TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 268, Number 2, December 1981

THE DROR-WHITEHEAD THEOREM IN PRO-HOMOTOPY AND SHAPE THEORIES

BY

S. SINGH

ABSTRACT. Many analogues of the classical Whitehead theorem from homotopy theory are now available in pro-homotopy and shape theories. E. Dror has significantly extended the homology version of the Whitehead theorem from the well-known simply connected case to the more general, for instance, nilpotent case. We prove a full analogue of Dror's theorems in pro-homotopy and shape theories. More specifically, suppose $\underline{f}$: $\underline{X} \to \underline{Y}$ is a morphism in the pro-homotopy category of pointed and connected topological spaces which induces isomorphisms of the integral homology pro-groups. Then $\underline{f}$ induces isomorphisms of the homotopy pro-groups, for instance, when $\underline{X}$ and $\underline{Y}$ are simple, nilpotent, complete, or $\underline{H}$-objects; these notions are well known in homotopy theory and we have naturally extended them to pro-homotopy and shape theories.

0. Introduction. The purpose of this paper is to establish an analogue of Dror's generalization of the Whitehead theorem (see **[DR]**), stated below as the Dror-Whitehead theorem, in the context of the pro-homotopy and shape theories. We shall elaborate on these matters in the next few paragraphs.

THE DROR-WHITEHEAD THEOREM. *Suppose a map* f: $X \to Y$ *induces isomorphisms of all the homology groups of spaces X and Y with integral coefficients. Then f induces isomorphisms of all the homotopy groups if and only if* $\Gamma_\omega \pi_* f$ *is an epimorphism,* $\Gamma'_\omega \pi_* f$ *is a monomorphism, and* $\Gamma \pi_* f$ *is a monomorphism.*

The functors Γ_ω, Γ'_ω, and Γ are defined by considering the action of the fundamental group on the homotopy groups; see **[DR]** or §2 of this paper. An important class of spaces to which the Dror-Whitehead theorem applies is nilpotent spaces; see **[DR]**, **[HI]**, **[BK]** or §3 of this paper.

Theorem (4.1.3) of this paper is our extension of the Dror-Whitehead theorem to pro-homotopy and shape theories; also, see Theorems (4.1.1), (4.1.2), (4.1.3), (4.5.1), and Corollary (4.3). Our Corollary (4.3) extends a theorem of Raussen **[RA]** in the same manner as Dror extends the Whitehead theorem. A parallel development of pivotal ingredients of "pro-algebra" is provided by **[SI$_1$]** which extends the work of Stallings **[ST]** and Dror **[DR]** concerning algebra; and our entire program is a natural extension of Dror's work.

As a concluding remark, we may add that many analogues of the various versions of the classical Whitehead theorem have been studied in pro-homotopy

Received by the editors September 3, 1980 and, in revised form, December 8, 1980.
1980 *Mathematics Subject Classification*. Primary 55P55, 55Q07; Secondary 54C56, 55N05.
Key words and phrases. Pro-homotopy, shape, nilpotent spaces, homology pro-groups, homotopy pro-groups, s-nilpotent continua, $\underline{H}$-structures.

and shape theories; for instance, see **[AM]**, **[DS]**, **[EH]**, **[MA]**, **[MO]**, **[MR]**, **[RA]** where many other references and related discussions may also be found.

The author wishes to thank R. Wells and S. Armentrout for their helpful comments and encouragement.

1. Notation and terminology.

(1.1) *Category theoretic conventions*. A diagram in a category is said to commute if every square or a triangle in the diagram commutes whenever appropriate. All categories will be denoted by script letters. By a map we mean a morphism in a category of interest which will be clear from the context. Suppose $F\colon \mathcal{D} \to \mathcal{E}$ is a functor between the categories $\mathcal{D}$ and $\mathcal{E}$. If $f\colon X \to Y$ is a map in $\mathcal{D}$, we denote the corresponding map by $Ff\colon FX \to FY$, rather than $F(f)\colon F(X) \to F(Y)$, i.e., we omit the cumbersome round brackets whenever convenient. The zero object in any category, whenever it exists, will be denoted by 0. For any category $\mathcal{D}$ we let pro-$\mathcal{D}$ denote the well-known pro-category constructed by Grothendieck **[GR]**. We shall assume familiarity with the construction of pro-$\mathcal{D}$ and related matters; see, for instance, **[AM]**, **[DS]**, **[MA]** for a relevant discussion.

