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COMPACT GROUPS OF HOMEOMORPHISMS ON TREE-LIKE CONTINUA

BY

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ABSTRACT. This paper is concerned with the fixed point sets of certain collections of homeomorphisms on a tree-like continuum. Extending a theorem of P. A. Smith, the authors prove that a periodic homeomorphism has a (nonvoid) continuum as its fixed point set. They then deduce possible periods for homeomorphisms on tree-like continua which satisfy certain decomposability or irreducibility conditions. The main result of the paper is that a compact group of homeomorphisms has a continuum as its fixed point set. This is applied to isometries. The paper concludes with sufficient conditions that a pointwise periodic homeomorphism have a fixed point.

Introduction. A metric topological space will be called a *space* and a *map* will be a continuous function from one space to another. By a *continuum*, we will mean a compact connected space. A *graph* will be a one-dimensional finite polyhedron while a *tree* is a simply connected graph. A continuum is *tree-like* if there exist finite open covers of arbitrarily small mesh whose nerves are trees. A covering is *one dimensional* if its nerve is a graph. We will say that a continuum M has the *fixed point property* provided that if $f\colon M \to M$ is a map, then there exists a point a of M for which $f(a) = a$. Then a is called a *fixed point* of M for f. The set of all such points is the *fixed point set* of M under f.

Recently, David Bellamy [**Be**] constructed an example of a tree-like continuum which admits a fixed-point-free map. Using this example and a theorem of Fugate and Mohler [**F-M**], one can construct a tree-like continuum which admits a fixed-point-free homeomorphism. Now tree-like continua are acyclic [**C-C**], and P. A. Smith has shown [**S**] that, on an acyclic continuum, a periodic homeomorphism whose period is a power of a prime must have a fixed point. Smith has examples which are not tree-like to show that the prime power restriction is needed.

In §1 of this paper, we show that the restriction can be removed in the tree-like case, and that the fixed point set is connected. We apply these results to the case where the continuum satisfies certain irreducibility or decomposability conditions. In §2, we extend our main theorems to compact groups, and to isometries. §3 is concerned with pointwise periodic homeomorphisms.

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1. Finite groups. If h is a homeomorphism on a continuum M and m is a positive integer, then h^m will mean the composition of h with itself m times, and $h^0 = 1$ will be the identity map on M. We will say that h is *periodic of period* n if n is the smallest positive integer for which $h^n = 1$. Naturally, if h is a homeomorphism of period n, then $G = \{h^j: M \to M: 0 \leqslant j < n\}$ is a finite topological group with the discrete topology that acts on M. If G is any topological group of homeomorphisms acting on a continuum M and $x \in M$, then the *orbit* of x under G is $G(x) = \{g(x): g \in G\}$ and the *fixed point set* of M under G will be $F_G = \{x \in M: G(x) = \{x\}\}$. Also, we will denote the open ball of radius $\delta > 0$ about a point x of M by $N(x, \delta)$. Before verifying the results for tree-like continua, we prove them for trees.

LEMMA 1.1. *A periodic homeomorphism on an arc has a connected fixed point set, and either it is the identity or it has period* 2.

PROOF. It suffices to demonstrate this for a periodic homeomorphism g of $[0, 1]$. Let $F_g = \{x \in [0, 1]: g(x) = x\}$. If F_g is not connected then there is a subarc $[a, b]$ of $[0, 1]$ such that $F_g \cap [a, b] = \{a, b\}$. Thus $g(a) = a$, $g(b) = b$ and if $x \in (a, b)$, then $g(x) \neq x$. Now one of the two sets $\{x \in (a, b): g(x) > x\}$, $\{x \in (a, b): g(x) < x\}$ must be empty, otherwise (a, b) is not connected. Without loss of generality, we may assume that if $x \in (a, b)$ then $g(x) > x$. It follows that, for all n, $g^{n+1}(x) > g^n(x) > \cdots > g(x) > x$. This contradicts the periodicity of g. Thus F_g is connected. Since F_g must be closed, F_g is an arc or a point.

Now either $g(0) = 0$ or $g(0) = 1$. If $g(0) = 0$, then $g(1) = 1$ and since F_g is connected, $F_g = [0, 1]$ so g is the identity. If $g(0) = 1$, then $g(1) = 0$ and g^2 fixes both 0 and 1. Since g^2 is periodic, g^2 is the identity, so g has period 2.

LEMMA 1.2. *Let G be a finite group of homeomorphisms on the tree T and $F_G = \{x \in T$: for each $g \in G$, $g(x) = x\}$. Then F_G is a nonempty continuum.*

PROOF. Each $g \in G$ is periodic. Let $F_g = \{x \in T: g(x) = x\}$. Since g is continuous, and trees have the fixed point property, F_g is a nonempty closed set. Let $a, b \in F_g$, and consider the arc $[a, b] \subset T$. Since $g[[a, b]]$ is an arc from $g(a) = a$ to $g(b) = b$, and T is uniquely arcwise connected, g maps $[a, b]$ onto itself. Since g restricted to $[a, b]$ is periodic, Lemma 1.1 guarantees that $[a, b] \subset F_g$.

Now $F_G = \bigcap \{F_g: g \in G\}$. Since T is hereditarily unicoherent, the intersection of any family of subcontinua of T is a continuum, so F_G is a continuum. To show that $F_G \neq 0$ we need only show, again because T is hereditarily unicoherent, that if $g, h \in G$, then $F_g \cap F_h \neq \varnothing$.

