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ARBORESCENT STRUCTURES. II: INTERPRETABILITY IN THE THEORY OF TREES

BY

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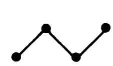
ABSTRACT. The first-order theory of arborescent structures is shown to be completely faithfully interpretable in the first-order theory of trees. It follows from this interpretation that Vaught's conjecture is true for arborescent structures, the theory of arborescent structures is decidable, and every $\aleph_0$-categorical arborescent structure has a decidable theory.

Arborescent structures were introduced in [8] by abstracting a very simple property that trees possess. The main result of [8] is that every consistent, recursively axiomatizable theory of arborescent structures has a recursive model. In this paper we continue the study of arborescent structures, showing as the principal result that the theory of arborescent structures is interpretable in the theory of trees in a very strong way. From this, many interesting results are deducible; for example, the theory of arborescent structures is decidable (Corollary 4.2) and Vaught's conjecture is true for arborescent structures (Corollary 4.4).

For a better appreciation of the significance of the notion of arborescence, some familiarity with the development leading to its isolation might be useful. Rabin [3] proved the decidability of $S2S$, and from this result the decidability of (more than) the first-order theory of trees easily follows by interpreting it in $S2S$. My interest in the theory of trees originated after proving in [5] that every $\aleph_0$-categorical theory of trees is decidable. This result was later shown to be connected with that of Rabin's when I proved in [4] that if a theory is interpretable in $S2S$, then so is every one of its $\aleph_0$-categorical completions. At this point I became interested in finding some rather simple conditions which implied interpretability in $S2S$, or in the theory of trees, possibly some sort of structural property. I first considered partially ordered sets, and was able to extend the above results for trees to reticles [7], which are partially ordered sets not embedding the 4-element partially ordered set



by showing that the theory of reticles is interpretable in the theory of trees. Reticles turned out to be a fascinating kind of partially ordered set which appeared to possess many of the nicer properties that trees do. I tried for some time to modify the interpretation of the theory of reticles in the theory of trees in order to

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generalize to reticles the theorem of Ershov [1] asserting that every recursively axiomatizable theory of trees has a recursive model. This proved unsuccessful, but I was finally able to prove directly the generalization to reticles. From this proof, given in [8], it became immediately apparent that the only property of reticles which was needed was arborescence, which, although trivial for trees, is not quite so for reticles. In fact, reticles turn out to be precisely those partially ordered sets which are arborescent. Those graphs which are arborescent are just the ones which do not have as an induced subgraph a four element path, and they are exactly the comparability graphs of reticles. These graphs were investigated in [6], where the decidability of the theory of these graphs as well as each of its $\aleph_0$-categorical completions was proved. Seeing the importance of arborescence, I returned to my original results about reticles, and considered whether these were true for all arborescent structures. The results of this paper imply they are by demonstrating that the theory of arborescent structures is interpretable in the theory of trees. This interpretation then yields many results for arborescent structures which follow immediately from the corresponding results for trees; these are to be found in §4 of this paper.

1. Definitions. We will consider throughout this paper a fixed finite language $\mathfrak{L} = \{R_0, \ldots, R_{m-1}, U_0, \ldots, U_{n-1}\}$, where each R_i is a binary predicate symbol and each U_j is a unary predicate symbol. Frequent use will be made of the 4-ary $\mathfrak{L}$-formula in the following definition.

DEFINITION 1.1. $(x, y) \equiv (u, v)$ will denote the 4-ary $\mathfrak{L}$-formula

$$x \neq y \wedge u \neq v \wedge \bigwedge_{i<m} \left[R_i(x, y) \leftrightarrow R_i(u, v) \wedge R_i(y, x) \leftrightarrow R_i(v, u) \right].$$

It is obvious that in any $\mathfrak{L}$-structure the formula $(x, y) \equiv (u, v)$ defines an equivalence relation on the set of ordered pairs of distinct elements. Also, the sentence $(x, y) \equiv (u, v) \rightarrow (y, x) \equiv (v, u)$ is true in every $\mathfrak{L}$-structure.

DEFINITION 1.2. Let $\mathfrak{A}$ be a $\mathfrak{L}$-structure and $B \subseteq A$.

(1) A partition $\{X, Y\}$ of B into two nonempty subsets is an *arborescent partition* of B iff whenever $a_1, a_2 \in X$ and $b_1, b_2 \in Y$ then $(a_1, b_1) \equiv (a_2, b_2)$.

(2) Two distinct elements $a, b \in B$ form an *arborescent pair* for B iff whenever $c \in B - \{a, b\}$, then $(a, c) \equiv (b, c)$.

The following gives two alternative definitions of an arborescent structure which are shown to be equivalent in Proposition 1.4 of [6].

DEFINITION/PROPOSITION 1.3. *For any $\mathfrak{L}$-structure $\mathfrak{A}$ the following are equivalent*:

(1) $\mathfrak{A}$ *is arborescent*;

(2) *for each finite $B \subseteq A$, if $|B| \geqslant 2$, then there is an arborescent partition of B*;

(3) *for each finite $B \subseteq A$, if $|B| \geqslant 2$, then there is an arborescent pair for B.*

The proof of the equivalence of (2) and (3), although very simple, is nevertheless instructive. Both directions of the equivalence are proved by induction on the cardinality of B, and both directions are trivial whenever $|B| = 2$. For (2) $\Rightarrow$ (3), if $|B| > 2$ and $\{X, Y\}$ is an arborescent partition of B with $|X| \geqslant 2$, then any arborescent pair for X is also an arborescent pair for B. For (3) $\Rightarrow$ (2), suppose that

$|B| > 2$ and that $a, b \in B$ is an arborescent pair for B. Let $\{X, Y\}$ be an arborescent partition of $B - \{a\}$ with $b \in Y$. Then $\{X, Y \cup \{a\}\}$ is an arborescent partition of B.

The definition of arborescence implies that a structure is arborescent iff each of its finite substructures is. One of the surprising results of this paper is Corollary 4.5 which asserts that in order to check for arborescence, only those substructures with at most 4 elements need be considered.

2. The canonical interpretation. The canonical interpretation Π of the theory of arborescent structures in the theory of trees will be described in this section. It will then be shown by an easy, but rather nonconstructive, argument that Π is a complete interpretation. From this fact we will be able to deduce several results about the theory of arborescent structures. Later, in §3, it will be shown by a much more involved construction that Π is even a completely faithful intrepretation of the theory of arborescent structures in the theory of trees. (For a very brief discussion of interpretability and the associated terminology, consult the Appendix to this paper.)

The interpretation Π which will be described here generalizes the interpretation of the theory of reticles in the theory of trees which was presented in [7]. In actuality, we will interpret the theory of arborescent structures in the theory of $(2m + n + 3)$-augmented trees, but this latter theory is completely faithfully interpretable in the theory of trees, as is shown in Proposition A of the Appendix.