(1.2) *Other conventions*. Our main reference concerning homotopy theory is **[WH]**. We let $\mathcal{Z}$ denote the additive group of integers. The basepoints will often be suppressed. For other related references and terminology, one may also consult **[SI$_1$]**. All the homology groups of spaces are the singular homology groups.

2. Algebraic preliminaries.

(2.1) *Group actions*. Let G and π be groups. We say G is a (left) π-group if there exists a homomorphism $\eta\colon \pi \to \operatorname{Aut} G$ into the group of automorphisms of G, or, equivalently, we say G is a (left) π-group if there exists a map $\alpha\colon \pi \times G \to G$ satisfying $e \cdot g = g$, $(xy) \cdot g = x \cdot (y \cdot g)$, and $x \cdot (gh) = (x \cdot g)(x \cdot h)$, where e is the identity of π, x, y belong to π, g, h belong to G, and $\alpha(x, g)$ is denoted by $x \cdot g$. The phrase "G is a π-group" will be replaced by "G is a π-module" only when G is abelian. Let $\mathcal{G}\mathcal{A}$ denote the category whose objects are the group actions, α's as above, and whose morphisms are pairs $(\phi, \psi)\colon \alpha \to \alpha'$ such that $\phi\colon \pi \times G \to \pi' \times G'$, $\psi\colon G \to G'$, $\psi\alpha = \alpha'\phi$, where α and α' are two objects of $\mathcal{G}\mathcal{A}$.

(2.1.0) *Pro-group actions*. Consider the category pro-$\mathcal{G}\mathcal{A}$ (cf. **[AM]**, **[GR]**). Let $\mathcal{G}$ and pro-$\mathcal{G}$ denote the categories of groups and pro-groups, respectively. Let $\underline{\pi} = (\pi_\lambda, r_{\lambda\lambda'}, \Lambda)$ and $\underline{G} = (G_\lambda, p_{\lambda\lambda'}, \Lambda)$ be two pro-groups. We say $\underline{G}$ is a $\underline{\pi}$-(pro-group) if for each λ we have an action $\alpha_\lambda\colon \pi_\lambda \times G_\lambda \to G_\lambda$ of π_λ on G_λ such that $\underline{\alpha}$: $\underline{\pi} \times \underline{G} \to \underline{G}$ is a special morphism in pro-$\mathcal{G}$; see **[MA]** for a definition of a special morphism and many other related matters concerning pro-$\mathcal{G}$. If G_λ is an abelian group, we shall substitute "$\underline{G}$ is a $\underline{\pi}$-(pro-module)" for "$\underline{G}$ is a $\underline{\pi}$-(pro-group)".

(2.1.1) *π-lower central series of G*. Let G be a π-group. Put $\Gamma_1 G = G$. Define $\Gamma_2 G$ to be the normal π-subgroup of G generated by elements of the form $(x \cdot g)g^{-1}$ for x in π and g in G. Define $\Gamma_\alpha G = \Gamma_2 \Gamma_{\alpha-1} G$ if α is not a limit ordinal and define $\Gamma_\beta G = \bigcap_{\alpha<\beta} \Gamma_\alpha G$ when β is a limit ordinal. This process defines a *lower central series*

(2.1.2) $$\Gamma_s G\colon \cdots \subset \Gamma_{\alpha+1} G \subset \Gamma_\alpha G \subset \cdots \subset \Gamma_2 G \subset \Gamma_1 G = G$$

of G which depends on the action of π on G.

If $\underline{G}$ is a $\underline{\pi}$-(pro-group), we have a pro-(lower central series)

(2.1.3) $$\Gamma_s\underline{G}: \cdots \subset \Gamma_{\alpha+1}\underline{G} \subset \Gamma_\alpha\underline{G} \subset \cdots \subset \Gamma_2\underline{G} \subset \Gamma_1\underline{G} = \underline{G}$$

of $\underline{G}$ which depends on the action of $\underline{\pi}$ on $\underline{G}$.

