups have the same Warfield invariants exactly when they contain decomposition bases which generate isomorphic valuated subgroups.

**5. Going from local to global.** We now present some general globalizing techniques that allow us to make full use of the local theory.

LEMMA 5.1. *If $G$ is a $p$-group, then the localization map from $H$ to $H_p$ induces an isomorphism from* $\mathrm{Ext}(G, H)$ *to* $\mathrm{Ext}(G, H_p)$.

PROOF. Let $K$ be the kernel of the localization map $H \to H_p$, and $C$ its cokernel. Then we have an exact sequence $\mathrm{Ext}(G, K) \to \mathrm{Ext}(G, H) \to \mathrm{Ext}(G, H/K) \to 0$. But $K$ is a torsion group with no $p$-torsion, so $\mathrm{Ext}(G, K) = 0$, whence $\mathrm{Ext}(G, H)$ is naturally isomorphic to $\mathrm{Ext}(G, H/K)$. We also have the exact sequence

$$\mathrm{Hom}(G, C) \to \mathrm{Ext}(G, H/K) \to \mathrm{Ext}(G, H_p) \to \mathrm{Ext}(G, C)$$

where the two ends are zero because $C$ is a torsion group with no $p$-torsion. Thus $\mathrm{Ext}(G, H/K)$ is naturally isomorphic to $\mathrm{Ext}(G, H_p)$, whence the lemma. □

LEMMA 5.2. *Suppose the following diagram is commutative with exact rows, $g_0$ is onto and $g_3$ is one-to-one.*

$$\begin{array}{ccccccc} A_0 & \to & A_1 & \xrightarrow{f} & A_2 & \to & A_3 \\ g_0\downarrow & & g_1\downarrow & & g_2\downarrow & & g_3\downarrow \\ B_0 & \to & B_1 & \xrightarrow[h]{} & B_2 & \to & B_3 \end{array}$$

*Then given $x$ in $B_1$ and $y$ in $A_2$ such that $g_2 y = hx$, there is $z$ in $A_1$ such that $fz = y$ and $g_1 z = x$.*

PROOF. Standard diagram chase. □

The following theorem has been called the abelian group theorist's Hasse principle [**WAR8**, p. 8].

THEOREM 5.3 (HASSE PRINCIPLE). *Let $G$ and $H$ be abelian groups, $A$ a subgroup of $G$ with $G/A$ torsion, and $f$ a map from $A$ to $H$. If, for each prime $p$, there is a map $\phi^p$ from $G_p$ to $H_p$ that extends $f_p$, then there is a map $f^*$ from $G$ to $H$, extending $f$, such that $f_p^* = \phi^p$.*

PROOF. Consider the commutative diagram

$$\begin{array}{ccc}
\operatorname{Hom}(G/A, H) & \simeq & \prod \operatorname{Hom}((G/A)_p, H_p) \\
\downarrow & & \downarrow \\
\operatorname{Hom}(G, H) & \to & \prod \operatorname{Hom}(G_p, H_p) \\
\downarrow & & \downarrow \\
\operatorname{Hom}(A, H) & \to & \prod \operatorname{Hom}(A_p, H_p) \\
\downarrow & & \downarrow \\
\operatorname{Ext}(G/A, H) & \simeq & \prod \operatorname{Ext}((G/A)_p, H_p) \\
\downarrow & & \downarrow \\
\operatorname{Ext}(G, H) & \to & \prod \operatorname{Ext}(G_p, H_p) \\
\downarrow & & \downarrow \\
\operatorname{Ext}(A, H) & \to & \prod \operatorname{Ext}(A_p, H_p) \\
\downarrow & & \downarrow \\
0 & \simeq & 0
\end{array}$$

The isomorphism on row one is a consequence of the fact that $G/A$ is torsion. The isomorphism on row four follows from Lemma 5.1, and the theorem is proved on applying Lemma 5.2 to the top four rows. □

The bottom four rows of the diagram in the preceding proof yield an analogous result for extensions. If $E$ is a short exact sequence, let $E_p$ denote the short exact sequence obtained by localizing each term in $E$. Suppose $E \in \operatorname{Ext}(A, H)$ and for each prime $p$ we are given an extension $E^p \in \operatorname{Ext}(G_p, H_p)$ which restricts to $E_p$. Then there is $E^* \in \operatorname{Ext}(G, H)$ such that, for each prime $p$, $E_p^* = E^p$. In particular, if $A = 0$, then we have:

LEMMA 5.4. *Suppose $G$ is a torsion group, $H$ is a group and, for each prime $p$, we are given an extension $E^p \in \operatorname{Ext}(G_p, H_p)$. Then there is a unique $E \in \operatorname{Ext}(G, H)$ such that, for each prime $p$, $E_p = E^p$.* □

This lemma is used to convert local embedding theorems to global ones.

**6. The isomorphism theorem.** Our definition of a Warfield group as a nice extension of a direct sum of valuated cyclics by a simply presented torsion group leads rapidly to the isomorphism theorem. Indeed, the proof which follows is almost entirely set theoretic in nature and does not require a close analysis of the group structure.

