

SPHERICAL MEANS AND GEODESIC CHAINS ON A RIEMANNIAN MANIFOLD¹

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ABSTRACT. Some spectral properties of spherical mean operators defined on a Riemannian manifold are given. As an application we deduce a statistic property of geodesic chains which is interesting from the view point of geometric probability.

1. Introduction. Let M be a complete Riemannian manifold, and r a positive real number. By SM , we shall denote the tangent unit sphere bundle of M . The spherical mean with radius r is an operator acting on continuous functions on M by

$$(L_r f)(x) = \int_{S_x M} f(\exp rv) dS_x(v) \quad (f \in C^0(M)),$$

where dS_x stands for the normalized canonical density on the unit sphere $S_x M$. When $M = \mathbf{R}^n$, this coincides with the classical spherical mean

$$(\widehat{L}_r f)(x) = \int_{S^{n-1}} f(x + rv) dS^{n-1}(v),$$

which, as is well known, has a continuous selfadjoint extension to $L^2(\mathbf{R}^n)$. Further taking the Fourier transform of both sides, one gets an equality

$$(*) \quad \begin{aligned} \widehat{L}_r \hat{f}(\xi) &= \int_{S^{n-1}} \exp(\sqrt{-1} r \langle v, \xi \rangle) dS^{n-1}(v) \hat{f}(\xi) \\ &= \Gamma(n/2) (\|\xi\| r/2)^{-n/2+1} J_{n/2-1}(\|\xi\| r) \hat{f}(\xi), \end{aligned}$$

$J_\nu(x)$ being the Bessel function. Using here the estimate $J_\nu(x) = O(x^{-1/2})$ ($x \uparrow \infty$), one gets an inequality

$$\int_{\mathbf{R}^n} (1 + \|\xi\|^2)^s |\widehat{L}_r \hat{f}(\xi)|^2 d\xi \leq c_r \int_{\mathbf{R}^n} (1 + \|\xi\|^2)^{s-(n-1)/2} |\hat{f}(\xi)|^2 d\xi,$$

from which it follows that L_r has a continuous extension to the Sobolev spaces: $H_{(s)}(\mathbf{R}^n) \rightarrow H_{(s+(n-1)/2)}(\mathbf{R}^n)$.

These properties can be generalized to spherical means defined on Riemannian manifolds as follows.

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THEOREM A. (i) L_r has a continuous selfadjoint extension: $L^2(M) \rightarrow L^2(M)$ with $\|L_r\|_2 \leq 1$.

(ii) If for any x in M , the restriction of the exponential mapping $\exp_x: rS_x M \rightarrow M$ is an immersion, then L_r has a continuous extension: $H_{(s)}^{\text{loc}}(M) \rightarrow H_{(s+(n-1)/2)}^{\text{loc}}(M)$. In particular, if, in addition, M is compact and $\dim M \geq 2$, then the operator $L_r: L^2(M) \rightarrow L^2(M)$ is compact, and any eigenfunction of L_r belonging to nonzero eigenvalue is smooth.

In case r is small enough, (ii) is due to Tsujishita [24], who proved in fact that the spherical mean with small radius is a *Fourier integral operator* of negative order. We will see in §2 that his idea still works well in the above situation, although L_r may be no longer a Fourier integral operator in the usual sense.

Let it be assumed that M is compact. Selfadjointness of L_r implies especially the equality $\int_M L_r f dg = \int_M f dg$ for any $f \in L^2(M)$, from which, using the general theory of Markov processes, one can equip a probability measure on the infinite product $M \times M \times M \times \dots$ such that the shift transformation is measure preserving and the coordinate process is a Markov process whose transition operator is just L_r . Ergodicity of this dynamical system is equivalent to the statement that 1 is simple eigenvalue of L_r . In this view, L_r will be called *ergodic* if the equality $L_r f = f$ ($f \in L^2(M)$) implies that f is constant.

Ergodicity of L_r means something more. To explain this, let us give here a notion of geodesic chains. Let k be a positive integer or ∞ . We will call a continuous mapping c of the interval $[0, k]$ into a complete Riemannian manifold M a *geodesic chain* of length k if each restriction $c_i = c[[i - 1, i]$ ($i = 1, 2, \dots$) is a geodesic curve in M . If, in addition, every c_i has a common length r , then such a curve will be called an *r-geodesic chain*. We denote by $\mathcal{C}_k(r; M)$ the set of all *r-geodesic chains* of length k in M . As will be seen in §3, the set $\mathcal{C}_k(r; M)$ is naturally identified with the fiber bundle $S^k M$ whose fiber $S_x^k M$ is the k -product $S_x M \times S_x M \times \dots \times S_x M$, so that $\mathcal{C}_k(r; M)$ has a measure which comes from the fiber product measure on $S^k M$. Suppose again M is compact. A geodesic chain $c \in \mathcal{C}_\infty(r; M)$ will be called *uniformly distributed* if for any Jordan measurable subset J in M , the time average

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=0}^{N-1} \chi_J(c(k)),$$

χ_J being the defining function, exists and equals the space average $\int_J dg / \int_M dg$ (cf. Weyl [25])

THEOREM B. If L_r is ergodic, then almost all *r-geodesic chains* are uniformly distributed. Here “almost all” means “except for a set of measure zero in $\mathcal{C}_\infty(r; M)$.”

The proof relies upon a property of the measure on $\mathcal{C}_\infty(r; M)$, which guarantees the existence of a geometric model of Markov process on $\mathcal{C}_\infty(r; M)$ with transition operator L_r .

By Theorem B, the ergodicity of L_r implies that there exists an *r-geodesic chain* c such that the set $\{c(k); k = 1, 2, \dots\}$ is dense in M . If M is a compact

homogeneous space, the converse holds (cf. [20]). In the general case, the situation seems more complicated. But as an application of Theorem A, we can prove

THEOREM C. *If $\dim M \geq 2$, and for any x in M , the mapping $\exp_x: rS_x M \rightarrow M$ is an immersion, then L_r is ergodic.*

In the course of proof, we require the fact that under the above condition, any two points in M can be joined by an r -geodesic chain of finite length.

As particular cases, we have

COROLLARY. *If one of the following conditions is satisfied, then L_r is ergodic.*

- (i) *The radius r is smaller than the injective radius of M .*
- (ii) *M has no conjugate point (r is arbitrary).*

Furthermore, we get some information about discrete spectrum.

THEOREM D. *Under the same conditions as Theorem C:*

- (i) *-1 is not an eigenvalue;*
- (ii) *there are infinitely many positive eigenvalues.*

If, in addition, M has no geodesic loop with length r , then there are infinitely many negative eigenvalues.

The first assertion implies the mixing property of the dynamical system. The fundamental idea of the proof of (ii) is due to M. Gromov.

We now return to the equality (*). The Fourier inversion gives rise to a formal expansion of L_r with respect to r :

$$L_r \sim I - (r^2/2n)\Delta + (r^4/8n(n+2))\Delta^2 + \dots$$

In §5, we observe that spherical means on Riemannian manifolds have also analogous expansions, which, as by-product, provide us with an approximation theorem for the fundamental solution for the heat equation by an iteration of L_r :

THEOREM E. *Let M be compact, and let K_t be the fundamental solution for the heat equation $(\partial/\partial t + \Delta)u = 0$. Then for any $f \in L^2(M)$, the iteration*

$$L_{r(t,N)}^N f \quad (r(t,N) = \sqrt{2nt/N}),$$

converges to $K_t f$ in L^2 -sense and in a.e. sense as $N \uparrow \infty$.

The formal expansion of L_r allows us also to prove the following.

THEOREM F. *If M is a symmetric space with rank ≥ 2 , then L_r is ergodic for any $r > 0$.*

In the case of rank one, we can give a condition for radius r in order that L_r is ergodic.