Given $g, h \in G$, we will say that a subcontinuum N of T has property P provided that $g[N] = N = h[N]$. It is easy to see that if $K_1, K_2, \ldots$ is a decreasing sequence of continua with property P, then $\bigcap K_i$ has property P. Since M has property P, it follows that there is a subcontinuum L of M such that L has property P, but no proper subcontinuum of L does. Notice that L has the fixed point property, so $F_g \cap L \neq \varnothing$ and $F_h \cap L \neq \varnothing$.

Let B denote the set of branch points of L; clearly, $g[B] = B = h[B]$ so if either B or L is degenerate, then $F_g \cap F_h \neq \varnothing$. If neither B nor L is degenerate, then we claim that L is an arc. For if this is not true, then B has at least two points. Let J denote the intersection of all subcontinua of L containing B. Then J is a continuum, and $g[J] = J = h[J]$. This says that J is a proper subcontinuum of L having property P, which is impossible.

Hence, we can assume that L is an arc $[a, b]$. Lemma 1.1 shows that either $h = 1$, or $g = 1$ (in either event, $F_g \cap F_n \neq \varnothing$) or $h^2 = g^2 = 1$, $h \neq 1$, $g \neq 1$. Then $g(a) = h(a) = b$ and $g(b) = h(b) = a$. Since $gh \in G$, gh is periodic, $gh(a) = a$ and $gh(b) = b$. From Lemma 1.1, $gh = 1$ so $g = g1 = ghh = 1h = h$, and $F_g = F_h$. This concludes the proof.

In extending these results to the tree-like case we will use letters at the beginning of the alphabet for open sets and covers for the principal tree-like continuum and letters towards the end for the quotient space that is also tree-like.

THEOREM 1.3. *If M is a tree-like continuum and G is a finite group of homeomorphisms acting on M, then the fixed point set of M under G is nonempty.*

PROOF. Let d' be a metric for M, and define d: $M \times M \to M$ by $d(x, y) = \max\{d'(g(x), g(y))$: $g \in G\}$. Then d is a metric for M that is topologically equivalent to d', and for each $g \in G$ and each $(x, y) \in M \times M$, $d(x, y) = d(g(x), g(y))$. Thus, every element of G is an *isometry* under d. Since the two metrics are equivalent, we will use the d-metric throughout the remainder of this section and we will denote the d-diameter of a set A by diam A.

Define $\varepsilon = \inf\{\text{diam } G(x)$: $x \in M\}$. Assuming that the conclusion of the theorem is false, then $\varepsilon > 0$. For each $x \in M$, $G(x)$ is finite, so there exists a δ_x, $0 < \delta_x < \varepsilon/2$, such that if $g, h \in G$ then $N(g(x), \delta_x) \cap N(h(x), \delta_x) \neq \varnothing$ if and only if $g(x) = h(x)$. Since g is an isometry, $g[N(h(x), \delta_x)] = N(gh(x), \delta_x)$. For each $x \in M$, define $A[x] = \bigcup \{N(g(x), \delta_x)$: $g \in G\}$ and note that $\mathcal{A} = \{A[x]$: $x \in M\}$ is an open cover of the compact space M. Let $\mathcal{B}$ be an essential finite subcover of $\mathcal{A}$ and choose an indexing set I for $\mathcal{B}$ with the property that $A[x] \in \mathcal{B}$ if and only if exactly one element of $G(x)$ belongs to I.

Define $M/G = \{G(x)$: $x \in M\}$ to be the *orbit space* with the quotient topology and let π: $M \to M/G$ be the projection map. Since M/G is a metric space and π is an open map [**M-Z**, Theorem 2, p. 232], M/G is one dimensional [**Co**] (in fact, M/G is tree-like [**M**]). Let $\gamma > 0$ be a Lebesgue number of the finite open cover $\pi[\mathcal{B}]$ of M/G and $\mathcal{U}$ a one-dimensional cover of M/G with mesh less than γ.

We now refine $\mathcal{B}$ with a finite open cover $\mathcal{C}$, so that $\mathcal{C}$ is one dimensional and invariant under each element of G (i.e., if $C \in \mathcal{C}$ and $g \in G$, then $g[C] \in \mathcal{C}$). For each $W \in \mathcal{U}$, choose exactly one $x \in I$ for which $W \subset \pi(A[x])$. For each $g \in G$, define $C(w, g) = \pi^{-1}[W] \cap N(g(x), \delta_x)$ and let $\mathcal{C} = \{C(w, g) \subset M$: $W \in \mathcal{U}$ and $g \in G\}$. For invariance, let $h \in G$ and $C(w, g) \in \mathcal{C}$. Since h is an isometry, $h[C(w, g)] = C(w, hg)$ and $C(w, hg) \in \mathcal{C}$. Clearly, $\mathcal{C}$ covers M, but to see that it is one dimensional, assume $\{C(u, g), C(v, h), C(w, k)\} \subset \mathcal{C}$ with a point $t \in M$

common to all. Then $\pi(t) \in U \cap V \cap W \subset M/G$, where $\{U, V, W\} \subset \mathcal{U}$. Since $\mathcal{U}$ was a one-dimensional cover, two of these open sets must be the same. Without loss of generality, we assume that $U = V$ and hence $C(u, g) = C(v, g)$. Let x be the element of I that was chosen for which $V \subset \pi[A[x]]$ and note that $t \in C(u, g) \cap C(v, h) = C(v, g) \cap C(v, h) = \pi^{-1}[V] \cap N(g(x), \delta_x) \cap N(h(x), \delta_x)$. Since these open balls have a point in common, $g(x) = h(x)$ and $C(v, g) = C(v, h) = C(u, g)$ and $\mathcal{C}$ is a one-dimensional open cover.