Consider a language

$$\mathfrak{L}^* = \{<, M, G, I, S_0, \ldots, S_{m-1}, T_0, \ldots, T_{m-1}, V_0, \ldots, V_{n-1}\},$$

where $<$ is a binary predicate symbol and each of the remaining symbols denotes a unary predicate. This language is essentially the language of $(2m + n + 3)$-augmented trees. Call an $\mathfrak{L}^*$-structure

$$\mathfrak{D} = (D, <, M, G, I, S_0, \ldots, S_{m-1}, T_0, \ldots, T_{m-1}, V_0, \ldots, V_{n-1})$$

a *determining tree* if it satisfies conditions (2.1)–(2.5) which follow:

(2.1) $(D, <)$ is a tree; M is the set of maximal elements; G is the set of infima of pairs of distinct elements in M; and $I = D - (M \cup G)$.

(2.2) Each element in I is the immediate successor of a (necessarily unique) member of G.

(2.3) Each element of G has at most one immediate successor whch is in I.

(2.4) $V_j \subseteq M$ for each $j < n$.

(2.5) $S_i \subseteq G$ and $T_i \subseteq G \cup M$ for each $i < m$.

We will now describe the interpretation Π^* of the theory of arborescent structures in the theory of $\mathfrak{L}^*$-structures which are $(2m + n + 3)$-augmented trees. The only $\mathfrak{L}^*$-structures which we will really be interested in are determining trees. Therefore, if $\mathfrak{D}$ is an $\mathfrak{L}^*$-structure which is not a determining tree, let $\mathfrak{D}^{\Pi^*} = \mathfrak{A}$, where $\mathfrak{A}$ is the arborescent structure such that $A = M^{\mathfrak{D}}$ and $R_0 = R_1 = \ldots = R_{m-1} = U_0 = U_1 = \ldots = U_{n-1} = \varnothing$.

Now, given a determining tree $\mathfrak{D}$, we will construct an arborescent $\mathfrak{L}$-structure $\mathfrak{A} = \mathfrak{D}^{\Pi^*}$ as follows:

(2.6) $A = M$;

(2.7) $U_j = V_j$ for each $j < n$;

(2.8) if $x, y \in A$ are distinct and $i < m$, then $(x, y) \in R_i$ iff one of the following holds, where $a = \inf(x, y)$:

(.1) a has no immediate successor in I, and $a \in S_i \cup T_i$;

(.2) a has an immediate successor $b \in I$, $b < y$, and $a \in T_i$;

(.3) a has an immediate successor $b \in I$, $b < x$, and $a \in S_i$;

(2.9) if $x \in A$ and $i < m$, then $(x, x) \in R_i$ iff $x \in S_i$.

Even though we have not explicitly defined Π^* syntactically, we have described it semantically. From this description, a syntactic definition is easily made. Composing Π^* with the interpretation of $(2m + n + 3)$-augmented trees in the theory of trees (see Proposition A in the Appendix), we obtain an interpretation Π, which we will refer to as the *canonical interpretation* of the theory of arborescent structures in the theory of trees. The next proposition justifies this terminology.

PROPOSITION 2.1. *The canonical interpretation* Π *is an interpretation of the theory of arborescent structures in the theory of trees.*

PROOF. Clearly, it suffices to show that Π^* is an interpretation of the theory of arborescent structures in the theory of determining trees. Let $\mathfrak{D}$ be a determining tree, let $\mathfrak{A} = \mathfrak{D}^{\Pi}$, and let $B \subseteq A$ be finite such that $|B| > 1$. Let a be a maximal element of the set $\{\inf(x, y) : x, y \in B,\ x \neq y\}$, and let $x, y \in B$ be such that $a = \inf(x, y)$. Then it is easy to see that x, y form an arborescent pair for B. $\square$

Next we show that every arborescent structure is determined in this manner from a determining tree.

PROPOSITION 2.2. *The canonical interpretation* Π *is complete.*

PROOF. By a standard application of the Compactness Theorem, we need consider only finite $\mathfrak{A}$. We will show that for every finite arborescent $\mathfrak{A}$ there is a determining tree $\mathfrak{D}$ such that $\mathfrak{D}^{\Pi^*} = \mathfrak{A}$. If $|A| < 1$, then the result is trivial, so suppose that $|A| > 1$, and that the result is true for all arborescent structures of smaller size. Let $x, y \in A$ be an arborescent pair for A, and let $\mathfrak{D}_0$ be a determining tree such that $\mathfrak{D}_0^{\Pi^*} = \mathfrak{A}_0 = \mathfrak{A}|(A - \{x\})$, where $x \notin D_0$. We construct a determining tree $\mathfrak{D} \supseteq \mathfrak{D}_0$ as follows. Let $D = D_0 \cup \{a, b, x\}$, where a and b are two new points. We let $a < b < x$, $x \in M$, $b \in I$, $a \in G$, and for each $z \in D_0$ we let the following hold:

(1) $z < a$ iff $z < b$ iff $z < x$ iff $z < y$;

(2) $z > a$ iff $z = y$;

(3) neither $z > b$ nor $z > x$.

Now let:

(4) $x \in V_j$ iff $x \in U_j$, for each $j < n$;

(5) $a \in S_i$ iff $(x, y) \in R_i$, for each $i < m$;

(6) $a \in T_i$ iff $(y, x) \in R_i$, for each $i < m$;

(7) $x \in S_i$ iff $(x, x) \in R_i$, for each $i < m$.
This $\mathfrak{D}$ is easily seen to work. $\square$

The easy construction of a determining tree in Proposition 2.2 yields quick proofs of Corollaries 4.1–4.3 stated in §4. However, this construction does have a defect: since the Compactness Theorem was used, the determining tree was obtained in a nonconstructive manner. It is desirable to have a more constructive method of obtaining a determining tree. In §3 we will present a much more lengthy construction in order to obtain a canonical determining tree for each arborescent structure, showing that the canonical interpretation Π is completely faithful.

3. The canonical interpretation is completely faithful. The canonical interpretation Π of the theory of arborescent structures in the theory of trees was defined in §2 and shown there to be complete. The purpose of this section is to improve that fact as follows.

THEOREM 3.1. *The canonical interpretation* Π *of the theory of arborescent structures in the theory of trees is completely faithful.*

In §2 we actually worked with an interpretation Π^* of the theory of arborescent structures in the theory of $(2m + n + 3)$-augmented trees. But since this latter theory is completely faithfully interpretable in the theory of trees it suffices to show that Π^* is completely faithful (see Proposition A in the Appendix).

Some formulas in the language $\mathfrak{L}$ will be defined; certain sentences constructed from these formulas will be shown to hold in all arborescent structures. The following expositional expedient will be employed: an $\mathfrak{L}$-sentence will sometimes be interpreted as an assertion, which is to be understood to hold in all arborescent structures. However, in proving such an assertion, we will have in mind a particular arborescent structure.

The following two sentences hold in all arborescent structures.

$$(x, y) \equiv (x, z) \vee (y, z) \equiv (y, x) \vee (z, x) \equiv (z, y) \vee x = y = z. \tag{3.1}$$

$$(x, y) \equiv (z, y) \equiv (z, w) \not\equiv (y, w) \equiv (x, w) \equiv (x, z) \rightarrow (z, w) \equiv (w, y). \tag{3.2}$$

Each of (3.1) and (3.2) is easily checked by, for example, appealing to Proposition 1.3. Considering (3) of that proposition and letting $B = \{x, y, z\}$ for (3.1) and $B = \{x, y, z, w\}$ for (3.2), one can easily see the proofs.

