TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 268, Number 1, November 1981

UNIFORM PARTITIONS OF AN INTERVAL

BY

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ABSTRACT. Let $\{x_n\}$ be a sequence of numbers in $[0, 1]$; for each n let $u_0(n), \ldots, u_n(n)$ be the lengths of the intervals resulting from partitioning of $[0, 1]$ by $\{x_1, x_2, \ldots, x_n\}$. For $p > 1$ put $A^{(p)}(n) = (n+1)^{p-1}\Sigma_0^n [u_j(n)]^p$; the paper investigates the behavior of $A^{(p)}(n)$ as $n \to \infty$ for various sequences $\{x_n\}$. THEOREM 1. *If* $x_n = n\theta \pmod 1$ *for an irrational* $\theta > 0$, *then* $\liminf A^{(p)}(n) < \infty$. *However* $\limsup A^{(p)} < \infty$ *if and only if the partial quotients of* θ *are bounded* (*in the continued fraction expansion of* θ). THEOREM 2 gives the exact values for lim inf and lim sup when $\theta = \frac{1}{2}(1 + \sqrt{5})$. THEOREM 3. *If* x_n*'s are random variables, uniformly distributed on* $[0, 1]$, *then* $\lim A^{(p)}(n) = \Gamma(p+1)$ *almost surely*.

1. Introduction. Let $x_1, x_2, \ldots$ be an infinite sequence of points between 0 and 1. For each n the points $x_1, x_2, \ldots, x_n$ partition the interval $[0, 1]$ into $n + 1$ subintervals. Extensive studies have been made of irregularities of such partitions by considering the quantity D_n, called discrepancy, defined by

$$D_n = \sup_{0<\alpha<\beta<1} \left| \frac{1}{n} \sum_{j=1}^{n} \chi_{(a,b)}(x_j) - (\beta - \alpha) \right|,$$

where $\chi_E(\cdot)$ is the characteristic function of a set E. (See [6].) In this paper we propose to study the problem by introducing a different measure of uniformity defined as follows. For each n, let $x_1(n), x_2(n), \ldots, x_n(n)$ be the points $x_1, x_2, \ldots, x_n$ arranged in nondecreasing order, let $x_0(n) \equiv 0$, $x_{n+1}(n) \equiv 1$, $u_j(n) = x_{j+1}(n) - x_j(n)$ ($j = 0, 1, \ldots, n$), and for each $p > 1$ consider

$$A^{(p)}(n) = (n+1)^{p-1} \sum_{j=0}^{n} \left[u_j(u)\right]^p. \tag{1}$$

The closer to 1 the value of $A^{(p)}(n)$ is, the more uniform is the partition (if the points $x_1, x_2, \ldots, x_n$ divide $[0, 1]$ into $n + 1$ equal parts then $A^{(p)}(n) = 1$). In this paper we investigate $\lim_n A^{(p)}(n)$ for various sequences $x_1, x_2, \ldots$. We begin with the classical case $x_n = n\theta \pmod 1$ for some irrational $\theta > 0$. It turns out that the limiting behavior of $A^{(p)}(n)$ strongly depends on the arithmetic character of θ. We have the following theorem.

THEOREM 1. *Let* $x_n = n\theta \pmod 1$ *for an irrational* $\theta > 0$ *and let* $A^{(p)}(n)$ *be defined by* (1). *We have for* $p > 1$:

I. $\liminf_n A^{(p)}(n) < \infty$;

II. $\limsup_n A^{(p)}(n) < \infty$ *if and only if the partial quotients of the continued fraction expansion of* θ *are bounded.*

Received by the editors June 23, 1980 and, in revised form, September 29, 1980.
AMS (MOS) subject classifications (1970). Primary 10K30.

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For certain values of θ both lim sup $A^{(p)}(n)$ and lim inf $A^{(p)}(n)$ can be evaluated as the following theorem shows:

THEOREM 2. *Let* $\theta = \frac{1}{2}(1 + \sqrt{5})$ *so that* $\theta^2 = \theta + 1$. *Let*

$$\psi(t) = 5^{-p/2}(\theta + t)^{p-1}\left[t(\theta - 1)^p + (1 - t)\theta^p + t + \theta - 1\right].$$

Let $A^{(p)}(n)$ *be obtained from the sequence* $n\theta$ (mod 1) *as in* (1). *We have*

$$\liminf_n A^{(p)}(n) = \psi(0) = \psi(1) = 5^{-p/2}(\theta^{2p-1} + \theta^{p-2}),$$

$$\limsup_n A^{(p)}(n) = \psi(t_0),$$

where

$$t_0 = (1 - p^{-1})(\theta^p - (\theta - 1)^p - 1)/(\theta^p + \theta - 1).$$

Special cases of Theorems 1 and 2 ($p = 3$) appear in **[2]**. We discuss next the behavior of $A^{(p)}(n)$ in case the sequence $\{x_n\}$ is chosen at random.

