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ON THE ZEROS OF DIRICHLET L-FUNCTIONS. II (WITH CORRECTIONS TO "ON THE ZEROS OF DIRICHLET L-FUNCTIONS. I" AND THE SUBSEQUENT PAPERS)

BY

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ABSTRACT. Some consequences of the main theorem of *On the zeros of Dirichlet L-functions*. I, Trans. Amer. Math. Soc. **196** (1974), 225–235 are proved.

1. Introduction. We shall give complete proofs of some consequences of the main theorem of [3]. They were first announced in [1] and have been improved in the form announced and used in [12]. In the meantime, [4]–[11] have appeared. We shall on this occasion give some corrections to [1]–[12].

Our basic estimates which we shall use below are the following (α'), (α'') and (β). Let $\zeta(s)$ be the Riemann zeta function and $S(t) = 1/\pi \arg \zeta(\frac{1}{2} + it)$ as usual. Let $N(t)$ be the number of the zeros of $\zeta(s)$ in $0 < \operatorname{Im} s < t$. Let $T > T_0$, k be an integer $\geqslant 1$ and h be a positive number. We shall denote positive absolute constants by A, A_1 and A_2. We have adapted Selberg's approach [15] to get the following (α) which is our main theorem of [3].

(α) If h is positive and bounded, then

$$\int_T^{2T} (S(t+h) - S(t))^{2k}\,dt = \frac{(2k)!}{(2\pi)^{2k}k!}T(2\log(3 + h\log T))^k + O\big((Ak)^{2k}T(\log(3 + h\log T))^{k-1/2}\big).$$

We remark that the condition for h, (namely, "bounded") has been remarked to the author by Professors Gallagher and Mueller (cf. Added in proof of Gallagher and Mueller [13]) and that the exponent to Ak is $2k$ as is remarked in p. 172 of [12]. (The right-hand side of 1.13 of p. 172 of [12] should be added by $HA^k(\log y)^{-\nu}$.) We shall use (α) in the following modified forms.

$$\int_T^{2T} (S(t+h) - S(t))^{2k}\,dt \ll (Ak)^k T\big(\log(((h \wedge 1)\log T + 3)\cdot e^k)\big)^k, \qquad (\alpha')$$

where $h \wedge 1 = \operatorname{Min}\{h, 1\}$.

$$\int_T^{2T} (S(t+h) - S(t))^{2k}\,dt \gg (Ak)^k T(\log((h \wedge 1)\log T + 3))^k$$

$$\text{if } k \ll \log\log((h \wedge 1)\log T + 3). \qquad (\alpha'')$$

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We shall also use the following estimate

$$\int_T^{2T} (S(t+h) - S(t))^{2k+1}\, dt \ll (Ak)^k T\left(\log(((h \wedge 1)\log T + 3)\cdot e^k)\right)^k. \quad (\beta)$$

We shall omit writing the proofs of these, since they can be derived similarly. We remark that the remainder term in (α) can be written as

$$O\left((Ak)^k\left(\log\left(e^k(3 + h\log T)\right)\right)^{k-1/2}\right).$$

Using (α'), (α'') and (β) we shall prove first

THEOREM 1.[1] *Let $T \geqslant T_0$ and C be a constant $> C_0$. Then for positive proportion of t in $T < t < 2T$,*

$$N\left(t + \frac{2\pi C}{\log T}\right) - N(t) > C + A\sqrt{\log C \cdot \log\log C}$$

and for positive proportion of t in $T < t < 2T$,

$$N\left(t + \frac{2\pi C}{\log T}\right) - N(t) < C - A\sqrt{\log C \cdot \log\log C}\ .$$

We denote the nth positive imaginary part of the zeros of $\zeta(s)$ by γ_n. Then as an immediate consequence of Theorem 1 we see that for positive proportion of γ_n in $T < \gamma_n < 2T$,

$$\frac{\gamma_{n+r} - \gamma_n}{r} \leqslant \frac{2\pi}{\log T}\left(1 - \exp(-Ar^2/\log(r+3))\right)$$

and for positive proportion of γ_n in $T < \gamma_n < 2T$,

$$\frac{\gamma_{n+r} - \gamma_n}{r} \geqslant \frac{2\pi}{\log T}\left(1 + \exp(-Ar^2/\log(r+3))\right),$$

where r is an integer $\geqslant 1$. We remark that this corollary for $r = 1$ was shown to the author by Professor Montgomery.

