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CONDITIONAL EXPECTATIONS IN C^*-CROSSED PRODUCTS

BY

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ABSTRACT. Let (A, G, α) be a C^*-dynamical system. Let B be a C^*-subalgebra of A and P be a conditional expectation of A onto B such that $\alpha_t P = P\alpha_t$ for each $t \in G$. Then it is proved that there exists a conditional expectation of $C^*(G, A, \alpha)$ onto $C^*(G, B, \alpha)$. In particular, if G is amenable and A is unital, then there always exists a conditional expectation of $C^*(G, A, \alpha)$ onto $C^*(G)$. Some related results are also obtained.

1. Introduction. Recently Anantharaman-Delaroche [1], [2] investigated the existence of a conditional expectation of a W^*-crossed product $W^*(G, M, \alpha)$ onto a W^*-crossed product $W^*(G, N, \alpha)$ under appropriate conditions, where N is a von Neumann subalgebra of a von Neumann algebra M.

In this paper corresponding results for C^*-crossed products are studied and analogous results are obtained. In particular, if a C^*-algebra A is unital and G is amenable, then the existence of a conditional expectation of $C^*(G, A, \alpha)$ onto $C^*(G)$ is shown.

2. Projections in C^*-algebras. Let A be a C^*-algebra, and let $M_n(A)$ be the C^*-algebra of $n \times n$ matrices $M = [a_{ij}]$ with entries a_{ij} in A (cf. Paschke [10, Appendix], Takesaki [17, IV. §3]).

LEMMA 2.1 (CF. PASCHKE [10, PROPOSITION 6.1], TAKESAKI [17, LEMMA IV.3.2]). *An element $M = [a_{ij}]$ of $M_n(A)$ is positive if and only if $\sum_{i,j} x_i^* a_{ij} x_j \geq 0$ in A for any $x_1, \ldots, x_n \in A$.*

Let B be a C^*-subalgebra of A. A bounded linear map $P\colon A \to B$ is called a conditional expectation if P has the following properties (cf. Umegaki [19]):

(i) P is an onto projection of norm one, that is, $P^2 = P$ and $\|P\| = 1$;

(ii) P is positive, that is, for any $x \in A$, $P(x^*x) \geq 0$;

(iii) for any $x \in A, y, z \in B$, $P(yxz) = yP(x)z$;

(iv) for any $x \in A$, $P(x^*)P(x) \leq P(x^*x)$.

If $P\colon A \to B$ is a conditional expectation, then, by (ii), $P(x^*) = P(x)^*$ for each $x \in A$. Tomiyama [18] proved that if $P\colon A \to B$ is an onto projection of norm one (that is P satisfies condition (i)), then P becomes a conditional expectation (cf. Takesaki [17, Theorem III.3.4]). In fact, in this case P satisfies conditions (ii) and (iv) above and

(iii)′ $P(xy) = P(x)y$ and $P(yx) = yP(x)$ for every $x \in A, y \in B$.

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PROPOSITION 2.2. *Let P: $A \to B$ be a conditional expectation. Then, for any positive integer n and any $x_1, \ldots, x_n \in A$, the $n \times n$ matrix $M = [P(x_i^* x_j) - P(x_i^*)P(x_j)]$ of $M_n(B)$ is positive.*

PROOF. For each $y_1, \ldots, y_n \in B$, $x = \sum_i x_i y_i$ is in A. By condition (iv) we have $P(x^*)P(x) \leqslant P(x^* x)$; hence

$$\sum_{i,j} y_i^* P(x_i^*) P(x_j) y_j \leqslant \sum_{i,j} y_i^* P(x_i^* x_j) y_j.$$

By Lemma 2.1, the matrix $M = [P(x_i^* x_j) - P(x_i^*)P(x_j)]$ is positive in $M_n(B)$.

COROLLARY 2.3 (NAKAMURA, TAKESAKI AND UMEGAKI [**9**], CF. STØRMER [**14**, THEOREM 4.1]). *Let A, B and P be as above. Then, for each positive integer n and each $x_1, \ldots, x_n \in A$, $M = [P(x_i^* x_j)]$ is a positive element of $M_n(B)$, that is, P is a completely positive map (cf. Takesaki* [**17**, IV. §3]).