THEOREM 3. *Let* $X_1, X_2, \ldots$ *be a sequence of independent random variables uniformly distributed on* [0, 1], *and let* $A^{(p)}(n)$ *be the random variable defined by* (1). *Then* $\lim_n A^{(p)}(n) = \Gamma(p + 1)$ *almost surely.*

We introduce the following definition.

DEFINITION. Let $\{x_n\}$ be a sequence of numbers in an interval [0, 1] and let $A^{(p)}(n)$ be given by (1). We say that this sequence p-partitions [0, 1] if $\lim_n A^{(p)}(n)$ exists.

COROLLARY. *For every* $p > 1$ *there is a sequence which p-partitions* [0, 1].

This is immediate from Theorem 3. We now proceed with the proofs of the theorems.

2. Proof of Theorems 1 and 2. We summarize first the basic facts about the distribution of the points $\{\theta\}, \{2\theta\}, \ldots, \{n\theta\}$ in [0, 1]. (Here $\{t\} = t$ (mod 1).) For the details and references see [7]. Let n be fixed, let $1 \leqslant a_n \leqslant n$ be such that $\{a_n\theta\}$ is the smallest among $\{\theta\}, \{2\theta\}, \ldots, \{n\theta\}$ and let $1 \leqslant b_n \leqslant n$ be such that $\{b_n\theta\}$ is the largest. Set $\alpha_n = \{a_n\theta\}$, $\beta_n = 1 - \{b_n\theta\}$. The interval [0, 1] is divided by $\{\theta\}, \{2\theta\}, \ldots, \{n\theta\}$ into $n + 1$ subintervals as follows: $n + 1 - a_n$ of them are of length α_n, $a_n + b_n - (n + 1)$ of them are of length $\alpha_n + \beta_n$ and $n + 1 - b_n$ have length β_n. The fact that $n + 1 \leqslant a_n + b_n$ can be deduced from the definitions of a_n and b_n. Thus with this notation,

$$\begin{aligned} A^{(p)}(n) = (n + 1)^{p-1}\big[&(n + 1 - a_n)\alpha_n^p \\ &+ (a_n + b_n - n - 1)(\alpha_n + \beta_n)^p + (n + 1 - b_n)\beta_n^p\big]. \end{aligned} \tag{2}$$

One can actually find a_n and b_n in terms of the continued fraction expansion of θ:

$$\theta = [d_0; d_1, d_2, \ldots] = d_0 + \frac{1}{d_1 +} \frac{1}{d_2 +} \cdots .$$

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As usual, we set $q_{-1} = 0, p_{-1} = 1, q_0 = 1, p_0 = d_0, q_{k+1} = d_{k+1}q_k + q_{k-1}, p_{k+1} = d_{k+1}p_k + p_{k-1}, \delta_k = (-1)^k(q_k\theta - p_k) > 0$. Given n, to find a_n and α_n express n as

$$n = q_{2m} + rq_{2m+1} + s, \qquad 0 \leqslant r < d_{2m+2}, 0 \leqslant s < q_{2m+1}, \tag{3}$$

so that $q_{2m} \leqslant n < q_{2m+2}$. Then

$$a_n = q_{2m} + rq_{2m+1}, \qquad \alpha_n = \delta_{2m} - r\delta_{2m+1}. \tag{4}$$

To find b_n and β_n we express n as

$$n = q_{2m-1} + uq_{2m} + v, \qquad 0 \leqslant u < d_{2m+1}, 0 \leqslant v < q_{2m}, \tag{5}$$

so that $q_{2m-1} \leqslant n < q_{2m+1}$. We have

$$b_n = q_{2m-1} + uq_{2m}, \qquad \beta_n = \delta_{2m-1} - u\delta_{2m}. \tag{6}$$

The following are standard facts about continued fractions (see [**4**]):

$$\begin{gathered} d_{k+1} = [\delta_{k-1}/\delta_k], \qquad \delta_{k+1} = \delta_{k-1} - d_{k+1}\delta_k, \\ d_{k+2}/q_{k+2} < \delta_k < 1/q_{k+1}. \end{gathered} \tag{7}$$