THEOREM 6.1. *Two Warfield groups are isomorphic if and only if they have the same Ulm and Warfield invariants.*

PROOF. We need only prove sufficiency. Let $A$ and $B$ be Warfield groups with the same Ulm and Warfield invariants. Let $X$ and $Y$ be nice decomposition bases for $A$ and $B$, respectively, such that $A/\langle X\rangle$ and $B/\langle Y\rangle$ are simply presented. By [**WAK**, Theorem 2.8], globalized via Theorem 5.3, it suffices to choose subordinates of $X$ and $Y$ so that $f_{A,\langle X\rangle} = f_{B,\langle Y\rangle}$ and $\langle X\rangle \simeq \langle Y\rangle$. For $x$ in $X$ let $F_x = \{(p, \alpha)\colon f_{\langle x\rangle}(p, \alpha) \neq 0\}$. By [**HRW3**, Lemma 9], for each $x$ in $X$ with $F_x$ nonempty, we can choose $\phi(x) \in F_x$ so that for every $(p, \alpha)$ we have

$$\operatorname{card}\{x\colon \phi(x) = (p, \alpha)\} = \operatorname{card}\{x\colon (p, \alpha) \in F_x\}$$

if the latter is infinite. If $\phi(x) = (p, \alpha)$, choose $x'$ in $\langle x\rangle$ so that $f_{\langle x'\rangle}(p, \alpha) = 0$. Doing the same for $Y$ and $B$, and replacing $X$ and $Y$ by the subordinates so obtained, we have

$$f_A(\alpha) = f_{A,\langle X\rangle}(\alpha) = f_B(\alpha) = f_{B,\langle Y\rangle}(\alpha) \quad \text{if } f_A(\alpha) \text{ is infinite.} \tag{$*$}$$

Taking further subordinates will not affect $(*)$, so by Theorem 4.2, we can assume $\langle X\rangle \simeq \langle Y\rangle$. But then $f_{A,\langle X\rangle}(\alpha) = f_{B,\langle Y\rangle}(\alpha)$ for all $\alpha$. By Lemma 4.1 the cokernels $A/\langle X\rangle$ and $B/\langle Y\rangle$ are Fuchsian while transitivity of niceness ensures that $\langle X\rangle$ and $\langle Y\rangle$ are nice in $A$ and $B$ respectively. □

**7. Warfield's lemma.** In Warfield's development of Warfield groups, a central role is played by a lemma, [**WAR1**, Lemma 6] and [**WAR7**, Lemma 4.8], which asserts that any two decomposition bases of a group have subordinates which generate the same subgroup. We provide a counterexample to this with:

EXAMPLE 7.1. There are decomposition bases $X$ and $Y$ of a countable rank free group $G$, such that no subgroup of $G$ is generated both by a subordinate to $X$ and a subordinate to $Y$.

PROOF. Let $G$ be free on $a_1, a_2, \ldots$. Then $b_1, b_2, \ldots$ is another basis for $G$ where

$$\begin{aligned} b_{2n-1} &= a_{2n-1} + a_{2n}, \\ b_{2n} &= a_{2n} + a_{2n+1} + a_{2n+2}, \\ a_{2n-1} &= b_{2n-1} - b_{2n} + b_{2n+1}, \\ a_{2n} &= b_{2n} - b_{2n+1}. \end{aligned}$$

Define the decomposition bases $X = \{x_1, x_2, \ldots\}$ and $Y = \{y_1, y_2, \ldots\}$ by

$$\begin{aligned} x_{2n-1} &= p^{n-1}a_{2n-1}, & y_{2n-1} &= p^{n-1}b_{2n-1}, \\ x_{2n} &= p^{n-1}a_{2n}, & y_{2n} &= p^{n-1}b_{2n}. \end{aligned}$$

Then the $x$'s and $y$'s are related by

$$\begin{aligned} y_{2n-1} &= x_{2n-1} + x_{2n}, \\ y_{2n} &= x_{2n} + p^{-1}(x_{2n+1} + x_{2n+2}), \\ x_{2n-1} &= y_{2n-1} - y_{2n} + p^{-1}y_{2n+1}, \\ x_{2n} &= y_{2n} - p^{-1}y_{2n+1}. \end{aligned}$$

Thus any multiple of an element in $X$, which is in the subgroup generated by $p^mY$, is in the subgroup generated by $p^{m+1}X$–and any multiple of an element in $Y$, which is in the subgroup generated by $p^{m+1}X$ is in the subgroup generated by $p^{m+1}Y$. Hence any decomposition basis subordinate to $X$ and contained in the subgroup generated by $p^mY$ is subordinate to $p^{m+1}X$, and any decomposition basis subordinate to $Y$ and contained in the subgroup generated by $p^{m+1}X$ is subordinate to $p^{m+1}Y$. By induction on $m$, any subgroup generated by a subordinate to $X$ and also by a subordinate to $Y$ is contained in the subgroup generated by $p^mX$. This is clearly impossible. $\square$

The failure of Warfield's lemma seeems to invalidate the proof of **[WAR7**, Theorem 5.2], which states that two local Warfield groups are isomorphic if they have the same Ulm and Warfield invariants. However, the proof of this isomorphism theorem given in [**HRW3**, Theorem 10] and the general development of that paper does not rely on Warfield's lemma. A global version of Warfield's lemma is assumed by Stanton in the proof of [**STA3**, Theorem 13] which he uses to prove that summands of global Warfield groups are Warfield [**STA6**, Theorem 8]. We prove that summands of Warfield groups are Warfield in Theorem 10.1 below.

Example 7.1 is best possible because Warfield's lemma is true in the finite case.