3. Projections in C^*-crossed products. Let G be a locally compact group and dt be the left Haar measure on G. Let A^* be the dual space of a C^*-algebra A and $\langle \cdot , \cdot \rangle$ be the duality pairing between A and A^*. Denote by Aut(A) the *-automorphism group of A. We suppose that α: $G \to \mathrm{Aut}(A)$ is a strongly continuous homomorphism. Then (A, G, α) is called a C^*-dynamical system.

Let $L^1(G, A)$ be the set of all (equivalence classes of) A-valued Bochner integrable functions on G with respect to dt. $L^1(G, A)$ is a Banach*-algebra with an approximate identity whose multiplication, involution and norm are respectively defined by

$$(xy)(t) = \int x(s)\alpha_s(y(s^{-1}t))\, ds, \qquad (x^*)(t) = \Delta(t)^{-1}\alpha_t(x(t^{-1}))^*,$$

$$\|x\|_1 = \int \|x(t)\|\, dt$$

for each x, $y \in L^1(G, A)$ and $t \in G$, where Δ is the modular function of G (Doplicher, Kastler and Robinson [**5**, §§II, III]; cf. Bratteli and Robinson [**3**, §2.7.1], Pedersen [**11**, §7.6]). We denote by $C^*(G, A, \alpha)$ the enveloping C^*-algebra of $L^1(G, A)$ (Doplicher, Kastler and Robinson [**5**, §IV]; cf. Bratteli and Robinson [**3**, §2.7.1], Pedersen [**11**, 7.6.5]). $C^*(G, A, \alpha)$ is called the C^*-crossed product (or the covariance algebra) of (A, G, α). We also denote by $C^*(G)$ the group C^*-algebra of G (cf. Dixmier [**4**, 13.9.1], Pedersen [**11**, 7.1.5]). $C^*(G)$ is nothing but $C^*(G, \mathbf{C}, \alpha_0)$, where $\mathbf{C}$ is the complex numbers and α_0: $G \to \mathrm{Aut}(\mathbf{C})$ is the unique trivial homomorphism.

Now let P be a conditional expectation of A onto a C^*-subalgebra B of A. Assume that $\alpha_t P = P\alpha_t$ for every $t \in G$. Then for each $t \in G$, $\alpha_t(B) \subset B$, hence α_t may also be considered as a *-automorphism of B.

PROPOSITION 3.1. *Let (A, G, α) be a C^*-dynamical system, B be a C^*-subalgebra of A, and P be a conditional expectation of A onto B. Suppose that for any $t \in G$, $\alpha_t P = P\alpha_t$. Then $C^*(G, B, \alpha)$ is a C^*-subalgebra of $C^*(G, A, \alpha)$.*

PROOF. If $y \in L^1(G, B)$, then $y \in L^1(G, A)$; thus it is sufficient to show that the norm $\|y\|_B$ of y in $C^*(G, B, \alpha)$ is equal to the norm $\|y\|_A$ of y in $C^*(G, A, \alpha)$. The inequality $\|y\|_A \leqslant \|y\|_B$ is clear from the definition. We must prove that $\|y\|_A \geqslant \|y\|_B$. Let Ψ be a (continuous) positive linear form on $L^1(G, B)$ with $\|\Psi\| \leqslant 1$. Then Ψ extends to a positive linear form on $C^*(G, B, \alpha)$ (cf. Dixmier [**4**, 2.7.5]). We use the same symbol Ψ for the extended linear form. To this Ψ, there corresponds a norm continuous positive definite function ψ: $G \to B^*$ by Pedersen [**11**, 7.6.7, 7.6.8]. Define ϕ: $G \to A^*$ by $\langle a, \phi(t)\rangle = \langle P(a), \psi(t)\rangle$ ($a \in A$, $t \in G$). Then ϕ is a norm continuous positive definite function. In fact, for any positive integer n, any $a_1, \ldots, a_n \in A$ and any $t_1, \ldots, t_n \in G$, the $n \times n$ matrix $M = [P(a_i^* a_j)]$ is a positive element in $M_n(B)$ by Corollary 2.3. Thus $M = N^*N$ for some $N = [b_{ij}] \in M_n(B)$, that is, $P(a_i^* a_j) = \sum_k b_{ki}^* b_{kj}$ $(1 \leqslant i, j \leqslant n)$. Since ψ is positive definite, it follows that

