UNIFORM PARTITIONS OF AN INTERVAL

BY

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ABSTRACT. Let \{x_n\} be a sequence of numbers in \([0, 1]\); for each \(n\) let \(u_0(n), \ldots, u_n(n)\) be the lengths of the intervals resulting from partitioning of \([0, 1]\) by \(\{x_1, x_2, \ldots, x_n\}\). For \(p > 1\) put \(A^{(p)}(n) = (n + 1)^{p-1} \sum_{j=0}^{n} [u_j(n)]^p\); the paper investigates the behavior of \(A^{(p)}(n)\) as \(n \to \infty\) for various sequences \(\{x_n\}\). Theorem 1. If \(x_n = n\theta \mod 1\) for an irrational \(\theta > 0\), then \(\lim \inf A^{(p)}(n) < \infty\). However \(\lim \sup A^{(p)}(n) < \infty\) if and only if the partial quotients of \(\theta\) are bounded (in the continued fraction expansion of \(\theta\)). Theorem 2 gives the exact values for \(\lim \inf\) and \(\lim \sup\) when \(\theta = \frac{1}{2}(1 + \sqrt{5})\). Theorem 3. If \(x_n's\) are random variables, uniformly distributed on \([0, 1]\), then \(\lim A^{(p)}(n) = \Gamma(p + 1)\) almost surely.

1. Introduction. Let \(x_1, x_2, \ldots\) be an infinite sequence of points between 0 and 1. For each \(n\) the points \(x_1, x_2, \ldots, x_n\) partition the interval \([0, 1]\) into \(n + 1\) subintervals. Extensive studies have been made of irregularities of such partitions by considering the quantity \(D_n\), called discrepancy, defined by

\[
D_n = \sup_{0 \leq \alpha < \beta < 1} \left| \frac{1}{n} \sum_{j=1}^{n} \chi_{(a, b)}(x_j) - (\beta - \alpha) \right|
\]

where \(\chi_E(\cdot)\) is the characteristic function of a set \(E\). (See [6].) In this paper we propose to study the problem by introducing a different measure of uniformity defined as follows. For each \(n\), let \(x_1(n), x_2(n), \ldots, x_n(n)\) be the points \(x_1, x_2, \ldots, x_n\) arranged in nondecreasing order, let \(x_0(n) = 0, x_{n+1}(n) = 1, u_j(n) = x_{j+1}(n) - x_j(n) (j = 0, 1, \ldots, n)\), and for each \(p > 1\) consider

\[
A^{(p)}(n) = (n + 1)^{p-1} \sum_{j=0}^{n} [u_j(u)]^p.
\]

The closer to 1 the value of \(A^{(p)}(n)\) is, the more uniform is the partition (if the points \(x_1, x_2, \ldots, x_n\) divide \([0, 1]\) into \(n + 1\) equal parts then \(A^{(p)}(n) = 1\)). In this paper we investigate \(\lim_n A^{(p)}(n)\) for various sequences \(x_1, x_2, \ldots\). We begin with the classical case \(x_n = n\theta \mod 1\) for some irrational \(\theta > 0\). It turns out that the limiting behavior of \(A^{(p)}(n)\) strongly depends on the arithmetic character of \(\theta\). We have the following theorem.

THEOREM 1. Let \(x_n = n\theta \mod 1\) for an irrational \(\theta > 0\) and let \(A^{(p)}(n)\) be defined by (1). We have for \(p > 1\):

I. \(\lim \inf_n A^{(p)}(n) < \infty\);

II. \(\lim \sup_n A^{(p)}(n) < \infty\) if and only if the partial quotients of the continued fraction expansion of \(\theta\) are bounded.

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For certain values of \( \theta \) both \( \limsup \ A^{(p)}(n) \) and \( \liminf \ A^{(p)}(n) \) can be evaluated as the following theorem shows:

**Theorem 2.** Let \( \theta = \frac{1}{2}(1 + \sqrt{5}) \) so that \( \theta^2 = \theta + 1 \). Let

\[
\psi(t) = 5^{-p/2}(\theta + t)^{p-1} \left[ t(\theta - 1)^p + (1 - t)\theta^p + t + \theta - 1 \right].
\]

Let \( A^{(p)}(n) \) be obtained from the sequence \( n\theta \pmod{1} \) as in (1). We have

\[
\liminf_n A^{(p)}(n) = \psi(0) = \psi(1) = 5^{-p/2}(\theta^{2p-1} + \theta^{p-2}),
\]

\[
\limsup_n A^{(p)}(n) = \psi(t_0),
\]

where

\[
t_0 = (1 - p^{-1})/(\theta^p - (\theta - 1)^p - 1)/(\theta^p + \theta - 1).
\]

Special cases of Theorems 1 and 2 (\( p = 3 \)) appear in [2]. We discuss next the behavior of \( A^{(p)}(n) \) in case the sequence \( \{x_n\} \) is chosen at random.