THEOREM 7.2. *Any two finite rank decomposition bases have subordinates that generate equal subgroups.*

PROOF. As two subgroups are equal if all their localizations are equal, and since any two finite decomposition bases $X$ and $Y$ have the property that $\langle X\rangle_p = \langle Y\rangle_p$ for all but finitely many primes $p$, it suffices to prove the theorem in the local case. We may assume $X \subset \langle Y\rangle$. Let $\alpha = \sup\{vz\colon 0 \neq z \in \langle Y\rangle\}$ and let $X' = \{x \in X\colon \sup_{n<\omega} vp^nx = \alpha\}$ and $Y' = \{y \in Y\colon \sup_{n<\omega} vp^ny = \alpha\}$. There is $\lambda < \alpha$ such that $\langle Y\rangle(\lambda) \subset \langle X'\rangle$ and so $\langle X'\rangle(\lambda) = \langle Y\rangle(\lambda) = \langle Y'\rangle(\lambda)$. Hence we may assume that $\langle X'\rangle = \langle Y'\rangle = \langle Y\rangle(\lambda)$. Passing to $Y\setminus Y'$ and $X\setminus X'$ in $\langle Y\rangle/\langle Y'\rangle$ we are done by induction on the number of elements in $Y$. $\square$

**8. Finite subsets of decomposition bases.** Wallace [**WAL**] showed that a cyclic decomposition basis is nice. More generally, any finite subset of a decomposition basis is nice. As far as we know the proof of this latter result has not appeared in print, although it was assumed in the proof of [**HRW3**, Lemma 30] whose consequences we will need shortly. We now give the proof.

LEMMA 8.1. *Let $Y$ be a finite subset of a decomposition basis $X$ of a group $G$. Then $\langle Y\rangle$ is nice.*

PROOF. We proceed by induction on the number of elements of $Y$, independent of $X$ and $G$. If $y \in Y$, then, by induction, $\langle Y\rangle/\langle y\rangle$ is nice in $G/\langle y\rangle$. By [**FUC**, Lemma 79.3], $\langle Y\rangle$ is nice in $G$ if $\langle y\rangle$ is. We shall choose $g'$ in $g + \langle y\rangle$ and fix nonnegative integers $i$ and $k$ such that if $vg' < v(g' + my) < v(g' + ny)$, then $v(g' + my) = vp^jy$ for some $j < i + k$. Hence there is an element of maximal value in $g + \langle y\rangle$. First choose $k$ such that $p^kg = ay + x$ with $x \in \langle X\setminus\{y\}\rangle$. If $p^k$ divides $a$, set $g' = g - (a/p^k)y$. Otherwise set $g' = g$. Then $p^kg' = a'y + x$ where $a' = 0$ or $a' = a$ is not divisible by $p^k$. If $vg' < v(g' + my)$, then $vg' = vmy = vp^iy$ for some $i$. Then $v(g' + my) < v(p^kg' + p^kmy) \leq v((a' + p^km)y + x) \leq vp^{k+i}y$. Moreover if $v(g' + my) < v(g' + ny)$, then $v(g' + my) = v((n - m)y) = vp^jy$ for some $j < i + k$. $\square$

A valuated group $G$ has *finite jump type* if, given an ordinal $\theta$, there are at most finitely many $\alpha < \theta$ such that $vx = \alpha$ and $vpx \geq \theta$ for some $x$ in $G$. Theorem 31 in **[HRW3]** states that any decomposition basis of finite jump type is nice. However the proof of [**HRW3**, Lemma 30] requires only that $X$ generate a direct sum of infinite cyclic valuated groups, and the proof of [**HRW3**, Theorem 31], which has a spurious $vp^n x_i$ in the last line, requires only that $X$ be a subset of a decomposition basis. Thus we have:

THEOREM 8.2. *Let $X$ be a subset of a decomposition basis of the valuated group $G$. If $X$ has finite jump type then $X$ is nice in $G$.* □

Note that this generalizes Lemma 8.1.

**9. Subordinates of decomposition bases.** In this section we prove that every decomposition basis of a Warfield group has a nice subordinate with simply presented cokernel (Corollary 9.4).

LEMMA 9.1. *Let $X$ and $Y$ be decomposition bases with $X \subset \langle Y \rangle$. Then there is a set of 'closed' subsets of $Y$ such that*

(1) *The union of any chain of closed sets is closed.*

(2) *Every infinite set is contained in a closed set of the same cardinality.*

(3) *If $S$ is closed, then*

(a) *$X \cap \langle S \rangle$ is a decomposition basis for $\langle S \rangle$,*

(b) *$S \cup (X \setminus \langle S \rangle)$ is a decomposition basis for $\langle Y \rangle$.*

PROOF. Given $S \subset Y$ we shall enlarge $S$ in an attempt to satisfy (3). For $x$ in $X$ let $P_x$ be the projection of $\langle X \rangle$ on $\langle x \rangle$. Let $\psi S = \{x \in X\colon P_x ns \neq 0$ for some $s \in S$ with $ns \in \langle X \rangle\}$. Clearly (3a) is equivalent to $\psi S \subset \langle S \rangle$. For $z$ in $\langle X \rangle$ and $p$ a prime, define the *$p$-value support* of $z$ in $X$ to be

$$p\text{-vspt}_X\, z = \{x \in X\colon P_x z \neq 0 \text{ and } v_p P_x z = v_p z\}$$

and set $\phi_p S = \bigcup \{p\text{-vspt}_X\, z\colon z \in \langle X \rangle \text{ and } p\text{-vspt}_Y\, z \subset S\}$. To see that $\phi_p S \subset \langle S \rangle$ for all $p$ implies (3b), note that if $z \in \langle X \rangle$ and $w \in \langle S \rangle$ are such that $v_p(w + z) \neq v_p w = v_p z$, then $p\text{-vspt}_Y\, z \subset S$.