(2.1.4) *The π-completions of G.* Let β be a limit ordinal. *The π-completion $\hat{\Gamma}_\beta G$ of G up to β* is the inverse limit of the inverse system

(2.1.5) $$\cdots \to G/\Gamma_\alpha G \to \cdots \to G/\Gamma_3 G \to G/\Gamma_2 G \to 0$$

where $\alpha < \beta$. The quotient maps of the form $G \to G/\Gamma_\alpha G$ determine a map i: $G \to \hat{\Gamma}_\beta G$ whose kernel is $\Gamma_\beta G$ and whose cokernel is denoted by $\Gamma'_\beta G = \hat{\Gamma}_\beta G/iG$. We have an exact sequence

(2.1.6) $$0 \to \Gamma_\beta G \to G \xrightarrow{i} \hat{\Gamma}_\beta G \to \Gamma'_\beta G \to 0$$

where $\Gamma'_\beta G$ is a pointed set if G is nonabelian.

Given a $\underline{\pi}$-(pro-group) $\underline{G}$. *The $\underline{\pi}$-completion* $\hat{\Gamma}_\beta\underline{G} = (\hat{\Gamma}_\beta G_\lambda, \hat{\Gamma}_\beta p_{\lambda\lambda'}, \Lambda)$ *of $\underline{G}$ up to a limit ordinal β* is defined as the (component-wise) inverse limit of

(2.1.7) $$\cdots \to \underline{G}/\Gamma_\alpha\underline{G} \to \cdots \to \underline{G}/\Gamma_3\underline{G} \to \underline{G}/\Gamma_2\underline{G} \to 0$$

where $\alpha < \beta$. We have an exact sequence

(2.1.8) $$0 \to \Gamma_\beta\underline{G} \to \underline{G} \xrightarrow{\underline{i}} \hat{\Gamma}_\beta\underline{G} \to \hat{\Gamma}'_\beta\underline{G} \to 0$$

where $\Gamma'_\beta\underline{G}$ is, in general, a (pointed) pro-set. It is easy to see that $\Gamma_\beta\underline{G}$ is the kernel of $\underline{i}$ in pro-$\mathcal{G}$ and $\Gamma'_\beta\underline{G}$ is the cokernel of $\underline{i}$ if each G_λ is abelian.

(2.2) *The maximal π-perfect subgroup of G.* A π'-group H is *π'-perfect* if $\Gamma_2 H = H$. Every π-group G contains a unique maximal π-perfect subgroup ΓG which is the π-subgroup of G generated by the family of π-perfect subgroups. Observe that $\Gamma G \subset \Gamma_\beta G$ for any ordinal β. Also, Γ can be viewed as a functor from $\mathcal{G}\mathcal{Q}$ into $\mathcal{G}$.

Given a $\underline{\pi}$-(pro-group) $\underline{G}$. The pro-group $\Gamma\underline{G} = (\Gamma G_\lambda, \Gamma p_{\lambda\lambda'}, \Lambda)$ will be called *the maximal $\underline{\pi}$-perfect pro-subgroup of $\underline{G}$.*

(2.3) *The functoriality.* Observe that Γ_s, Γ_β, Γ'_β, and $\hat{\Gamma}_\beta$ are functors defined on $\mathcal{G}\mathcal{Q}$: Our constructions $\Gamma_s\underline{G}$, $\Gamma_\beta\underline{G}$, $\Gamma'_\beta\underline{G}$, and $\hat{\Gamma}_\beta$ are the pro-functors defined on pro-$\mathcal{G}\mathcal{Q}$ which are the natural extensions of these functors.

3. Nilpotent spaces and related matters.

(3.1) *The action of $\pi_1 X$.* All spaces considered in the sequel are pointed and the basepoint is often suppressed. It is well known that the fundamental group $\pi_1 X$ of a space X acts on the homotopy group $\pi_i X$ where $i = 1, 2, \ldots$; and furthermore, this action is natural (cf. **[WH]**). By "$\pi_i X$ is a $\pi_1 X$-group" we shall always mean this natural action and we often leave the map, $\pi_1 X \times \pi_i X \to \pi_i X$, determining this action unlabelled.