Let K be the one-dimensional simplicial complex which is the nerve of $\mathcal{C}$. We will also use K to denote the space of this complex, and $f\colon M \to K$ will be the canonical barycentric map. Recall that one first defines, for each $C \in \mathcal{C}$, a map $a_C\colon M \to \text{Reals}$, by $a_C(x) = d(x, M - C)$. Then

$$f(x) = \sum_{C \in \mathcal{C}} \left[\frac{a_C(x)}{\sum_{B \in \mathcal{C}} a_B(x)} \right] C.$$

For each $g \in G$, define $g'\colon K \to K$ so that if $s \in K$ and $s = tB + (1 - t)D$, then $g'(s) = tg[B] + (1 - t)g[D]$. Since $\mathcal{C}$ is invariant under g, g' is also a homeomorphism. Also g is an isometry, so for each $m \in M$, $a_{g[C]}(g(m)) = a_C(m)$, making g' periodic and $fg = g'f$ (see Figure 1). Since M is tree-like and K is a graph, the Case and Chamberlin [**C-C**] characterization of tree-like curves guarantees that f is homotopic to a constant map. Thus [**Cu**] if J is the universal covering space of K with projection map p, there exists a map $\phi\colon M \to J$ for which $p\phi = f$.

$$\begin{tikzcd}
M \arrow[r, "f"] \arrow[d, "g"] \arrow[rr, bend left, "\phi"] & K \arrow[d, "g'"] & J \arrow[l, "p"'] \arrow[d, "\bar{g}"] \\
M \arrow[r, "f"'] \arrow[rr, bend right, "\phi"'] & K & J \arrow[l, "p"]
\end{tikzcd}$$

FIGURE 1

Let $a \in M$. For each $g \in G$, $\phi g(a)$ lies above $p\phi g(a) = fg(a) = g'f(a) = g'p\phi(a)$. Also, $g'p\colon \phi[M] \to K$ is a map from a tree (hence from a tree-like continuum) to a one-dimensional polyhedron. Once again, the main result of [**C-C**] guarantees the existence of a lift $\bar{g}\colon \phi[M] \to J$ for which $\bar{g}\phi(a) = \phi g(a)$ [**M**, Lemma 1.2] and $p\bar{g} = g'p$.

To show that $\bar{g}$ is a homeomorphism on $\phi[M]$, notice that ϕg and $\bar{g}\phi$ are each lifts of fg that agree at a. Since M is connected, the unique lifting property [**Sp**, Theorem 2, p. 67] guarantees that $\bar{g}\phi = \phi g$ and so $\bar{g}\phi[M] = \phi g[M] = \phi[M]$; thus $\bar{g}$ is onto. Now let $b, c \in \phi[M]$ and assume that $\bar{g}(b) = \bar{g}(c)$. If $b \neq c$, there exists an arc $A \subset \phi[M]$ from b to c. Then $p\bar{g}[A] = g'p[A]$ is a closed loop in K and $\bar{g}[A]$ is a loop in J. Thus $p\bar{g}[A]$ is homotopic to a constant map relative to $p\bar{g}(b) = p\bar{g}(c)$ [**M**, Lemma 1.1]. Since g' is a homeomorphism, $p[A]$ is also homotopic to a constant map leaving $p(b) = p(c)$ fixed throughout the homotopy. Now A is a lift of $p[A]$ and $b = c$ [**M**, Lemma 1.1]. This contradiction shows that $\bar{g}$ is one-to-one.

Let g and $h \in G$, g' and h' be the induced homeomorphisms on K, and $\bar{g}$ and $\bar{h}$ be the induced homeomorphisms on $\phi[M]$. Now $\bar{g}$ and $\bar{h}$ are lifts of $g'p$ and $h'p$ respectively, so $p\bar{g}\bar{h} = g'p\bar{h} = g'h'p = (gh)'p$, the last equality following from the definition of g' and h'. Thus both $\bar{g}\bar{h}$ and $\overline{gh}$ are lifts of gh. Moreover, $\bar{g}\bar{h}\phi(a) = \bar{g}\phi gh(a) = \phi gh(a) = \overline{gh}\phi(a)$, so we may apply the unique lifting property to conclude that $\overline{gh} = \bar{g}\bar{h}$. It follows that the correspondence $g \to \bar{g}$ is a homomorphism, and so $\bar{G} = \{\bar{g}: g \in G\}$ is a (finite) group of homeomorphisms acting on the tree $\phi[M]$. Thus by Lemma 1.2, there exists a $\phi(m) \in \phi[M]$ for which $\bar{G}(\phi(m)) = \{\phi(m)\}$. For all $g \in G$, $fg(m) = g'f(m) = g'p\phi(m) = p\bar{g}\phi(m) = p\phi(m) = f(m)$. Thus, $f[G(m)] = \{f(m)\}$ and there must exist an $x \in I$ and a $C(w, g) \in \mathcal{C}$ for which $G(m) \subset C(w, g) \subset N(g(x), \delta_x)$. Thus diam $G(m) \leqslant$ diam $N(g(x), \delta_x) = 2\delta_x < \varepsilon$. This contradiction to the way that ε was chosen proves that the fixed point set of M under G is nonempty.

THEOREM 1.4. *If G is a finite group of homeomorphisms acting on a tree-like curve M, then*

(i) *the fixed point set of M under each element of G is tree-like, and*

(ii) *the fixed point set of M under G is tree-like.*

PROOF. Suppose that there exists a $g \in G$ for which the fixed point set F_g of M under g is not tree-like. Since F_g is nonempty by the previous theorem and clearly compact, there must exist disjoint closed sets A and B for which $F_g = A \cup B$. Let U and V be open in M such that $A \subset U$, $B \subset V$, and cl $U \cap$ cl $V = \varnothing$. Let $\delta = \inf\{d(x, y): x \in \text{cl } U, y \in \text{cl } V\}$; clearly $\delta > 0$. Since g moves every element of the compact set $M - (U \cup V)$, there is an $\varepsilon > 0$ so that if $x \in M - (U \cup V)$, then $d(x, g(x)) > \varepsilon$. Without loss of generality, we may take $\varepsilon < \delta$.