Call $S$ *$p$-closed* if $\phi_p S \subset \langle S \rangle$ and $\psi S \subset \langle S \rangle$. Since $\phi_p$ and $\psi$ commute with unions of chains, the union of $p$-closed sets is $p$-closed. Call $S$ *closed* if it is $p$-closed for each $p$. Note that we have verified (1) and (3). To verify (2) we must first check that $\psi S$ and $\phi_p S$ are not too large. Clearly $|\psi S| \leq |\langle S \rangle|$. On the other hand $p\text{-vspt}_X\, z$ is finite for all $z$ in $\langle X \rangle$ and if $z_1$ and $z_2$ have the same projection on $\langle S \rangle$, and $p\text{-vspt}_Y\, z_i \subset S$ for $i = 1, 2$, then $p\text{-vspt}_X\, z_1 = p\text{-vspt}_X\, z_2$, so $|\phi_p S| \leq |\langle S \rangle|$. For $S$ an infinite set, let $S_p$ be the smallest subset of $Y$ such that $\langle S_p \rangle \supset S \cup \phi_p S \cup \psi S$. Note that $|S_p| = |S|$. Let $p(1), p(2), p(3), \ldots$ be an enumeration of the primes such that each prime occurs infinitely often. Then $S_{p(1)} \cup S_{p(1)p(2)} \cup \ldots$ is a closed set of the same cardinality as $S$. □

LEMMA 9.2. *Every countable decomposition basis has a nice subordinate.*

PROOF. First observe that a decomposition basis is nice if and only if some subordinate of finite index is nice. With this, the lemma follows from diagonalizing the local version [**HRW3**, Theorem 33] to avoid using more than a finite number of primes on each basis element. □

THEOREM 9.3. *Let $X$ and $Y$ be decomposition bases with $X \subset \langle Y \rangle$. Then there exists a subordinate $Z$ of $X$ such that $\langle Z \rangle$ is nice in $\langle Y \rangle$, and $\langle Y \rangle / \langle Z \rangle$ is Fuchsian.*

PROOF. Using Lemma 9.2 and the fact that any countable torsion valuated group is Fuchsian we may assume that $Y$ is uncountable. Let $\lambda$ be the least ordinal of cardinality $|Y|$, and let $\{y_\alpha\}_{\alpha<\lambda}$ be a well ordering of $Y$. Define $Y_\alpha$ inductively by

(i) $Y_{\alpha+1}$ is the closure of $Y_\alpha \cup \{y_\alpha\}$ as in Lemma 9.1,

(ii) $Y_\alpha = \bigcup_{\theta<\alpha} Y_\theta$ for $\alpha$ a limit ordinal

and let $X_\alpha = X \cap \langle Y_\alpha \rangle$. It is immediate from Lemma 9.1 that

(1) $X_\lambda = X$ and $Y_\lambda = Y$.

(2) $X_\alpha = \bigcup_{\theta<\alpha} X_{\theta+1}$ and $Y_\alpha = \bigcup_{\theta<\alpha} Y_{\theta+1}$.

(3) $X_\alpha$ is a decomposition basis of $\langle Y_\alpha \rangle$.

(4) $|Y_\alpha| < |Y|$ for $\alpha < \lambda$.

(5) $Y_\alpha \cup (X \setminus X_\alpha)$ is a decomposition basis.

Now $X^\alpha = X_{\alpha+1} \setminus X_\alpha$ represents a decomposition basis in $\langle Y_{\alpha+1} \rangle / \langle Y_\alpha \rangle$ so, by induction on the cardinality of $|Y|$, we can find a subordinate $Z^\alpha$ of $X^\alpha$ such that $\langle Y_\alpha \cup Z^\alpha \rangle$ is nice in $\langle Y_{\alpha+1} \rangle$ and $\langle Y_{\alpha+1} \rangle / \langle Y_\alpha \cup Z^\alpha \rangle$ is Fuchsian. Then $Z = \bigcup_{\alpha<\lambda} Z^\alpha$ is a decomposition basis of $\langle Y \rangle$. Note that $Z_\alpha = \bigcup_{\theta<\alpha} Z^\theta = Z \cap \langle Y_\alpha \rangle$.

We shall show, by induction on $\alpha$, that if $\rho \leqslant \alpha$, then $\langle Y_\rho \cup Z_\alpha \rangle$ is nice in $\langle Y_\alpha \rangle$. Note that this is trivial if $\rho = \alpha$. If $\langle Y_\rho \cup Z_\alpha \rangle$ is nice in $\langle Y_\alpha \rangle$, then $\langle Y_\rho \cup Z_{\alpha+1} \rangle = \langle Y_\rho \cup Z_\alpha \cup Z^\alpha \rangle$ is nice in $\langle Y_\alpha \cup Z^\alpha \rangle$ which is nice in $\langle Y_{\alpha+1} \rangle$. Suppose $\alpha$ is a limit ordinal, and $\langle Y_\rho \cup Z_\theta \rangle$ is nice in $\langle Y_\theta \rangle$ for all $\rho \leqslant \theta < \alpha$. If $w \in \langle Y_\alpha \rangle$, then $w \in \langle Y_\theta \rangle$ for some $\theta < \alpha$. We may assume $w$ has maximal value in $w + \langle Y_\rho \cup Z_\theta \rangle$. But $Y_\theta \cup (Z \setminus Z_\theta)$ is a decomposition basis, so $w$ has maximal value in $w + \langle Y_\rho \cup Z \rangle$.