Next, using (α'), we shall prove

THEOREM 2. *Suppose that $T \geqslant T_0$, j is an integer $\geqslant 1$, k is an integer $\geqslant j$, r is an integer $\geqslant 1$ and h is a positive number $\gg (\log T)^{-1}$. We put*

$$d(\gamma_n, r) = (\gamma_{n+r} - \gamma_n)/r.$$

Then we have

$$\frac{1}{N(T)} \sum_{\substack{d(\gamma_n, r) > h \\ \sqrt{T} < \gamma_n < T}} d(\gamma_n, r)^j$$

$$\ll \frac{(Ak)^{k+j-1}(1 + 1/k)^{(j-1)(2k-j/2)}\left(\log\left(e^k rh \log T/(1 + 1/k)^{j-1}\right)\right)^k}{B(k,j)(r\log T)^{j-1}\log T \cdot (rh\log T)^{2k-j+1}},$$

where $B(k, 1) = 1$ and $B(k,j) = (2k-1)(2k-2)\cdots(2k-j+1)$ for $j \geqslant 2$.

Using Theorem 2 we shall prove the following two corollaries.

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[1]The interval $(T, 2T)$ may be replaced by $(0, T)$, since the intervals $(T, 2T)$ in (α), (α'), (α'') and (β) may be replaced by $(0, T)$.

COROLLARY 1. *For each integral $k \geqslant 1$ and integral $r \geqslant 1$, we have*

$$\frac{1}{N(T)} \sum_{\gamma_n < T} d(\gamma_n, r)^k \ll \frac{1}{(\log T)^k}.$$

COROLLARY 2. *If $C > C_0$ and r is an integer $\geqslant 1$, then*

$$\frac{1}{N(T)} \sum_{\substack{\gamma_n < T \\ d(\gamma_n, r) > C/\log T}} \cdot 1 \leqslant e^{-ArC}.$$

We shall also prove the following theorem using (α').

THEOREM 3. *Let $K > K_0$. Then "the number of the zeros of $\zeta(s)$ in $0 < \operatorname{Im} s < T$ whose multiplicities are $\geqslant K$" $\leqslant e^{-AK}N(T)$.*

We shall prove Theorem 1 in §2, Theorem 2 and its corollaries in §3 and Theorem 3 in §4. In §5 we shall give some corrections and complements to **[1]**–**[12]**. We remark finally that the results above can be proved also for Dirichlet L-functions if we suppose that the modulus is $\ll T^{(1/4)-\varepsilon}$, $\varepsilon > 0$.

Finally, the author wishes to express his thanks to Professor Gallagher, Professor Montgomery and Professor Mueller for their valuable suggestions.

2. Proof of Theorem 1. We put $f(t) = S(t + h) - S(t)$ and $h = 2\pi C/\log T$. We put $E_M = \{t \in (T, 2T); f(t) > M\}$ for $M \geqslant 0$. Let $\varphi_M(t)$ be the characteristic function of E_M. Let $C > C_0$ and let $k = [A \log\log C]$ with an appropriate positive absolute constant A. We consider the integral $I = \int_T^{2T} f^{2k+1}(t)\varphi_0(t)\,dt$.

$$\begin{aligned} I &= \int_T^{2T} f^{2k+1}(t)\varphi_0(t)\varphi_M(t)\,dt \\ &\quad + \int_T^{2T} f^{2k+1}(t)\varphi_0(t)(1 - \varphi_M(t))\,dt \\ &\leqslant \sqrt{E_M}\left(\int_T^{2T} |f(t)|^{2(2k+1)}\,dt\right)^{1/2} + M^{2k+1}T \\ &\leqslant \sqrt{E_M}\,(Ak)^{k+1/2}(\log C)^{k+1/2}\sqrt{T} + M^{2k+1}T, \end{aligned}$$

by (α'). On the other hand,

$$\begin{aligned} I &= \frac{1}{2}\int_T^{2T} |f(t)|^{2k+1}\,dt + \frac{1}{2}\int_T^{2T} f(t)^{2k+1}\,dt \\ &= \frac{1}{2}I_1 + \frac{1}{2}I_2, \end{aligned}$$

say. By (α'), (α'') and (β), we get

$$\begin{aligned} I_1 &\geqslant \frac{\left(\int_T^{2T} |f(t)|^{2k}\,dt\right)^{(2k-1)/2(k-1)}}{\left(\int_T^{2T} |f(t)|^2\,dt\right)^{1/2(k-1)}} \\ &\gg \left(T(Ak)^k(\log C)^k\right)^{(2k-1)/2(k-1)}(T\log C)^{-1/2(k-1)} \\ &\gg T(A_1k)^{k(2k-1)/2(k-1)}(\log C)^{k+1/2} \\ &\gg T(A_2k)^k(\log C)^k \\ &\gg I_2. \end{aligned}$$