Using the techniques of the proof of Theorem 1.3, construct a one-dimensional finite open cover $\mathcal{C}$ of M, with mesh $< \varepsilon$, which is invariant under g. With the same notation as in Figure 1, let $\bar{F}$ denote the fixed point set of $\phi[M]$ under $\bar{g}$. It follows from Lemma 1.2 that $\bar{F}$ is connected, so $p[\bar{F}]$ is a connected subset of the polyhedron K. Moreover, if $x \in A \cup B$, then $\bar{g}\phi(x) = \phi g(x) = \phi(x)$, so $\phi[A \cup B] \subset \bar{F}$ and $f[A] \cup f[B] = f[A \cup B] = p\phi[A \cup B] \subset p[\bar{F}]$.

Now since f is an ε-map and $\varepsilon < \delta$, $f[\text{cl } U] \cap f[\text{cl } V] = \varnothing$. Thus $f[\text{cl } U]$ and $f[\text{cl } V]$ are separated sets containing $f[A]$ and $f[B]$ respectively. Since $p[\bar{F}]$ is connected and contains $f[A] \cup f[B]$, $p[\bar{F}] \not\subset f[U] \cup f[V]$. Thus there exists $m \in M$ such that $\phi(m) \in \bar{F}$ and $p\phi(m) \in p[\bar{F}] - (f[U] \cup f[V])$; thus $m \notin U \cup V$. Since $m \in M - U \cup V$, $d(m, g(m)) > \varepsilon$ and so $f(g(m)) \neq f(m)$. However, $f(g(m)) = g'f(m) = g'p\phi(m) = p\bar{g}\phi(m) = p\phi(m)$ because $\phi(m) \in \bar{F}$. But $p\phi(m) = f(m)$, so $f(g(m)) = f(m)$, which is a contradiction. Hence F_g is connected.

To establish (ii), let $F_G = \{x \in M$: for each $g \in G$, $g(x) = x\}$. Theorem 1.3 shows that $F_G \neq \varnothing$. Clearly, $F_G = \bigcap \{F_g: g \in G\}$ and we have just seen that each F_g is a continuum. Since M is hereditarily unicoherent, the intersection of any family of subcontinua of M is connected. Thus, F_G is connected.

COROLLARY 1.5. *If h is a periodic homeomorphism or a tree-like continuum M, then the fixed point set of h is a nonvoid continuum.*

THEOREM 1.6. *Suppose that M is tree-like, h: $M \to M$ is periodic and F_h is the fixed point set of h. If $x \in M - F_h$, then no continuum in $M - F_h$ contains both x and $h(x)$.*

PROOF. Suppose, to the contrary, that there is a point $x \in M - F_h$ and a continuum L such that $\{x, h(x)\} \subset L \subset M - F_h$. Suppose that n is the period of h and let $D = \bigcup_{i=1}^{n} h^i[L]$. Clearly D is a continuum and $h[D] = D$. Moreover, since $L \cap F_h = \varnothing$ and h is one-to-one, $D \cap F_h = \varnothing$. Now D is a tree-like continuum which is sent into itself by the periodic homeomorphism h, so h has a fixed point in D. But then $D \cap F_h \neq \varnothing$. This contradiction establishes the theorem.

COROLLARY 1.7. *If h is a periodic homeomorphism on the indecomposable tree-like continuum M, and h is not the identity, then h sends exactly one composant of M into itself.*

PROOF. Since h has a fixed point x, the composant C containing x is sent into itself. Since h is not the identity, F_h is a proper subcontinuum of M, so $F_h \subset C$. If D is another composant and $y \in D$, then $h(y) \notin D$, otherwise we contradict the preceding theorem.

The following result extends a theorem of Brechner **[B]**, for chainable continua, to the tree-like case. It should be contrasted with a result of Wayne Lewis **[L1]**, that there is a nontrivial homeomorphism of the pseudo arc which fixes each point in some open set.

COROLLARY 1.8. *If h is a periodic homeomorphism on an indecomposable tree-like continuum and h fixes each point in some open set, then h is the identity.*

Given a tree-like continuum M, the question arises of the possible periods for a periodic homeomorphism on M. Lewis has shown **[L2]** that if p is a prime then there is a homeomorphism on the pseudo arc of period p. The next theorems, which extend results of Smith and Young **[S-Y]**, show that if M satisfies sufficiently strong decomposability and irreducibility conditions, then the possible periods are quite restricted. The first result extends Lemma 1.1.

THEOREM 1.9. *Suppose that M is an irreducible tree-like continuum and each indecomposable subcontinuum of M has void interior in M. If h is a periodic homeomorphism of M, then h^2 is the identity.*

PROOF. It follows from **[T]** that there are disjoint subcontinua A and B of M such that M is irreducible from each point of A to each point of B. Moreover, if M is irreducible from x to y, then $\{x, y\} \subset A \cup B$. It follows that any homeomorphism of M must send $A \cup B$ onto itself.

If $h[A] = A$ then $h[B] = B$ and h must have a fixed point in A and a fixed point in B. Since F_h is a continuum meeting both A and B, $F_h = M$ and h is the identity. If $h[A] = B$ then $h[B] = A$ and so $h^2[A] = A$ and $h^2[B] = B$. Then h^2 must fix every point of M.