The following terminology will be used in regard to (3.2). If x, y, z, w are distinct elements of an $\mathfrak{L}$-structure, and (3.2) is false for these elements, then we will say that $\langle x, y, z, w\rangle$ *contradicts* (3.2). Notice that if $\langle x, y, z, w\rangle$ contradicts (3.2), then so does each of $\langle w, z, y, x\rangle$, $\langle y, w, x, z\rangle$ and $\langle z, x, w, y\rangle$.

We now define a ternary formula $\varphi_1(x, y, z)$:

$$\varphi_1(x, y, z) =_{\mathrm{df}} (x, y) \not\equiv (x, z) \equiv (y, z) \not\equiv (y, x). \tag{3.3}$$

The following three sentences follow easily from the definition of φ_1.

$$\varphi_1(x, y, z) \rightarrow \varphi_1(y, x, z), \tag{3.4}$$

$$x \neq y \rightarrow \varphi_1(x, x, y), \tag{3.5}$$

$$\neg\varphi_1(x, y, y) \wedge \neg\varphi_1(y, x, y). \tag{3.6}$$

The sentences (3.7)–(3.9) will be used later on.

$$\varphi_1(x, y, z) \wedge \varphi_1(x, z, w) \to \varphi_1(y, z, w). \tag{3.7}$$

To prove (3.7) assume $\varphi_1(x, y, z)$ and $\varphi_1(x, z, w)$. We can furthermore assume that x, y, z, w are pairwise distinct as otherwise the conclusion follows immediately. From $\varphi_1(x, y, z)$ and $\varphi_1(x, z, w)$ it follows that $(z, y) \equiv (z, x)$ and $(z, w) \not\equiv (z, x)$, respectively. Thus $(z, w) \not\equiv (z, y)$, so that according to (3.1) either $(y, w) \equiv (y, z)$ or else $(y, w) \equiv (z, w)$. We will show that $(y, w) \not\equiv (y, z)$, so that $(y, w) \equiv (z, w)$ will follow.

Suppose, by way of contradiction, that $(y, w) \equiv (y, z)$. Then $(y, w) \not\equiv (x, w)$, so that according to (3.1) either $(y, x) \equiv (w, x)$ or $(y, x) \equiv (y, w)$. But, if $(y, x) \equiv (w, x)$, then $\langle y, x, w, z\rangle$ contradicts (3.2); and if $(y, x) \equiv (y, w)$, then $(y, x) \equiv (y, z)$ so that $\neg\varphi_1(x, y, z)$, contradicting the hypothesis. Thus $(y, w) \not\equiv (y, z)$, so that $(y, w) \equiv (z, w)$. Hence, definition (3.3) implies $\varphi_1(y, z, w)$. This proves (3.7).

$$[\varphi_1(x, y, u) \wedge \varphi_1(x, u, z) \wedge \varphi_1(x, z, w)] \to [\varphi_1(x, y, w) \vee \varphi_1(x, y, z)]. \tag{3.8}$$

To prove (3.8) assume $\varphi_1(x, y, u) \wedge \varphi_1(x, u, z) \wedge \varphi_1(x, z, w)$. We can further assume that all of x, y, u, z, w are pairwise distinct, for if they were not, then the result would follow easily from (3.5) and (3.6). (The case $y = w$ is slightly different–the hypothesis being a contradiction in this situation.) By (3.7) we get $\varphi_1(y, u, z)$ and $\varphi_1(u, z, w)$, so that the following hold:

$$(w, x) \equiv (w, z) \equiv (w, u), \quad (u, y) \equiv (u, x), \quad (z, x) \equiv (z, y) \equiv (z, u).$$

Now, in order to derive a contradiction, let us assume the conclusion to (3.8) to be false. Notice that $\neg\varphi_1(x, u, w)$ implies $(x, w) \equiv (x, u) \vee (u, x) \equiv (u, w)$ and $\neg\varphi_1(x, y, u)$ implies $(x, w) \equiv (x, y) \vee (y, x) \equiv (y, w)$. However, $(x, w) \not\equiv (x, y)$, for $(x, w) \equiv (x, y)$ together with either $(x, w) \equiv (x, u)$ or $(u, x) \equiv (u, w)$ yields $(x, y) \equiv (x, u)$, and this contradicts $\varphi_1(x, y, u)$. Thus, we have

$$(y, x) \equiv (y, w).$$

This leaves two possibilities: $(x, w) \equiv (x, u)$ or $(u, x) \equiv (u, w)$.

First consider $(x, w) \equiv (x, u)$. By (3.1), one of the following holds: $(w, y) \equiv (w, z)$, $(y, w) \equiv (y, z)$, or $(z, y) \equiv (z, w)$. But $(w, y) \equiv (w, z)$ implies $(y, x) \equiv (y, u)$, which contradicts $\varphi_1(x, y, u)$; and $(z, y) \equiv (z, w)$ implies $(u, z) \equiv (u, x)$, contradicting $\varphi_1(x, u, z)$. Thus, $(y, w) \equiv (y, z)$; but this implies that $(u, y) \equiv (w, y)$ (else $\langle z, w, u, y\rangle$ contradicts (3.2)), and this implies $(w, y) \equiv (w, z)$, which we have already seen to be impossible.

Next consider $(u, x) \equiv (u, w)$. By (3.1), one of the following holds: $(w, y) \equiv (w, z)$, $(z, y) \equiv (z, w)$, or $(y, w) \equiv (y, z)$. But $(w, y) \equiv (w, z)$ implies $(x, y) \equiv (x, u)$, which contradicts $\varphi_1(x, u, z)$; and $(z, y) \equiv (z, w)$ implies $(z, x) \equiv (u, x)$, which also contradicts $\varphi_1(x, u, z)$. Thus, $(y, w) \equiv (y, z)$; but this implies $(u, y) \equiv (w, y)$ (else $\langle z, w, u, y\rangle$ contradicts (3.2)), and $(u, y) \equiv (w, y)$ implies $(y, x) \equiv (y, u)$, contradicting $\varphi_1(x, y, u)$. This proves (3.8).

$$\varphi_1(x, y, z) \to [\varphi_1(x, y, w) \vee \varphi_1(x, w, y) \vee \varphi_1(x, z, w) \vee \varphi_1(x, w, z)]. \tag{3.9}$$

To check that (3.9) holds in each arborescent structure, assume $\varphi_1(x, y, z)$. We can assume that x, y, z, w are pairwise distinct, as otherwise the conclusion is immediate from (3.5) and (3.6). If either $(x, y) \equiv (x, w)$ or $(x, y) \equiv (w, x)$, then from (3.3) it easily follows that either $\varphi_1(x, z, w)$ or $\varphi_1(x, w, z)$. If $(x, y) \not\equiv (x, w) \not\equiv (y, x)$, then similarly $\varphi_1(x, y, w)$ or $\varphi_1(x, w, y)$. This ends the proof of (3.9).