Hence we get

$$|E_M| \geqslant \left(\frac{T(Ak)^{k(2k-1)/2(k-1)}(\log C)^{k+1/2} - M^{2k+1}T}{\sqrt{T}\,(Ak)^{k+1/2}(\log C)^{k+1/2}} \right)^2$$
$$\geqslant Te^{-A \log\log C},$$

provided that $M \ll \sqrt{\log C \log\log C}$. This proves the first part of Theorem 1. The second part of Theorem 1 can be derived similarly.

3. Proof of Theorem 2 and its corollaries.

3-1. PROOF OF THEOREM 2. We shall prove our theorem by induction on j. We remark that $d(\gamma_n, r) \ll (\log\log\log T)^{-1}$ for $\sqrt{T} \leqslant \gamma_n \leqslant T$ by 9.12 of **[17]**. Suppose that $d(\gamma_n, r) \geqslant h$. Then

$$\int_{\gamma_n}^{\gamma_{n+r}-rh/2}\left(S\left(t+\frac{hr}{2}\right) - S(t)\right)^{2k} dt \gg (hr\log T)^{2k}\left(\gamma_{n+r} - \gamma_n - \frac{rh}{2}\right)$$
$$\gg (\gamma_{n+r} - \gamma_n)(hr\log T)^{2k}.$$

Hence we have

$$\sum_{\substack{d(\gamma_n, r) > h \\ \sqrt{T} < \gamma_n < T}} d(\gamma_n, r) \ll \frac{1}{r(rh\log T)^{2k}} \sum_{\substack{d(\gamma_n, r) > h \\ \sqrt{T} < \gamma_n < T}} \int_{\gamma_n}^{\gamma_{n+r}-rh/2}\left(S\left(t+\frac{rh}{2}\right) - S(t)\right)^{2k} dt$$
$$\ll \frac{1}{(hr\log T)^{2k}} \int_{\sqrt{T}}^{AT}\left(S\left(t+\frac{rh}{2}\right) - S(t)\right)^{2k} dt$$
$$\ll \frac{(Ak)^k T(\log(e^k rh\log T))^k}{(hr\log T)^{2k}}.$$

Next, suppose that our theorem is correct for $j \geqslant 1$. Then,

$$\frac{1}{N(T)} \sum_{\substack{d(\gamma_n, r) > h \\ \sqrt{T} < \gamma_n < T}} d(\gamma_n, r)^{j+1}$$
$$\leqslant \frac{1}{N(T)} \sum_{\substack{d(\gamma_n, r) > h \\ \sqrt{T} < \gamma_n < T}} d(\gamma_n, r)^j(1+k)\left(d(\gamma_n, r) - \frac{h}{1+1/k}\right)$$
$$= \frac{(1+k)}{N(T)} \sum_{\substack{d(\gamma_n, r) > h \\ \sqrt{T} < \gamma_n < T}} d(\gamma_n, r)^j \int_{h/(1+1/k)}^{d(\gamma_n, r)} du$$
$$\leqslant \frac{(1+k)}{N(T)} \int_{h/(1+1/k)}^{A/\log\log\log T} \left[\sum_{\substack{d(\gamma_n, r) > u \\ \sqrt{T} < \gamma_n < T}} d(\gamma_n, r)^j \right] du$$

$$\ll \frac{k(Ak)^{k+j-1}(1+1/k)^{(j-1)(2k-j/2)}}{B(k,j)(\log T)^j r^{j-1}}$$

$$\cdot \int_{h/(1+1/k)}^{A/\log\log\log T} \frac{\left(\log\left(e^k ru \log T/\left(1+1/k\right)^{j-1}\right)\right)^k}{(ru\log T)^{2k-(j-1)}}\, du$$

$$\ll \frac{(Ak)^{k+j}(1+1/k)^{j(2k-(j+1)/2)}\left(\log\left(e^k rh \log T/\left(1+1/k\right)^j\right)\right)^k}{B(k,j+1)(\log T)^{j+1} r^j (rh\log T)^{2k-j}}.$$

This proves our theorem.