**Theorem 3.** Let \( X_1, X_2, \ldots \) be a sequence of independent random variables uniformly distributed on \([0, 1]\), and let \( A^{(p)}(n) \) be the random variable defined by (1). Then \( \lim_n A^{(p)}(n) = \Gamma(p + 1) \) almost surely.

We introduce the following definition.

**Definition.** Let \( \{x_n\} \) be a sequence of numbers in an interval \([0, 1]\) and let \( A^{(p)}(n) \) be given by (1). We say that this sequence \( p \)-partitions \([0, 1]\) if \( \lim_n A^{(p)}(n) \) exists.

**Corollary.** For every \( p > 1 \) there is a sequence which \( p \)-partitions \([0, 1]\).

This is immediate from Theorem 3. We now proceed with the proofs of the theorems.

**2. Proof of Theorems 1 and 2.** We summarize first the basic facts about the distribution of the points \( \{\theta\}, \{2\theta\}, \ldots, \{n\theta\} \) in \([0, 1]\). (Here \( \{t\} = t \pmod{1} \).) For the details and references see [7]. Let \( n \) be fixed, let \( 1 < a_n < n \) be such that \( \{a_n\theta\} \) is the smallest among \( \{\theta\}, \{2\theta\}, \ldots, \{n\theta\} \) and let \( 1 < b_n < n \) be such that \( \{b_n\theta\} \) is the largest. Set \( a_n = \{a_n\theta\} \), \( \beta_n = 1 - \{b_n\theta\} \). The interval \([0, 1]\) is divided by \( \{\theta\}, \{2\theta\}, \ldots, \{n\theta\} \) into \( n + 1 \) subintervals as follows: \( n + 1 - a_n \) of them are of length \( \alpha_n \), \( a_n + b_n - (n + 1) \) of them are of length \( \alpha_n + \beta_n \) and \( n + 1 - b_n \) have length \( \beta_n \). The fact that \( n + 1 < a_n + b_n \) can be deduced from the definitions of \( a_n \) and \( b_n \). Thus with this notation,

\[
A^{(p)}(n) = (n + 1)^{p-1} \left[ (n + 1 - a_n)\alpha_n^p + (a_n + b_n - n - 1)\alpha_n^p + (n + 1 - b_n)\beta_n^p \right].
\]

One can actually find \( a_n \) and \( b_n \) in terms of the continued fraction expansion of \( \theta \):

\[
\theta = [d_0; d_1, d_2, \ldots] = d_0 + \frac{1}{d_1 + \frac{1}{d_2 + \cdots}}.
\]
As usual, we set \( q_{-1} = 0, p_{-1} = 1, q_0 = 1, p_0 = d_0, q_{k+1} = d_{k+1} q_k + q_{k-1}, p_{k+1} = d_{k+1} p_k + p_{k-1}, \) \( \delta_k = (-1)^k (q_k \theta - p_k) > 0. \) Given \( n, \) to find \( a_n \) and \( \alpha_n \) express \( n \) as

\[
\begin{aligned}
n &= q_{2m} + r q_{2m+1} + s, \\
0 &< r < d_{2m+2}, \quad 0 < s < q_{2m+1},
\end{aligned}
\]

so that \( q_{2m} < n < q_{2m+2}. \) Then

\[
\begin{aligned}
a_n &= q_{2m} + r q_{2m+1}, \\
\alpha_n &= \delta_{2m} - r \delta_{2m+1}.
\end{aligned}
\]

To find \( b_n \) and \( \beta_n \) we express \( n \) as

\[
\begin{aligned}
n &= q_{2m-1} + u q_{2m} + v, \\
0 &< u < d_{2m+1}, \quad 0 < v < q_{2m},
\end{aligned}
\]

so that \( q_{2m-1} < n < q_{2m+1}. \) We have

\[
\begin{aligned}
b_n &= q_{2m-1} + u q_{2m}, \\
\beta_n &= \delta_{2m-1} - u \delta_{2m}.
\end{aligned}
\]

The following are standard facts about continued fractions (see [4]):

\[
\begin{aligned}
d_{k+1} &= \left\lfloor \frac{\delta_k}{\delta_{k-1}} \right\rfloor, \\
\delta_{k+1} &= \delta_k - d_k \delta_{k-1}, \\
d_{k+2} / q_{k+2} < \delta_k < 1 / q_{k+1}.
\end{aligned}
\]

