ON THE ZEROS OF DIRICHLET L-FUNCTIONS. II
(WITH CORRECTIONS TO “ON THE ZEROS OF
DIRICHLET L-FUNCTIONS. I” AND
THE SUBSEQUENT PAPERS)

BY

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Abstract. Some consequences of the main theorem of On the zeros of Dirichlet

1. Introduction. We shall give complete proofs of some consequences of the main
theorem of [3]. They were first announced in [1] and have been improved in the
form announced and used in [12]. In the meantime, [4]–[11] have appeared. We
shall on this occasion give some corrections to [1]–[12].

Our basic estimates which we shall use below are the following (α'), (α'') and
(β). Let ζ(s) be the Riemann zeta function and S(t) = 1/π arg ζ(½ + it) as usual.
Let N(t) be the number of the zeros of ζ(s) in 0 < Im s < t. Let T > T₀, k be an
integer > 1 and h be a positive number. We shall denote positive absolute
constants by A, A₁ and A₂. We have adapted Selberg’s approach [15] to get the
following (α) which is our main theorem of [3].

(α) If h is positive and bounded, then

\[ \int_0^{2T} (S(t + h) - S(t))^{2k} \, dt \ll \frac{(2k)!}{(2\pi)^{2k} k!} (\log (3 + h \log T))^k \]

\[ + O((Ak)^{2k} T (\log (3 + h \log T))^{k-1/2}) \]

We remark that the condition for h, (namely, “bounded”) has been remarked to the
author by Professors Gallagher and Mueller (cf. Added in proof of Gallagher and
Mueller [13]) and that the exponent to Ak is 2k as is remarked in p. 172 of [12].
(The right-hand side of 1.13 of p. 172 of [12] should be added by HA^2(\log y)^T.)

We shall use (α) in the following modified forms.

\[ \int_0^{2T} (S(t + h) - S(t))^{2k} \, dt \ll (Ak)^k T (\log ((h \land 1) \log T + 3) \cdot e^k))^k \quad (α') \]

where h \land 1 = \text{Min}\{h, 1\}.

\[ \int_0^{2T} (S(t + h) - S(t))^{2k} \, dt \gg (Ak)^k T (\log ((h \land 1) \log T + 3))^k \]

if k \ll \log \log ((h \land 1) \log T + 3). \quad (α'')

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We shall also use the following estimate

\[ \int_T^{2T} (S(t + h) - S(t))^{2k+1} \, dt \ll (Ak)^k T \log((h \wedge 1) \log T + 3) \cdot e^k)^k. \quad (\beta) \]

We shall omit writing the proofs of these, since they can be derived similarly. We remark that the remainder term in (a) can be written as

\[ O((Ak)^k (\log(e^k(3 + h \log T)))^{k-1/2}). \]

Using (α'), (α'') and (β) we shall prove first

**Theorem 1.** Let \( T > T_0 \) and \( C \) be a constant \( > C_0 \). Then for positive proportion of \( t \) in \( T < t < 2T \),

\[ N\left(t + \frac{2\pi C}{\log T}\right) - N(t) > C + A \sqrt{\log C \cdot \log \log C} \]

and for positive proportion of \( t \) in \( T < t < 2T \),

\[ N\left(t + \frac{2\pi C}{\log T}\right) - N(t) < C - A \sqrt{\log C \cdot \log \log C}. \]

We denote the \( n \)th positive imaginary part of the zeros of \( \xi(s) \) by \( \gamma_n \). Then as an immediate consequence of Theorem 1 we see that for positive proportion of \( \gamma_n \) in \( T < \gamma_n < 2T \),

\[ \frac{\gamma_{n+r} - \gamma_n}{r} \ll \frac{2\pi}{\log T} \left( 1 - \exp(-Ar^2/\log(r + 3)) \right) \]

and for positive proportion of \( \gamma_n \) in \( T < \gamma_n < 2T \),

\[ \frac{\gamma_{n+r} - \gamma_n}{r} \gg \frac{2\pi}{\log T} \left( 1 + \exp(-Ar^2/\log(r + 3)) \right), \]

where \( r \) is an integer \( \geq 1 \). We remark that this corollary for \( r = 1 \) was shown to the author by Professor Montgomery.