We should point out that some of the results about L_r had been obtained by P. Günther [9] when M is a harmonic Riemannian space, and that an expansion theorem for a similar mean valued operator has been proved by A. Gray and T. J. Willmore [7].

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2. Spherical means. Let M be a connected n -dimensional complete Riemannian manifold, and $\pi: SM \rightarrow M$ its tangent unit sphere bundle. The canonical density associated with the metric of M is denoted dg . The bundle SM possesses a measure given by the density dSM which takes the form

$$\int_{SM} f dSM = \int_M dg(x) \int_{S_x M} f(v) dS_x(v).$$

The density dSM coincides with the one associated with the standard metric on SM (see [18]).

Let L_r be the spherical mean with radius r . It is easy to see that L_r is an operator of the space of bounded continuous functions into itself with $\sup|L_r f| \leq \sup|f|$.

LEMMA 2.1. L_r can be extended to a continuous operator $L_r: L^p(M) \rightarrow L^p(M)$, $p \geq 1$, with $\|L_r\|_p \leq 1$. Further, in case $p = 2$, the extension is a selfadjoint operator of $L^2(M)$.

PROOF. Let $\{\varphi_t\}$ ($-\infty < t < \infty$) be the geodesic flow on SM , and $\pi_*: C^0(SM) \rightarrow C^0(M)$ the fiber integral

$$(\pi_* h)(x) = \int_{S_x M} h dS_x.$$

Then we easily find $L_r = \pi_* \varphi_r^* \pi^*$, where φ_r^* , π^* denote the pull backs by φ_r , π , respectively. Since $\{\varphi_t\}$ is a volume preserving flow (see [18]), we get

$$\begin{aligned} \int_M L_r f \cdot h dg &= \int_M h(x) dg(x) \int_{S_x M} \varphi_r^* \pi^* f dS_x \\ &= \int_{SM} \varphi_r^*(\pi^* f \cdot \varphi_{-r}^* \pi^* h) dSM \\ &= \int_{SM} \pi^* f \cdot \varphi_{-r}^* \pi^* h dSM = \int_M f \cdot L_r h dg, \end{aligned}$$

which proves the last half of our assertion. The rest follows from the estimates $\|\pi^*\|_p = 1$, $\|\pi_*\|_p \leq 1$.

REMARK. In his paper [9], P. Günther considered an operator relating to a mean valued operator treated by Ruse, Walker and Willmore [16], which takes the form

$$(M_r f)(x) = \int_{rS_x M} f(\exp v) d\sigma'_x,$$

where $d\sigma'_x$ is a volume element on the r -sphere satisfying $\exp_x^*(dg) = dr \wedge d\sigma'_x$. He proved that M_r is selfadjoint. But the mean valued operator \tilde{L}_r of the form

$$(\tilde{L}_r f)(x) = M_r(1)(x)^{-1} (M_r f)(x)$$

is not in general selfadjoint. If M is a harmonic Riemannian space, \tilde{L}_r coincides with L_r .

We now proceed to the proof of Theorem A(ii). By ∇ , we shall denote the covariant differentiation associated with the metric. We assume hereafter that for any x in M , the mapping $\exp_x: rS_x M \rightarrow M$ is an immersion. We should note that this assumption is equivalent to the statement that the mapping $\pi_r: SM \rightarrow M \times M$ given by $\pi_r(v) = (\pi(v), \exp rv)$ is an immersion, or that for any $v \in SM$, the point $\exp rv$ is not conjugate to $\pi(v)$ along the geodesic $\exp tv$ ($0 < t \leq r$).

The following is easy to see.

LEMMA 2.2. *The normal bundle $N(\pi_r)$ of the immersion π_r is given by*

$$\bigcup_{v \in SM} R(v \oplus -\varphi_r(v)).$$

To prove Theorem A, let $\{U_\alpha\}$ be a locally finite open covering of M such that the inverse image $\pi_r^{-1}(U_\alpha \times U_\beta)$ is a finite disjoint union of open sets $\{W_i\}$ and each restriction $\pi_r: W_i \rightarrow U_\alpha \times U_\beta$ is injective. Then using partitions of unity $\{\varphi_\alpha\}$ associated with $\{U_\alpha\}$, we have

$$L_r f = \sum_{\alpha, \beta} \varphi_\alpha L_r \varphi_\beta f \quad \text{for } f \in C_0^\infty(M).$$

We will prove that the operator $L_{\alpha\beta} = \varphi_\alpha L_r \varphi_\beta: C_0^\infty(U_\beta) \rightarrow C_0^\infty(U_\alpha)$ is written as a finite sum of Fourier integral operators possibly with distinct Lagrange manifolds. To this end, let $K_{\alpha\beta}$ be the distribution kernel of $L_{\alpha\beta}$, which has the form

$$\begin{aligned} \langle K_{\alpha\beta}, h \rangle &= \int_{SM} \pi_r^*(\varphi_\alpha \varphi_\beta h) dSM \\ &= \sum_i \int_{W_i} \pi_r^*(\varphi_\alpha \varphi_\beta h) dSM \quad (h \in C_0^\infty(U_\alpha \times U_\beta)). \end{aligned}$$

Hence it suffices to show that each distribution $K_{\alpha\beta}^i$ given by

$$\langle K_{\alpha\beta}^i, h \rangle = \int_{W_i} \pi_r^*(\varphi_\alpha \varphi_\beta h) dSM$$

belongs to the class $I^{-(n-1)/2}(U_\alpha \times U_\beta, \Lambda_{W_i})$ (see [11] for the notation), where $\Lambda_{W_i} = T_{W_i}(U_\alpha \times U_\beta) \setminus 0$ is the conormal bundle of W_i minus zero section. For this, consider W_i as a submanifold of $U_\alpha \times U_\beta$. We may assume there exists a local coordinate (x_1, \dots, x_{2n}) of $U_\alpha \times U_\beta$ such that $W_i = \{(x_1, \dots, x_{2n}); x_{2n} = 0\}$. Put $dSM|_{W_i} = f dx_1 \cdots dx_{2n-1}$. Then

$$\begin{aligned} \langle K_{\alpha\beta}^i, h \rangle &= \int_{W_i} \varphi_{\alpha\beta}(x_1, \dots, x_{2n-1}, 0) h(x_1, \dots, x_{2n-1}, 0) f dx_1 \cdots dx_{2n-1} \\ &= (2\pi\sqrt{-1})^{-1} \int_{\mathbb{R}^n \times \mathbb{R}} h(x_1, \dots, x_{2n}) a(x_1, \dots, x_{2n}, \theta) \\ &\quad \cdot \exp(x_{2n}\theta) dx_1 \cdots dx_{2n} d\theta \end{aligned}$$

where $\varphi_{\alpha\beta} = \varphi_\alpha \cdot \varphi_\beta$, $a(x, \theta) = f(x_1, \dots, x_{2n-1}) \varphi_{\alpha\beta}(x_1, \dots, x_{2n})$. Hence, $K_{\alpha\beta}^i$ is expressed as an oscillatory integral which belongs to $I^m(U_\alpha \times U_\beta, \Lambda)$ with

$$\Lambda = \{(x_1, \dots, x_{2n-1}, 0; \theta dx_{2n})\}, \quad m = -\frac{1}{2}(n-1).$$

We now let $L_{\alpha\beta}^i: \mathcal{D}(U_\beta) \rightarrow \mathcal{D}'(U_\alpha)$ be a linear mapping with the distribution kernel $K_{\alpha\beta}^i$. Then the regularity of L_r reduces to that of $L_{\alpha\beta}^i$. Noting that the mappings

$$\begin{aligned} N(\pi_r) \setminus 0 &\rightarrow TM \setminus 0, & N(\pi_r) \setminus 0 &\rightarrow TM \setminus 0, \\ c(v \oplus -\varphi_r(v)) &\mapsto cv, & c(v \oplus -\varphi_r(v)) &\mapsto -c\varphi_r(v) \end{aligned}$$

are two-fold coverings, we see that the restrictions to $\Lambda_{\mathcal{W}_i}$ of the natural projections: $T^*(U_\alpha \times U_\beta) \rightarrow T^*U_\alpha, T^*U_\beta$ are local diffeomorphisms, which implies that $L_{\alpha\beta}^i$ is extended to: $H_{(s)}^{\text{comp}}(U_\alpha) \rightarrow H_{(s+(n-1)/2)}^{\text{comp}}(U_\beta)$ (see [11]). This proves Theorem A(ii).