We are now ready to prove Theorem 1. Set \( x_n = n \theta (\text{mod } 1), p > 1 \) and let

\[
A(n) = A^{(p)}(n) = (n + 1)^{p-1} \prod_{j=0}^{n} \left[ u_j(n) \right]^{p}
\]

be given as in (1). To show I we will show that \( A(q_{2m} + q_{2m+1} - 1) \) is bounded. From the discussion above and (2) we see that for \( n = q_{2m} + q_{2m+1} - 1 \) the following hold:

\[
\begin{aligned}
a_n &= q_{2m}, \\
b_n &= q_{2m+1}, \\
\alpha_n &= \delta_{2m}, \\
\beta_n &= \delta_{2m+1},
\end{aligned}
\]

and using (7) we get

\[
A(n) = (q_{2m} + q_{2m+1})^{p-1} (q_{2m} \delta_{2m+1} + q_{2m+1} \delta_{2m})
\]

\[
< (q_{2m} + q_{2m+1})^{p-1} (q_{2m} q_{2m+2} + q_{2m+1} q_{2m+1})
\]

\[
= O(1) \quad (m \to \infty).
\]

Hence I follows. To show II assume that all the partial quotients \( d_k \) of \( \theta \) are bounded by \( D, \) say. We wish to show that \( A(n) \) is bounded. We claim that a number \( c > 0 \) can be chosen such that the following three inequalities hold:

\[
\begin{aligned}
c^{-1} &< \delta_{k-1} / \delta_k < c, \\
c^{-1} &< a_n / b_n < c, \\
c^{-1} &< \alpha_n / \beta_n < c.
\end{aligned}
\]

The first inequality holds for some \( c > 0 \) because \( [\delta_{k-1} / \delta_k] = d_{k+1} \) and we are assuming that \( d_k \)'s are bounded. Suppose next that \( q_{2m} < n < q_{2m+1} \) for some \( m \) so that \( a_n = q_{2m} \) (see (4)). Let \( u \) and \( v \) be determined by (5) so that

\[
\begin{aligned}
a_n &= \frac{q_{2m}}{u q_{2m} + q_{2m-1}} < \frac{q_{2m}}{q_{2m-1}} = d_{2m} q_{2m-1} + q_{2m-2} \\
&< d_{2m} + 1 < D + 1.
\end{aligned}
\]

On the other hand,

\[
\begin{aligned}
a_n &= \frac{q_{2m}}{d_{2m+1} q_{2m} + q_{2m-1}} \leq \frac{1}{d_{2m+1} + 1} < \frac{1}{D + 1}.
\end{aligned}
\]
If \( n \) satisfies \( q_{2m-1} \leq n < q_{2m} \) then the analysis is based on (5) and (3) and leads to the same conclusion. As for the ratio of \( \alpha_n \) and \( \beta_n \) we have the following: For \( q_{2m} \leq n < q_{2m+1} \) and \( u \) given by (5),

\[
1 \leq \frac{\alpha_n}{\beta_n} = \frac{\delta_{2m}}{\delta_{2m-1} - u\delta_{2m}} \leq \frac{\delta_{2m}}{\delta_{2m+1}} < d_{2m+2} + 1 \leq D + 1.
\]

For \( q_{2m-1} \leq n < q_{2m} \) and \( r \) given by (3),

\[
1 \leq \frac{\beta_n}{\alpha_n} = \frac{\delta_{2m-1}}{\delta_{2m-2} - r\delta_{2m-1}} \leq \frac{\delta_{2m-1}}{\delta_{2m}} < d_{2m+1} + 1 \leq D + 1.
\]

This establishes (8). To show now that \( A(n) \) is bounded, we take (2) and bound all the terms by \( \alpha_n\beta_n \) using (8):

\[
A(n) \leq (a_n + b_n)^p - 1 [ b_n\alpha_n^p + (b_n - 1) (\alpha_n + \beta_n)^p + a_n\beta_n^p ]
\]

\[
\leq M(a_n\beta_n)^p
\]

where \( M \) depends only on the constant \( c \) from (8) (and hence on \( \max d_k \)). Since \( \alpha_n\beta_n + b_n\alpha_n = 1 \), the first part of II follows. We next show the converse of II, that is, if the partial quotients are unbounded then \( \lim \sup A(n) = \infty \).