Taking $\alpha = \lambda$ we see that $\langle Y_\rho \cup Z \rangle$ is nice in $\langle Y \rangle$. In particular, taking $\rho = 0$ shows $\langle Z \rangle$ is nice in $\langle Y \rangle$. Since

$$\begin{aligned}\langle Y_{\rho+1} \cup Z \rangle / \langle Y_\rho \cup Z \rangle &= \langle Y_{\rho+1} \cup Z \setminus \langle Y_{\rho+1} \rangle \rangle / \langle Y_\rho \cup Z^\rho \cup Z \setminus \langle Y_{\rho+1} \rangle \rangle \\ &\simeq \langle Y_{\rho+1} \rangle / \langle Y_\rho \cup Z^\rho \rangle\end{aligned}$$

is Fuchsian, this yields a nice composition series for $\langle Y \rangle / \langle Z \rangle$. □

COROLLARY 9.4. *Each decomposition basis of a Warfield group has a subordinate which generates a nice subgroup with simply presented cokernel.*

PROOF. Let $X$ be an arbitrary decomposition basis of $G$ and $Y$ a nice decomposition basis with simply presented cokernel. We may assume that $X \subset \langle Y \rangle$. Theorem 9.3 and Lemma 4.1 complete the proof. □

Another consequence of Theorem 9.3 is

COROLLARY 9.5. *Let $G$ be a group with a decomposition basis which generates a nice subgroup. Then every decomposition basis of $G$ has a subordinate which generates a nice subgroup.* $\square$

One might conjecture that the cokernel of any subgroup of a direct sum of valuated cyclics generated by a nice decomposition basis is Fuchsian. We give an example to show that this is not the case.

EXAMPLE 9.6. Let $G$ be an abelian $p$-group without elements of infinite height which is not a direct sum of cyclics, that is, is not Fuchsian. Let $\Sigma C_i \to G$ be a pure resolution of $G$ with each $C_i$ a finite cyclic, and let $F$ be a direct sum of infinite cyclic valuated groups with generators $x_i$ such that $vp^n x_i = n$ if $p^n$ is less than the order of $C_i$, and $vp^{n+m} x_i = \omega + n$ if $p^m$ is the order of $C_i$. Consider the composite map given by $F \to \Sigma C_i \to G$ where $x_i$ is mapped onto a generator of $C_i$. This map is a nice cokernel since it is the composite of two nice cokernels. Its kernel $K$ contains $F(\omega)$. Since $K/F(\omega) \subset \Sigma C_i$ is a direct sum of cyclics, the local stacked basis theorem says we can find a (group) basis $\{y_\alpha\}$ of $K$ so that $\{n_\alpha y_\alpha\}$ is a basis for $F(\omega)$. The purity of $K/F(\omega)$ in $\Sigma C_i$, and the stacking of $F(\omega)$ in $K$, imply that the $y_\alpha$ constitute a decomposition basis of $F$. But $F/K = G$ is not Fuchsian. $\square$

**10. Summands of Warfield groups are Warfield.** The closure of the class of Warfield groups under taking summands is automatic under Warfield's original definition of these groups as summands of simply presented groups, but the difficulties then translate into the equivalent problem of whether such summands contain nice decomposition bases with simply presented cokernels. The global Azumaya theorems of **[AHR]** provide the decomposition bases and Corollary 9.4 makes them nice with simply presented cokernel.

THEOREM 10.1. *A summand of a Warfield group is Warfield.*

PROOF. Let $A = B \oplus C$ be a Warfield group. By **[AHR**, Theorem 6.6] $B$ and $C$ have decomposition bases $X$ and $Y$. By Corollary 9.4 we may assume $\langle X \cup Y\rangle$ is nice and has simply presented cokernel. Therefore $\langle X\rangle$ is nice in $B$ with simply presented cokernel. $\square$

**11. A local fine embedding theorem.** For our global existence theorem we need to generalize the countable local fine embedding theorem **[HRW2**, Theorem 12] to allow for uncountable ordinals. Let $H$ be a direct sum of valuated cyclics with basis $X$. For each ordinal $\alpha$ define the *finite jump invariants of $H$ at $\alpha$* by

$$FJ_H(\alpha) = \operatorname{card}\{x \in X\colon vp^n x = \alpha \text{ for some } n \text{ and } \alpha + \omega > vp^{n+1}x > \alpha + 1\}.$$

The $\alpha$th derived Ulm invariant of $H$ is easily computed to be

$$g_H(\alpha) = \operatorname{card}\{x \in X\colon vp^n x = \alpha \neq vp^m x + 1 \text{ for some } n \text{ and all } m\}.$$

An *OC function* $f$ is a cardinal valued function on an ordinal together with a cardinal $f(\infty)$. We will freely extend the domain of an OC function to larger ordinals by defining it to be zero outside of its original domain. If $f_1$ and $f_2$ are OC functions, then we say that $f_1$ *dominates* $f_2$ if $\Sigma_{n<\omega} f_1(\alpha + n) \geqslant \Sigma_{\alpha+\omega<\lambda} f_2(\lambda)$ for all ordinals $\alpha$. A function is *admissible* if it dominates itself. A direct sum of cyclics $H$ *admits* an OC function $f$ if $f + FJ_H$ dominates $f + g_H$. Note that when everything is countable, this definition of 'admits' agrees with that in [**HRW2**, p. 357].