We are now ready to prove Theorem 1. Set $x_n = n\theta \pmod 1)$, $p > 1$ and let

$$A(n) = A^{(p)}(n) = (n+1)^{p-1}\sum_{j=0}^{n}[u_j(n)]^p$$

be given as in (1). To show I we will show that $A(q_{2m} + q_{2m+1} - 1)$ is bounded. From the discussion above and (2) we see that for $n = q_{2m} + q_{2m+1} - 1$ the following hold:

$$a_n = q_{2m}, \quad b_n = q_{2m+1}, \quad \alpha_n = \delta_{2m}, \quad \beta_n = \delta_{2m+1},$$

and using (7) we get

$$\begin{aligned} A(n) &= (q_{2m} + q_{2m+1})^{p-1}(q_{2m}\delta_{2m+1}^p + q_{2m+1}\delta_{2m}^p) \\ &\leqslant (q_{2m} + q_{2m+1})^{p-1}(q_{2m}q_{2m+2}^{-p} + q_{2m+1}q_{2m+1}^{-p}) \\ &= O(1) \qquad (m \to \infty). \end{aligned}$$

Hence I follows. To show II assume that all the partial quotients d_k of θ are bounded by D, say. We wish to show that $A(n)$ is bounded. We claim that a number $c > 0$ can be chosen such that the following three inequalities hold:

$$c^{-1} < \delta_{k-1}/\delta_k < c, \quad c^{-1} < a_n/b_n < c, \quad c^{-1} < \alpha_n/\beta_n < c. \tag{8}$$

The first inequality holds for some $c > 0$ because $[\delta_{k-1}/\delta_k] = d_{k+1}$ and we are assuming that d's are bounded. Suppose next that $q_{2m} \leqslant n < q_{2m+1}$ for some m so that $a_n = q_{2m}$ (see (4)). Let u and v be determined by (5) so that

$$\begin{aligned} \frac{a_n}{b_n} &= \frac{q_{2m}}{uq_{2m} + q_{2m-1}} \leqslant \frac{q_{2m}}{q_{2m-1}} = \frac{d_{2m}q_{2m-1} + q_{2m-2}}{q_{2m-1}} \\ &\leqslant d_{2m} + 1 \leqslant D + 1. \end{aligned}$$

On the other hand,

$$\frac{a_n}{b_n} \geqslant \frac{q_{2m}}{d_{2m+1}q_{2m} + q_{2m-1}} \geqslant \frac{1}{d_{2m+1} + 1} \geqslant \frac{1}{D+1}.$$

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If n satisfies $q_{2m-1} \leqslant n < q_{2m}$ then the analysis is based on (5) and (3) and leads to the same conclusion. As for the ratio of α_n and β_n we have the following: For $q_{2m} \leqslant n < q_{2m+1}$ and u given by (5),

$$1 \leqslant \frac{\alpha_n}{\beta_n} = \frac{\delta_{2m}}{\delta_{2m-1} - u\delta_{2m}} \leqslant \frac{\delta_{2m}}{\delta_{2m+1}} < d_{2m+2} + 1 \leqslant D + 1.$$

For $q_{2m-1} \leqslant n < q_{2m}$ and r given by (3),

$$1 \leqslant \frac{\beta_n}{\alpha_n} = \frac{\delta_{2m-1}}{\delta_{2m-2} - r\delta_{2m-1}} \leqslant \frac{\delta_{2m-1}}{\delta_{2m}} \leqslant d_{2m+1} + 1 \leqslant D + 1.$$

This establishes (8). To show now that $A(n)$ is bounded, we take (2) and bound all the terms by $a_n\beta_n$ using (8):

$$\begin{aligned} A(n) &\leqslant (a_n + b_n)^{p-1}\left[b_n\alpha_n^p + (b_n - 1)(\alpha_n + \beta_n)^p + a_n\beta_n^p\right] \\ &\leqslant M(a_n\beta_n)^p \end{aligned}$$

where M depends only on the constant c from (8) (and hence on max d_k). Since $a_n\beta_n + b_n\alpha_n = 1$, the first part of II follows. We next show the converse of II, that is, if the partial quotients are unbounded then lim sup $A(n) = \infty$.