Using φ_1 we define another formula φ_2.

$$\varphi_2(x, y, z) =_{\mathrm{df}} \varphi_1(x, y, z) \vee \exists w(\varphi_1(x, y, w) \wedge \varphi_1(x, w, z)). \tag{3.10}$$

The next sentence is the analogue of (3.4).

$$\varphi_2(x, y, z) \rightarrow \varphi_2(y, x, z). \tag{3.11}$$

To prove (3.11), assume $\varphi_2(x, y, z)$. If $\varphi_1(x, y, z)$, then the conclusion follows trivially from (3.4), so suppose that $\varphi_1(x, y, w) \wedge \varphi_1(x, w, z)$ for some w. Clearly, $(w, z) \not\equiv (w, y)$, so that from (4.1) either $(y, z) \equiv (y, w)$ or $(y, z) \equiv (w, z)$. However, if $(y, z) \equiv (y, w)$, then $(y, x) \equiv (z, x)$ (as otherwise $\neg\varphi_1(x, y, w)$), and then $\langle w, z, x, y\rangle$ contradicts (3.2). Thus $(y, z) \equiv (w, z)$, so that $\varphi_1(y, x, w)$ and $\varphi_1(y, w, z)$. This proves (3.11).

The sentence (3.11) will be frequently used without specific reference.

The formula $\varphi_2(x, y, z)$ defines a partial order which depends on the parameter x. This fact is expressed by the next two sentences.

$$\neg\varphi_2(x, y, y), \tag{3.12}$$

$$\varphi_2(x, y, z) \wedge \varphi_2(x, z, w) \rightarrow \varphi_2(x, y, w). \tag{3.13}$$

To show (3.12), suppose to the contrary that $\varphi_2(x, y, y)$. Then according to (3.10) either $\varphi_1(x, y, y)$ or $\exists w(\varphi_1(x, y, w) \wedge \varphi_1(x, w, y))$. The first alternative contradicts (3.6), so suppose $\varphi_1(x, y, w) \wedge \varphi_1(x, w, y)$. Then $\varphi_1(x, y, w)$ implies $(w, y) \equiv (w, x)$ and $\varphi_1(x, w, y)$ implies $(w, y) \not\equiv (w, x)$, a contradiction, proving (3.12).

Sentence (3.13) follows quite easily from (3.8).

If we think of the partial order $\varphi_2(x, y, z)$ in a way so that y is greater than z, then x is the unique greatest element; that is,

$$x \neq y \rightarrow \varphi_2(x, x, y). \tag{3.14}$$

Sentence (3.14) is immediate from its analogue (3.5).

The partial order turns out to be a weak order. (A *weak order* is a partial order in which incomparability is an equivalence relation.) This fact is captured by the following sentence which will play a key role in the sequel.

$$\varphi_2(x, y, z) \rightarrow \varphi_2(x, w, z) \vee \varphi_2(x, y, w). \tag{3.15}$$

This sentence follows quite easily from (3.9).

The following sentence is an improvement of (3.7).

$$\varphi_2(x, y, z) \wedge \varphi_1(x, z, w) \rightarrow \varphi_1(y, z, w). \tag{3.16}$$

To prove (3.16), assume that $\varphi_2(x, y, z)$ and $\varphi_1(x, z, w)$. If $\varphi_1(x, y, z)$, apply (3.7). Otherwise, let u be such that $\varphi_1(x, y, u)$ and $\varphi_1(x, u, z)$. Then (3.7) implies $\varphi_1(y, u, z)$, which, by (3.4), is equivalent to $\varphi_1(u, y, z)$. Apply (3.7) again to $\varphi_1(x, u, z)$ and $\varphi_1(x, z, w)$ to obtain $\varphi_1(u, z, w)$. Applying (3.7) one more time to $\varphi_1(u, y, z)$ and $\varphi_1(u, z, w)$ yields the desired conclusion.

$$\varphi_2(x, y, z) \wedge \varphi_2(x, z, w) \rightarrow \varphi_2(y, z, w). \tag{3.17}$$

To prove (3.17) assume that $\varphi_2(x, y, z)$ and $\varphi_2(x, z, w)$. If $\varphi_1(x, z, w)$ apply (3.16). Otherwise, let v be such that $\varphi_1(x, z, v)$ and $\varphi_1(x, v, w)$. It follows from (3.13) that $\varphi_2(x, y, v)$. Applying (3.16) we get $\varphi_1(y, v, w)$ and $\varphi_1(y, z, v)$. Hence $\varphi_2(y, z, w)$ by (3.10).

We now define two more 4-ary formulas.

$$\varphi_4(x, y; u, v) =_{\text{df}} \forall z(\varphi_2(x, y, z) \rightarrow \varphi_2(u, v, z)), \tag{3.18}$$

$$\varphi_3(x, y; u, v) =_{\text{df}} \varphi_4(x, y; u, v) \wedge \varphi_4(u, v; x, y). \tag{3.19}$$

Clearly, formula φ_3 defines an equivalence relation on the set of unordered pairs of (not necessarily distinct) elements. This is expressed by the following three sentences.

$$\varphi_3(x, y; y, x), \tag{3.20}$$

$$\varphi_3(x, y; u, v) \rightarrow \varphi_3(u, v; x, y), \tag{3.21}$$

$$\varphi_3(x, y; u, v) \wedge \varphi_3(u, v; z, w) \rightarrow \varphi_3(x, y; z, w). \tag{3.22}$$

It is also clear that the formula φ_4 defines a partial order on the set of equivalence classes defined by φ_3. This is expressed by the following two sentences together with (3.19).

$$\varphi_4(x, y; u, v) \wedge \varphi_4(u, v; z, w) \rightarrow \varphi_4(x, y; z, w). \tag{3.23}$$

$$\varphi_4(x, y; x, y). \tag{3.24}$$

The following sentence is an immediate consequence of the definition of φ_3 and sentences (3.13) and (3.17).

$$\varphi_2(x, y, z) \rightarrow \varphi_3(x, z; y, z). \tag{3.25}$$

$$\varphi_4(x, y; u, v) \rightarrow \varphi_3(x, y; x, u) \vee \varphi_3(x, y; y, u). \tag{3.26}$$

To prove (3.26), assume $\varphi_4(x, y; u, v)$. Notice that by (3.25) we can assume $\neg\varphi_2(x, u, y)$ and $\neg\varphi_2(y, u, x)$. We will now show that $\varphi_3(x, y; x, u)$. To show $\varphi_4(x, y; x, u)$, assume $\varphi_2(x, y, z)$. Then by (3.15) either $\varphi_2(x, y, u)$, or $\varphi_2(x, u, z)$. If the first alternative holds, then the original assumption implies that $\varphi_2(u, v, u)$, contradicting (3.11) and (3.12). Hence, $\varphi_2(x, u, z)$, showing $\varphi_4(x, y; x, u)$. Conversely, to show $\varphi_4(x, u; x, y)$, suppose $\varphi_2(x, u, z)$. Then by (3.15) either $\varphi_2(x, u, y)$ or $\varphi_2(x, y, z)$. The first alternative was assumed not to hold, so that $\varphi_2(x, y, z)$, thereby showing that $\varphi_4(x, u; x, y)$. Thus, $\varphi_3(x, y; x, u)$, proving (3.26).

$$\varphi_4(x, y; y, z) \leftrightarrow \neg\varphi_2(x, y, z). \tag{3.27}$$

To show one implication of (3.27) assume $\varphi_4(x, y; y, z)$. If $\varphi_2(x, y, z)$, then $\varphi_2(y, z, z)$ by (3.18), but this contradicts (3.12).