3-2. PROOF OF COROLLARY 1.

$$S \equiv \sum_{\gamma_n<T} d(\gamma_n, r)^k$$

$$= \sum_{\substack{d(\gamma_n, r)>C/\log T\\ \gamma_n<T}} d(\gamma_n, r)^k + \sum_{\substack{d(\gamma_n, r)<C/\log T\\ \gamma_n<T}} d(\gamma_n, r)^k$$

$$= S_1 + S_2,$$

say, where we suppose that $C > C_0$. By Theorem 2,

$$S_1 \ll \frac{(Ak)^k(\log(e^k rC))^k N(T)}{(\log T)^k r^{k-1}(rC)^{k+1}} + \frac{N(T)}{\sqrt{T}}.$$

On the other hand,

$$S_2 \ll \frac{C^k}{(\log T)^k} N(T).$$

Hence we get, by taking $C = (Ak/r^2)^{k/(2k+1)}$,

$$S \ll \frac{N(T)}{(\log T)^k}\left[\frac{(Ak)^k(\log(e^k rC))^k}{r^{k-1}(rC)^{k+1}} + C^k\right]$$

$$\ll \frac{N(T)(Ak)^{3k/2}}{(\log T)^k r^{k-1}},$$

provided that $r^2 \ll k$. Thus we get Corollary 1.

3-3. PROOF OF COROLLARY 2. Suppose that $1 \ll rC \ll A^k$.

$$S \equiv \sum_{\substack{d(\gamma_n, r)>C/\log T\\ \gamma_n<T}} \cdot 1$$

$$< \left(\frac{\log T}{C}\right)^k \sum_{\substack{d(\gamma_n, r)>C/\log T\\ \sqrt{T}<\gamma_n<T}} d(\gamma_n, r)^k + \frac{N(T)}{\sqrt{T}}$$

$$\ll \frac{(Ak)^k(\log(e^k rC))^k N(T)}{(rC)^{2k}C} \quad \text{(by Theorem 2)}$$

$$\ll \frac{(Ak)^{2k}N(T)}{(rC)^{2k}C} \ll e^{-ArC}N(T)$$

by taking $k = [A_1 rC]$.

4. Proof of Theorem 3. Let $h \log T = f(k)$, where f will be chosen later. We consider the integral $I \equiv \int_T^{2T} (N(t+h) - N(t))^{2k}\, dt$.

$$
\begin{aligned}
I &= \int_T^{2T} \sum_{t<\gamma^{(1)}, \gamma^{(2)}, \dots, \gamma^{(2k)}<t+h} \cdot 1\, dt \\
&= \sum_{\substack{T<\gamma^{(1)}, \dots, \gamma^{(2k)}<2T+h \\ \mathrm{Min}_j(\gamma^{(j)}) > \mathrm{Max}_j(\gamma^{(j)})-h}} \int_{\mathrm{Max}_j(\gamma^{(j)})-h}^{\mathrm{Min}_j(\gamma^{(j)})} dt \\
&\geqslant \sum_{\substack{T<\gamma^{(1)}, \dots, \gamma^{(2k)}<2T+h \\ \mathrm{Min}_j(\gamma^{(j)}) > \mathrm{Max}_j(\gamma^{(j)})-h}} \left(h - \left(\mathrm{Max}_j(\gamma^{(j)}) - \mathrm{Min}_j(\gamma^{(j)})\right)\right) \\
&\geqslant h \sum_{T<\gamma^{(1)} = \dots = \gamma^{(2k)}<2T+h} \cdot 1 \\
&\geqslant h \sum_{l=1}^{\infty} l^{2k} M_l(2T, T),
\end{aligned}
$$

where $\gamma^{(j)}$ runs over the imaginary parts of the zeros of $\zeta(s)$ for $j = 1, 2, \dots, 2k$ and $M_l(2T, T)$ is the number counted simply of the zeros of $\zeta(s)$ in $T < \mathrm{Im}\, s < 2T$ whose multiplicities are exactly l.