There are two cases: either \( \{d_{2m}\} \) is unbounded or \( \{d_{2m+1}\} \) is unbounded. We present the arguments in the first case only, the second is completely analogous. For each \( m \) let \( y_m = \lfloor \frac{q_{2m+2}}{2} \rfloor - 2 \), so that \( y_m > 0 \) for infinitely many \( m \)'s and \( \lim \sup y_m = \infty \). Let

\[
n = n_m = q_{2m} + (y_m + 1)q_{2m+1} - 1
\]

\[
= q_{2m+1} + (q_{2m} + y_mq_{2m+1} - 1).
\]

For those \( m \)'s for which \( y_m > 1 \) we have from (3)-(6),

\[
a = q_{2m} + y_mq_{2m+1}, \quad \alpha = \delta_{2m} - y_m\delta_{2m+1} > y_m\delta_{2m+1},
\]

\[
b = q_{2m+1}, \quad \beta = \delta_{2m+1}, \quad a + b = n + 1.
\]

Thus, substituting in (2),

\[
A(n_m) = (q_{2m} + (y_m + 1)q_{2m+1})^{p-1} (q_{2m+1}\alpha^p + (q_{2m} + y_mq_{2m+1})\beta^p)
\]

\[
> (y_mq_{2m+1})^{p-1} (q_{2m+1}y_m^p\delta_{2m+1})^p = y_m^{2p-1}q_{2m+1}\delta_{2m+1}^p.
\]

It follows from (7) that

\[
(q_{2m+1}\delta_{2m+1})^p > \left[ \frac{q_{2m+1}d_{2m+2} + q_{2m+3}}{d_{2m+3}q_{2m+2} + q_{2m+1}} \right]^p \geq \left[ \frac{q_{2m+1}}{d_{2m+2} + q_{2m+1}} \right]^p > (d_{2m+2} + 3)^{-p}.
\]

Thus \( A(n_m) > y_m^{2p-1}(d_{2m+2} + 3)^{-p} \), so \( \lim \sup A(n) = \infty \).

This completes the proof of Theorem 1 and we take up Theorem 2. For \( \theta = \frac{1}{2}(1 + \sqrt{5}) \) all partial quotients are equal to 1 and we have

\[
q_{-1} = 0, \quad q_0 = 1, \quad q_{k+1} = q_k + q_{k-1},
\]

\[
p_{-1} = 1, \quad p_0 = 1, \quad p_{k+1} = p_k + p_{k-1},
\]

\[
\delta_{k+1} = \delta_{k-1} - \delta_k, \quad q_{k-1} = 5^{-1/2}\left( \theta^k - (-1)^k \theta^{-k} \right), \quad \delta_k = \theta^{-(k+1)}.
\]
so that

\[
\lim_k \left( \frac{q_k}{\theta^k} \right) = \theta / \sqrt{\theta}.
\]

Let \( 0 < t < 1 \) be given, and suppose \( 0 < t_k < 1 \) are such that \( t_k q_{2k-1} - 1 \) is a positive integer and \( t_k \to t \). We will show that if \( n_k = q_{2k} + t_k q_{2k-1} - 1 \) then

\[
\lim_k A(n_k) = 5^{-p/2}(\theta + i)^{p-1} \left[ \frac{1}{i(t - 1)\theta^p + (1 - i)\theta^p + t + \theta - 1} \right] = \psi(t).
\]

Similarly, if \( 0 < s_k < 1 \) is such that \( s_k q_{2k} - 1 \) is a positive integer and \( s_k \to t \), then with \( n_k = q_{2k+1} + s_k q_{2k} - 1 \),

\[
\lim_k A(n_k) = \psi(t).
\]

To show (11) we have from (3)-(6):

\[
\begin{align*}
a &= q_{2k}, & a &= \delta_{2k}, & b &= q_{2k-1}, & b &= \delta_{2k-1}, & n + 1 - a &= t_k q_{2k-1}, \\
a + b - (n + 1) &= (1 - t_k) q_{2k-1}, & n + 1 - b &= q_{2k} + (t_k - 1) q_{2k-1}.
\end{align*}
\]

Substituting these values into (2) we get

\[
A(n_k) = (q_{2k} + t_k q_{2k-1})^{p-1} \left[ t_k q_{2k} \theta^{p-1} + (1 - t_k) q_{2k-1} (\delta_{2k} + \delta_{2k-1}) \right.
\]

\[
+ (q_{2k} + (t_k - 1) q_{2k-1}) \delta_{2k-1} - 1].
\]

Substituting values for \( \delta \)'s from (9) we obtain

\[
A(n_k) = (q_{2k} + t_k q_{2k-1})^{p-1} \left[ t_k q_{2k-1} (\theta^{-(2k+1)p} + (1 - t_k) q_{2k-1} (1 + \theta^{-1})^p \theta^{-2kp} + (q_{2k} + (t_k - 1) q_{2k-1}) \theta^{-2kp} \right].
\]