Next, using (α'), we shall prove

**Theorem 2.** Suppose that \( T > T_0, j \) is an integer \( \geq 1, k \) is an integer \( \geq j, r \) is an integer \( \geq 1 \) and \( h \) is a positive number \( \gg (\log T)^{-1} \). We put

\[ d(\gamma_n, r) = (\gamma_{n+r} - \gamma_n)/r. \]

Then we have

\[ \frac{1}{N(T)} \sum_{d(\gamma_n, r) > h} d(\gamma_n, r)^j \ll \left( Ak \right)^{k+j-1} \left( 1 + \frac{1}{k} \right)^{j-1} (2k-j+1) \left( \log(e^k rh \log T / (1 + 1/k)^{j-1}) \right)^k, \]

\[ B(k, j)(r \log T)^{j-1} \log T \cdot (rh \log T)^{2k-j+1}, \]

where \( B(k, 1) = 1 \) and \( B(k, j) = (2k-1)(2k-2) \cdots (2k-j+1) \) for \( j \geq 2 \).

Using Theorem 2 we shall prove the following two corollaries.

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1 The interval \((T, 2T)\) may be replaced by \((0, T)\), since the intervals \((T, 2T)\) in (α), (α'), (α'') and (β) may be replaced by \((0, T)\).
Corollary 1. For each integral $k > 1$ and integral $r > 1$, we have
\[
\frac{1}{N(T)} \sum_{\gamma_n < T} d(\gamma_n, r)^k \ll \frac{1}{(\log T)^k}.
\]

Corollary 2. If $C > C_0$ and $r$ is an integer $> 1$, then
\[
\frac{1}{N(T)} \sum_{\gamma_n < T} 1 < e^{-\Lambda C}.
\]

We shall also prove the following theorem using ($\alpha'$).

Theorem 3. Let $K > K_0$. Then "the number of the zeros of $\xi(s)$ in $0 < \text{Im } s < T$ whose multiplicities are $> K" < e^{-\Lambda K N(T)}$.

We shall prove Theorem 1 in §2, Theorem 2 and its corollaries in §3 and Theorem 3 in §4. In §5 we shall give some corrections and complements to [1]–[12]. We remark finally that the results above can be proved also for Dirichlet $L$-functions if we suppose that the modulus is $< T^{1/4 - \varepsilon}$, $\varepsilon > 0$.

Finally, the author wishes to express his thanks to Professor Gallagher, Professor Montgomery and Professor Mueller for their valuable suggestions.

2. Proof of Theorem 1. We put $f(t) = S(t + h) - S(t)$ and $h = 2\pi C / \log T$. We put $E_M = \{ t \in (T, 2T); f(t) > M \}$ for $M > 0$. Let $\varphi_M(t)$ be the characteristic function of $E_M$. Let $C > C_0$ and let $k = \lfloor A \log \log C \rfloor$ with an appropriate positive absolute constant $A$. We consider the integral $I = \int_T^{2T} f^{2k+1}(t) \varphi_0(t) \varphi_M(t) dt$.

\[
I = \int_T^{2T} f^{2k+1}(t) \varphi_0(t) \varphi_M(t) dt + \int_T^{2T} f^{2k+1}(t) \varphi_0(t)(1 - \varphi_M(t)) dt \leq \sqrt{E_M} \left( \int_T^{2T} |f(t)|^{2k+1} dt \right)^{1/2} + M^{2k+1}T,
\]