$$\begin{aligned}
\sum_{i,j} \left\langle \alpha_{t_i^{-1}}(a_i^* a_j), \phi\left(t_i^{-1} t_j\right)\right\rangle &= \sum_{i,j} \left\langle P\left(\alpha_{t_i^{-1}}(a_i^* a_j)\right), \psi\left(t_i^{-1} t_j\right)\right\rangle \\
&= \sum_{i,j} \left\langle \alpha_{t_i^{-1}}\left(P(a_i^* a_j)\right), \psi\left(t_i^{-1} t_j\right)\right\rangle \\
&= \sum_{i,j} \left\langle \alpha_{t_i^{-1}}\left(\sum_k b_{ki}^* b_{kj}\right), \psi\left(t_i^{-1} t_j\right)\right\rangle \\
&= \sum_k \sum_{i,j} \left\langle \alpha_{t_i^{-1}}(b_{ki}^* b_{kj}), \psi\left(t_i^{-1} t_j\right)\right\rangle \geqslant 0.
\end{aligned}$$

Hence ϕ is positive definite. Let Φ be a positive linear form on $C^*(G, A, \alpha)$ corresponding to ϕ by Pedersen [**11**, 7.6.7, 7.6.8]. Then $\|\Phi\| = \|\phi(e)\| = \|\psi(e)\| = \|\Psi\| \leqslant 1$, where e is the identity of G (cf. Pedersen [**11**, 7.6.7]), and

$$\Phi(y^*y) = \int \langle (y^*y)(t), \phi(t)\rangle\, dt = \int \langle (y^*y)(t), \psi(t)\rangle\, dt = \Psi(y^*y).$$

This implies that $\|y\|_B \leqslant \|y\|_A$. Therefore, $\|y\|_B = \|y\|_A$ for any $y \in L^1(G, B)$, and $C^*(G, B, \alpha)$ is a C^*-subalgebra of $C^*(G, A, \alpha)$. (In the sequel we denote by $\|x\|$ the norm of $x \in L^1(G, A)$ in $C^*(G, A, \alpha)$.)

THEOREM 3.2. *Let* (A, G, α), B *and* P *be as in Proposition* 3.1. *Then there exists a conditional expectation of* $C^*(G, A, \alpha)$ *onto* $C^*(G, B, \alpha)$.

PROOF. Define Q: $L^1(G, A) \to L^1(G, B)$ by $(Q(x))(t) = P(x(t))$ ($x \in L^1(G, A)$, $t \in G$). Then Q is a linear map and $Q(y) = y$ for every $y \in L^1(G, B)$. Take a positive linear form Ψ on $L^1(G, B)$ with $\|\Psi\| \leqslant 1$. As in the proof of Proposition 3.1, to this Ψ there correspond a norm continuous positive definite function ψ: $G \to B^*$, a norm continuous positive definite function ϕ: $G \to A^*$ and a positive linear form Φ on $C^*(G, A, \alpha)$ with $\|\Phi\| \leqslant 1$. Let $K(G, A)$ be the set of all A-valued continuous functions on G with compact support. Then for $x \in K(G, A)$,

$$\begin{aligned}\Psi(Q(x)^*Q(x)) &= \int \langle (Q(x)^*Q(x))(t), \psi(t)\rangle\, dt\\ &= \int\int \Delta(s^{-1})\big\langle \alpha_s\big(P(x(s^{-1})^*)P(x(s^{-1}t))\big), \psi(t)\big\rangle ds\, dt\\ &= \int\int \big\langle \alpha_{s^{-1}}(P(x(s)^*)P(x(t))), \psi(s^{-1}t)\big\rangle\, ds\, dt\end{aligned}$$

and

$$\begin{aligned}\Phi(x^*x) &= \int \langle (x^*x)(t), \phi(t)\rangle\, dt\\ &= \int\int \Delta(s^{-1})\big\langle \alpha_s\big(P(x(s^{-1})^*x(s^{-1}t))\big), \psi(t)\big\rangle\, ds\, dt\\ &= \int\int \big\langle \alpha_{s^{-1}}(P(x(s)^*x(t))), \psi(s^{-1}t)\big\rangle\, ds\, dt.\end{aligned}$$