From (10) it follows then that

\[
\lim_k A(n_k) = 5^{-p/2}(\theta + i)^{p-1} \left[ \frac{1}{i(t - 1)\theta^p + (1 - i)\theta^p + t + \theta - 1} \right],
\]

which implies (11) since \( 1/\theta = \theta = 1 \) and \( 1 + 1/\theta = \theta \). Equation (12) follows pretty much the same way. Let now \( \{n_j\} \) be such that \( A(n_j) \) converges to \( \xi \), say. Clearly \( n_j \) belongs infinitely often to an interval of the form \([q_{2k+1}, q_{2k+1}+1)\) or infinitely often to an interval of the form \([q_{2k+1}, q_{2k+2})\). In the first case \( n_j = q_{2k} + t_k q_{2k-1} - 1 \) for some \( 0 < t_k < 1 \) and \( k = k(j) \), depending on \( j \); in the second case \( n_j = q_{2k+1} + s_k q_{2k} - 1 \), \( 0 < s_k < 1 \), \( k = k(j) \). By taking subsequences, if needed, we may assume that \( t_k \) (or \( s_k \)) converges to \( t \), say. Thus in both cases \( \lim_j A(n_j) = \xi = \psi(t) \) for some \( 0 < t < 1 \). Hence \( \lim \sup A(n) \) and \( \lim \inf A(n) \) are, respectively, the maximum and the minimum of \( \psi(t) \) for \( 0 < t < 1 \). By direct calculation we can obtain that

\[
\psi(0) = \psi(1) = 5^{-p/2}(\theta_{2p-1} + \theta_{p-2}).
\]

The simplification is based on the fact that \( \theta^2 = \theta + 1 \). Also

\[
5^{p/2} \psi'(t) = (d/dt)(\theta + i)^{p-1} \left[ t(-\theta^p + (\theta - 1)^p + 1) + \theta^p + \theta + t - 1 \right]
\]

\[
= (d/dt)(\theta + i)^{p-1} \left[ Et + f \right]
\]

\[
= (\theta + i)^{p-2} \left[ (p - 1)(Et + F) + E(\theta + t) \right];
\]

\[
E = -\theta^p + (\theta - 1)^p + 1, \quad F = \theta^p + \theta + t - 1.
\]
Solving the equation $\psi'(t) = 0$ gives the only solution between 0 and 1:

$$t_0 = \frac{(1 - p^{-1})(\theta p - (\theta - 1)p - 1)}{\theta p + \theta - 1}.$$ 

To finish the proof we will show that $\psi'(0) > 0$, which is certainly sufficient since $\psi(0) = \psi(1)$ Now, from (14),

$$5^p/2\psi'(0) = \theta p^{-2}[(p - 1)(\theta p + \theta - 1) + \theta(-\theta p + (\theta - 1)p + 1)]$$

$$= \theta p^{-2}f(p).$$

Thus it is enough to show that $f(p) > 0$ for $p > 1$. Direct calculation gives $f(1) = 0$ and

$$f'(p) = (p - 1)\theta p\log \theta + \theta p + \theta - 1$$

$$+ \theta[-\theta p\log \theta + (\theta - 1)p\log(\theta - 1)]$$

$$= \theta p[(p - 1)\log \theta + 1 - \theta \log \theta] - (\log \theta)/\theta p - 1 + \theta - 1.$$

Since $1 - \theta \log \theta = 0.221 \ldots$, $f'(p)$ is an increasing function for $p > 1$. Also,

$$f'(1) = \theta(1 - \theta \log \theta) - \log \theta + \theta - 1 = 0.495 \ldots$$

so that $f(p)$ is positive for $p > 1$. The proof of Theorem 2 is now complete.

3. Proof of Theorem 3. Since $p$ is going to be fixed throughout, we will write $A_n$ for $A^{(p)}(n)$. The basic tool to be used is the martingale convergence theorem: Let $F_n$ be an increasing sequence of $\sigma$-fields, $Z_n$ a random variable measurable with respect to $F_n$. If $E(Z_{n+1}|F_n) = Z_n$ and $\sup_n E(|Z_n|) < \infty$, then the sequence $Z_n$ converges almost surely. $E(Z|F)$ is the conditional expectation of $Z$ relative to $F$. (See J. L. Doob [1] for the details.) We let $(X_1^{(n)}, \ldots, X_n^{(n)})$ be the order statistic of size $n$, i.e. the values of $X_1, X_2, \ldots, X_n$ arranged in increasing order, put $X_0^{(n)} = 0, X_{n+1}^{(n)} = 1$ and introduce random variables $U_j(n) = X_j^{(n)} - X_{j-1}^{(n)}$ so that once again

$$A_n = (n + 1)p^{-1}\sum_{j=0}^{n} [U_j(n)]^p$$

is a random variable. We take our $\sigma$-fields to be $F_n = F(U_0, U_1, \ldots, U_n)$, the $\sigma$ fields generated by the random variables $U_0, U_1, \ldots, U_n$ and consider the random variable

$$Z_n = A_n + \sum_{j=1}^{n-1} [A_j - E(A_{j+1}|F_j)].$$

We will show the following:

(a) $\{Z_n\}$ is a martingale relative to $\{F_n\}$.