REMARK. If there is no closed geodesic with length $2r$ then L_r is a F.I.O. in the sense of Hörmander, because $N(\pi_r) \setminus 0$ has no intersection in $TM \times TM$, and hence the conormal bundle $\Lambda(SM)$ is a Lagrange manifold in $T^*M \times T^*M$.

3. Geodesic chains. The set of all geodesic segments with length r in a complete manifold M is just $\mathcal{C}_1(r; M)$, which is identified with the total space SM by a bijection $\tau_1: SM \rightarrow \mathcal{C}_1(r; M)$ defined by $\tau_1(v)(t) = \exp tv, 0 \leq t \leq 1$. Using this identification, we can introduce a measure P_1 onto the set $\mathcal{C}_1(r; M)$, which is natural from a geometric point of view. For instance, when $M = \mathbf{R}^n$, this is just the usual measure on the set of oriented line segments with constant length, introduced in the theory of geometric probability [17].

For general k , we can equip a measure on $\mathcal{C}_k(r; M)$ in a similar way. To explain this, let S^kM be the fiber bundle on M whose fiber at x is the product $S_xM \times \cdots \times S_xM$ (k -times). The set $\mathcal{C}_k(r; M)$ is then identified with S^kM as follows. For a chain $c = (c_1, \dots, c_k), c_i = c[[i-1, i]$, one can associate a series of tangent vectors $(v_1, v'_2, \dots, v_k^{(k-1)})$ such that $v_i^{(i-1)}$ is the unit velocity vector of c_i at the point $c(i-1)$. Translating the vector $v_i^{(i-1)}$ parallelly along the curve (c_1, \dots, c_{i-1}) , one can get a sequence $(v_1, \dots, v_k) \in S_x^kM$. It is easy to see that this correspondence is bijective. We let $\tau_k: S^kM \rightarrow \mathcal{C}_k(r; M)$ be the inverse of this bijection. Note that the subset $\mathcal{C}_k(r)_x$ in $\mathcal{C}_k(r)$, which consists of all r -geodesic chains c with $c(0) = x$, corresponds to the fiber S_x^kM .

Define a probability measure on $\mathcal{C}_k(r)_x$ to be the product of normalized measure on the Euclidean sphere S_x . We shall denote it by $P_{k,x}$. We can then equip a measure P_k on $\mathcal{C}_k(r)$ so that the following relation holds:

$$\int_{\mathcal{C}_k} f dP_k = \int_M dg(x) \int_{\mathcal{C}_{k,x}} f dP_{k,x}.$$

Consider a mapping $p_k: \mathcal{C}_\infty(r)_x \rightarrow \mathcal{C}_k(r)_x$ defined by $p_k(c) = c[[0, k]$. Since p_k corresponds, in the identification, to the projection $S_x^\infty M \rightarrow S_x^k M$, the inverse image $p_k^{-1}(c)$ is identified with the product $\prod_{k+1}^\infty S_x M$, which has a product measure to be denoted $P_x(\cdot|c)$. We easily observe

$$P_{\infty,x}(\mathbf{A}) = \int_{\mathcal{C}_{k,x}} P_x(\mathbf{A} \cap p_k^{-1}(c)|c) dP_{k,x}(c)$$

for any measurable set \mathbf{A} in $\mathcal{C}_\infty(r)_x$.

It is worthwhile to note that the compact open topology of $\mathcal{C}_k(r)$ coincides with the induced one from S^kM , and that if \mathbf{A} is measurable in $\mathcal{C}_k(r)_x$ with measure 1, then \mathbf{A} is dense in $\mathcal{C}_k(r)_x$.

From now on, let us assume M is compact and $\int_M dg = 1$, so that P_k is a probability measure. We write, for brevity, $\mathcal{C}_x(r)$, $\mathcal{C}(r)$, P_x , P for $\mathcal{C}_\infty(r; M)_x$, $\mathcal{C}_\infty(r; M)$, $P_{\infty,x}$, P_∞ , respectively. Define a mapping $\pi_k: \mathcal{C}(r) \rightarrow M$ by $\pi_k(c) = c(k)$. Let \mathfrak{B} express the σ -ring generated by $\pi_k^{-1}(A)$ ($k = 1, 2, \dots$) where A runs over all Borel subsets in M . The shift T is the mapping of $\mathcal{C}(r)$ into itself given by $(Tc)(t) = c(t + 1)$. Since $\pi_{k+1} = \pi_k \circ T$, T is \mathfrak{B} -measurable.

LEMMA 3.1. *Let $c \in \mathcal{C}_k(r; M)$, and $\mathbf{A} \in \mathfrak{B}$. Then*

$$P_x(T^{-k}\mathbf{A} \cap p_k^{-1}(c)|c) = P_{c(k)}(\mathbf{A}).$$

PROOF. Suppose $c = \tau_k(v_1^0, \dots, v_k^0)$. On the subset $(v_1^0, \dots, v_k^0) \times \prod_{k+1} S_x M$ in $S_x M$, the k th iteration of T is expressed as $(v_1^0, \dots, v_k^0, v_{k+1}, \dots) \mapsto (P_c v_{k+1}, P_c v_{k+2}, \dots)$, P_c being the parallel translation along the curve c , which deduces

$$P_x(T^{-k}\mathbf{A} \cap p_k^{-1}(c)|c) = \text{measure of } \{(v_{k+1}, v_{k+2}, \dots); (P_c v_{k+1}, \dots) \in \mathbf{A}\}.$$

Since the parallel translation is volume preserving, the last expression is equal to $P_{c(k)}(\mathbf{A})$ as desired.

The shift operator and the spherical mean L_r have a close connection with each other:

LEMMA 3.2. *For $\mathbf{A} \in \mathfrak{B}$, put $P_x(\mathbf{A}) = P_x(\mathbf{A} \cap \mathcal{C}_x(r))$. Then the function $x \mapsto P_x(\mathbf{A})$ belongs to $L^1(M)$, and $P_x(T^{-1}\mathbf{A}) = (L_r P(\mathbf{A}))(x)$.*

PROOF. The first assertion is obvious. The second follows from the above lemma. Indeed

$$\begin{aligned} P_x(T^{-1}\mathbf{A}) &= \int_{\mathcal{C}_{1,x}} P_x(T^{-1}\mathbf{A} \cap p_1^{-1}(c)|c) dP_{1,x}(c) \\ &= \int_{S_x M} P_{\text{exp}(rv)}(\mathbf{A}) dS_x(v) = (L_r P(\mathbf{A}))(x). \end{aligned}$$

Let $\mathbf{A} \in \mathfrak{B}$. By definition of P ,

$$P(\mathbf{A}) = \int_M dg(x) \int_{\mathcal{C}_x(r)} \chi_{\mathbf{A}} dP_x = \int_M P_x(\mathbf{A}) dg(x).$$

Hence from Lemma 2.1, we find

$$\begin{aligned} P(T^{-1}\mathbf{A}) &= \int_M P_x(T^{-1}\mathbf{A}) dg(x) = \int_M L_r(P(\mathbf{A})) dg \\ &= \int_M P_x(\mathbf{A}) dg(x) = P(\mathbf{A}). \end{aligned}$$

Thus we have proved

PROPOSITION 3.3. *The triple $(\mathcal{C}(r), P, T)$ is a dynamical system.*

REMARK. (i) Instead of the set $\mathcal{C}_\infty(r; M)$, one can consider a set $\mathcal{C}_{-\infty, \infty}(r; M)$ consisting of geodesic chains $c: (-\infty, \infty) \rightarrow M$. In the same way, we can introduce a measure on this set such that the shift is a bijective measure preserving transformation.