(3.1.1) *The case when $i = 1$.* The group $\pi_1 X$ is considered as a $\pi_1 X$-group, and this action is by conjugation (cf. **[WH]**); in this case, one obtains the classical lower central series $\Gamma_s \pi_1 X$.

(3.1.2) *The case when $i \geqslant 2$.* Since $\pi_i X$ is abelian, we often say "$\pi_i X$ is as $\pi_1 X$-module" or we regard $\pi_i X$ as a module over the integral group ring $\mathfrak{Z}\pi_1 X$. The filteration of $\Gamma_s \pi_i X$ is induced by the powers, $(I\pi_1 X)^\alpha$, of the augmentation ideal $I\pi_1 X$ of $\mathfrak{Z}\pi_1 X$ for finite α and this can be appropriately interpreted for infinite α.

(3.2) *A pro-homotopy category*. Let $\mathcal{H}\mathcal{T}_0$ denote the homotopy categories of pointed and connected topological spaces. We are interested in the category $\mathcal{C} = \text{pro-}\mathcal{H}\mathcal{T}_0$. Let $\underline{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ be an object of $\mathcal{C}$. Now, $\pi_i\underline{X}$ for $1 \leqslant i < \infty$ is a $\pi_1\underline{X}$-(pro-group).

(3.2.1) *The case when* $i = 1$. We obtain the pro-(lower central series) $\Gamma_s\pi_1\underline{X} = (\Gamma_s\pi_1 X_\lambda, \Gamma_s p_{\lambda\lambda'}, \Lambda)$. Observe that the sequence (2.1.8) and $\Gamma\pi_1\underline{X}$ (see (2.2)) are meaningfully defined in this setting.

(3.2.2) *The case when* $i \geqslant 2$. The (abelian) pro-group $\pi_i\underline{X}$ is a $\pi_1\underline{X}$-module and we obtain the lower central series $\Gamma_s\pi_i\underline{X}$, $\Gamma\pi_i\underline{X}$, and the exact sequence (2.1.8).

Observe that $\Gamma_s\pi_i\underline{X}$, $\Gamma\pi_i\underline{X}$, $\Gamma_\beta\pi_i\underline{X}$, $\Gamma'_\beta\pi_i\underline{X}$, and $\hat{\Gamma}_\beta\pi_i\underline{X}$ are functors for $i \geqslant 1$; hence, they are invariants of the isomorphism class of $\underline{X}$ in $\mathcal{C}$.

(3.2.3) *Nilpotent objects in* $\mathcal{C}$. An object $\underline{X}$ in $\mathcal{C}$ is *nilpotent in* $\mathcal{C}$ if for each $i \geqslant 1$ there exists an integer $k = k(i)$ such that $\Gamma_k\pi_i\underline{X} \approx 0$ in pro-$\mathcal{G}$. An object $\underline{X}$ in $\mathcal{C}$ is called *simple in* $\mathcal{C}$ if $\Gamma_2\pi_i\underline{X} \approx 0$ in pro-$\mathcal{G}$ for all $i \geqslant 1$.

(3.2.4) *Complete objects in* $\mathcal{C}$. An object $\underline{X}$ is called *complete in* $\mathcal{C}$ if the natural map $\pi_i\underline{X} \to \hat{\Gamma}_\omega\pi_i\underline{X}$ is an isomorphism in pro-$\mathcal{G}$ for $i \geqslant 1$.

(3.3) *Some preliminary lemmas*. We assume familiarity with the matters related to the "singular functor" and the "geometric realization functor"; see, for instance, [**AM**], [**BK**], [**WH**]. Let $\underline{f}$: $\underline{X} \to \underline{Y}$ be a special map in $\mathcal{C}$. We are only interested in the maps induced by $\underline{f}$ on the homotopy and homology pro-groups; hence, we shall assume without loss of generality that Postnikov decompositions and related constructions can be appropriately applied; see (3.3.2) below and [**AM**, Chapter 1]. The following extends Lemma (6.1) of [**DR**].