The function

$$(s, t) \to \big\langle \alpha_{s^{-1}}(P(x(s)^*)P(x(t))), \psi(s^{-1}t)\big\rangle$$

on $G \times G$ is continuous and of compact support S_1. Similarly the function

$$(s, t) \to \big\langle \alpha_{s^{-1}}(P(x(s)^*)x(t)), \psi(s^{-1}t)\big\rangle$$

on $G \times G$ is continuous and of compact support S_2. Then S_1 and S_2 are contained in a set $K \times K$ for some compact subset K of G. The measure on K induced by the left Haar measure dt is a finite measure, hence it is the weak*-limit of positive measures m_l of finite support. Then $m_l = \sum_i c_i\delta_{t_i}$ for some positive numbers $c_1, \ldots, c_n$ and $t_1, \ldots, t_n \in G$, where δ_t is the point measure at $t \in G$. We have

$$\begin{aligned}&\int\int \big\langle \alpha_{s^{-1}}(P(x(s)^*)P(x(t))), \psi(s^{-1}t)\big\rangle\, dm_l(s)\, dm_l(t)\\ &\qquad = \sum_{i,j} \big\langle \alpha_{t_i^{-1}}\big(P(x(t_i)^*)P(x(t_j))\big), \psi\big(t_i^{-1}t_j\big)\big\rangle c_ic_j\end{aligned}$$

and

$$\begin{aligned}&\int\int \big\langle \alpha_{s^{-1}}(P(x(s)^*x(t))), \psi(s^{-1}t)\big\rangle\, dm_l(s)\, dm_l(t)\\ &\qquad = \sum_{i,j} \big\langle \alpha_{t_i^{-1}}\big(P\big(x(t_i)^*x(t_j)\big)\big), \psi\big(t_i^{-1}t_j\big)\big\rangle c_ic_j.\end{aligned}$$

By Proposition 2.2, $M = [c_ic_jP(x(t_i)^*x(t_j)) - c_ic_jP(x(t_i)^*)P(x(t_j))]$ is positive in $M_n(B)$. Thus, $M = N^*N$ for some $N = [b_{ij}] \in M_n(B)$. It follows that

$$\begin{aligned}&\sum_{i,j} \big\langle \alpha_{t_i^{-1}}\big(P\big(x(t_i)^*x(t_j)\big)\big), \psi\big(t_i^{-1}t_j\big)\big\rangle c_ic_j - \sum_{i,j} \big\langle \alpha_{t_i^{-1}}\big(P(x(t_i)^*)P(x(t_j))\big), \psi\big(t_i^{-1}t_j\big)\big\rangle c_ic_j\\ &\quad = \sum_{i,j} \Big\langle \alpha_{t_i^{-1}}\Big(\sum_k b_{ki}^*b_{kj}\Big), \psi\big(t_i^{-1}t_j\big)\Big\rangle = \sum_k \sum_{i,j} \big\langle \alpha_{t_i^{-1}}(b_{ki}^*b_{kj}), \psi\big(t_i^{-1}t_j\big)\big\rangle \geqslant 0.\end{aligned}$$

Therefore

$$\iint \langle \alpha_{s^{-1}}(P(x(s)^*)P(x(t))), \psi(s^{-1}t)\rangle \, dm_l(s)\, dm_l(t)$$

$$\leqslant \iint \langle \alpha_{s^{-1}}(P(x(s)^* x(t))), \psi(s^{-1}t)\rangle \, dm_l(s)\, dm_l(t)$$

and, in the limit,

$$\iint \langle \alpha_{s^{-1}}(P(x(s)^*)P(x(t))), \psi(s^{-1}t)\rangle \, ds\, dt$$

$$\leqslant \iint \langle \alpha_{s^{-1}}(P(x(s)^* x(t))), \psi(s^{-1}t)\rangle \, ds\, dt.$$