by ($\alpha'$). On the other hand,
\[
I = \frac{1}{2} \int_T^{2T} |f(t)|^{2k+1} dt + \frac{1}{2} \int_T^{2T} |f(t)|^{2k+1} dt = \frac{1}{2} I_1 + \frac{1}{2} I_2,
\]
say. By ($\alpha''$) and ($\beta$), we get
\[
I_1 > \frac{(f^{2T}_T |f(t)|^{2k} dt)^{(2k-1)/2(k-1)}}{(f^{2T}_T |f(t)|^{2} dt)^{1/2(k-1)}} \geq T(Ak)^k (\log C)^k \geq T(A_1k)^{k(2k-1)/2(k-1)}(\log C)^{k+1/2} \geq T(A_2k)^k (\log C)^k \geq I_2.
\]
Hence we get

\[ |E_M| \geq \left( \frac{T(Ak)^{k(2k-1)/2(k-1)}(\log C)^{k+1/2} - M^{2k+1}T}{\sqrt{T} (Ak)^{k+1/2}(\log C)^{k+1/2}} \right)^2 \geq Te^{-A \log \log C}, \]

provided that \( M \ll \sqrt{\log C \log C} \). This proves the first part of Theorem 1. The second part of Theorem 1 can be derived similarly.

3. Proof of Theorem 2 and its corollaries.

3-1. Proof of Theorem 2. We shall prove our theorem by induction on \( j \). We remark that \( d(\gamma_n, r) < \log \log T \) for \( \sqrt{T} < \gamma_n < T \) by 9.12 of [17]. Suppose that \( d(\gamma_n, r) > h \). Then

\[
\int_{\gamma_n}^{\gamma_n + rh/2} \left( S \left( t + \frac{hr}{2} \right) - S(t) \right)^{2k} dt \gg (hr \log T)^{2k} (\gamma_n + r - \gamma_n - \frac{rh}{2})
\]

Hence we have

\[
\sum_{d(\gamma_n, r) > h} d(\gamma_n, r) < \frac{1}{r(rh \log T)^{2k}} \sum_{d(\gamma_n, r) > h} \int_{\gamma_n}^{\gamma_n + rh/2} \left( S \left( t + \frac{hr}{2} \right) - S(t) \right)^{2k} dt
\]

\[
= \frac{1}{(hr \log T)^{2k}} \int_{\sqrt{T}}^{\gamma_n} \left( S \left( t + \frac{rh}{2} \right) - S(t) \right)^{2k} dt
\]

\[
= \frac{(Ak)^k T (\log(e^{rh \log T})^k}{(hr \log T)^{2k}}.
\]

Next, suppose that our theorem is correct for \( j > 1 \). Then,

\[
\frac{1}{N(T)} \sum_{d(\gamma_n, r) > h} d(\gamma_n, r)^{j+1}
\]

\[
< \frac{1}{N(T)} \sum_{d(\gamma_n, r) > h} d(\gamma_n, r)^j (1 + k) \left( d(\gamma_n, r) - \frac{h}{1 + 1/k} \right)\]

\[
= \frac{(1 + k)}{N(T)} \sum_{d(\gamma_n, r) > h} d(\gamma_n, r)^j \int_{h/(1 + 1/k)}^{d(\gamma_n, r)} du
\]

\[
< \frac{(1 + k)}{N(T)} \int_{h/(1 + 1/k)}^{A/\log \log log T} \left( \sum_{d(\gamma_n, r) > u} d(\gamma_n, r)^j \right) du
\]
ON THE ZEROS OF DIRICHLET L-FUNCTIONS. II

\[ \frac{1}{B(k, j)(\log T)^{r^{j-1}}} \cdot \int_{\log \log T}^{\log T} \frac{\left(\log(e^{r} T / (1 + 1/k)^{j-1})\right)^{k} d\mu}{(ru \log T)^{2k-j}} \]

\[ \frac{(Ak)^{k+j}(1 + 1/k)^{(2k-j)/2}(\log(e^{r} rh \log T / (1 + 1/k)^{j})^{k}}{B(k, j + 1)(\log T)^{r^{j+1}}(rh \log T)^{2k-j}}. \]

This proves our theorem.