(3.3.1) LEMMA. *In addition, suppose* $\pi_i\underline{f}$: $\pi_i\underline{X} \to \pi_i\underline{Y}$ *is an isomorphism for* $0 \leqslant i \leqslant (n-1)$, *and suppose that* $H_i\underline{f}$: $H_i\underline{X} \to H_i\underline{Y}$ *is an isomorphism for* $i = n$ *and an epimorphism for* $i = n + 1$. *Then the following conclusions hold*:

(C_1) *In the case* $n = 1$, $H_i\underline{f}$: $H_i\underline{X} \to H_i\underline{Y}$ *is an isomorphism for* $i = n$ *and an epimorphism for* $i = n + 1$ (*this conclusion merely restates the hypothesis*).

(C_n: $n \geqslant 2$) *In the case* $n \geqslant 2$, *the map* $H_j(\pi_1\underline{X}; \pi_n\underline{X}) \to H_j(\pi_1\underline{Y}; \pi_n\underline{Y})$ *induced by* $\underline{f}$ *is an isomorphism for* $j = 0$ *and an epimorphism for* $j = 1$.

(3.3.2) *Some preliminary discussions*. For each CW-complex X the coskeleton, P_nX, is a functor defined on the homotopy category of CW-complexes [**AM**], [**PO**]. The following is a characterization of P_nX: The homotopy groups of P_nX, $n \geqslant 1$, vanish in dimension $> n$ and the canonical map $X \to P_nX$ is universal with respect to maps of X into CW-complexes with vanishing homotopy groups in dimensions $> n$; see [**AM**]. Observe that Artin and Mazur [**AM**] denote P_nX by $\cos k_{n+1}X$.

(3.3.3) *Proof of Lemma* (3.3.1): *A sketch*. Consider the system $K(\pi_n\underline{X}, n) \to P_n\underline{X} \to P_{n-1}\underline{X}$ of fibrations of the nth stage Postnikov system, i.e., for each λ in Λ, we have a fibration $K(\pi_nX_\lambda, n) \to P_nX_\lambda \to P_{n-1}X_\lambda$. Consider the pro-(spectral sequence)

$$(3.3.4) \qquad \underline{E}^2_{p,q} = \left\{{}_\lambda E^2_{p,q} = H_p\left(P_{n-1}X_\lambda; \left\{H_qK(\pi_nX_\lambda, n)\right\}\right) \Rightarrow H_{p+q}P_nX_\lambda\right\}_\lambda,$$

where the bracket around $H_q K(\pi_n X_\lambda, n)$ signifies that the homology is with "twisted coefficients", i.e., $H_q K(\pi_n X_\lambda, n)$ is considered as a $\pi_1 P_{n-1} X_\lambda$-module. It follows that for each λ in Λ, the sequence

$$(3.3.5)\quad \begin{cases} H_{n+2}P_n X_\lambda \to H_{n+2}P_{n-1}X_\lambda \to H_1(\pi_1 X_\lambda; \pi_n X_\lambda) \to H_{n+1}P_n X_\lambda \\ \qquad \to H_{n+1}P_{n-1}X_\lambda \to H_0(\pi_1 X_\lambda; \pi_n X_\lambda) \to H_n X_\lambda \to H_n P_{n-1} X_\lambda \to 0 \end{cases}$$

is exact.

Observe that for each λ in Λ, $H_{n+1}X_\lambda \to H_{n+1}P_n X_\lambda$ is an epimorphism; hence, $H_{n+1}\underline{X} \to H_{n+1}P_n\underline{X}$ is an epimorphism. It follows that $H_{n+1}P_n\underline{X} \to H_{n+1}P_n\underline{Y}$ is an epimorphism since $H_{n+1}\underline{X} \to H_{n+1}\underline{Y}$ is an epimorphism.