(ii) Lemma 3.2 implies, for any Borel set A , $P_x(\pi_k^{-1}(A)) = (L_r^k \chi_A)(x)$. Therefore the k th iteration L_r^k of spherical mean is an integral operator associated with the double fibering

$$\begin{array}{ccc} & S^k M & \\ \pi_0 \swarrow & & \searrow \pi_k \\ M & & M, \end{array}$$

that is, $L_r^k f = \int_{S^k} \pi_k^* f dP_{x, k}$.

As is usual, the dynamical system $(\mathcal{C}(r), P, T)$ is said to be *ergodic* if any invariant \mathfrak{B} -measurable set has measure zero, or its complement does. Ergodicity is equivalent to either one of the following conditions.

(i) For any Borel set A in M ,

$$\lim_{N \rightarrow \infty} \frac{1}{N} \left(\sum_{k=0}^{N-1} \chi_A(c(k)) \right) = \int_A dg \quad \text{a.e. } c.$$

(ii) For any $f \in L^2(M)$,

$$\lim_{N \rightarrow \infty} \frac{1}{N} \left(\sum_{k=0}^{N-1} L_r^k f(x) \right) = \int_M f dg \quad \text{a.e. } x.$$

(iii) If $L_r f = f$, $f \in L^2(M)$, then $f = \text{constant}$.

For instance, the equivalence of (i) and ergodicity comes from

$$\frac{1}{N} \sum_{k=0}^{N-1} \chi_{\pi_0^{-1}(A)}(T^k c) = \frac{N+l}{N} \frac{1}{N+l} \left(\sum_{k=0}^{N+l-1} \chi_A(c(k)) - \sum_{k=0}^{l-1} \chi_A(c(k)) \right).$$

Further, the equivalence of (i) and (ii) comes from the equality

$$\int_{\mathcal{C}(r)} (\chi_{\pi_0^{-1}(A)} \circ T^k) \chi_{\pi_0^{-1}(B)} dP = \int_M L_r^k \chi_A \cdot \chi_B dg,$$

which is deduced as

$$\begin{aligned} \text{the left-hand side} &= \int_{\mathcal{C}(r)} \chi_{\pi_k^{-1}(A) \cap \pi_0^{-1}(B)} dP \\ &= \int_M P_x(\pi_k^{-1}(A) \cap \pi_0^{-1}(B)) dg(x) \\ &= \int_M P_x(\pi_k^{-1}(A)) \cdot \chi_B(x) dg(x) \\ &= \text{the right-hand side.} \end{aligned}$$

We are in a position to prove Theorem B. Suppose that $(\mathcal{C}(r), P, T)$ is ergodic. We will construct a measurable set \mathbf{A} in $\mathcal{C}(r)$ with $P(\mathbf{A}) = 1$ such that any geodesic chains in \mathbf{A} are uniformly distributed. Choose a system of C^∞ -functions $\{f_i\}$ ($i = 1, 2, \dots$) such that any bounded Riemannian integrable function can be approximated from above and below by finite linear combinations of $\{f_i\}$ in the

sense that given f we can find, for any $\epsilon > 0$, $c_1, \dots, c_m, d_1, \dots, d_m (\in \mathbf{R})$ such that

$$\sum c_i f_i \leq f \leq \sum d_i f_i, \quad \int_M \sum d_i f_i - \sum c_i f_i \, dg < \epsilon.$$

For instance we may choose for $\{f_i\}$ a complete orthonormal basis of eigenfunctions of the Laplacian. The subset

$$A = \left\{ c \in C(r); \lim_{N \rightarrow \infty} \frac{1}{N} \left(\sum_{k=0}^{N-1} f_i(c(k)) \right) = \int_M f_i \, dg \text{ for any } i \right\}$$

satisfies our demand.

4. Ergodicity and eigenvalues of spherical means. Let M be a complete manifold. Let k be a positive integer. We endow M with an equivalence relation by setting $x \sim_k^r y$ if and only if x and y are joined by an r -geodesic chain of length kl for an integer l . The following is obvious.

LEMMA 4.1 (MAXIMAL PRINCIPLE). *Let f be a continuous function on M satisfying $L_r^k f = f$. Suppose that f attains the maximum, and that an equivalence class of the relation \sim_k^r containing a point at which f takes the maximum is dense in M . Then $f \equiv \text{constant}$. In particular, if M is compact, and if any two points of M are joined by an r -geodesic chain, then any continuous eigenfunction of L_r , with eigenvalue 1 must be constant.*

To prove Theorem C, it is enough to see the following (taking account of the above and Theorem A(ii)).

LEMMA 4.2. *Suppose $\dim M \geq 2$, and the mapping π_r is immersion. Then any two points of M can be joined by an r -geodesic chain of even length.*

PROOF. We will show that any equivalence classes of the relation \sim_2^r are open. We make use of the following notations:

$$B_\epsilon(x) = \{y \in M; d(x, y) < \epsilon\},$$

$$\Delta = \{(u, v) \in S_x M \times S_x M\},$$

$$\Delta_\delta = \{(u, v) \in S_x M \times S_x M; g(u, v) > 1 - \delta\},$$

$$\partial\Delta_\delta = \{(u, v) \in S_x M \times S_x M; g(u, v) = 1 - \delta\}.$$

Let $x \in M$, and let $F: S_x M \times S_x M \rightarrow M$ be a mapping given by $F(u, v) = \pi_2(\tau_2(u, -v))$. From the assumption, we easily see that there exist small positive numbers ϵ, δ such that

$$F(\partial\Delta_\delta) \subset M \setminus B_\epsilon(x), \quad F(\Delta_\delta) \cap (B_\epsilon(x) \setminus x) = F(\Delta_\delta \setminus \Delta) \cap (B_\epsilon(x) \setminus x).$$

Since the set $A = F(\bar{\Delta}_\delta) \cap (B_\epsilon(x) \setminus x)$ is closed in the connected space $B_\epsilon(x) \setminus x$, we have only to prove that A is open in $B_\epsilon(x) \setminus x$. In fact, this being the case, the point x and any other point in $B_\epsilon(x)$ can be joined by an r -geodesic chain of length two, so that the open neighborhood $B_\epsilon(x)$ of x is contained in the equivalence class of x .

Our claim comes from the assertion that the restriction $F: \Delta_\delta \setminus \Delta \rightarrow M$ is a submersion, which is proved in the following way. Let $(u, v) \in \Delta_\delta \setminus \Delta$, and let $dF: T_u S_x M \oplus T_v S_x M \rightarrow T_{F(u, v)} M$ be the differential of F at (u, v) . We denote by

$v^* \in T_{\exp(ru)}M$ the parallel translation of v along the geodesic $\tau_1(u)$, and by $v^{**} \in T_{F(u,v)}M$ that of v along the chain $\tau_1(u, -v)$. Using Gauss' lemma, we get

$$dF(0 \oplus T_v S_x M) = \text{the orthogonal complement of } v^{**} \text{ in } T_{F(u,v)}M.$$