3-2. Proof of Corollary 1.

\[ S = \sum_{\gamma_{n} < T} d(\gamma_{n}, r)^{k} \]

\[ = \sum_{d(\gamma_{n}, r) > C/\log T} d(\gamma_{n}, r)^{k} + \sum_{d(\gamma_{n}, r) < C/\log T} d(\gamma_{n}, r)^{k} \]

\[ = S_{1} + S_{2}, \]

say, where we suppose that \( C > C_{0} \). By Theorem 2,

\[ S_{1} \ll \frac{(Ak)^{k}(\log(e^{r} C)^{k} N(T)}{(\log T)^{k} C^{k-1}(rC)^{k+1}} + \frac{N(T)}{\sqrt{T}}. \]

On the other hand,

\[ S_{2} \ll \frac{C^{k}}{(\log T)^{k}} N(T). \]

Hence we get, by taking \( C = (Ak/r^{2})^{k/(2k+1)} \),

\[ S \ll \frac{N(T)}{(\log T)^{k}} \left( \frac{(Ak)^{k}(\log(e^{r} C)^{k} r^{k-1}(rC)^{k+1} + C^{k}}{N(T)(Ak)^{3k/2}(\log T)^{k} r^{k-1}} \right) \]

provided that \( r^{2} \ll k \). Thus we get Corollary 1.

3-3. Proof of Corollary 2. Suppose that \( 1 \ll rC \ll A^{k} \).

\[ S = \sum_{d(\gamma_{n}, r) > C/\log T} \cdot 1 \]

\[ \ll \left( \frac{\log T}{C} \right)^{k} \sum_{d(\gamma_{n}, r) > C/\log T} \frac{d(\gamma_{n}, r)^{k} + \frac{N(T)}{\sqrt{T}}}{\sqrt{T} < \gamma_{n} < T} \]

\[ \ll \frac{(Ak)^{k}(\log(e^{r} C)^{k} N(T)}{(rC)^{2k} C} \] (by Theorem 2)

\[ \ll \frac{(Ak)^{k} N(T)}{(rC)^{2k} C} \ll e^{-A_{1}C} N(T) \]

by taking \( k = [A_{1}rC] \).
4. Proof of Theorem 3. Let \( h \log T = f(k) \), where \( f \) will be chosen later. We consider the integral \( I \equiv \int_T^{2T} (N(t + h) - N(t))^{2k} \, dt \).

\[
I = \int_T^{2T} \sum_{t < \gamma^{(1)} < \gamma^{(2)} < \cdots < \gamma^{(2k)} < t + h} \cdot 1 \, dt
\]

\[
= \sum_{T < \gamma^{(1)} < \cdots < \gamma^{(2k)} < 2T + h} \int_{\text{Min}(\gamma^{(j)})}^{\text{Max}(\gamma^{(j)}) - h} dt
\]

\[
> \sum_{T < \gamma^{(1)} < \cdots < \gamma^{(2k)} < 2T + h} \left( h - \left( \text{Max}(\gamma^{(j)}) - \text{Min}(\gamma^{(j)}) \right) \right)
\]

\[
> \sum_{T < \gamma^{(1)} = \cdots = \gamma^{(2k)} < 2T + h} 1
\]

\[
> h \sum_{l=1}^{\infty} l^{2k} M_l(2T, T),
\]

where \( \gamma^{(j)} \) runs over the imaginary parts of the zeros of \( \xi(s) \) for \( j = 1, 2, \ldots, 2k \) and \( M_l(2T, T) \) is the number counted simply of the zeros of \( \xi(s) \) in \( T < \text{Im} \, s < 2T \) whose multiplicities are exactly \( l \).

On the other hand, by \( (a') \), we get

\[
I \ll T \left( f(k) + \sqrt{k} \sqrt{\log(e^{k^2 f(k)})} \right)^{2k} A^{2k}
\]

\[
\ll N(T) h f(k)^{-1} \left( f(k) + \sqrt{k} \sqrt{\log(e^{k^2 f(k)})} \right)^{2k} A^{2k}.
\]

Here we take \( f(k) = k \). Then \( I \ll N(T) h(Ak)^{2k} \). Hence we get

\[
\sum_{l=1}^{\infty} l^{2k} M_l(2T, T) \ll N(T)(Ak)^{2k}.
\]

Now

\[
K^{2k-1} \sum_{l=K}^{\infty} l M_l(2T, T) \ll \sum_{l=K}^{\infty} l^{2k} M_l(2T, T) \ll N(T)(Ak)^{2k}.
\]

Hence we get

\[
\sum_{l=K}^{\infty} l M_l(2T, T) \ll \frac{(Ak)^{2k} N(T)}{K^{2k-1}} \ll e^{-AKN(T)}
\]

if \( K > K_0 \).