We have $\Psi(Q(x)^* Q(x)) \leqslant \Phi(x^* x)$. Since $K(G, A)$ is dense in $L^1(G, A)$, the above inequality holds for any $x \in L^1(G, A)$. This implies that $\|Q(x)\| < \|x\|$ for every $x \in L^1(G, A)$, and Q extends to a conditional expectation of $C^*(G, A, \alpha)$ onto $C^*(G, B, \alpha)$.

Let ϕ be a state of A. If $\langle \alpha_t(a), \phi\rangle = \langle a, \phi\rangle$ for any $a \in A$, $t \in G$, then ϕ is said to be an α-invariant state. Let $CB(G)$ be the Banach algebra of all complex-valued bounded continuous functions on G with supremum norm. For each state ϕ of A and $a \in A$, define $\phi_a \in CB(G)$ by $(\phi_a)(t) = \langle \alpha_t(a), \phi\rangle$. If A is unital and G is amenable, then for any (left and right) invariant mean m (cf. Greenleaf [**7**, p. 29], Pedersen [**11**, 7.3.3]), the state ϕ_m of A defined by $\langle a, \phi_m\rangle = m(\phi_a)$ is α-invariant (cf. Emch [**6**, p. 173]). Notice that, if G is amenable, the reduced C^*-crossed product $C_r^*(G, A, \alpha)$ (Zeller-Meier [**20**, Définition 4.6 (for G discrete)], Takai [**15**]; cf. Pedersen [**11**, 7.7.4]) is equal to the C^*-crossed product $C^*(G, A, \alpha)$ for any C^*-dynamical system (A, G, α) (Zeller-Meier [**20**, Théorème 5.1 (for G discrete)], Takai [**15**, Proposition 2.2]; cf. Pedersen [**11**, 7.7.7]). It is known that G is amenable if and only if $C^*(G) = C_r^*(G)$ (cf. Dixmier [**4**, §18.3], Greenleaf [**7**, §3.5], Pedersen [**11**, §7.3]).

COROLLARY 3.3. *Let (A, G, α) be a C^*-dynamical system. Suppose that A is unital and has an α-invariant state. Then there exists a conditional expectation of $C^*(G, A, \alpha)$ onto its C^*-subalgebra $C^*(G)$.*

PROOF. Let 1 be the identity of A and identify $\mathbf{C}$ with $\mathbf{C}1$. Let ϕ be an α-invariant state of A. Define $P\colon A \to \mathbf{C}1$ by $P(a) = \phi(a)1$ $(a \in A)$. Then P is a conditional expectation and, since ϕ is α-invariant, $P\alpha_t = \alpha_t P$ for every $t \in G$. By Theorem 3.2 there exists a conditional expectation of $C^*(G, A, \alpha)$ onto the C^*-subalgebra $C^*(G)$ of $C^*(G, A, \alpha)$.

COROLLARY 3.4. *Let (A, G, α) be a C^*-dynamical system with A unital. If G is amenable, then there exists a conditional expectation of $C^*(G, A, \alpha)$ onto $C^*(G)$.*

Now we consider the case where G is abelian. Denote by $\hat{G}$ the dual group of G. For each $\sigma \in \hat{G}$, define $\hat{\alpha}_\sigma\colon L^1(G, A) \to L^1(G, A)$ by

$$(\hat{\alpha}_\sigma(x))(t) = \overline{(t, \sigma)}\, x(t) \qquad (x \in L^1(G, A), t \in G),$$

where (t, σ) is the value of the character σ at t. Then $\hat{\alpha}_\sigma$ can be extended to a $*$-automorphism of $C^*(G, A, \alpha)$ and, denoting it by the same symbol $\hat{\alpha}_\sigma$, $\hat{\alpha}$: $G \to \mathrm{Aut}(C^*(G, A, \alpha))$ is shown to be a strongly continuous homomorphism (Takai [**15**, pp. 30–31]; cf. Pedersen [**11**, 7.8.3]). $(C^*(G, A, \alpha), \hat{G}, \hat{\alpha})$ is called the dual C^*-dynamical system of (A, G, α).