For each ordinal $\alpha$ define *band*$(\alpha)$ to be the set of ordinals $\lambda$ such that $\alpha + \omega = \lambda + \omega$. A valuated group $A$ is said to be a *group on* $(\alpha, \theta)$ if, for each $x$ in $A$ with $\alpha < vx < \theta$, there is a $y$ in $A$ such that $\alpha \leqslant vy$ and $py = x$.

LEMMA 11.1. *Let $H$ be a countable direct sum of local cyclic reduced valuated groups, and $f$ an OC function such that $\Sigma f(\alpha)$ is countable. If $H$ admits $f$, then $H$ can be embedded nicely in a countable valuated group $A$ with Fuchsian cokernel, such that $f_{A,H} = f$. Moreover if $W = \bigcup \{\text{band}(\alpha)\colon \alpha \in vH \text{ or } f(\alpha) \neq 0\}$, then $A$ is a group on $(\alpha, \theta)$ if for all $\rho_1 < \rho_2 < \rho_3$ in $(\alpha, \theta)$ we have $\rho_1$ and $\rho_3$ in $W$ implies $\rho_2$ in $W$.*

PROOF. Let $\phi$ be an isomorphism of $W$ onto a countable ordinal. Let $K$ be $H$ valuated with $\phi \circ v$. Then $K$ admits $f \circ \phi^{-1}$ since band$(\phi\alpha) = \phi(\text{band }\alpha)$. By [**HRW2**, Theorem 12] we can construct a countable group $B$ with $K$ as a nice subgroup such that $f \circ \phi^{-1} = f_{B,K}$. Let $A$ be $B$ valuated with $\phi^{-1} \circ v$. $\square$

In [**RWA**, Theorem 1] it was shown how to embed an arbitrary valuated group $A$ in a group $T(A)$. If $A$ is already a group on some intervals $(\alpha, \lambda)$, then we can modify this construction to exercise control over the relative Ulm invariants of the embedding.

LEMMA 11.2. *Let $A$ be a reduced valuated group which is a group on $(\alpha, \alpha + \omega)$ for all $\alpha$, and $X$ a set of ordinals such that $A$ is a group on $(\alpha, \infty)$ whenever $[\alpha, \alpha + \omega) \cap X$ is empty. Then there is a reduced group $B$ containing $A$ as a nice subgroup with simply presented torsion cokernel, such that $\alpha \in X$ if $f_{B,A}(\alpha) \neq 0$. Moreover* card $B(\alpha) \leqslant \omega \times$ card $A(\alpha) \times$ *(length$(A) - \alpha$) for all $\alpha <$ length $A$, and $B(\alpha) = 0$ if $\alpha \geqslant$ length $A$.*

PROOF. Let $Y$ be the set of ordinals $\alpha$ such that $(\theta, \theta + \omega) \cap X$ is nonempty whenever $\theta + \omega \leqslant \alpha$. Let $B$ have generators the set of strings $\{\alpha_1\alpha_2 \ldots \alpha_n a\colon 0 \neq a \in A,\ \alpha_i \in Y,\ \alpha_1 < \cdots < \alpha_n < va$, and either $\alpha_i + 1 = \alpha_{i+1}$ or $\alpha_i \in X\}$ subject to the relations on $A$ and the relations $p\alpha_1 \ldots \alpha_n a = \alpha_2 \ldots \alpha_n a$ and $p\alpha_1 a = a$. Since $B$ is a subgroup of $T(A)$, the embedding is nice. The cokernel is obviously simply presented. Any nonzero element of $f_{B,A}(\alpha)$ can be represented as a sum $z$ of distinct terms $\alpha_1 \ldots \alpha_n a$ with unit coefficients, and $\alpha_1 = \alpha$. But if $\alpha \notin X$, then $\alpha_2 = \alpha + 1$ so $vpz = \alpha + 1$, a contradiction. Finally $B(\alpha)$ is generated by the set of strings of the form $\alpha_1 \ldots \alpha_n a$ with $n \geqslant 0$, $\alpha_1 \geqslant \alpha$ and $a \in A(\alpha)$, the cardinality of which is no greater than indicated. $\square$

THEOREM 11.3. *Let $H$ be a countable direct sum of local reduced valuated cyclics and $f$ an OC function. If $H$ admits $f$, then there is a local Warfield group $G$ containing $H$ as a nice valuated subgroup with simply presented torsion cokernel such that $f_{G,H} = f$.*

PROOF. Let $f^*$ be the OC function given by $f^*(\alpha) = \inf_{m<\omega} \Sigma_{n<\omega} f(\alpha + m + n)$, and let $\lambda$ be the least ordinal such that $f^*(\lambda)$ is countable. Define $f'$ by

$$f'(\alpha) = \begin{cases} \min(f(\alpha), \omega) & \text{if } \alpha \in \text{band}(vh) \text{ for } h \in H \text{ or } \alpha \geqslant \lambda, \\ 0 & \text{otherwise.} \end{cases}$$