(b) $\sum_{n=1}^{\infty} E(|A_n - E(A_{n+1}|F_n)|) < \infty$.

(c) $\lim_n E(|A_n|) = \Gamma(p + 1)$.

Note that (b), (c) imply $\lim \sup E(|Z_n|) < \infty$ so that $Z_n$ converges a.e. by the martingale convergence theorem. In addition, (b) shows $\Sigma [A_n - E(A_{n+1}|F_n)]$ converges a.e. and thus $A_n$ converges a.e. The limit will be identified later. The proof of (a) is straightforward:

$$Z_{n+1} = Z_n + A_{n+1} - A_n + A_n - E(A_{n+1}|F_n)$$

$$= Z_n + A_{n+1} - E(A_{n+1}|F_n)$$

so that $E(Z_{n+1}|F_n) = Z_n$. 

Before we take up (b) we recall facts regarding random variables $U_0(n), U_1(n), \ldots, U_n(n)$ (see [5, Chapter 9]). Since $U_0(n) + \cdots + U_n(n) \equiv 1$, the $U$'s are certainly not independent, but if we delete one of them, the remaining $n$ are "uniformly" distributed on the simplex

$$T_n = \{(t_1, t_2, \ldots, t_n); t_j > 0, t_1 + t_2 + \cdots + t_n < 1\};$$

more precisely, the joint density function of the remaining $n$ is given by

$$f_n(t_1, t_2, \ldots, t_n) = \begin{cases} n! & \text{if } (t_1, t_2, \ldots, t_n) \in T_n, \\ 0 & \text{otherwise.} \end{cases}$$

Thus for any $\alpha > 0$ and any $i$,

$$E\left(\left[ U_i(n) \right]^{\alpha} \right) = n! \int_{T_n} x_i^\alpha dx_1dx_2 \ldots dx_n = \frac{n!\Gamma(\alpha + 1)}{\Gamma(n + \alpha + 1)}. \quad \text{(15)}$$

Similarly for any $\alpha > 0, \beta > 0, i \neq j$,

$$E\left(\left[ U_i(n) \right]^{\alpha}\left[ U_j(n) \right]^{\beta} \right) = n! \int_{T_n} x_i^\alpha x_j^\beta dx_1 \ldots dx_n = \frac{n!\Gamma(\alpha + 1)\Gamma(\beta + 1)}{\Gamma(n + \alpha + \beta + 1)}. \quad \text{(16)}$$

The values of these integrals can be either evaluated directly or looked up in [3]. Notice that the right-hand side of both of the above formulas is independent of $i$ and $j$.

To prove (b) we use the Cauchy-Schwarz inequality $E(\left| W \right|) < \left[ E(W^2) \right]^{1/2}$ with $W = E(A_{n+1}|F_n) - A_n$ and show that for some constant $c(p)$, depending only on $p$, we have

$$E\left(\left[ E\left( A_{n+1}|F_n \right) - A_n \right]^2 \right) < c(p)n^{-3}. \quad \text{(17)}$$

This will certainly prove (b) since $\sum n^{-3/2}$ converges. We derive now the formula for $E(A_{n+1}|F_n)$. Since $X_{n+1}$ is uniformly distributed on $[0, 1]$ and independent of $U_0, \ldots, U_n$, we have

$$E(A_{n+1}|U_0(n) = u_0, U_1(n) = u_1, \ldots, U_n(n) = u_n)$$

$$= (n + 2)^p - 1 \sum_{j=0}^n \left[ \sum_{i=0}^n u_i^p + t^p + (u_j - t)^p \right] dt$$

$$= (n + 2)^p - 1 \sum_{j=0}^n \left[ \sum_{i=0, i \neq j}^n (u_iu_j) + \frac{2}{p + 1} u_j^{p+1} \right]$$

$$= (n + 2)^p - 1 \left[ \frac{1}{(n + 1)^{p-1}} A_n - \sum_{j=0}^n \left( 1 - \frac{2}{p + 1} \right) u_j^{p+1} \right]$$