There are two cases: either $\{d_{2m}\}$ is unbounded or $\{d_{2m+1}\}$ is unbounded. We present the arguments in the first case only, the second is completely analogous. For each m let $y_m = [d_{2m+2}/2] - 2$, so that $y_m > 0$ for infinitely many m's and lim sup $y_m = \infty$. Let

$$\begin{aligned} n = n_m &= q_{2m} + (y_m + 1)q_{2m+1} - 1 \\ &= q_{2m+1} + (q_{2m} + y_m q_{2m+1} - 1). \end{aligned}$$

For those m's for which $y_m > 1$ we have from (3)–(6),

$$\begin{gathered} a = q_{2m} + y_m q_{2m+1}, \quad \alpha = \delta_{2m} - y_m\delta_{2m+1} > y_m\delta_{2m+1}, \\ b = q_{2m+1}, \quad \beta = \delta_{2m+1}, \quad a + b = n + 1. \end{gathered}$$

Thus, substituting in (2),

$$\begin{aligned} A(n_m) &= (q_{2m} + (y_m + 1)q_{2m+1})^{p-1}(q_{2m+1}\alpha^p + (q_{2m} + y_m q_{2m+1})\beta^p) \\ &\geqslant (y_m q_{2m+1})^{p-1}(q_{2m+1}y_m^p\delta_{2m+1}^p) = y_m^{2p-1}q_{2m+1}^p\delta_{2m+1}^p. \end{aligned}$$

It follows from (7) that

$$\begin{aligned} (q_{2m+1}\delta_{2m+1})^p &\geqslant \left[\frac{q_{2m+1}d_{2m+3}}{d_{2m+3}q_{2m+2} + q_{2m+1}}\right]^p \geqslant \left[\frac{q_{2m+1}}{q_{2m+2} + q_{2m+1}}\right]^p \\ &= \left[\frac{q_{2m+1}}{d_{2m+2}q_{2m+1} + q_{2m} + q_{2m+1}}\right]^p > (d_{2m+2} + 3)^{-p}. \end{aligned}$$

Thus $A(n_m) \geqslant y_m^{2p-1}(d_{2m+2} + 3)^{-p}$, so lim sup $A(n) = \infty$.

This completes the proof of Theorem 1 and we take up Theorem 2. For $\theta = \frac{1}{2}(1 + \sqrt{5}\,)$ all partial quotients are equal to 1 and we have

$$\begin{aligned} &q_{-1} = 0, \quad q_0 = 1, \quad q_{k+1} = q_k + q_{k-1}, \\ (9) \qquad &p_{-1} = 1, \quad p_0 = 1, \quad p_{k+1} = p_k + p_{k-1}, \\ &\delta_{k+1} = \delta_{k-1} - \delta_k, \quad q_{k-1} = 5^{-1/2}(\theta^k - (-1)^k\theta^{-k}), \quad \delta_k = \theta^{-(k+1)}, \end{aligned}$$

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so that

$$\lim_k \left(q_k/\theta^k\right) = \theta/\sqrt{5}\,. \tag{10}$$

Let $0 \leqslant t \leqslant 1$ be given, and suppose $0 < t_k < 1$ are such that $t_k q_{2k-1} - 1$ is a positive integer and $t_k \to t$. We will show that if $n_k = q_{2k} + t_k q_{2k-1} - 1$ then

$$\begin{aligned}\lim_k A(n_k) &= 5^{-p/2}(\theta + t)^{p-1}\left[t(\theta - 1)^p + (1 - t)\theta^p + t + \theta - 1\right]\\ &= \psi(t).\end{aligned} \tag{11}$$

Similarly, if $0 < s_k < 1$ is such that $s_k q_{2k} - 1$ is a positive integer and $s_k \to t$, then with $n_k = q_{2k+1} + s_k q_{2k} - 1$,

$$\lim_k A(n_k) = \psi(t). \tag{12}$$

To show (11) we have from (3)–(6):

$$a = q_{2k},\quad \alpha = \delta_{2k},\quad b = q_{2k-1},\quad \beta = \delta_{2k-1},\quad n + 1 - a = t_k q_{2k-1},$$
$$a + b - (n + 1) = (1 - t_k)q_{2k-1},\quad n + 1 - b = q_{2k} + (t_k - 1)q_{2k-1}.$$

Substituting these values into (2) we get

$$\begin{aligned}A(n_k) = (q_{2k} + t_k q_{2k-1})^{p-1}\big[&t_k q_{2k}\delta_{2k}^p + (1 - t_k)q_{2k-1}(\delta_{2k} + \delta_{2k-1})^p\\ &+ (q_{2k} + (t_k - 1)q_{2k-1})\delta_{2k-1}^p\big].\end{aligned}$$

Substituting values for δ's from (9) we obtain

$$\begin{aligned}A(n_k) = (q_{2k} + t_k q_{2k-1})^{p-1}\big[&t_k q_{2k-1}\theta^{-(2kp+p)} + (1 - t_k)q_{2k-1}(1 + \theta^{-1})^p\theta^{-2kp}\\ &+ (q_{2k} + (t_k - 1)q_{2k-1})\theta^{-2kp}\big].\end{aligned}$$