THEOREM 1.10. *Suppose that M is a tree-like continuum which is hereditarily decomposable and irreducible about a set containing q points. If h is a periodic homeomorphism on M of period k, then the symmetric group of permutations of q objects contains an element of order k.*

PROOF. There is a positive integer n, $n \leqslant q$, so that M is irreducible about a set with n points, but M is reducible about any set with fewer than n points. It suffices to show that the symmetric group on n objects contains an element of order k.

It follows from **[R]** that there is a collection $\mathcal{E} = \{E_1, \ldots, E_n\}$ of exactly n mutually disjoint subcontinua of M such that each E_i has void interior and M is irreducible about the finite set K if, and only if, K meets each E_i. Moreover, this collection $\mathcal{E}$ is unique, so any homeomorphism of M induces a permutation of $\mathcal{E}$. Let j be the order of the permutation induced by h; $j \leqslant k$. For each i, $h^j[E_i] = E_i$ and so each E_i contains a fixed point of h^j. Since h^j is periodic, its fixed point set is a continuum, which must be M. It follows that $j = k$. This concludes the proof.

We note that the preceding result could be extended to the class of tree-like continua such that each indecomposable subcontinuum has void interior, if the cited theorem in **[R]** can be so extended.

We conclude this section by showing that a fixed-point-free map on a tree-like continuum cannot be induced by maps between one-dimensional polyhedra. The proof is a simplified version of the proof of Theorem 1.3.

THEOREM 1.11. *Suppose that, for each positive integer i, K_i is a one-dimensional polyhedron and h_i: $K_{i+1} \to K_i$, g_i: $K_i \to K_i$ are maps such that $g_i h_i = h_i g_{i+1}$. Let M denote the continuum which is the inverse limit of K_i, with bonding maps h_i, and let g: $M \to M$ be the map induced by g_i.*

If M is tree-like, then g has a fixed point.

PROOF. For each i, let f_i: $M \to K_i$ denote the projection map. Since g is induced by g_i, $g_i f_i = f_i g$. Fix i. We shall show that g_i has a fixed point. Let J be the universal covering space of K_i with covering projection p: $J \to K_i$, and proceed as in the sixth paragraph of the proof of Theorem 1.3. We obtain maps ϕ: $M \to J$ and $\bar{g}$: $\phi[M] \to J$ so that $p\bar{g} = g_i p$ on $\phi[M]$, $p\phi = f_i$, and $\bar{g}\phi = \phi g$. Thus we have the commutative diagram of Figure 1, with f, g' and K replaced by f_i, g_i and K_i respectively.

Now $\bar{g}\phi[M] = \phi g[M] \subset \phi[M]$. Since $\phi[M]$ is a tree, there is a point $m \in M$ so that $\bar{g}\phi(m) = \phi(m)$. Because $\bar{g}$ is a lift of $g_i p$, $g_i p\phi(m) = p\bar{g}\phi(m) = p\phi(m)$. Thus g_i fixes $p\phi(m)$.

Let d denote a metric for M. To show that g has a fixed point, it suffices to show that, for each $\varepsilon > 0$, there is an $m \in M$ such that $d(m, g(m)) < \varepsilon$. Given such an ε, there is a positive integer i such that, for each $y \in K_i$, diam $f_i^{-1}(y) < \varepsilon$. As we have shown above, there is an $m \in M$ such that $g_i p\phi(m) = p\phi(m)$. Thus $f_i(m) = p\phi(m) = g_i p\phi(m) = g_i f_i(m) = f_i g(m)$, so $d(m, g(m)) < \varepsilon$, concluding the proof.

2. Compact groups. We extend the principal results of the first section to include infinite groups of homeomorphisms, if they are compact. We shall assume that the

metric for the compact group G is given by $\bar{d}(f, g) = \sup\{d(f(x), g(x)): x \in M\}$. We are indebted to Carl Eberhart for the main idea in the proof of the following theorem.

THEOREM 2.1. *If G is a compact group of homeomorphisms on the tree-like continuum M, then*

(i) *for each $g \in G$, F_g, the fixed point set of M under g, is a nonvoid continuum, and*

(ii) *F_G, the fixed point set of M under G, is a nonvoid continuum.*

PROOF. *Case* 1. G is connected. For each $x \in M$, define $G_x = \{g \in G: g(x) = x\}$ to be the closed *isotropy* subgroup of G at x. Then the map $\alpha_x: G/G_x \to G(x) \subset M$, defined by $\alpha_x(gG_x) = g(x)$, is a homeomorphism, because G is compact [**Br**, p. 40]. Since G/G_x is connected, $G(x)$ is also and must be tree-like. Since trees are contractible and AWS cohomology commutes with inverse limits, $G(x)$ also has trivial cohomology in all positive dimensions. Then by a theorem of Borel [**H-M**, p. 310], $G = G_x$. Thus every element of G fixes every element of M and $G = \{1\}$.

Case 2. G is not connected. If we let H be the component of G that contains 1, then H is a closed connected subgroup of G that acts on M. Then by Case 1, $H = \{1\}$. Hence G is totally disconnected and since G is compact, must be zero dimensional.

We now define a decreasing sequence of invariant (normal in the group sense) subgroups of G in the obvious way. Let H_1 be a compact, open, invariant subgroup of G for which diam $H_1 < 1$ and G/H_1 is finite [**M-Z**, p. 56]. Let n be a positive integer and if m is a positive integer, $m \leqslant n$, assume H_m has been defined in such a way that $H_1 \supset H_2 \supset \cdots \supset H_m \supset \cdots \supset H_n$, H_m is a compact, open invariant subgroup of G for which diam $H_m < 1/m$ and G/H_m is a finite group. Now H_n is an open set containing 1 and G is zero dimensional, so there exists an open, compact, invariant subgroup H_{n+1} of G for which diam $H_{n+1} < 1/(n+1)$, $H_{n+1} \subset H_n$ and G/H_{n+1} is finite [**M-Z**, p. 56].