$$= A_n + \left[ \frac{(n + 2)^p - (n + 1)^p}{(n + 1)^{p-1}} \right] A_n - (n + 2)^p - 1 \frac{p - 1}{p + 1} \sum_{j=0}^n u_j^{p+1}$$

$$= A_n + \left[ (n + 2)^p - (n + 1)^p \right] \sum_{j=0}^n u_j^p - (n + 2)^p - 1 \frac{p - 1}{p + 1} \sum_{j=0}^n u_j^{p+1}. \quad \text{(18)}$$
Thus

\[(17)\]

\[E(A_{n+1}|F_n) - A_n = \sum_{j=0}^{n} \left[ a_n U^p_j(n) - b_n U^{p+1}_j(n) \right],\]

where

\[a_n = (n + 2)^{p-1} - (n + 1)^{p-1} = (p - 1)n^{p-2}(1 + o(1))\]

and

\[b_n = (n + 2)^{p-1} \frac{p - 1}{p + 2} = n^{p-1} \frac{p - 1}{p + 1}(1 + o(1)).\]

In view of remarks after (16) we get (writing \(U_k\) for \(U_k(n)\))

\[E\left[ \left( E(A_{n+1}|F_n) - A_n \right)^2 \right] = E\left( \left[ \sum_{j=0}^{n} a_n U^p_j - b_n U^{p+1}_j \right]^2 \right)\]

\[= nE\left[ \left( a_n U^p_0 - b_n U^{p+1}_0 \right)^2 \right]\]

\[+ n(n + 1)E\left[ \left( a_n U^p_0 - b_n U^{p+1}_0 \right) \left( a_n U^p_1 - b_n U^{p+1}_1 \right) \right]\]

\[= n\left[ a_n^2 E(U_0^{2p}) - 2a_n b_n E(U_0^{2p+1}) + b_n^2 E(U_1^{2p+1}) \right]\]

\[+ n(n + 1)\left[ a_n^2 E(U_0^{2p} U_1^{p+1}) - a_n b_n \left( E(U_0^{p} U_1^{p+1}) + E(U_0^{p+1} U_1^p) \right) \right]\]

\[+ b_n^2 E(U_0^{p+1} U_1^{p+1}) \]

\[= nP_n + n(n + 1)Q_n.\]

We will show that both \(nP_n\) and \(n(n + 1)Q_n\) are \(O(n^{-3})\), the implicit constant depending on \(p\) only. Before we do that we need the following estimate:

\[(18)\]

\[n! \Gamma(n + 2p + 1) < C_1(p)n^{-2p},\]

\(C_1(p)\) depending on \(p\) alone. Indeed, using Stirling's formula

\[\log \Gamma(x) = (x - \frac{1}{2})\log x - x + \log \sqrt{2\pi} + o(1)\]

we get

\[\log(n! \Gamma(n + 2p + 1)) = \log \Gamma(n + 1) - \log \Gamma(n + 2p + 1)\]

\[= (n + \frac{1}{2})\log(n + 1) - (n + 1) - (n + 2p + \frac{1}{2})\log(n + 2p + 1)\]

\[+ n + 2p + 1 + O(1)\]

\[= n[\log(n + 1) - \log(n + 2p + 1)]\]

\[+ \frac{1}{2}[\log(n + 1) - \log(n + 2p + 1)] - 2p\log n + O(1)\]

Since for any fixed \(d\), \(x(\log(x + d) - \log x) \to d (x \to \infty)\), the result follows. We now estimate \(nP_n\). From the definition of \(P_n\) and (15)–(17) we have

\[nP_n = n!\left[ (p - 1)^2 n^{2p-4} \frac{\Gamma(2p + 1)}{\Gamma(n + 2p + 1)} - (p - 1)^2 \frac{p + 1}{n^{2p-3}} \right]\]

\[\times \left\{ \frac{\Gamma(2p + 2)}{\Gamma(n + 2p + 2)} + \left( \frac{p - 1}{p + 1} \right)^2 n^{2p-2} \frac{\Gamma(2p + 3)}{\Gamma(n + 2p + 3)} \right\}(1 + o(1)).\]
Using the identity \( x \Gamma(x) = \Gamma(x + 1) \) several times we get

\[
n P_n = \frac{n^{2p-3}(p - 1)^2 n!}{\Gamma(n + 2p + 1)} \Gamma(2p + 1)
\]

\[
\times \left[ 1 - \frac{2p + 1}{p + 1} \frac{n}{n + 2p + 1} + \frac{(2p + 1)n^2}{(p + 1)(n + 2p + 1)(n + 2p + 2)} \right] (1 + o(1)),
\]