From (10) it follows then that

$$\lim_k A(n_k) = 5^{-p/2}(\theta + t)^{p-1}\left[t/\theta^p + (1 - t)(1 + \theta^{-1})^p + \theta + t - 1\right],$$

which implies (11) since $1/\theta = \theta - 1$ and $1 + 1/\theta = \theta$. Equation (12) follows pretty much the same way. Let now $\{n_j\}$ be such that $A(n_j)$ converges to ξ, say. Clearly n_j belongs infinitely often to an interval of the form $[q_{2k+1}, q_{2k+1})$ or infinitely often to an interval of the form $[q_{2k+1}, q_{2k+2})$. In the first case $n_j = q_{2k} + t_k q_{2k-1} - 1$ for some $0 < t_k < 1$ and $k = k(j)$, depending on j; in the second case $n_j = q_{2k+1} + s_k q_{2k} - 1$, $0 < s_k < 1$, $k = k(j)$. By taking subsequences, if needed, we may assume that t_k (or s_k) converges to t, say. Thus in both cases $\lim_j A(n_j) = \xi = \psi(t)$ for some $0 \leqslant t \leqslant 1$. Hence lim sup $A(n)$ and lim inf $A(n)$ are, respectively, the maximum and the minimum of $\psi(t)$ for $0 \leqslant t \leqslant 1$. By direct calculation we can obtain that

$$\psi(0) = \psi(1) = 5^{-p/2}(\theta^{2p-1} + \theta^{p-2}). \tag{13}$$

The simplification is based on the fact that $\theta^2 = \theta + 1$. Also

$$\begin{aligned}5^{p/2}\psi'(t) &= (d/dt)(\theta + t)^{p-1}\left[t(-\theta^p + (\theta - 1)^p + 1) + \theta^p + \theta - 1\right]\\ &= (d/dt)(\theta + t)^{p-1}\left[Et + f\right]\\ &= (\theta + t)^{p-2}\left[(p - 1)(Et + F) + E(\theta + t)\right];\\ E &= -\theta^p + (\theta - 1)^p + 1, \qquad F = \theta^p + \theta - 1.\end{aligned} \tag{14}$$

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Solving the equation $\psi'(t) = 0$ gives the only solution between 0 and 1:

$$t_0 = (1 - p^{-1})(\theta^p - (\theta - 1)^p - 1)/(\theta^p + \theta - 1).$$

To finish the proof we will show that $\psi'(0) > 0$, which is certainly sufficient since $\psi(0) = \psi(1)$ Now, from (14),

$$\begin{aligned} 5^{p/2}\psi'(0) &= \theta^{p-2}\left[(p-1)(\theta^p + \theta - 1) + \theta(-\theta^p + (\theta-1)^p + 1)\right] \\ &= \theta^{p-2}f(p). \end{aligned}$$

Thus it is enough to show that $f(p) > 0$ for $p > 1$. Direct calculation gives $f(1) = 0$ and

$$\begin{aligned} f'(p) &= (p-1)\theta^p \log\theta + \theta^p + \theta - 1 \\ &\quad + \theta\left[-\theta^p \log\theta + (\theta-1)^p \log(\theta - 1)\right] \\ &= \theta^p\left[(p-1)\log\theta + 1 - \theta\log\theta\right] - (\log\theta)/\theta^{p-1} + \theta - 1. \end{aligned}$$

Since $1 - \theta\log\theta = 0.221\ldots$, $f'(p)$ is an increasing function for $p > 1$. Also,

$$f'(1) = \theta(1 - \theta\log\theta) - \log\theta + \theta - 1 = 0.495\ldots$$

so that $f(p)$ is positive for $p > 1$. The proof of Theorem 2 is now complete.

3. Proof of Theorem 3. Since p is going to be fixed throughout, we will write A_n for $A^{(p)}(n)$. The basic tool to be used is the martingale convergence theorem: Let F_n be an increasing sequence of σ-fields, Z_n a random variable measurable with respect to F_n. If $E(Z_{n+1}|F_n) = Z_n$ and $\sup_n E(|Z_n|) < \infty$, then the sequence Z_n converges almost surely. $E(Z|F)$ is the conditional expectation of Z relative to F. (See J. L. Doob [**1**] for the details.) We let $(X_1^{(n)}, \ldots, X_n^{(n)})$ be the order statistic of size n, i.e. the values of $X_1, X_2, \ldots, X_n$ arranged in increasing order, put $X_0^{(n)} \equiv 0$, $X_{n+1}^{(n)} \equiv 1$ and introduce random variables $U_j(n) = X_{j+1}^{(n)} - X_j^{(n)}$ so that once again