As $\Sigma f'(\alpha)$ is countable and $H$ admits $f'$, Lemma 11.1 yields a countable valuated group $A$ containing $H$ nicely with $f_{A,H} = f'$, such that $A$ is a group on every band and on $(\lambda, \infty)$. Let $X = \{\alpha\colon \alpha < \lambda \text{ and } f^*(\alpha) = f(\alpha)\}$. Then $X$ and $A$ satisfy the conditions of Lemma 11.2 so we have a group $B$ containing $A$, and hence $H$, as a nice subgroup with simply presented torsion cokernel, such that $f_{B,H}(\alpha) = f'(\alpha)$ if $\alpha \notin X$ and $f_{B,H}(\alpha) \leqslant f_B(\alpha) \leqslant \text{card } B(\alpha) \leqslant \omega \times \text{card } A(\alpha) \times (\text{length } A - \alpha) \leqslant \omega \times (\lambda - \alpha) \leqslant f(\alpha)$ if $\alpha$ is in $X$. Thus $f_{B,H}(\alpha) \leqslant f(\alpha)$ for all $\alpha$, and $f_{B,H}(\alpha) = f(\alpha)$ if $\alpha \geqslant \lambda + \omega$. It now suffices to add a simply presented $p$-group to $B$. Since $f^*(\alpha)$ is uncountable for $\alpha < \lambda$, we may choose $f''$ admissible so that length$(f'') < \lambda + \omega$ and $f = f_{B,H} + f''$. Let $C$ be a simply presented $p$-group with Ulm invariants $f''$. Then $G = B \oplus C$. $\square$

A direct sum of valuated cyclics $H$ *allows* an OC function $F$ if there is an $f$ such that $F = f + f_H$ and $H$ admits $f$.

THEOREM 11.4. *Let $H$ be a countable direct sum of infinite cyclic reduced valuated groups and $F$ an OC function. If $F \geqslant f_H$ and $F$ dominates $F + g_H$ then there is a subordinate of $H$ which allows $F$.*

PROOF. Let $\lambda$ be the least ordinal such that $\Sigma_{n<\omega} F(\lambda + n)$ is countable. We may assume that $H = M \oplus L$ where $L(\lambda) = L$ and $vx < \lambda$ for $0 \neq x \in M$. Let $F'$ be defined by $F'(\alpha) = F(\lambda + \alpha)$ and let $L'$ be $L$ valued with $v'$ such that $\lambda + v'(x) = v(x)$. Since $F'$ is admissible, $v'x$ is countable for all $x$ in $L'$. By **[HRW2**, Theorem 13], there is a subordinate $N'$ of $L'$ admitting an OC function $f'$ such that $F' = f' + f_{N'}$. Let $N$ be the corresponding subordinate of $L$, set $f(\alpha + \lambda) = f'(\alpha)$ and, for $\alpha < \lambda$ let $f(\alpha)$ be anything satisfying $F(\alpha) = f(\alpha) + f_M$. Set $K = M \oplus N$. Then $F = f + f_K$. The proof that $K$ admits $f$ has two cases. If $\alpha < \lambda$ then

$$\sum_{n<\omega} f(\alpha + n) + FJ_K(\alpha + n) = \sum_{n<\omega} F(\alpha + n) \geqslant \sum_{\theta \geqslant \alpha+\omega} F(\theta) + g_H(\theta)$$

$$\geqslant \sum_{\theta \geqslant \alpha+\omega} F(\theta) + g_K(\theta) \geqslant \sum_{\theta \geqslant \alpha+\omega} f(\theta) + g_K(\theta).$$

If $\alpha = \lambda + \rho$ then

$$\sum_{n<\omega} f(\alpha + n) + FJ_K(\alpha + n) = \sum_{n<\omega} f'(\rho + n) + FJ_{N'}(\rho + n)$$

$$= \sum_{\theta \geqslant \rho+\omega} f'(\theta) + g_{N'}(\theta) = \sum_{\theta \geqslant \alpha+\omega} f(\theta) + g_K(\theta). \quad \square$$

The restriction to reduced valuated groups in this section may be removed. The following theorem shows that the elements of value $\infty$ can be taken care of separately.

THEOREM 11.5. *Let $H$ be a direct sum of cyclic $p$-local valuated groups. Then $H$ can be embedded as a nice subgroup with simply presented torsion cokernel in $G = A \oplus D$, where $A$ is a direct sum of reduced cyclic valuated groups and $D$ is a divisible group. Moreover $G$ and $H$ have the same Ulm, derived Ulm, and Warfield invariants.*

PROOF. Let $D$ be a divisible hull of $H(\infty)$ and $G$ the amalgamated sum of $H$ and $D$ over $H(\infty)$. A reduced cyclic in $H$ becomes a cyclic in $A$, while a nonreduced cyclic in $H$ is embedded in a torsion cyclic of $A$ plus a rank-one summand of $D$. Clearly all invariants are preserved. $\square$

**12. Global existence.** A *global* OC function $F$ is a family of OC functions $F_p$, one for each prime $p$. We say a property defined for local OC functions holds for $F$ if it holds for each $F_p$. A direct sum of valuated cyclics $K$ allows a global OC function $F$ if $K_p$ allows $F_p$ for all $p$.