To compute $dF(X \oplus 0)$, ($X \in T_u S_x M$), we choose a curve $u(s)$ in $S_x M$ with $u(0) = u$, $du(0)/ds = X$, and consider a family of geodesics $\{c_s(t)\}_s$ given by $c_s(t) = \tau_2(u(s), -v)(t + 1)$ ($0 \leq t \leq 1$). The vector field $J_X(t) = (\partial/\partial s)|_{s=0} c_s(t)$ is then a Jacobi field along the curve $c(t) = c_0(t)$, satisfying

$$dF(X \oplus 0) = J_X(1), \quad d_u(\pi \circ \varphi_r)(X) = J_X(0).$$

Further we have

$$\begin{aligned} (\nabla_{\dot{c}} J(0), \dot{c}(0)) &= g\left(\frac{D}{\partial t}\Big|_{t=0} \frac{\partial}{\partial s}\Big|_{s=0} c_s(t), \dot{c}(0)\right) \\ &= g\left(\frac{D}{\partial s}\Big|_{s=0} \frac{\partial}{\partial t}\Big|_{t=0} c_s(t), \dot{c}(0)\right) \\ &= \frac{1}{2} \frac{\partial}{\partial s}\Big|_{s=0} g\left(\frac{\partial}{\partial t}\Big|_{t=0} c_s(t), \frac{\partial}{\partial t}\Big|_{t=0} c_s(t)\right) \\ &= 0, \end{aligned}$$

because $g(\dot{c}_s(0), \dot{c}_s(0)) \equiv r^2$. We now suppose that $g(J_X(1), v^{**}) = 0$ for any X in $T_u S_x M$. Since $(d^2/dt^2)g(J_X(t), \dot{c}(t)) = 0$, we have constants a, b such that

$$g(J_X(t), \dot{c}(t)) = at + b.$$

But $a + b = g(J_X(1), \dot{c}(1)) = -g(J_X(1), v^{**}) = 0$, and $a = g(\nabla_{\dot{c}} J(0), \dot{c}(0)) = 0$, which implies

$$g(J_X(0), v^*) = -r^{-1}g(J_X(0), \dot{c}(0)) = 0.$$

On the other hand, we have from Gauss' lemma,

$$\{J_X(0); X \in T_u S_x M\} = \text{the orthogonal complement of } \varphi_r(u) \text{ in } T_{\exp(ru)}M,$$

which contradicts the fact that $\varphi_r(u)$ is not any constant multiple of v^* . Hence we can find a vector $X \in T_u S_x M$ such that $J_X(1)$ is not contained in the hyperplane $dF(0 \oplus T_v S_x M)$, which implies $\text{Im } dF = T_{F(u,v)}M$, as desired.

The above lemma leads also to Theorem D(i). Indeed, if $L_r f = -f$, then $f \in C^\infty(M)$, and $L_r^2 f = f$; hence by the maximum principle, f must be constantly zero.

As a corollary, we have the following which strengthens the assertion for ergodicity.

PROPOSITION 4.3. *Under the same assumption as Theorem C, the dynamical system $(\mathcal{C}(r), P, T)$ has the mixing property, namely, for any pair \mathbf{A}, \mathbf{B} in \mathfrak{B} , $\lim_{N \rightarrow \infty} P(T^{-N}\mathbf{A} \cap \mathbf{B}) = P(\mathbf{A}) \cdot P(\mathbf{B})$.*

PROOF. Our assertion is obviously equivalent to: For any $f, h \in L^2(M)$,

$$\lim_{N \rightarrow \infty} \int_M L_r^N f \cdot h \, dg = \int_M f \, dg \int_M h \, dg.$$

This comes from a stronger assertion:

$$\lim_{N \rightarrow \infty} L_r^N f = \int_M f \, dg \quad \text{in } L^2\text{-sense,}$$

which, looking at the L^2 -expansion of f into eigenfunctions, is easily deduced.

We now focus our attention on eigenvalues other than ± 1 . We continue to assume that the mapping $\pi_r: SM \rightarrow M \times M$ is an immersion. Note that the singular support of the distribution kernel of L_r is just the image of π_r , therefore from smoothness of eigenfunctions, it follows immediately that L_r has infinitely many eigenvalues.

To prove the existence of infinitely many positive eigenvalues, it suffices to see that, for any positive integer N , there exist smooth functions Φ_1, \dots, Φ_N such that

$$\begin{aligned} \langle \Phi_i, \Phi_j \rangle &= 0 \text{ for } i \neq j, \\ \langle L_r \Phi_i, \Phi_j \rangle &= 0 \text{ for } i \neq j, \\ \langle L_r \Phi_i, \Phi_i \rangle &> 0 \text{ for any } i, \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes the L^2 -inner product $\int_M (\cdot, \cdot) dg$. For this we first show that we can take a set $\{x_1, \dots, x_N, y_1, \dots, y_N\}$ so that

$$\begin{aligned} \exp(rS_{x_i}M) \ni y_i \quad \text{for any } i, \quad \exp(rS_{x_i}M) \not\ni x_j \quad \text{for } i \neq j, \\ \exp(rS_{y_i}M) \not\ni y_j \quad \text{for } i \neq j, \quad \exp(rS_{x_i}M) \not\ni y_j \quad \text{for } i \neq j. \end{aligned}$$

This is obviously possible if r is small enough. In the general case, we have to choose x_i, y_i somewhat carefully. The following is useful in the argument.

LEMMA 4.4. *Let $x \neq y$. Suppose $(\exp(rS_xM)) \cup (\exp(2rS_xM)) \not\ni y$. Then the immersed manifold $\exp(rS_xM)$ intersects transversally with $\exp(rS_yM)$.*

PROOF. Suppose that $(\exp(rS_xM)) \cap (\exp(rS_yM)) \neq \emptyset$, and that at a point z in the intersection, $\exp(rS_xM)$ does not transversally intersect with $\exp(rS_yM)$. This implies these tangent spaces at z coincide with each other, or equivalently, there are vectors $u \in S_xM, v \in S_yM$ such that $\varphi_r(u) = \pm \varphi_r(v)$. Since $x \neq y$, we have $\varphi_r(u) = -\varphi_r(v)$, which means $y = \exp(2ru)$, and this is a contradiction.

This lemma guarantees that the set $\exp(rS_xM) \setminus \exp(rS_yM)$ is open dense in $\exp(rS_xM)$.

We now take points $x_1 \neq y_1$ with $y_1 \in \exp(rS_{x_1}M)$, and take a point x_2 in the complement of the union

$$\exp(rS_{x_1}M) \cup \exp(2rS_{x_1}M) \cup \exp(rS_{y_1}M) \cup \exp(2rS_{y_1}M).$$

From what has been said, we can take a point y_2 in

$$\exp(rS_{x_2}M) \setminus (\exp(rS_{x_1}M) \cup \exp(rS_{y_1}M)).$$

Proceeding with this argument, we can get a set $\{x_i, y_i\}$ with the desired property.

In order to construct functions Φ_i , take C^∞ -functions φ_i, ψ_i ($i = 1, \dots, N$) such that $\varphi_i \geq 0, \psi_i \geq 0, \varphi_i > 0$ on $B_\epsilon(x_i), \psi_i > 0$ on $B_\epsilon(y_i), \text{supp } \varphi_i = \overline{B_\epsilon(x_i)}, \text{supp } \psi_i = \overline{B_\epsilon(y_i)}$. Choosing a small $\epsilon > 0$, we can assume

$$\begin{aligned} B_\epsilon(x_i) \cap B_\epsilon(x_j) &= \emptyset \text{ for } i \neq j, \\ B_\epsilon(x_i) \cap B_\epsilon(y_j) &= \emptyset \text{ for any } i, j, \\ B_\epsilon(y_i) \cap B_\epsilon(y_j) &= \emptyset \text{ for } i \neq j, \\ \{\exp(rS_xM); x \in B_\epsilon(x_i)\} \cap B_\epsilon(x_j) &= \emptyset \text{ for } i \neq j, \\ \{\exp(rS_yM); y \in B_\epsilon(y_i)\} \cap B_\epsilon(y_j) &= \emptyset \text{ for } i \neq j, \\ \{\exp(rS_xM); x \in B_\epsilon(x_i)\} \cap B_\epsilon(y_j) &= \emptyset \text{ for } i \neq j, \end{aligned}$$

$\{\exp(rS_y M); y \in B_\epsilon(y_i)\} \cap B_\epsilon(x_j) = \emptyset$ for $i \neq j$, so that

$$\begin{aligned} \langle \varphi_i, \varphi_j \rangle &= 0, \quad \langle \varphi_i, \psi_j \rangle = 0, \quad \langle \psi_i, \psi_j \rangle = 0, \\ \langle L_r \varphi_i, \varphi_j \rangle &= 0, \quad \langle L_r \varphi_i, \psi_j \rangle = 0, \\ \langle L_r \psi_i, \varphi_j \rangle &= 0, \quad \langle L_r \psi_i, \psi_j \rangle = 0. \end{aligned}$$

Since $(L_r \varphi_i)(y_i) > 0$, $(L_r \psi_i)(x_i) > 0$, we have $\langle L_r \varphi_i, \psi_i \rangle > 0$. Putting $\Phi_i = \varphi_i + \psi_i$, we get the desired functions.