$$A_n = (n+1)^{p-1} \sum_{j=0}^{n} \left[U_j(n)\right]^p$$

is a random variable. We take our σ-fields to be $F_n = F(U_0, U_1, \ldots, U_n)$, the σ fields generated by the random variables $U_0, U_1, \ldots, U_n$ and consider the random variable

$$Z_n = A_n + \sum_{j=1}^{n-1} \left[A_j - E(A_{j+1}|F_j)\right].$$

We will show the following:

(a) $\{Z_n\}$ is a martingale relative to $\{F_n\}$.

(b) $\sum_{n=1}^{\infty} E(|A_n - E(A_{n+1}|F_n)|) < \infty$.

(c) $\lim_n E(|A_n|) = \Gamma(p+1)$.

Note that (b), (c) imply $\limsup E(|Z_n|) < \infty$ so that Z_n converges a.e. by the martingale convergence theorem. In addition, (b) shows $\sum[A_n - E(A_{n+1}|F_n)]$ converges a.e. and thus A_n converges a.e. The limit will be identified later. The proof of (a) is straightforward:

$$\begin{aligned} Z_{n+1} &= Z_n + A_{n+1} - A_n + A_n - E(A_{n+1}|F_n) \\ &= Z_n + A_{n+1} - E(A_{n+1}|F_n) \end{aligned}$$

so that $E(Z_{n+1}|F_n) = Z_n$.

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Before we take up (b) we recall facts regarding random variables $U_0(n), U_1(n), \dots, U_n(n)$ (see [5, Chapter 9]). Since $U_0(n) + \cdots + U_n(n) \equiv 1$, the U's are certainly not independent, but if we delete one of them, the remaining n are "uniformly" distributed on the simplex

$$T_n = \{(t_1, t_2, \dots, t_n) : t_j \geq 0, t_1 + t_2 + \cdots + t_n \leq 1\};$$

more precisely, the joint density function of the remaining n is given by

$$f_n(t_1, t_2, \dots, t_n) = \begin{cases} n! & \text{if } (t_1, t_2, \dots, t_n) \in T_n, \\ 0 & \text{otherwise.} \end{cases}$$

Thus for any $\alpha > 0$ and any i,

$$E([U_i(n)]^\alpha) = n! \int_{T_n} x_i^\alpha \, dx_1 dx_2 \dots dx_n = \frac{n!\Gamma(\alpha + 1)}{\Gamma(n + \alpha + 1)}. \tag{15}$$

Similarly for any $\alpha > 0$, $\beta > 0$, $i \neq j$,

$$\begin{aligned} E([U_i(n)]^\alpha [U_j(n)]^\beta) &= n! \int_{T_n} x_i^\alpha x_j^\beta \, dx_1 \dots dx_n \\ &= n!\Gamma(\alpha + 1)\Gamma(\beta + 1)/\Gamma(n + \alpha + \beta + 1). \end{aligned} \tag{16}$$

The values of these integrals can be either evaluated directly or looked up in [3]. Notice that the right-hand side of both of the above formulas is independent of i and j.

To prove (b) we use the Cauchy-Schwarz inequality $E(|W|) \leq [E(W^2)]^{1/2}$ with $W = E(A_{n+1}|F_n) - A_n$ and show that for some constant $c(p)$, depending only on p, we have

$$E([E(A_{n+1}|F_n) - A_n]^2) \leq c(p)n^{-3}.$$

This will certainly prove (b) since $\sum n^{-3/2}$ converges. We derive now the formula for $E(A_{n+1}|F_n)$. Since X_{n+1} is uniformly distributed on [0, 1] and independent of $U_0, \dots, U_n$ we have