LEMMA 12.1. *Let $H$ be a countable direct sum of infinite valuated cyclic groups, and $F$ an OC function. If, for each prime $p$, the localization $H_p$ has a subordinate allowing $F_p$, then $H$ has a subordinate allowing $F$.*

PROOF. Let $K(p)$ be a subordinate of $H_p$ allowing $F_p$. If $K(p)$ is of finite index in some subordinate $L(p)$ of $H_p$, then $L(p)$ allows $F_p$. So we may choose $K(p)$ so that the first $p$ generators of $K(p)$ are the first $p$ generators of $H$. Then $K = \bigcap K(p)$ is a subordinate of $H$ such that $K_p = K(p)$. $\square$

THEOREM 12.2. *Let $H$ be a countable direct sum of infinite cyclic valuated groups and $F$ a global OC function. If $F \geqslant f_H$ and $F$ dominates $F + g_H$ then there is a subordinate of $H$ which allows $F$. Moreover there is a Warfield group $G$ with the Warfield invariants of $H$ and Ulm invariants $F$.*

PROOF. By Theorem 11.4 each $H_p$ has a subordinate allowing $F_p$, so Lemma 12.1 yields a subordinate $K$ which allows $F$. For each prime $p$, Theorem 11.3 provides a nice sequence $0 \to K_p \to G(p) \to T(p) \to 0$ where $G(p)$ is a $p$-local Warfield group with Ulm invariants $F_p$, and $T(p)$ is simply presented torsion. We glue these sequences together using Lemma 5.4 to obtain $G$. $\square$

THEOREM 12.3. *Let $L$ be a direct sum of valuated cyclics and $f$ a global OC function which dominates $f + g_L$. Then $L$ can be nicely embedded in a simply presented group $A$ such that $f_{A,L} = f$ and $A/L$ is simply presented torsion.*

PROOF. Let $X$ be a basis of $L$. By [**HRW2**, Theorem 3] we can, for each prime $p$, embed the valuated $p$-tree $\{p^n x\colon x \in X\}$ in a $p$-tree $T(p)$ with relative Ulm invariants $f_p$. Note that $X \subset T(p)$ for all $p$. Let $A$ be the group generated by the nodes of the $T(p)$'s subject to the relations of the $T(p)$'s. $\square$

COROLLARY 12.4. *Every Warfield group is isomorphic to a summand of a simply presented group with a simply presented torsion complement.*

PROOF. Let $G$ be a Warfield group and $X$ a decomposition basis for $G$ such that $L = \langle X \rangle$ is nice and has simply presented cokernel. Let $f(\alpha) = \omega \times f_A(\alpha)$. Then $f$

dominates $f + g_L$ by [**HRW2**, Theorem 3] so Theorem 12.3 provides a simply presented group $A$ containing $L$ as a nice subgroup with simply presented torsion cokernel such that $f_{A,L} = f$. Let $B$ be a simply presented torsion group with Ulm invariants $f$. Then Theorem 6.1 says that $G \oplus B$ is isomorphic to $A$. $\square$

THEOREM 12.5. *Let $H$ be a direct sum of infinite cyclic valuated groups and $F$ a global OC function. If $F$ dominates $F + g_H$, and $F \geqslant f_H$, then there exists a Warfield group $G$ with Ulm function $F$ and Warfield invariants equal to those of $H$. Furthermore, $G = A \oplus B$ where $A$ is simply presented and $B$ has countable torsion-free rank.*

PROOF. Let $X$ be a basis for $H$. For each prime $p$, let $\lambda_p$ be the least ordinal such that $\Sigma_{n<\omega} F_p(\lambda_p + n)$ is countable. Since $F$ dominates $g_H$ and $F \geqslant f_H$, then for each $p$ there is a partition of $X$ into subsets $L(p)$ and $M(p)$ so that

$$M(p) \text{ is countable,} \tag{1}$$

$$\langle L(p)\rangle(\lambda_p + \omega) = 0, \tag{2}$$

$$f_{\langle L(p)\rangle}(p, \alpha) = 0 \text{ if } \alpha \geqslant \lambda_p. \tag{3}$$

Let $M = \bigcup_p M(p)$ and $L = X \setminus M$.

Define $f$ and $h$ by

$$f_p(\alpha) = \begin{cases} F_p(\alpha) & \text{if } F_p(\alpha) \text{ is infinite and } \alpha < \lambda_p, \\ 0 \quad \text{if } \alpha \geqslant \lambda_p, \\ F_p(\alpha) - f_H(p, \alpha) & \text{otherwise;} \end{cases}$$

$$h_p(\alpha) = \begin{cases} F_p(\alpha) & \text{if } F_p(\alpha) \text{ is infinite and } \alpha + \omega < \lambda_p, \\ F_p(\alpha) & \text{if } \alpha \geqslant \lambda_p, \\ f_{\langle M\rangle}(p, \alpha) & \text{otherwise.} \end{cases}$$

Then $f + f_{\langle L\rangle} + h = F$, $f$ dominates $f + g_{\langle L\rangle}$ and $h$ dominates $h + g_{\langle M\rangle}$. By Theorem 12.3 there is a simply presented group $A$ containing $\langle L\rangle$ such that $f_{A,\langle L\rangle} = f$, and Theorem 12.2 gives a Warfield group $B$ with Ulm function $h$ and Warfield invariants the same as those of $\langle M\rangle$. Since $f_A + f_B = F$ and $w_A + w_H = w_{\langle L\rangle} + w_{\langle M\rangle} = w_H$, the group $A \oplus B$ is the one sought. $\square$

The conditions $F$ dominates $F + g_H$ and $F \geqslant f_H$ are certainly necessary in Theorem 12.5 because of [**HRW2**, Theorem 6].

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