Next we deal with negative eigenvalues. Take a set $\{x_i, y_i\}$ in the same way as above. The assumption that there is no geodesic loop with length r means that $\exp(rS_{x_i} M) \ni x_i$, $\exp(rS_{y_i} M) \ni y_i$. Hence we can assume, in addition,

$\{\exp(rS_x M); x \in B_\epsilon(x_i)\} \cap B_\epsilon(x_i) = \emptyset$, $\{\exp(rS_y M); y \in B_\epsilon(y_i)\} \cap B_\epsilon(y_i) = \emptyset$, so that

$$\langle L_r \varphi_i, \varphi_i \rangle = 0, \quad \langle L_r \psi_i, \psi_i \rangle = 0.$$

Putting $\Psi_i = \varphi_i - \psi_i$, we obtain

$$\begin{aligned} \langle \Psi_i, \Psi_j \rangle &= 0, \quad \langle L_r \Psi_i, \Psi_j \rangle = 0 \quad \text{for } i \neq j, \\ \langle L_r \Psi_i, \Psi_i \rangle &< 0 \quad \text{for any } i, \end{aligned}$$

which guarantees the existence of infinitely many eigenvalues.

REMARK. We can apply the argument in §2 and this section to the operator M_r (see the remark after Lemma 2.1). The same statements as Theorems A(ii) and D(ii) hold for M_r . (Note that M_r is well defined if (and only if) $\pi_r: SM \rightarrow M \times M$ is an immersion.)

5. Expansion of L_r as $r \downarrow 0$. In this section we consider a formal Taylor expansion of L_r with respect to the radius r . The method we will take is different from [7].

We denote by Z the vector field on SM generating the geodesic flow $\{\varphi_t\}$. For any C^∞ -function f on M , the function $r \mapsto L_r f(x)$ is smooth, and

$$\frac{d^k}{dr^k} \Big|_{r=0} L_r f(x) = \int_{S_x M} (Z^k \pi^* f)(v) \, dS_x(v).$$

In order to compute more explicitly the function $Z^k(\pi^* f)$, let us choose a local coordinate (x^1, \dots, x^n) of M , and by Γ_{jk}^i we denote the Christoffel symbols. We write ∇_α for $\nabla_{\partial/\partial x^\alpha}$.

LEMMA 5.1. *Let $v = \sum v^i(\partial/\partial x^i)$. Then*

$$Z^k(\pi^* f)(v) = \nabla_{v, \dots, v}^k f = \sum_{\alpha_1 \dots \alpha_k} v^{\alpha_1} \dots v^{\alpha_k} \nabla_{\alpha_k} \dots \nabla_{\alpha_1} f.$$

PROOF. For a local coordinate of TM , choose $(x^1, \dots, x^n, v^1, \dots, v^n)$. The vector field Z is then expressed as

$$\sum v^i \frac{\partial}{\partial x^i} - \sum \Gamma_{jk}^i v^j v^k \frac{\partial}{\partial v^i}$$

(see S. Sasaki [18]), so we find

$$Z(\pi^* f)(v) = \sum v^i \nabla_i f.$$

Supposing now that we have proved the assertion for k , we obtain

$$\begin{aligned} Z(Z^k \pi^* f)(v) &= \sum v^i v^{\alpha_1} \dots v^{\alpha_k} \frac{\partial}{\partial x^i} \nabla_{\alpha_k} \dots \nabla_{\alpha_1} f \\ &\quad - \sum_{i=1}^k \sum \Gamma_{jk}^{\alpha_i} v^j v^k v^{\alpha_1} \dots \widehat{v^{\alpha_i}} \dots v^{\alpha_k} \nabla_{\alpha_k} \dots \nabla_{\alpha_1} f \\ &= \sum v^{\alpha_1} \dots v^{\alpha_k} v^i \left(\frac{\partial}{\partial x^i} \nabla_{\alpha_k} \dots \nabla_{\alpha_1} f \right. \\ &\quad \left. - \sum_{h=1}^k \Gamma_{i\alpha_h}^h \nabla_{\alpha_k} \dots \nabla_{\alpha_{h+1}} \nabla_h \nabla_{\alpha_{h-1}} \dots \nabla_{\alpha_1} f \right) \\ &= \sum v^{\alpha_1} \dots v^{\alpha_{k+1}} \nabla_{\alpha_{k+1}} \dots \nabla_{\alpha_1} f, \end{aligned}$$

which implies that the lemma holds for $k + 1$. Hence the proof is completed.

We put

$$\begin{aligned} h^{\alpha_1 \dots \alpha_k}(x) &= \int_{S_x M} v^{\alpha_1} \dots v^{\alpha_k} dS_x(v), \\ P_k &= \sum h^{\alpha_1 \dots \alpha_k} \nabla_{\alpha_k} \dots \nabla_{\alpha_1}. \end{aligned}$$

Since $h^{\alpha_1 \dots \alpha_k} = 0$ for odd k , the formal Taylor expansion of L_r is given by

$$L_r \sim \sum_{k=0}^{\infty} \frac{1}{(2k)!} P_{2k} r^{2k}.$$

Correctly speaking, we have

LEMMA 5.2. *Let $f \in C^\infty(M)$. For each k , we can write*

$$L_r f = f + \frac{1}{2!} r^2 P_2 f + \dots + \frac{1}{(2k)!} r^{2k} P_{2k} f + o(r^{2(k+1)}).$$

Here $o(r^{2(k+1)})$ is a C^∞ -function on M satisfying

$$\lim_{r \rightarrow 0} \frac{1}{r^{2k}} o(r^{2(k+1)}) = 0$$

uniformly on any compact set in M .

REMARK. If (M, g) is real analytic, and if $f \in C^\omega(M)$, then $L_r f(x)$ is a real analytic function of the variable r , and the above expansion converges for small r .

We next calculate the leading symbols of the differential operators P_{2k} . To facilitate doing this, we introduce the following notation. For a formal power series with linear differential operator coefficients, $D_r = \sum_{k=0}^{\infty} r^k D_k$, we define a symbol series $\sigma(D_r)$ to be

$$\sigma(D_r) = \sum_{k=0}^{\infty} \sigma(D_k) r^k,$$

where $\sigma(D_k)$ stands for the leading symbol of D_k .

LEMMA 5.3. Let $Q_\nu(x) = 2^\nu \Gamma(\nu + 1)x^{-\nu} J_\nu(x)$. Then

$$\sigma\left(\sum_{k=0}^{\infty} \frac{1}{(2k)!} P_{2k} r^{2k}\right) = \sigma(Q_{(n-2)/2}(\sqrt{-\Delta} r)).$$

Here the right-hand side stands for the symbol series of what is obtained by expanding $Q_\nu(\sqrt{-\Delta} r)$ ($\nu = (n - 2)/2$), into a series of powers of Δ .