For the other direction, assume $\neg\varphi_2(x, y, z)$. We want to show $\varphi_4(x, y; y, z)$, so suppose $\varphi_2(x, y, u)$. Then $\varphi_2(y, x, u)$ by (3.11). Following (3.15) either $\varphi_2(y, z, u)$ or $\varphi_2(y, x, z)$. The second alternative contradicts $\neg\varphi_2(x, y, z)$ by (3.11), so that $\varphi_2(y, z, u)$. This proves (3.27).

Returning to the partial order defined by φ_4, we note that this partial order turns out to be a tree as is asserted by the following sentence.

$$\varphi_4(x, y; u, v) \wedge \varphi_4(z, w; u, v) \rightarrow \varphi_4(x, y; z, w) \vee \varphi_4(z, w; x, y). \tag{3.28}$$

Assume that $\varphi_4(x, y; u, v)$ and $\varphi_4(z, w; u, v)$. Without loss of generality, using (3.26), we have that $\varphi_3(x, y; x, u)$ and $\varphi_3(z, w; z, u)$. We know that not both $\varphi_2(x, u, z)$ and $\varphi_2(z, u, x)$ cannot hold (as otherwise (3.11) and (3.13) would imply $\varphi_2(u, x, x)$, contradicting (3.12)); so without loss of generality assume $\neg\varphi_2(x, u, z)$. By (3.27) this is equivalent to $\varphi_4(x, u; u, z)$. But then, since $\varphi_3(x, y; x, u)$ and $\varphi_3(z, w; z, u)$, this gives $\varphi_4(x, y; z, w)$, completing the proof of (3.28).

Notice that the maximal elements of the tree correspond to points, and that each nonmaximal element is the infimum of two maximal points. All this is expressed by the following sentences.

$$\varphi_4(x, x; u, v) \rightarrow x = u = v, \tag{3.29}$$

$$\varphi_4(x, y; x, x), \tag{3.30}$$

$$\varphi_4(u, v; x, x) \wedge \varphi_4(u, v; y, y) \rightarrow \varphi_4(u, v; x, y). \tag{3.31}$$

Both sentences (3.29) and (3.30) follow easily from (3.12) and (3.14). To show (3.31), suppose that $\varphi_4(u, v; x, x)$ and $\varphi_4(u, v; y, y)$. Clearly, these are equivalent to $\neg\varphi_2(u, v, x)$ and $\neg\varphi_2(u, v, y)$, respectively. Now suppose $\varphi_2(u, v, z)$. By (3.15) this implies $\varphi_2(u, x, z)$, and then $\varphi_2(x, u, z)$ by (3.11). Applying (3.15) again, we get that either $\varphi_2(x, y, z)$ (in which case we are done) or $\varphi_2(x, u, y)$. Then we can similarly get $\varphi_2(y, u, x)$. By (3.11) we then have that both $\varphi_2(u, x, y)$ and $\varphi_2(u, y, x)$, and this contradicts (3.12) and (3.13). This proves (3.31).

The equivalence relation defined by φ_3 respects $\equiv$ in the sense of the next sentence.

$$\varphi_3(x, y; u, v) \rightarrow (x, y) \equiv (u, v) \vee (x, y) \equiv (v, u) \vee x = y = u = v. \tag{3.32}$$

Sentence (3.32) is clearly true if $x = y$, for then $x = y = u = v$ by (3.29); so assume $\varphi_3(x, y; u, v)$ and $x \neq y$ and consequently $u \neq v$. Using (3.29)–(3.31) and the fact that φ_4 defines a tree on the equivalence classes defined by φ_3, we can assume, without loss of generality, that $\varphi_3(x, y; x, u)$ and $\varphi_3(x, u; u, v)$. Thus it suffices to prove (3.32) under the additional assumption that $x = u$. So now suppose $\varphi_3(x, y; x, v)$ and $(x, y) \not\equiv (x, v)$. From (3.3) we get that either $\varphi_1(x, y, v)$, $\varphi_1(x, v, y)$ or $(x, y) \equiv (v, x)$. The first two alternatives imply $\varphi_2(x, v, v)$ and $\varphi_2(x, y, y)$, respectively, and these contradict (3.12). Hence $(x, y) \equiv (v, x)$.

The tree defined by the formula φ_4 is not quite adequate as a determining tree. Some modification of it will be necessary.

The formula φ_4 defines a reflexive partial order (on the set of equivalence classes of the relation defined by φ_3). We define φ_5 to be the "irreflexivization" of φ_4.

$$\varphi_4(x, y; u, v) =_{\mathrm{df}} \varphi_4(x, y; u, v) \wedge \neg\varphi_3(x, y; u, v). \tag{3.33}$$

This formula defines an equivalence relation (which depends on the parameters x and y) as is asserted by the following sentences:

$$\varphi_5(x, y; u, v) \rightarrow \varphi_5(x, y; v, u), \tag{3.34}$$

$$\varphi_5(x, y; u, v) \wedge \varphi_5(x, y; v, w) \rightarrow \varphi_5(x, y; u, w). \tag{3.35}$$

Another formula is now defined.

$$\varphi_6(u, v) =_{\text{df}} \bigvee_{i<m} \Big[R_i(u, v) \wedge \neg R_i(v, u) \wedge \bigwedge_{j<i} \big(R_j(u, v) \leftrightarrow R_j(v, u)\big) \Big]. \tag{3.36}$$

The following sentences are all obvious consequences of (3.36).

$$\varphi_6(u, v) \rightarrow (u, v) \not\equiv (v, u) \wedge u \neq v, \tag{3.37}$$

$$\varphi_6(u, v) \rightarrow \neg \varphi_6(v, u), \tag{3.38}$$

$$\big[(u, v) \not\equiv (v, u) \wedge u \neq v\big] \rightarrow \big[\varphi_6(u, v) \vee \varphi_6(v, u)\big], \tag{3.39}$$

$$(u, v) \equiv (x, y) \wedge \varphi_6(u, v) \rightarrow \varphi_6(x, y). \tag{3.40}$$

For fixed x, y such that $\varphi_6(x, y)$, the formula φ_6 defines a linear order on the set of equivalence classes defined by φ_5. We define a new formula which will facilitate the assertion of this fact.