For each positive integer n, let M/H_n be the orbit space of H_n with the quotient topology and $\pi_n: M \to M/H_n$ be the projection map. Since G is compact and M is metric, M/H_n is metric and π_n is an open map [**M-Z**, p. 232]. Now, a theorem of McLean [**M**] tells us that M/H_n is tree-like; moreover, G/H_n may be regarded as a finite group of homeomorphisms on M/H_n. If $g \in G$, then $gH_n(H_n(x)) = H_n(g(x))$ [**M-Z**, p. 61].

For each positive integer n, let $F(G/H_n)$ be the fixed point set of this group of homeomorphisms. It is easy to see that $F(G/H_n) = \{H_n(x): x \in M$ and for each $g \in G$, $H_n(g(x)) = H_n(x)\}$. It follows from Theorem 1.3 that, for each n, $F(G/H_n)$ is a nonvoid closed subset of M/H_n; hence $\pi_n^{-1}[F(G/H_n)]$ is a closed subset of M. Moreover, $\pi_n^{-1}[F(G/H_n)] = \{x \in M$: for each $g \in G$, $H_n(g(x)) = H_n(x)\}$. Using this, we can see that, for each n, $\pi_{n+1}^{-1}[F(G/H_{n+1})] \subset \pi_n^{-1}[F(G/H_n)]$; for if $x \in \pi_{n+1}^{-1}[F(G/H_{n+1})]$ then, for each $g \in G$, $H_{n+1}(g(x)) = H_{n+1}(x)$. Since $H_{n+1} \subset H_n$, it follows that $H_{n+1}(g(x)) \subset H_n(g(x))$ and $H_{n+1}(x) \subset H_n(x)$. Thus $H_n(g(x)) \cap H_n(x) \neq \varnothing$, and since these sets are decomposition elements, $H_n(g(x)) = H_n(x)$, so $x \in \pi_n^{-1}[F(G/H_n)]$.

Thus we have a decreasing sequence of closed subsets of M and so there is a $t \in \bigcap \{\pi_n^{-1}[F(G/H_n)]$: n is a positive integer$\}$. This means that for each $g \in G$ and each n, $H_n(t) = H_n(g(t))$. Since each subgroup H_n has diameter less than $1/n$, it follows that each orbit $H_n(x)$ has diameter less than $1/n$. So for each $g \in G$ and each n, $d(t, g(t)) < 1/n$ and thus $t \in F_G$. Inasmuch as $F_G = \bigcap \{F_g: g \in G\}$, each F_g is also nonvoid.

Fix $g \in G$. We will show that F_g is connected. For each positive integer n, let $F(gH_n)$ be the fixed point set of the homeomorphism gH_n on M/H_n. From Theorem 1.4, we conclude that $F(gH_n)$ is a nonvoid continuum. Also, $F(gH_n) = \{H_n(x)\colon\ x \in M,\ H_n(g(x)) = H_n(x)\}$ and $\pi_n^{-1}[F(gH_n)] = \{x \in M\colon\ H_n(g(x)) = H_n(x)\}$. Arguments similar to those above show that $\pi_{n+1}^{-1}[F(gH_{n+1})] \subset \pi_n^{-1}[F(gH_n)]$ and $F_g = \bigcap \{\pi_n^{-1}[F(gH_n)]$: n is a positive integer$\}$. If F_g fails to be connected, then there are disjoint closed sets A and B so that $F_g = A \cup B$. There are open sets U and V, containing A and B respectively, such that $\operatorname{cl} U \cap \operatorname{cl} V = \varnothing$. There is a positive integer N so that if $n > N$ then $\pi_n^{-1}[F(gH_n)] \subset U \cup V$. Choose $i > N$ such that $1/i < \inf\{d(u, v)\colon\ u \in \operatorname{cl} U,\ v \in \operatorname{cl} V\}$. Since π_i is a $(1/i)$-map, it follows that $\pi_i[\operatorname{cl} U] \cap \pi_i[\operatorname{cl} V] = \varnothing$. This shows that

$$\pi_i\left[\pi_i^{-1}\left[F(gH_n)\right] \cap \operatorname{cl} U\right] \quad \text{and} \quad \pi_i\left[\pi_i^{-1}\left[F(gH_n)\right] \cap \operatorname{cl} V\right]$$

are disjoint closed sets, each meeting $F(gH_n)$, and their union contains $F(gH_n)$. But this says that $F(gH_n)$ is not connected, which is a contradiction. Thus, F_g is connected.

Since $F_G = \bigcap \{F_g: g \in G\}$ and M is hereditarily unicoherent, we see that F_G is a continuum. This concludes the proof.

THEOREM 2.2. *If M is a tree-like continuum with metric ρ and $H = \{g\colon g$ is a ρ-isometry, $g\colon M \to M\}$ then*

(i) *for each $g \in H$, F_g, the fixed point set of M under g, is a nonvoid continuum, and*

(ii) *F_H, the fixed point set of M under H, is a nonvoid continuum.*

PROOF. To say that g is a ρ-isometry means that, for all $x, y \in M$, $\rho(g(x), g(y)) = \rho(x, y)$. Let $\operatorname{cl} H$ denote the closure of H in the function space M^M, with the sup-metric topology. Clearly, H is an equicontinuous family. Using the Arzela-Ascoli Theorem [**D**, p. 267], $\operatorname{cl} H$ is compact. Since the uniform limit of a sequence of ρ-isometries is a ρ-isometry, $\operatorname{cl} H = H$. It is easy to see that if g and h are ρ-isometries, so are gh and g^{-1}. Thus H is a group. We may now apply the preceding theorem to conclude the proof.