so the estimate \( n P_n = O(n^{-3}) \) follows from (18). Next we estimate \( n(n + 1)Q_n \), again using (15)-(17):

\[
n(n + 1)Q_n = n^2 n! \left[ (p - 1)^2 n^{2p-4} \frac{\Gamma^2(p + 1)}{\Gamma(n + 2p + 1)}
\right.
\]

\[
- 2 \frac{(p - 1)^2}{p + 1} n^{2p-3} \frac{\Gamma(p + 1)\Gamma(p + 2)}{\Gamma(n + 2p + 2)}
\]

\[
+ \left( \frac{p - 1}{p + 1} \right)^2 n^{2p-2} \frac{\Gamma^2(p + 2)}{\Gamma(n + 2p + 3)} \right] (1 + o(1))
\]

\[
= \frac{n! n^{2p-2}(p - 1)^2 \Gamma^2(p + 1)}{\Gamma(n + 2p + 1)}
\]

\[
\times \left[ 1 - \frac{2n}{2p + n + 1} + \frac{n^2}{(n + 2p + 1)(n + 2p + 2)} \right] (1 + o(1)).
\]

The expression in square brackets is equal to

\[
\frac{(-n + (2p + 1)(2p + 2))}{(n + 2p + 1)(n + 2p + 2)} = O(n^{-1}).
\]

Hence it follows from (18) that

\[
n(n + 1)Q_n = C(p) \frac{n! n^{2p-3}}{\Gamma(n + 2p + 1)} (1 + o(1)) = O(n^{-3}).
\]

Thus (b) is proved.

To show (c) we evaluate \( E(A_n) \) directly from (15):

\[
E(A_n) = (n + 1)^p \sum_{j=0}^p E(U^n_p(n)) = \frac{(n + 1)^p n! \Gamma(p + 1)}{\Gamma(n + p + 1)}.
\]

We show now that \( \gamma_n = (n + 1)^p \Gamma(n + p + 1) \rightarrow 1 \) \( (n \rightarrow \infty) \): Clearly \( \gamma_n \sim \beta_n = n! n^{2p-1} \Gamma(n + p + 1) \), so using Stirling’s formula,

\[
\log \beta_n = p \log n + \left( n + \frac{1}{2} \right) \log(n + 1) - (n + 1) + \frac{1}{2} \log(2\pi)
\]

\[
- \left( n + p + \frac{1}{2} \right) \log(n + p + 1) + n + p + 1 - \frac{1}{2} \log(2\pi) + o(1)
\]

\[
= n \left[ \log(n + 1) - \log(n + 1 + p) \right] + p + o(1).
\]

Again, \( x \log(x + d) - \log(x) \rightarrow d \) \( (x \rightarrow \infty) \), so \( \log \beta_n \rightarrow 0 \) \( (n \rightarrow \infty) \), proving the assertion. Thus \( \lim_n E(A_n) = \Gamma(p + 1) \). This completes the proof of (a)-(c) and shows that \( A_n^{(p)} \) converges almost surely. What remains is the identification of the limit. Since \( E(A_n) \rightarrow \Gamma(p + 1) \) it is reasonable to expect that \( A_n \rightarrow \Gamma(p + 1) \) since
the limit should be constant a.e. To establish it rigorously we show that $A_n \to \Gamma(p + 1)$ in probability. This proof is due to Professor Boris Pittel. Let $Y_0, Y_1, \ldots$ be a sequence of exponentially distributed independent random variables, so that $P(Y_t < t) = 1 - e^{-t}$. Let $S_n = Y_0 + Y_1 + \cdots + Y_n$. It is known that the vectors $(U_0(n), U_1(n), \ldots, U_n(n))$ and $(Y_0/S_n, Y_1/S_n, \ldots, Y_n/S_n)$ have the same distribution (see [5, p. 242]). Therefore the distributions of $A_n$ and

$$(n + 1)^{p-1} \left[ \sum_{j=0}^{n} Y_j^p \right] / S_n^p$$

are also the same. By the strong law of large numbers,

$$\frac{(n + 1)^{p-1} \sum_{j=0}^{n} Y_j^p}{S_n^p} = \frac{(n + 1)^{-1} \sum_{j=0}^{n} Y_j^p}{(S_n/n + 1)^p} \frac{E(Y_0^p)}{(E(Y_0))^p} \to \Gamma(p + 1)$$

almost everywhere, and thus in probability. Hence $A_n^{(p)}$ also converges to $\Gamma(p + 1)$ in probability. The proof of Theorem 3 is thus completed.

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