$$\varphi_7(x, y; u, v) =_{\text{df}} \varphi_3(x, y; u, v) \wedge \varphi_6(x, y) \wedge \varphi_6(u, v). \tag{3.41}$$

$$\varphi_7(x, y; u, v) \wedge \varphi_7(x, y; v, w) \rightarrow \varphi_7(x, y; u, w), \tag{3.42}$$

$$\begin{aligned}\big[\varphi_6(x, y) \wedge \varphi_5(x, y; u, u) \wedge \varphi_5(x, y; v, v) \wedge \neg \varphi_5(x, y; u, v)\big]& \\ \rightarrow \big[\varphi_7(x, y; u, v) \vee \varphi_7(x, y; v, u)\big].&\end{aligned} \tag{3.43}$$

To prove (3.42) assume $\varphi_7(x, y; u, v)$ and $\varphi_7(x, y; v, w)$. It follows from (3.41), (3.37) and (3.32) that $(y, x) \not\equiv (x, y) \equiv (u, v)$ and also $(x, y) \equiv (v, w)$. Then $(v, u) \not\equiv (u, v) \equiv (v, w)$, so it follows from (3.1) that $(u, w) \equiv (v, w)$. From (3.40) we get that $\varphi_6(u, w)$ and from the fact that φ_4 defines a tree that $\varphi_3(x, y; u, w)$. Hence $\varphi_7(x, y; u, w)$, proving (3.42).

We now prove (3.43). The hypothesis of (3.43) implies that $\varphi_3(x, y; u, v)$, and (3.32) implies that either $(x, y) \equiv (u, v)$ or $(x, y) \equiv (v, u)$. Then (3.40) implies either $\varphi_6(u, v)$ or $\varphi_6(v, u)$, so that by (3.41) either $\varphi_7(x, y; u, v)$ or $\varphi_7(x, y; v, u)$.

The above formula will now be used to define a certain structure.

Consider the equivalence relations defined by φ_3 (see sentences (3.20)–(3.22)). Let $[x, y]$ denote the equivalence class to which the pair (x, y) belongs. Let D be the set of these equivalence classes. Define $<$ on D by

$$[x, y] < [u, v] \text{ iff } \varphi_4(x, y; u, v).$$

Sentences (3.19), (3.23), (3.24), (3.28) assert that $<$ is a well-defined partial order and, in fact, is a tree ordering on D. Sentences (3.29)–(3.31) assert that the set M of maximal elements is

$$M = \{[x, y] \in D : x = y\},$$

and that the inf of two maximal elements $[x, x]$ and $[y, y]$ is $[x, y]$. Sentence (3.29) assures that

$$G = \{[x, y] \in D : x \neq y\}$$

is the set of nonmaximal elements.

Now let $Q \subseteq D^3$ be such that

$$\begin{aligned} Q = \{\langle [x, y], [u, u], [v, v]\rangle \in D^3 &: \\ &\varphi_7(x, y; u, v) \vee (u \neq v \wedge [x, y] < [u, v])\},\end{aligned}$$

and let

$$G_0 = \{[x,y] \in D : x \neq y \wedge (x,y) \not\equiv (y,x)\}.$$

Finally let

$$S_i = \{[x,y] \in D : [x,y] \in G_0 \text{ and } (x,y) \in R_i\},$$

$$T_i = \{[x,y] \in G_0 : \langle [x,y], [x,x], [y,y] \rangle \in Q \text{ and } (x,y) \in R_i\} \cup \{[x,x] \in M : (x,x) \in R_i\},$$

$$V_j = \{[x,x] \in M : x \in U_j\}.$$

Notice that by setting $I = \varnothing$ we obtain a determining tree

$$\mathfrak{D} = (D, <, M, G, I, S_0, \ldots, S_{m-1}, T_0, \ldots, T_{m-1}, V_0, \ldots, V_{n-1}).$$

The structure $(\mathfrak{D}, Q)$ satisfies the following sentences:

$$Q(x,y,z) \to x < y \wedge x < z \wedge M(y) \wedge M(z), \tag{3.44}$$

$$Q(x,y,z) \wedge x < w \wedge M(w) \to Q(x,w,w), \tag{3.45}$$

$$Q(x,y,z) \wedge Q(x,z,w) \to Q(x,y,w), \tag{3.46}$$

$$Q(x,y,z) \wedge Q(x,z,y) \to \exists w(x < w \leqslant y \wedge w \leqslant z). \tag{3.47}$$

The point of sentences (3.44)–(3.47) is this. For each $x \in G$ let $M_x = \{y \in M: x < y\}$. There is an equivalence relation on M_x in which two elements are equivalent iff $x <$ their inf. The relation Q either does nothing on M_x or else it linearly orders the set of these equivalence classes.

Starting with a structure $(\mathfrak{D}, Q)$, in which $\mathfrak{D}$ is a determining tree with $I = \varnothing$ and satisfies sentences (3.44)–(3.47), we will obtain a determining tree $\mathfrak{D}'$ from it. In case $(\mathfrak{D}, Q)$ is the structure just constructed, then $\mathfrak{D}'$ will be a determining tree which determines our original arborescent structure.

Partition G into two subsets G_0 and G_1 by

$$G_0 = \{x \in G: \neg\exists y Q(x,y,y)\}, \quad G_1 = \{x \in G: \exists y Q(x,y,y)\}.$$

For each $x \in G_1$, let

$$N_x = \{y \in M_x: \exists z \neg Q(x,y,z)\},$$

and for each $y \in N_x$, let

$$E_{x,y} = \{z \in N_x: Q(x,y,z) \wedge Q(x,z,y)\},$$

$$F_{x,y} = \{z \in N_x: Q(x,z,y)\}.$$

Now let

$M' = \{\langle x, \phi \rangle: x \in M\}$,
$G' = \{\langle x, F_{x,y} \rangle: x \in G_1 \text{ and } y \in N_x\} \cup \{\langle x, M_x \rangle: x \in G_0\}$,
$I' = \{\langle x, E_{x,y} \rangle: x \in G_1 \text{ and } y \in N_x\}$,
$D' = M' \cup G' \cup I'$.

Define $<'$ on D' so that $\langle x, A \rangle <' \langle y, B \rangle$ iff each of the following holds:

$\langle x, A \rangle \neq \langle y, B \rangle$,
$x \leqslant y$ and $A \supseteq B$,
if $B = \varnothing$, then $y \in A$.

Finally, define S_i', T_i' and V_j' so that

$S_i' = \{\langle x, A\rangle \in G' \colon x \in S_i\}$,

$T_i' = \{\langle x, A\rangle \in G' \cup M' \colon x \in T_i\}$,

$V_i' = \{\langle x, A\rangle \in M' \colon x \in V_j\}$.

It is now an easy matter to show that

$$\mathfrak{D}' = (D', <', M', G', I', S_0', \ldots, S_{m-1}', T_0', \ldots, T_{m-1}', V_0', \ldots, V_{n-1}')$$

is a determining tree. There are two key facts that are used. (1) Whenever $x \in G_1$ and $X, Y \in \{E_{x,y} \colon y \in N_x\} \cup \{F_{x,y} \colon y \in N_x\}$, then either $X \subseteq Y$, $Y \subseteq X$ or $X \cap Y = \varnothing$. (2) Whenever $z > x \in G_1$ and $X \in \{E_{x,y} \colon y \in N_x\} \cup \{F_{x,y} \colon y \in N_x\}$, then either $M_z \cap X = \varnothing$ or $M_z \subseteq X$.