3. Pointwise periodic homeomorphisms. If M is a continuum and $h\colon M \to M$ is a homeomorphism, then h is *pointwise periodic* if, for each $x \in M$, the h orbit of x, $o(x) = \{h^n(x)\colon\ n \geqslant 0\}$, is finite. For the other definitions in this section we refer the reader to a paper by Hall and Kelley [**H-K**]. In this paper, they prove the following.

THEOREM 3.1. *If M is a compact metric space and $h: M \to M$ is a pointwise periodic homeomorphism, then the following are equivalent*:

(i) *the collection of point orbits under h is a continuous collection,*

(ii) *h is strongly almost periodic,*

(iii) *h is regular,*

(iv) *the orbit under h of every closed subset of M is closed in M.*

In a remark following this theorem, Hall and Kelley also verify that if M is a compact metric space and h is a pointwise periodic homeomorphism from M onto M that is also regular, then h is an isometry. This, together with Theorem 2.2, proves the following corollary.

COROLLARY 3.2. *If M is a tree-like continuum and h: $M \to M$ is a pointwise periodic homeomorphism satisfying any one of* (i), (ii), (iii) *or* (iv), *then the fixed point set of M under h is tree-like.*

Our final result establishes another setting in which pointwise periodic homeomorphisms on tree-like continua have fixed points.

A continuum M is *decomposable* provided there exists a pair of proper subcontinua of M whose union is M and *indecomposable* if it is not decomposable. If each subcontinuum of M is indecomposable, then M is *hereditarily indecomposable*. We shall use the following property of hereditarily indecomposable continua: if two subcontinua of M meet, then one is a subset of the other. For concepts concerning hyperspaces, used in the proof of the next theorem, see [N].

THEOREM 3.3. *Suppose that M is a hereditarily indecomposable tree-like continuum, and h: $M \to M$ is a pointwise periodic homeomorphism. Then h has a fixed point.*

PROOF. If the theorem fails, then there is a nondegenerate subcontinuum M' of M which is minimal with respect to being mapped into itself. Thus, M' is a hereditarily indecomposable tree-like curve and h/M' is pointwise periodic. Therefore we may assume that $M' = M$ and no proper subcontinuum of M' is mapped into itself.

Let $C(M)$ be the hyperspace of subcontinua of M, with the Hausdorff metric; $C(M)$ is a continuum. Let μ: $C(M) \to [0, 1]$ be a Whitney map; μ is monotone, and if $A, B \in C(M)$ and A is a proper subset of B, then $\mu(A) < \mu(B)$. Without loss of generality, we may assume that $\mu(M) = 1$ [N, p. 67].

Let $\bar{h}$: $C(M) \to C(M)$ be the map induced by h; $\bar{h}$ is defined by $\bar{h}(A) = h[A]$. It follows from [N] that $\bar{h}$ is a homeomorphism. We shall show that $\bar{h}$ is pointwise periodic. Suppose that $A \in C(M)$. Since each $x \in A$ has a finite orbit, it follows that there is a positive integer n so that $h^n[A] \cap A \neq \varnothing$. We claim that in fact $h^n[A] = A$. If the claim fails, then since M is hereditarily indecomposable, either $h^n[A]$ is a proper subset of A or A is a proper subset of $h^n[A]$. Assume that the latter condition holds, and choose $x \in h^n[A] - A$. Then for each positive integer i, $h^{in}(x) \in h^{(i+1)n}[A] - h^{in}[A]$ and so $o(x)$ is infinite. This is impossible, so $h^n[A] = A$. The other case follows similarly. Thus $\bar{h}^n(A) = A$ and $\bar{h}$ is pointwise periodic.

Our goal is to produce a Whitney map σ: $C(M) \to [0, 1]$, and a real number t, $0 < t < 1$, so that $\bar{h}$ is a periodic homeomorphism of $\sigma^{-1}(t)$ onto itself.

For each positive integer i, let $J_i = \{x \in M\colon \operatorname{card} o(x) \leqslant i\}$. Each J_i is closed and $M = \bigcup \{J_i\colon 1 \leqslant i\}$, since $\operatorname{card} o(x)$ is finite for each $x \in M$. By the Baire Category Theorem, there is a positive integer n for which $\operatorname{int} J_n \neq \varnothing$. We claim that there is a number s, $0 < s < 1$, such that if $K \in C(M)$ and $\mu(K) \geqslant s$, then $K \cap \operatorname{int} J_n \neq \varnothing$. For if this is not true, then for each positive integer i, there is a $K_i \in C(M)$ such that $\mu(K_i) > 1 - 1/i$ and $K_i \cap \operatorname{int} J_n = \varnothing$. Since $C(M)$ is compact, by passing to a subsequence, we may assume that $\{K_i\}$ converges to K_0. Since $\{\mu(K_i)\}$ converges to 1, $\mu(K_0) = 1$ and $K_0 = M$. Let V be open in M such that $V \subset \operatorname{cl} V \subset \operatorname{int} J_n$ and let $\mathfrak{U} = \{A \in C(M)\colon A \subset (M - \operatorname{cl} V) \cup \operatorname{int} J_n, A \cap (M - \operatorname{cl} V) \neq \varnothing, A \cap \operatorname{int} J_n \neq \varnothing\}$. Then $\mathfrak{U}$ is a basic open set in $C(M)$ and $M \in \mathfrak{U}$. Hence there is a positive integer N such that if $i > N$, then $K_i \in \mathfrak{U}$, and so $K_i \cap \operatorname{int} J_n \neq \varnothing$. This contradiction establishes the claim.