5. Some corrections and complements. In this section we shall give some corrections and complements to [1]–[12] using the same notations as in [1]–[12].

5-1. As we have noticed in §1, \( h \) in (a) must be bounded. Similarly, \( h \)'s in 1.2, 1.15 and 1.17 of p. 140 of [1], 1.10 of p. 348 of [4], 1.20 of p. 51 of [7], 1.7 of p. 70 of [10] and 1.24 of p. 417 and 1.27 of p. 424 of [11] must be bounded. We may remark that since we have used bounded \( h \) in the applications, these corrections are harmless.
5-2. In p. 228 of [3], the remainder term in 1.11 should be multiplied by \(k^2\) and \(k!\) of 1.12 should be \((Ak)^k\). The remainder term in 1.4 of p. 230 of [3] should be multiplied by \(k^2\). The proof of Lemma 2 in p. 228 of [3] should be simplified and corrected as follows. We may suppose that \(k > 2\) and \(a(p) = 1\), for simplicity. We put \(F_1(x) = \sum_{p < x} 1/p\). Then

\[
\sum_{p_i < x} \frac{1}{p_1 p_2 \cdots p_k} - k! F_1^k(x) \ll k! \sum_{p_i < x} \frac{1}{p_1 p_2 \cdots p_k} \ll k! k^2 F_1^{k-2}(x),
\]

where * indicates that we sum over all primes \(p_1, p_2, \ldots, p_{2k} < x\) such that \(p_1 p_2 \cdots p_k = p_{k+1} p_{k+2} \cdots p_{2k}\) and the prime () indicates that we sum over all \(p_1, p_2, \ldots, p_k\) such that some \(p_i\) and some \(p_j\) are equal for \(1 < i < j < k\).

5-3. (iii) and (iv) of p. 234 of [3] should be erased.

5-4. \(c^n q \cdot \log \log q / \sqrt{n} \log C\) in 1.18 of p. 61, 1.4 of p. 62 and 1.5 of p. 63 of [8] should be replaced by \(C \log c^n q \cdot \log^2 q\).

5-5. As we have seen in the proof of Theorem 1 in the present paper, \(\epsilon\) in 1.15 and 1.17 of p. 140 of [1], 1.11 and 1.13 of p. 399 of [6], 1.5 and 1.12 of p. 50 and 1.4, 1.6, 1.18 and 1.23 of p. 56 of [7] should be omitted.

5-6. [4] is generalized in [12].

5-7. [9] should be corrected and improved as in the present paper.

5-8. (1) in p. 52 of [7] can be improved as follows.

\[
\sum_{x} (R_n(t, \chi))^{2k} \ll (Ak)^{2k} q .
\]

We may omit writing the proofs of (1') above and (2) in p. 52 of [7], (even though it was announced that these proofs would be published).

5-9. In the statement of Theorem 2 of [5], the condition \(\lambda(q) < t(q)\) should be replaced by \(\lambda(q) = o(t(q))\), because \((q - 2)/2\pi\) should be multiplied to the first term of 1.16, the first two terms of 1.17 and the term of 1.18 of p. 142 and the final result is

\[
\sum_{|\gamma| < |\tau(q)|} e^{i\sigma y} = (q - 2)\sin(at(q))(\log(qt(q)/2\pi))/\pi a + O(q(at(q) + a^{-1})).
\]

We remark also that \(O(q)\) should be added to the right-hand side of 1.13, 1.14 and 1.15 in p. 142.

**REFERENCES**


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