PROOF. Let $\xi = \sum \xi_i dx^i$ be any covector in T_x^*M . Since

$$(P_{2k})(\xi) = \sum h^{\alpha_1 \dots \alpha_{2k}} \xi_{\alpha_1} \dots \xi_{\alpha_{2k}},$$

the left side equals

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{1}{(2k)!} r^{2k} \sum \int_{S_x M} v^{\alpha_1} \xi_{\alpha_1} \dots v^{\alpha_{2k}} \xi_{\alpha_{2k}} dS_x(v) \\ = \sum_{k=0}^{\infty} \frac{1}{(2k)!} r^{2k} \int_{S_x M} (v^1 \xi_1 + \dots + v^n \xi_n)^{2k} dS_x(v). \end{aligned}$$

By a well-known formula (cf. F. John [12]), the integral becomes

$$\frac{\Gamma(n/2)\Gamma(k + \frac{1}{2})}{\sqrt{\pi} \Gamma((n + 2k)/2)} \|\xi\|^{2k},$$

so that the last expression is equal to

$$\begin{aligned} \Gamma(n/2)(\sqrt{-1} \|\xi\| r/2)^{-(n-2)/2} \\ \cdot \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma((n-2)/2 + k + 1)} \left(\frac{\sqrt{-1} \|\xi\| r}{2}\right)^{(n-2)/2 + 2k} \\ = Q_{(n-2)/2}(\sqrt{-1} \|\xi\| r). \end{aligned}$$

Since the leading symbol of the Laplacian Δ is $-\|\xi\|^2$, we have proved the lemma.

COROLLARY. P_{2k} is a formally selfadjoint elliptic differential operator whose principal symbol coincides with that of

$$\frac{\Gamma(n/2)(2k)!}{k! \Gamma((n-2)/2 + k + 1) 2^{2k}} (-\Delta)^k.$$

A direct computation yields

$$\begin{aligned} P_2 f &= -(1/n)\Delta f, \\ P_4 f &= \frac{3}{n(n+2)} \left(\Delta^2 f + \frac{1}{3} \langle \nabla f, \nabla \tau \rangle + \frac{2}{3} \langle \nabla^2 f, \rho \rangle\right), \end{aligned}$$

where ρ is the Ricci curvature, and τ denotes the scalar curvature (cf. [7]).

We now suppose M is compact, and let

$$K_t f(x) = \int_M k_t(x, y) f(y) dg(y),$$

$k_t(x, y)$ being the fundamental solution of the heat operator $\partial/\partial t + \Delta$. From the probabilistic view point, K_t is considered as the transition operator associated with the Brownian motions on M .

To prove Theorem E, we recall a theorem of P. Chernoff [4], which says that if a one-parameter family $F(t)$ ($t \geq 0$) of bounded linear operators of a Banach space \mathfrak{X} satisfies $\|F(t)\| \leq 1$, $0 \leq t < \infty$, $F(0) = I$, and there exists an infinitesimal generator A of a contraction semigroup R^t on \mathfrak{X} such that

$$\lim_{t \rightarrow 0} \frac{F(h) - I}{h} u = Au$$

for each u in a core of A , then for all u in \mathfrak{X} ,

$$\lim_{N \rightarrow \infty} F(t/N)^N u = R^t u.$$

In view of Lemmas 2.1 and 5.2, this theorem can be applied to the cases $\mathfrak{X} = L^2(M)$ or $C^0(M)$, $F(t) = L_{\sqrt{2nt}}$, $A = \Delta$, $R^t = K_t$, thereby proving Theorem E.

REMARK. C^0 -convergence comes from a more general result proved by Jorgensen [13].

The following two lemmas are obvious from the definition of L_r .

LEMMA 5.4. *Let φ be an isometry of M . Then for any $r > 0$, $L_r \varphi^* = \varphi^* L_r$.*

LEMMA 5.5. *Let $\tilde{\omega}: \tilde{M} \rightarrow M$ be a Riemannian covering. Let \tilde{L}_r be the spherical mean on \tilde{M} . Then $\tilde{L}_r \tilde{\omega}^* = \tilde{\omega}^* L_r$.*

PROPOSITION 5.6. *Let M be a complete locally Riemannian symmetric space. Then $L_r \Delta = \Delta L_r$.*

PROOF. In view of Lemma 5.5, we may assume, without loss of generality, that M is globally symmetric, so that we may write $M = G/H$, where G is the largest connected group of isometries acting on M . From commutativity of L_r with each element in G , it follows that P_{2k} is a G -invariant differential operator. We know from general theory of symmetric spaces [10] that G -invariant differential operators on M commute with each other, hence $L_r \Delta$ and ΔL_r have the same expansion with respect to r . We note here that M is a C^ω -manifold, so that for any $f \in C^\omega(M)$ we have $L_r \Delta f = \Delta L_r f$. Since $C^\omega(M)$ is dense in $C^\infty(M)$, we get the lemma.

REMARK. The above proof implies $L_r L_s = L_s L_r$ for any $s, r > 0$.

We can now prove Theorem F. Let M be a symmetric space with rank > 2 , and let $f = \sum f_\lambda$ be the L^2 -expansion of $f \in L^2(M)$ into eigenfunctions of the Laplacian. If $L_r f = f$, it follows from the above lemma that $L_r f_\lambda = f_\lambda$ for any λ . Since $f_\lambda \in C^\infty$, it suffices to show that any two points can be joined by an r -geodesic chain. But this is clear from the fact that two points lie always in a two-dimensional totally geodesic flat subspace in M , and any two points in a two-dimensional flat space can be joined by an r -geodesic chain, irrespective of the value r .

Let us consider the case M is a rank one symmetric space of compact type. Denote by l the common length of primitive closed geodesics. Let $v \in S_x M$, and let c be a closed geodesic with $\dot{c}(0) = v$. Choosing an orthogonal basis v_1, \dots, v_n of $T_x M$ such that $v = v_1$ and v_i is the eigenvector of the linear transformation of $T_x M$ given by $u \rightarrow R(v, u)v$, we extend these to the orthogonal frame $v_i(t)$ along c

by the parallel translation. Then any Jacobi field X with $X(0) = 0$ can be written in the form

$$X(t) = \sum_{i=2}^n a_i \sin(\sqrt{\lambda_i} t) v_i(t),$$

where $R(v, v)v = \lambda_i v_i$. We can assume the following relations for each case (after normalizing the curvature).

M	l	λ_i
$S^n (n \geq 2)$	2π	$\lambda_1 = \dots = \lambda_{n-1} = 1$
$P^n(\mathbf{R}) (n \geq 2)$	π	:
$P^n(\mathbf{C})$	2π	$\lambda_1 = 1, \lambda_2 = \dots = \lambda_{2n-1} = \frac{1}{4}$
$P^n(\mathbf{Q})$	2π	$\lambda_1 = \lambda_2 = \lambda_3 = 1, \lambda_4 = \dots = \lambda_{4n-1} = \frac{1}{4}$
$P^2(\mathbf{C}_a)$	2π	$\lambda_1 = \dots = \lambda_7 = 1, \lambda_8 = \dots = \lambda_{15} = \frac{1}{4}$

Let $\mathbf{K} = \mathbf{R}, \mathbf{C}, \mathbf{Q}$ or \mathbf{C}_a . Any two totally geodesic codim $_{\mathbf{K}}$ -one submanifolds in the projective space $P^n(\mathbf{K})$ have nonempty intersection. This implies that any two points in $P^n(\mathbf{K})$ can be joined by an $(l/2)$ -geodesic chain. Hence, in conjunction with the above table we have

PROPOSITION 5.7. *The spherical mean L_r on $P^n(\mathbf{K})$ (resp. on S^n) is ergodic if and only if $r \notin \mathbf{Z}l$ (resp. $r \notin \mathbf{Z}\frac{1}{2}l$).*

REMARK.

$$L_r = L_{r-kl/2} \quad \text{if } k = [2r/l] \text{ is even,}$$

$$L_r = L_{(k+1)/2-r} \quad \text{if } k = [2r/l] \text{ is odd,}$$

$$L_{kl} = \text{Id.}$$

We conclude this section by proposing the following open problem.