$$\begin{aligned} &E(A_{n+1}|U_0(n) = u_0, U_1(n) = u_1, \dots, U_n(n) = u_n) \\ &= (n + 2)^{p-1} \sum_{j=0}^{n} \int_0^{u_j} \left[\sum_{\substack{i=0 \\ i \neq j}}^{n} u_i^p + t^p + (u_j - t)^p \right] dt \\ &= (n + 2)^{p-1} \sum_{j=0}^{n} \left[\sum_{\substack{i=0 \\ i \neq j}}^{n} (u_i^p u_j) + \frac{2}{p + 1} u_j^{p+1} \right] \\ &= (n + 2)^{p-1} \left[\frac{1}{(n + 1)^{p-1}} A_n - \sum_{j=0}^{n} \left(1 - \frac{2}{p + 1}\right) u_j^{p+1} \right] \\ &= A_n + \left[\frac{(n + 2)^{p-1} - (n + 1)^{p-1}}{(n + 1)^{p-1}} \right] A_n - (n + 2)^{p-1} \frac{p - 1}{p + 1} \sum_{j=0}^{n} u_j^{p+1} \\ &= A_n + [(n + 2)^{p-1} - (n + 1)^{p-1}] \sum_{j=0}^{n} u_j^p - (n + 2)^{p-1} \frac{p - 1}{p + 1} \sum_{j=0}^{n} u_j^{p+1}. \end{aligned}$$

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Thus

$$E(A_{n+1}|F_n) - A_n = \sum_{j=0}^{n} \left[a_n U_j^p(n) - b_n U_j^{p+1}(n) \right], \tag{17}$$

where

$$a_n = (n+2)^{p-1} - (n+1)^{p-1} = (p-1)n^{p-2}(1+o(1))$$

and

$$b_n = (n+2)^{p-1}\frac{p-1}{p+2} = n^{p-1}\frac{p-1}{p+1}(1+o(1)).$$

In view of remarks after (16) we get (writing U_k for $U_k(n)$)

$$\begin{aligned}
E\left(\left[E(A_{n+1}|F_n) - A_n\right]^2\right) &= E\left(\left[\sum_{j=0}^{n} a_n U_j^p - b_n U_j^{p+1}\right]^2\right) \\
&= nE\left(\left[a_n U_0^p - b_n U_0^{p+1}\right]^2\right) \\
&\quad + n(n+1)E\left(\left[a_n U_0^p - b_n U_0^{p+1}\right]\left[a_n U_1^p - b_n U_1^{p+1}\right]\right) \\
&= n\left[a_n^2 E\left(U_0^{2p}\right) - 2a_n b_n E\left(U_0^{2p+1}\right) + b_n E\left(U_0^{2p+2}\right)\right] \\
&\quad + n(n+1)\left[a_n^2 E\left(U_0^p U_1^p\right) - a_n b_n\left\{E\left(U_0^p U_1^{p+1}\right) + E\left(U_0^{p+1} U_1^p\right)\right\}\right. \\
&\qquad\qquad \left. + b_n^2 E\left(U_0^{p+1} U_1^{p+1}\right)\right] \\
&= nP_n + n(n+1)Q_n.
\end{aligned}$$

We will show that both nP_n and $n(n+1)Q_n$ are $O(n^{-3})$, the implicit constant depending on p only. Before we do that we need the following estimate:

$$n!/\Gamma(n+2p+1) \leqslant C_1(p)n^{-2p}, \tag{18}$$

$C_1(p)$ depending on p alone. Indeed, using Stirling's formula

$$\log\Gamma(x) = \left(x - \tfrac{1}{2}\right)\log x - x + \log\sqrt{2\pi} + o(1)$$

we get

$$\begin{aligned}
\log(n!/\Gamma(n+2p+1)) &= \log\Gamma(n+1) - \log\Gamma(n+2p+1) \\
&= \left(n+\tfrac{1}{2}\right)\log(n+1) - (n+1) - \left(n+2p+\tfrac{1}{2}\right)\log(n+2p+1) \\
&\quad + n + 2p + 1 + O(1) \\
&= n\left[\log(n+1) - \log(n+2p+1)\right] \\
&\quad + \tfrac{1}{2}\left[\log(n+1) - \log(n+2p+1)\right] - 2p\log n + O(1)
\end{aligned}$$

Since for any fixed d, $x(\log(x+d) - \log x) \to d$ $(x \to \infty)$, the result follows. We now estimate nP_n. From the definition of P_n and (15)–(17) we have

$$\begin{aligned}
nP_n = n!n\Bigg[&(p-1)^2 n^{2p-4}\frac{\Gamma(2p+1)}{\Gamma(n+2p+1)} - \frac{(p-1)^2}{p+1}n^{2p-3} \\
&\times \frac{\Gamma(2p+2)}{\Gamma(n+2p+2)} + \left(\frac{p-1}{p+1}\right)^2 n^{2p-2}\frac{\Gamma(2p+3)}{\Gamma(n+2p+3)}\Bigg](1+o(1)).
\end{aligned}$$