It follows that if $\mu(A) \geqslant s$, then A contains a point of order n; hence $h^n[A] \cap A \neq \varnothing$, and as demonstrated above, $\bar{h}^n(A) = A$. We now define $\sigma\colon C(M) \to [0, 1]$ by $\sigma(A) = \sup\{\mu(\bar{h}^i(A))\colon 1 \leqslant i \leqslant n\}$. Clearly σ is continuous; $\sigma(\{x\}) = 0$, for each $x \in M$; and if A is a proper subcontinuum of B, then $\sigma(A) < \sigma(B)$. Thus σ is a Whitney map. Moreover, $\sigma(M) = 1$. Notice that if $\mu(A) \geqslant s$, then $\sigma(\bar{h}(A)) = \sigma(A)$.

We wish to show that there is a number t, $0 < t < 1$, so that if $\sigma(A) \geqslant t$, then $\mu(A) \geqslant s$. If this is not true, then we obtain a sequence $A_1, A_2, \ldots$ of subcontinua of M such that $\sigma(A_i) > 1 - 1/i$ but $\mu(A_i) < s$. As before, we can assume that $\{A_i\} \to A_0$ and so $\{\sigma(A_i)\} \to \sigma(A_0) = 1$; thus $A_0 = M$. But $\{\mu(A_i)\} \to \mu(A_0)$, so $\mu(M) = \mu(A_0) \leqslant s < 1$, which contradicts the fact that $\mu(M) = 1$. Thus a number t exists as claimed, and we now know that if $\sigma(A) \geqslant t$, then $\sigma(\bar{h}(A)) = \sigma(A)$. Since Whitney maps are monotone [N, Theorem 14.2], $\bar{h}|_{\sigma^{-1}(t)}$ is a periodic homeomorphism of the continuum $\sigma^{-1}(t)$ onto itself.

Since M is hereditarily indecomposable, each $x \in M$ belongs to a unique continuum K_x such that $\sigma(K_x) = t$ (for if two continua meet, one contains the other). This says that the function $\eta\colon M \to \sigma^{-1}(t)$ given by $\eta(x) = K_x$ is well defined. It is shown in [N, Theorem 1.80] that η is a monotone and open mapping. It follows from [M] that $\sigma^{-1}(t)$ is a tree-like continuum.

Thus $\bar{h}|_{\sigma^{-1}(t)}$ is a periodic homeomorphism on a tree-like continuum. Moreover, this homeomorphism is fixed-point-free, since $\bar{h}(A) = A$ means that $h[A] = A$ and no proper subcontinuum of M is sent into itself by h. This contradicts Theorem 1.3 and concludes the proof.

We conclude by asking if the preceding result can be extended to all tree-like continua. That is, if M is a tree-like continuum and $h\colon M \to M$ is a pointwise periodic homeomorphism, does h have a fixed point?

References

[Be] David P. Bellamy, *A tree-like continuum without the fixed point property*, Houston J. Math. **6** (1979), 1–14.

[B] Beverly Brechner, *Extendable periodic homeomorphisms on chainable continua*, preprint.

[Br] Glen E. Bredon, *Introduction to compact transformation groups*, Academic Press, New York, 1972.

[C-C] J. H. Case and R. E. Chamberlin, *Characterizations of tree-like continua*, Pacific J. Math. **10** (1960), 73–84.

[Co] H. Cook, *Tree-likeness of dendroids and λ-dendroids*, Fund. Math. **68** (1970), 19–22.

[Cu] M. L. Curtis, *The covering homotopy theorem*, Proc. Amer. Math. Soc. **7** (1956), 682–684.

[D] James Dugundji, *Topology*, Allyn and Bacon, Boston, Mass., 1966.

[F-M] J. B. Fugate and L. Mohler, *A note on fixed points in tree-like continua*, Topology Proc. **2** (1977), 457–460.

[H-K] D. W. Hall and J. L. Kelley, *Periodic types of transformations*, Duke Math. J. **8** (1941), 625–630.

[H-M] K. H. Hoffman and P. S. Mostert, *Elements of compact semigroup*, Merrill, Columbus, Ohio, 1966.

[K] J. Krasinkiewicz, *On the hyperspaces of hereditarily indecomposable continua*, Fund. Math. **84** (1974), 175–186.

[L1] Wayne Lewis, *Stable homeomorphisms of the pseudo arc*, Canad. J. Math. **31** (1979), 363–374.

[L2] _____, *Periodic homeomorphisms on chainable continua*, preprint.

[M] T. B. McLean, *Confluent images of tree-like curves are tree-like*, Duke Math. J. **39** (1972), 465–473.

[M-Z] D. Montgomery and L. Zippin, *Topological transformation groups*, Interscience, Huntington, N.Y., 1974.

[N] S. Nadler, Jr., *Hyperspaces of sets*, Dekker, New York, 1978.

[R] M. J. Russell, *Monotone decompositions of continua irreducible about a finite set*, Fund. Math. **72** (1971), 255–264.

[S] P. A. Smith, *Fixed point theorems for periodic transformations*, Amer. J. Math. **6** (1941), 1–8.

[Sp] E. H. Spanier, *Algebraic topology*, McGraw-Hill, New York, 1966.

[S-Y] Michel Smith and Sam Young, *Periodic homeomorphisms on T-like continua*, Fund. Math. **104** (1979), 221–224.

[T] E. S. Thomas, Jr., *Monotone decompositions of irreducible continua*, Rozprawy Mat. **50** (1966).

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