We leave to the reader the easy verification that, given an arborescent $\mathfrak{A}$, the determining tree so obtained from the structure $(\mathfrak{D}, Q)$ defined in this section is a determining tree for the structure $\mathfrak{A}$. We call this determining tree the *canonical determining* tree of $\mathfrak{A}$. Also, if $\mathfrak{D}$ is the canonical determining tree of $\mathfrak{A}$, and if $\mathfrak{A}'$ is the arborescent structure determined, then $\mathfrak{A} \cong \mathfrak{A}'$. There is a natural and obvious isomorphism from $\mathfrak{A}$ to $\mathfrak{A}'$, and this isomorphism is definable in $\mathfrak{A}$. Thus, we have proved Theorem 3.1.

4. Consequence of the interpretability. In this section we gather together some of the interesting properties of arborescent structures which can be deduced from the results of §§2 and 3.

We are going to refer to theories being interpretable in $S2S$ (the monadic second-order theory of the complete binary ω-tree as studied by Rabin [**3**]) in the strong sense of [**4**].

COROLLARY 4.1. *The theory of arborescent structures is interpretable in* $S2S$.

PROOF. This follows from Propositions 2.1 and 2.2 and the fact that the theory of trees is interpretable in $S2S$. □

COROLLARY 4.2. *The weak monadic second-order theory of arborescent structures and the monadic second-order theory of countable arborescent structures are both decidable.*

PROOF. Follows from Corollary 4.1 and Rabin [**3**]. □

COROLLARY 4.3. *Every* $\aleph_0$*-categorical arborescent structure has a decidable theory.*

PROOF. Follows from Corollary 4.1 and [**4**]. □

COROLLARY 4.4 (VAUGHT'S CONJECTURE FOR ARBORESCENT STRUCTURES). *Suppose* $\varphi \in \mathfrak{L}_{\omega_1,\omega}$ *has uncountably many pairwise nonisomorphic, countable, arborescent models. Then* φ *has* $2^{\aleph_0}$ *nonisomorphic countable models.*

PROOF. Follows from Theorem 3.1 and Steel's proof of Vaught's conjecture for trees [**10**]. □

COROLLARY 4.5. *A structure is arborescent iff every substructure with at most* 4 *elements is arborescent.*

PROOF. A check of the construction of the canonical determining tree reveals that only 4-element substructures were considered. □

The special case of the previous corollary applied only to partially ordered sets was proved by more direct means in **[8]**.

COROLLARY 4.6. *A partially ordered set is arborescent iff it is a reticle.*

The next corollary was also proved in **[8]**, but by a different, less constructive method.

COROLLARY 4.7. *The theory of arborescent structures is finitely axiomatizable.*

For the next definition recall the definition of the formula φ_2 presented in (3.10) of §3.

DEFINITION 4.8. Let $\mathfrak{A}$ be arborescent and $W \subseteq A$. Then W is *weakly indiscernible* iff the following two properties hold:

(1) whenever $a, b \in W$ are distinct, then $\mathfrak{A} \models (a, b) \equiv (b, a)$;

(2) whenever $a, b, c \in W$ are distinct, then $\mathfrak{A} \models \neg\varphi_2(a, b, c)$.

The notion of weak indiscernibility is very natural when considering trees. A tree is n-branching iff there are no weakly indiscernible sets with more than n elements. This motivates the next definition.

DEFINITION 4.9. Suppose $n < \omega$ and $\mathfrak{A}$ is an arborescent structure. Then $\mathfrak{A}$ is *n-branching* if it has no weakly indiscernible sets with more than n elements. It is *finite-branching* if it has no infinite weakly indiscernible sets.

PROPOSITION 4.10. *Let $\mathfrak{A}$ be arborescent and $\mathfrak{D}$ its canonical determining tree. Then $\mathfrak{A}$ is finite-branching iff $\mathfrak{D}$ is finite-branching. If $2 \leq n < \omega$, then $\mathfrak{A}$ is n-branching iff $\mathfrak{D}$ is n-branching.*

PROOF. If $X \subseteq A$ has at least 3 elements and is weakly indiscernible in $\mathfrak{A}$, then it is clearly weakly indiscernible in $\mathfrak{D}$. Conversely, suppose that $Y \subseteq D$ has at least 3 elements and is weakly indiscernible in $\mathfrak{D}$. Let $X \subseteq D$ be a set of maximal elements such that for each $x \in X$ there is a unique $y \in Y$ such that $y \leq x$. Then X is weakly indiscernible in $\mathfrak{D}$, and clearly also is weakly indiscernible in $\mathfrak{A}$. Finally notice that X can be chosen so that for each $y \in Y$ there is an $x \in X$ such that $x \geq y$. □

COROLLARY 4.11. *If σ is an $\mathfrak{L}$-sentence which is consistent with the theory of arborescent structures, then there is some $n < \omega$ and an n-branching arborescent $\mathfrak{A} \models \sigma$.*

PROOF. Follows from Proposition 4.10 and Corollary 2.6 of **[5]**.

COROLLARY 4.12. *If σ is an $\mathfrak{L}$-sentence which has as a model an $\aleph_0$-categorical arborescent structure, then there is some $n < \omega$ and an $\aleph_0$-categorical n-branching arborescent $\mathfrak{A} \models \sigma$.*

PROOF. Follows from Proposition 4.10 and Theorem 2.8 of **[5]**. □

For the definition of nuclearity and its important properties consult §1 of **[5]** or §5 of **[7]**.

COROLLARY 4.13. *If* $4 \leqslant n < \omega$, *then every* $\aleph_0$-*categorical* n-*branching arborescent structure is* $(n-1)$-*nuclear.*

PROOF. Let $\mathfrak{A}$ be an $\aleph_0$-categorical n-branching arborescent structure. Suppose $X \subseteq A$ is finite and $a \in A$. Then $X \cup \{a\} \subseteq D$, where $\mathfrak{D}$ is the canonical determining tree of $\mathfrak{A}$. Then $\mathfrak{D}$ is $\aleph_0$-categorical, and, by Proposition 4.10, is also n-branching, so by Lemma 2.3 of **[5]** there is a nucleus $Y \subseteq X$ for a (with respect to the structure $\mathfrak{D}$) and $|Y| \leqslant n-1$. Clearly, this Y is also a nucleus with respect to $\mathfrak{A}$. $\square$

COROLLARY 4.14. *An* $\aleph_0$-*categorical arborescent structure has a finitely axiomatizable theory iff it is finite-branching.*

PROOF. Follows from the corresponding fact for trees (Theorem 2.2 of **[5]**) and Corollaries 4.12 and 4.13. $\square$

Appendix: Interpretability. Let $\mathfrak{L}_0$ and $\mathfrak{L}_1$ be languages which, for the sake of convenience, are finite and contain only relation symbols. The literature (e.g. **[2]**, **[9]**) usually formulates the notion of interpretability of $\mathfrak{L}_0$ in $\mathfrak{L}_1$ in the following way. The interpretation Π consists of a unary $\mathfrak{L}_1$-formula $\delta(x)$, which is the universe of Π, and an n-ary $\mathfrak{L}_1$-formula $\theta_R(\bar{x})$ for each n-ary relation symbol R in $\mathfrak{L}_0$. In the standard way any $\mathfrak{L}_0$-formula φ yields an $\mathfrak{L}_1$-formula φ^Π; also, in the standard way, any $\mathfrak{L}_1$-structure $\mathfrak{A}$ yields an $\mathfrak{L}_0$-structure $\mathfrak{A}^\Pi$.