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Using the identity $x\Gamma(x) = \Gamma(x+1)$ several times we get

$$nP_n = \frac{n^{2p-3}(p-1)^2 n!}{\Gamma(n+2p+1)}\Gamma(2p+1)$$
$$\times\left[1 - \frac{2p+1}{p+1}\frac{n}{n+2p+1} + \frac{(2p+1)n^2}{(p+1)(n+2p+1)(n+2p+2)}\right](1+o(1)),$$

so the estimate $nP_n = O(n^{-3})$ follows from (18). Next we estimate $n(n+1)Q_n$, again using (15)–(17):

$$\begin{aligned} n(n+1)Q_n = n^2 n!\Bigg[&(p-1)^2 n^{2p-4}\frac{\Gamma^2(p+1)}{\Gamma(n+2p+1)} \\ &-2\frac{(p-1)^2}{p+1}n^{2p-3}\frac{\Gamma(p+1)\Gamma(p+2)}{\Gamma(n+2p+2)} \\ &+\left(\frac{p-1}{p+1}\right)^2 n^{2p-2}\frac{\Gamma^2(p+2)}{\Gamma(n+2p+3)}\Bigg](1+o(1)) \\ = \frac{n!n^{2p-2}(p-1)^2\Gamma^2(p+1)}{\Gamma(n+2p+1)}& \\ \times\left[1 - \frac{2n}{2p+n+1} + \frac{n^2}{(n+2p+1)(n+2p+2)}\right]&(1+o(1)). \end{aligned}$$

The expression in square brackets is equal to

$$(-n + (2p+1)(2p+2))/\,(n+2p+1)(n+2p+2) = O(n^{-1}).$$

Hence it follows from (18) that

$$n(n+1)Q_n = C(p)\frac{n!n^{2p-3}}{\Gamma(n+2p+1)}(1+o(1)) = O(n^{-3}).$$

Thus (b) is proved.

To show (c) we evaluate $E(A_n)$ directly from (15):

$$E(A_n) = (n+1)^{p-1}\sum_{j=0}^{n} E(U_n^p(n)) = \frac{(n+1)^{p-1}nn!\Gamma(p+1)}{\Gamma(n+p+1)}.$$

We show now that $\gamma_n = (n+1)^{p-1}nn!/\Gamma(n+p+1) \to 1$ $(n\to\infty)$: Clearly $\gamma_n \sim \beta_n = n^p n!/\Gamma(n+p+1)$, so using Stirling's formula,

$$\begin{aligned} \log\beta_n &= p\log n + \left(n+\tfrac{1}{2}\right)\log(n+1) - (n+1) + \tfrac{1}{2}\log(2\pi) \\ &\quad - \left(n+p+\tfrac{1}{2}\right)\log(n+p+1) + n+p+1 - \tfrac{1}{2}\log(2\pi) + o(1) \\ &= n\left[\log(n+1) - \log(n+1+p)\right] + p + o(1). \end{aligned}$$

Again, $x[\log(x+d) - \log(x)] \to d$ $(x\to\infty)$, so $\log\beta_n \to 0$ $(n\to\infty)$, proving the assertion. Thus $\lim_n E(A_n) = \Gamma(p+1)$. This completes the proof of (a)–(c) and shows that $A_n^{(p)}$ converges almost surely. What remains is the identification of the limit. Since $E(A_n) \to \Gamma(p+1)$ it is reasonable to expect that $A_n \to \Gamma(p+1)$ since

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the limit should be constant a.e. To establish it rigorously we show that $A_n \to \Gamma(p+1)$ in probability. This proof is due to Professor Boris Pittel. Let $Y_0, Y_1, \ldots$ be a sequence of exponentially distributed independent random variables, so that $P(Y_s < t) = 1 - e^{-t}$. Let $S_n = Y_0 + Y_1 + \cdots + Y_n$. It is known that the vectors $(U_0(n), U_1(n), \ldots, U_n(n))$ and $(Y_0/S_n, Y_1/S_n, \ldots, Y_n/S_n)$ have the same distribution (see [5, p. 242]). Therefore the distributions of A_n and

$$(n+1)^{p-1}\left[\sum_{j=0}^{n} Y_j^p\right]/S_n^p$$

are also the same. By the strong law of large numbers,

$$\frac{(n+1)^{p-1}\sum_{j=0}^n Y_j^p}{S_n^p} = \frac{(n+1)^{-1}\sum_{j=0}^n Y_j^p}{(S_n/n+1)^p} \to \frac{E(Y_0^p)}{(E(Y_0))^p} = \Gamma(p+1)$$

almost everywhere, and thus in probability. Hence $A_n^{(p)}$ also converges to $\Gamma(p+1)$ in probability. The proof of Theorem 3 is thus completed.

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