On the other hand, by (α'), we get

$$
\begin{aligned}
I &\ll T\left(f(k) + \sqrt{k}\sqrt{\log(e^k f(k))}\right)^{2k} A^{2k} \\
&\ll N(T) h f(k)^{-1} \left(f(k) + \sqrt{k}\sqrt{\log(e^k f(k))}\right)^{2k} A^{2k}.
\end{aligned}
$$

Here we take $f(k) = k$. Then $I \ll N(T)h(Ak)^{2k}$. Hence we get

$$\sum_{l=1}^{\infty} l^{2k} M_l(2T, T) \ll N(T)(Ak)^{2k}.$$

Now

$$K^{2k-1} \sum_{l=K}^{\infty} l M_l(2T, T) \leqslant \sum_{l=K}^{\infty} l^{2k} M_l(2T, T) \ll N(T)(Ak)^{2k}.$$

Hence we get

$$\sum_{l=K}^{\infty} l M_l(2T, T) \ll \frac{(Ak)^{2k} N(T)}{K^{2k-1}} \leqslant e^{-AK} N(T)$$

if $K > K_0$.

5. Some corrections and complements. In this section we shall give some corrections and complements to **[1]**–**[12]** using the same notations as in **[1]**–**[12]**.

5-1. As we have noticed in §1, h in (α) must be bounded. Similarly, h's in 1.2, 1.15 and 1.17 of p. 140 of **[1]**, 1.10 of p. 348 of **[4]**, 1.20 of p. 51 of **[7]**, 1.7 of p. 70 of **[10]** and 1.24 of p. 417 and 1.27 of p. 424 of **[11]** must be bounded. We may remark that since we have used bounded h in the applications, these corrections are harmless.

5-2. In p. 228 of [3], the remainder term in 1.11 should be multiplied by k^2 and $k!$ of 1.12 should be $(Ak)^k$. The remainder term in 1.4 of p. 230 of [3] should be multiplied by k^2. The proof of Lemma 2 in p. 228 of [3] should be simplified and corrected as follows. We may suppose that $k \geqslant 2$ and $a(p) = 1$, for simplicity. We put $F_1(x) = \sum_{p<x} 1/p$. Then

$$\sum_{p_i<x}{}^{*} \frac{1}{p_1p_2 \cdots p_k} - k!F_1^k(x) \ll k! \sum_{p_i<x}{}' \frac{1}{p_1p_2 \cdots p_k} \ll k!k^2F_1^{k-2}(x),$$

where $*$ indicates that we sum over all primes $p_1, p_2, \ldots, p_{2k} < x$ such that $p_1p_2 \cdots p_k = p_{k+1}p_{k+2} \cdots p_{2k}$ and the prime ($'$) indicates that we sum over all $p_1, p_2, \ldots, p_k$ such that some p_i and some p_j are equal for $1 \leqslant i < j \leqslant k$.

5-3. (iii) and (iv) of p. 234 of [3] should be erased.

5-4. $C\sqrt{q} \cdot \log\log q/\sqrt{\log C}$ in 1.18 of p. 61, 1.4 of p. 62 and 1.5 of p. 63 of [8] should be replaced by $C \log C\sqrt{q}\ \log^2 q$.

5-5. As we have seen in the proof of Theorem 1 in the present paper, ε in 1.15 and 1.17 of p. 140 of [1], 1.11 and 1.13 of p. 399 of [6], 1.5 and 1.12 of p. 50 and 1.4, 1.6, 1.18 and 1.23 of p. 56 of [7] should be omitted.

5-6. [4] is generalized in [12].

5-7. [9] should be corrected and improved as in the present paper.

5-8. (1) in p. 52 of [7] can be improved as follows.

$$\sum_{\chi}{}' (R_x(t, \chi))^{2k} \ll (Ak)^{2k}q. \tag{1$'$}$$

We may omit writing the proofs of (1$'$) above and (2) in p. 52 of [7], (even though it was announced that these proofs would be published).

5-9. In the statement of Theorem 2 of [5], the condition $\lambda(q) \leqslant t(q)$ should be replaced by $\lambda(q) = o(t(q))$, because $(q-2)/2\pi$ should be multiplied to the first term of 1.16, the first two terms of 1.17 and the term of 1.18 of p. 142 and the final result is

$$\sum_{|\gamma| \leqslant |t(q)|} e^{ia\gamma} = (q-2)\sin(at(q))(\log(qt(q)/2\pi))/\pi a + O(q(at(q) + a^{-1})).$$

We remark also that $O(q)$ should be added to the right-hand side of 1.13, 1.14 and 1.15 in p. 142.

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