COROLLARY 3.5. *Let (A, G, α) be a C^*-dynamical system with A unital and G abelian. Then there exists a conditional expectation of $C^*(\hat{G}, C^*(G, A, \alpha), \hat{\alpha})$ onto $C^*(\hat{G}, C^*(G), \hat{\alpha})$.*

PROOF. By Corollary 3.4 there exists a conditional expectation P of $C^*(G, A, \alpha)$ onto $C^*(G)$. In view of the construction of P, it is easy to see that $\hat{\alpha}_\sigma P = P\hat{\alpha}_\sigma$ for all $\sigma \in \hat{G}$. Hence, by Theorem 3.2 there exists a conditional expectation of $C^*(\hat{G}, C^*(G, A, \alpha), \hat{\alpha})$ onto $C^*(\hat{G}, C^*(G), \hat{\alpha})$.

4. Projections in C^*-crossed products with discrete groups. We now treat the case where G is discrete. The following theorem is essentially due to Zeller-Meier [**20**].

THEOREM 4.1. *Let (A, G, α) be a C^*-dynamical system with G discrete. Then*
(i) *there exists a conditional expectation of $C_r^*(G, A, \alpha)$ onto A, and*
(ii) *there exists a conditional expectation of $C^*(G, A, \alpha)$ onto A.*

PROOF. (i) Since G is discrete, the Haar measure dt on G is a counting measure, that is, for any finite subset F of G, the measure of F is the number of elements in F. By the correspondence $a \to \delta_e a$ $(a \in A)$, A may be considered as a C^*-subalgebra of $C_r^*(G, A, \alpha)$ [**20**, p. 171], where δ_e is the characteristic function of $\{e\}$, that is, $\delta_e(t) = 1$ if $t = e$, or 0 if $t \neq e$. Define P: $L^1(G, A) \to A$ by $P(x) = x(e)$ $(x \in L^1(G, A))$. Then P can be extended to a conditional expectation of $C_r^*(G, A, \alpha)$ onto A. In fact, for any state ϕ of A, let $(\pi_\phi, H_\phi, \xi_\phi)$ be the GNS representation of A induced by ϕ and $\eta_\phi = \delta_e \xi_\phi \in L^2(G, H_\phi)$, where $L^2(G, H_\phi)$ is the Hilbert space of all H_ϕ-valued functions η on G such that $\int \|\eta(t)\|^2\, dt < \infty$. Denote $\Pi_\phi = \mathrm{Ind}\, \pi_\phi$ [**20**, Définition 4.1]. Then $(\Pi_\phi, L^2(G, H_\phi), \eta_\phi)$ is a cyclic $*$-representation of $L^1(G, A)$, and for each $x \in L^1(G, A)$, $(\Pi_\phi(x)\eta_\phi, \eta_\phi) = \phi(x(e))$ [**20**, Proposition 4.2(ii)]. It follows that

$$(\Pi_\phi(x^*x)\eta_\phi, \eta_\phi) = \phi((x^*x)(e)) \geqslant \phi(x(e)^*x(e)) = \phi(P(x)^*P(x)).$$

This implies that $\|P(x)\| \leqslant \|x\|_r$ (the norm of x in $C_r^*(G, A, \alpha)$) and P extends to a conditional expectation of $C_r^*(G, A, \alpha)$ onto A.

(ii) By the correspondence $a \to \delta_e a$ $(a \in A)$, A is also a C^*-subalgebra of $C^*(G, A, \alpha)$ [**20**, p. 146]. Since $\|x\|_r \leqslant \|x\|$ for any $x \in L^1(G, A)$, P in the proof of (i) can be extended to a conditional expectation of $C^*(G, A, \alpha)$ onto A.

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