PROBLEM. Let E denote the set of positive numbers r such that L_r is ergodic. Then is E always dense in \mathbf{R}_+ ?

6. Generalizations and remarks. This section gives generalizations of some results concerning r -geodesic chains. Since it seems unnecessary to repeat all the argument here, only frameworks will be stated.

Let M be a compact Riemannian manifold, and let $\{\mu_x\} (x \in M)$ be a family of probability measure such that each μ_x is assigned to the tangent space $T_x M$, which is supposed to satisfy:

(†) $\{\mu_x\}$ is invariant under parallel translations,

that is, for any piecewise smooth curve $\gamma: [a, b] \rightarrow M$, the parallel translation $P_\gamma: (T_{\gamma(a)}M, \mu_{\gamma(a)}) \rightarrow (T_{\gamma(b)}M, \mu_{\gamma(b)})$ along γ is measure preserving.

This condition is equivalent to: for any geodesic curve γ the P_γ is measure preserving. In fact any piecewise smooth curve can be approximated by broken geodesics.

We let $N_x, x \in M$, be supports of measures μ_x such that $P(N_{\gamma(a)}) = N_{\gamma(b)}$ for any curve $\gamma: [a, b] \rightarrow M$. We denote by $\mathcal{C}(\mu)$ the totality of geodesic chains $c: [0, \infty) \rightarrow M$ with $\dot{c}(k) = dc(k)/dt \in N_{c(k)}$. In the same vein as §3, we can identify $\mathcal{C}(\mu)$ with the fiber space N whose fiber is $N_x = N_x \times N_x \times \dots$, and can equip a measure P_μ on $\mathcal{C}(\mu)$. The condition (†) guarantees an analogous result to Lemma 3.1, from which it follows at once that the stochastic process $\pi_k: \mathcal{C}(\mu) \rightarrow M$ given by $\pi_k(c) = c(k)$ has a Markov property

$$P_\mu(T^{-k}A|\mathfrak{B}_k) = P_\mu(A|\pi_k) \quad \text{for } A \in \mathfrak{B},$$

where $T: \mathcal{C}(\mu) \rightarrow \mathcal{C}(\mu)$ is the shift, \mathfrak{B}_k is the σ -ring generated by $\pi_k^{-1}(\cdot)$, $\mathfrak{B} = \cup \mathfrak{B}_k$, and the symbol $P_\mu(\cdot | \cdot)$ stands for the conditional probability.

Letting

$$(L_\mu f)(x) = \int_{T_x M} f(\exp v) d\mu_x(v) = \int_{N_x} f(\exp v) d\mu_x(v),$$

we have

$$P_\mu(T^{-1}A|x) = P_{\mu,x}(T^{-1}A) = (L_\mu P_\mu; (A))(x),$$

so that $P_{\mu,x}(\pi_k^{-1}A) = (L_\mu^k \chi_A)(x)$ for each k . Usually $P_{\mu,x}(\pi_k^{-1}A)$ is called the k -step transition probability of the Markov process $\{\pi_k\}$ and frequently written as $P_k(x; A)$. Using the same argument as §3, we can prove that if the fiber product measure μ on N is invariant under the geodesic flows, then the shift T is measure preserving, and that the dynamical system $(\mathcal{C}(\mu), P, T)$ is ergodic if and only if 1 is a simple eigenvalue of L_μ .

EXAMPLE (i). For each positive r , we have a measure μ_x given by

$$\mu_x(A) = \int_{S_x M \cap (A/r)} dS_x.$$

It is easy to see that $\mathcal{C}(\mu)$ is identified with $\mathcal{C}_\infty(r; M)$, and $L_\mu = L_r$.

EXAMPLE (ii). Define a measure by

$$\mu_x(A) = \text{Vol}(B_x^r)^{-1} \int_{A \cap B_x^r} dv,$$

where $B_x^r = \{v \in T_x M; \|v\| \leq r\}$. The family $\{\mu_x\}$ obviously satisfies the condition (†). In this case, N is the ball bundle $B^r = \cup B_x^r$ on M , and the measure μ is invariant under the geodesic flows.

Let it be assumed that radius r is smaller than the injective radius of M , so that the mapping $\Pi(v) = (\pi v, \exp v)$ is a diffeomorphism of B^r into $M \times M$. Since the distribution kernel K_μ of the associated operator L_μ is given by

$$\langle K, h \rangle = \int_{B^r} \Pi^* h dB^r \quad (dB^r = \text{Vol}(B_x^r)^{-1} dv dg),$$

K_μ is an L^2 -function on $M \times M$, and hence it follows that L_μ is of Hilbert-Schmidt type. Further we easily contend that $L_\mu(L^2(M)) \subset C^0(M)$. Thus L_μ is ergodic.

EXAMPLE (iii). Put

$$\mu_x(A) = (4\pi r)^{-n/2} \int_A \exp(-\|v\|^2/4r) dv,$$

which satisfies (†), and μ is invariant under the geodesic flow. When M has no conjugate point, the associated operator L_μ has the smooth distribution kernel, because the mapping $\Pi: TM \rightarrow M \times M (v \mapsto (\pi v, \exp v))$ is a covering mapping. From this fact, we can prove that L_μ is ergodic.

EXAMPLE (iv). Another important example arises in the theory of symmetric spaces. Let $M = G/K$ be a globally symmetric space. Fix an element a in G . We let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition of the Lie algebra \mathfrak{g} corresponding to G , and write $a = \exp(rv_0)$, $v_0 \in \mathfrak{g}$ with $\|v_0\| = 1$. Putting $N_{gK} = dg(\text{Ad}(K)v_0)$, we have a subbundle $N = \cup N_{gK}$ of SM . We set, for $A \subset \mathfrak{p} = T_{eK}M$,

$$\mu_{eK}(A) = \int_{\{k \in K; \text{Ad}(k)v_0 \in A\}} dk,$$

where dk denotes the normalized Haar measure on K . The measure can be extended in an obvious way to a measure on each fiber of N , and the family μ_x satisfies the condition (†), thanks to the symmetricity. Further we easily find

$$(L_\mu f) = \int_K f(gkaK) dk.$$

If M is a rank one symmetric space, K acts transitively on the sphere $\{v \in \mathfrak{p}; \|v\| = 1\}$, so one concludes that L_μ coincides with the spherical mean L_r .

REMARK. As was pointed out in the introduction, there is another way to construct a Markov process with the transition probability $P_k(x, A) = (L_\mu^k \chi_A)(x)$, which is standard to probabilist, but not geometrical. Namely, for a probability space, we choose an infinite product $M^\infty = M \times M \times \dots$, and define a probability measure P on M^∞ so that, for any $k \geq 2$, and for any Borel set A_0, A_1, \dots, A_k of M , the following relation holds:

$$\begin{aligned} P(X_0 \in A_0, X_1 \in A_1, \dots, X_k \in A_k) \\ = \int_{A_0} dg(x_0) \int_{A_1} P_1(x_0, dx_1) \dots \int_{A_k} P_1(x_{k-1}, dx_k) \end{aligned}$$

where $x_i: M \rightarrow M$ is the i th coordinate. It is a standard fact that the coordinate process $\{x_k\}$ is a Markov process with transition operator L_μ (see [5]). Usually, we call such a process “random walks” because the process provides us with a mathematical model for motions of drunken walkers who walk very randomly on the manifold. The process $(\mathcal{C}(\mu), \pi_k)$ constructed above is considered as a geometrical model of random walks.

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