We wish to consider a slight generalization of this. A *weak interpretation* Π (*of degree* k) of $\mathfrak{L}_0$ in $\mathfrak{L}_1$ consists of a k-ary $\mathfrak{L}_1$-formula $\delta(\bar{x})$ which is the *universe* of Π, a $2k$-ary $\mathfrak{L}_1$-formula $\varepsilon(\bar{x}, \bar{y})$ which is the *equality* of Π, and finally an nk-ary $\mathfrak{L}_1$-formula $\theta_R(\bar{x}, \bar{y}, \ldots)$ for each n-ary predicate symbol R in $\mathfrak{L}_0$ such that ε defines a congruence relation relative to it. Again, any $\mathfrak{L}_0$-formula φ naturally yields an $\mathfrak{L}_1$-formula φ^Π and an $\mathfrak{L}_1$-structure $\mathfrak{A}$ naturally yields an $\mathfrak{L}_0$-structure $\mathfrak{A}^\Pi$.

A weak interpretation is an *interpretation* if it is of degree 1 and $\varepsilon(x, y)$ is the formula $x = y \wedge \delta(x)$.

If Π is a weak interpretation of $\mathfrak{L}_0$ in $\mathfrak{L}_1$ and σ is a weak interpretation of $\mathfrak{L}_1$ in $\mathfrak{L}_2$, then there is an obvious way to compose these so as to obtain a weak interpretation $\Pi\sigma$ of $\mathfrak{L}_0$ in $\mathfrak{L}_2$.

Now suppose T_0 is an $\mathfrak{L}_0$-theory and T_1 is an $\mathfrak{L}_1$-theory, and suppose Π is a (weak) interpretation of $\mathfrak{L}_0$ in $\mathfrak{L}_1$. Then Π is a (weak) interpretation of T_0 in T_1 if whenever φ is an $\mathfrak{L}_0$-sentence such that $T_0 \vdash \varphi$, then $T_1 \vdash \varphi^\Pi$. A (weak) interpretation Π of T_0 in T_1 is usually referred to as being *faithful* if whenever φ is an $\mathfrak{L}_0$-sentence such that $T_1 \vdash \varphi^\Pi$, then $T_0 \cup \{\exists x \exists y (x \neq y)\} \vdash \varphi$. (The sentence $\exists x \exists y (x \neq y)$ is adjoined to avoid some anomalous situations.)

The definitions of weak interpretation and faithful weak interpretation are both presented syntactically; they can be recast in more semantical terms. Thus, Π is a weak interpretation of T_0 in T_1 iff whenever $\mathfrak{A}_1 \models T_1$, then $\mathfrak{A}_1^\Pi \models T_0$. Such a Π is faithful iff whenever $\mathfrak{A}_0 \models T_0$ and $|A| \geqslant 2$, then there is an $\mathfrak{A}_1 \models T_1$ such that $\mathfrak{A}_0 \equiv \mathfrak{A}_1^\Pi$. An obvious strengthening of the notion of faithfulness can be obtained by replacing the elementary equivalence in the characterization by isomorphism. If Π is a weak interpretation of T_0 in T_1, then Π is *complete* iff whenever $\mathfrak{A}_0 \models T_0$, and

$|A| \geqslant 2$, then there is an $\mathfrak{A}_1 \models T_1$ such that $\mathfrak{A}_0 \cong \mathfrak{A}_1^{\Pi}$. A defect in this definition of completeness is that it is not syntactic; however, by a further strengthening, the syntactic nature can be recovered. A weak interpretation is *completely faithful* if there is an interpretation σ and an $\mathfrak{L}_0$-formula α such that whenever $\mathfrak{A} \models T_0$ and $|A| \geqslant 2$, then α defines an isomorphism between $\mathfrak{A}$ and $\mathfrak{A}^{\Pi_\sigma}$.

We now present a very simple example of a completely faithful interpretation. Fix some $s < \omega$. Let $\mathfrak{L}_0 = \{<, U_0, \ldots, U_{s-1}\}$ and $\mathfrak{L}_1 = \{<\}$, where $<$ is a binary relation symbol and $U_0, \ldots, U_{s-1}$ are unary relation symbols. Consider the $\mathfrak{L}_1$-theory which is the theory of trees and the $\mathfrak{L}_0$-theory which is the theory of s-augmented trees (i.e. $\mathfrak{L}_0$-structures $\mathfrak{A}$ such that $\mathfrak{A} \restriction \mathfrak{L}_1$ is a tree).

PROPOSITION A. *There is a completely faithful interpretation of the theory of s-augmented trees in the theory of trees.*

PROOF. Let T_0 be the theory of s-augmented trees and T_1 the theory of trees. Let $k = 2^s$, and let $P_0, P_1, \ldots, P_{k-1}$ enumerate all the subsets of s. The interpretation Π of T_0 in T_1 will have as its universe $\delta(x)$ the formula $\exists y(x < y)$. Let $\theta_<(x, y)$ be the formula $x < y$ and for each $i < s$ let $\theta_{U_i}(x)$ be the formula

$$\bigvee \left\{\exists^j y(x < y \wedge \neg\delta(y)) : j < k \text{ and } i \in P_j\right\}.$$

We now indicate what weak interpretation σ will yield definably isomorphic $\mathfrak{A}$ and $\mathfrak{A}^{\Pi\sigma}$ whenever $\mathfrak{A} \models T_0$ and $|A| > 2$. This weak interpretation σ will be of degree $k + 1$. We need to define $\mathfrak{L}_0$-formulas $\delta(\bar{x})$, $\varepsilon(\bar{x}, \bar{y})$ and $\theta_<(\bar{x}, \bar{y})$. Let $\delta(\bar{x})$ be

$$\left[\bigwedge_{i<j<k} (x_i \neq x_{i+1} \rightarrow x_j = x_{j+1})\right] \wedge \left[\bigwedge_{i<n} x_i \neq x_{i+1} \rightarrow \bigwedge_{j<s} \left(U_j(x_i) \leftrightarrow j \in P_i\right)\right].$$

Let $\varepsilon(\bar{x}, \bar{y})$ be

$$x_0 = y_0 \wedge \bigwedge_{i<k} (x_i = x_{i+1} \leftrightarrow y_i = y_{i+1}).$$

Let $\theta_<(\bar{x}, \bar{y})$ be

$$(x_0 = x_k = y_0 \neq y_k) \vee (x_0 = x_k < y_0).$$

The reader will easily verify that this is the desired interpretation. $\square$

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