Since (3.3.5) is exact, the corresponding sequences of pro-groups for $\underline{X}$ and $\underline{Y}$ are exact. Form a diagram whose rows are these exact sequences of pro-groups for $\underline{X}$ and $\underline{Y}$ and whose vertical maps are induced by $\underline{f}$. Our proof is finished by the "Five Lemma"; see (3.6.6). □

(3.4) Lemma. *Suppose* $\underline{f}$: $\underline{X} \to \underline{Y}$ *satisfies the hypotheses of Lemma* (3.3.1). *Then the following conclusion holds*:

($\hat{\mathrm{C}}_n$: $n \geqslant 1$) *The induced map* $\hat{\Gamma}_\omega \pi_n \underline{f}$: $\hat{\Gamma}_\omega \pi_n \underline{X} \to \hat{\Gamma}_\omega \pi_n \underline{Y}$ *is an isomorphism.*

Proof. Observe that $n \geqslant 1$ is some fixed integer. We assume (C_1) or (C_n: $n \geqslant 2$) holds: In either case, we conclude that the map $\pi_n\underline{X}/\Gamma_i\pi_n\underline{X} \to \pi_n\underline{Y}/\Gamma_i\pi_n\underline{Y}$, induced by $\underline{f}$ is an isomorphism. This follows from Theorem (3.3.1) or Theorem (4.2) of **[SI$_1$]**, respectively. This suffices to prove the result. □

(3.4.1) Remark. In the setting described above, for each $n \geqslant 1$, the conclusion (C_n) implies ($\hat{\mathrm{C}}_n$).

(3.5) Lemma. *Suppose* $\underline{f}$: $\underline{X} \to \underline{Y}$ *satisfies the hypotheses of Lemma* (3.3.1). *Then the following assertions are equivalent*:

(a) *The map* $\pi_n\underline{f}$ *is an isomorphism.*

(b) $\Gamma_\omega\pi_n\underline{f}$ *is an epimorphism*, $\Gamma'_\omega\pi_n\underline{f}$ *is a monomorphism, and* $\Gamma\pi_n\underline{f}$ *is a monomorphism.*

Proof. It suffices to show that (b) implies (a). Consider the commutative diagram

$$(3.5.1)\quad \begin{array}{ccccccccc} 0 & \to & \Gamma_\omega\pi_n\underline{X} & \to & \pi_n\underline{X} & \to & \hat{\Gamma}_\omega\pi_n\underline{X} & \to & \Gamma'_\omega\pi_n\underline{X} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \Gamma_\omega\pi_n\underline{Y} & \to & \pi_n\underline{Y} & \to & \hat{\Gamma}_\omega\pi_n\underline{Y} & \to & \Gamma'_\omega\pi_n\underline{Y} & \to & 0. \end{array}$$

The unlabelled maps $\Gamma_\omega\pi_n\underline{f}$, $\hat{\Gamma}_\omega\pi_n\underline{f}$, and $\Gamma'_\omega\pi_n\underline{f}$ are an epimorphism, an isomorphism, and a monomorphism, respectively. It follows from a "Five Lemma" (this is discussed in (3.6)) that $\pi_n\underline{f}$ is an epimorphism. In the next few paragraphs, we shall show that $\pi_n\underline{f}$ is a monomorphism.

It is easy to see that $\pi_n\underline{X}/\Gamma_\omega\pi_n\underline{X} \to \pi_n\underline{Y}/\Gamma_\omega\pi_n\underline{Y}$ is an epimorphism since $\pi_n\underline{X} \to \pi_n\underline{Y}$ is an epimorphism. It follows from (3.5.1) that both $\pi_n\underline{X}/\Gamma_\omega\pi_n\underline{X} \to \hat{\Gamma}_\omega\pi_n\underline{X}$ and $\pi_n\underline{Y}/\Gamma_\omega\pi_n\underline{Y} \to \hat{\Gamma}_\omega\pi_n\underline{Y}$ are monomorphisms, and $\hat{\Gamma}_\omega\pi_n\underline{f}$ is an isomorphism by Lemma (3.4). It is now easy to see that the map $\pi_n\underline{X}/\Gamma_\omega\pi_n\underline{X} \to \pi_n\underline{Y}/\Gamma_\omega\pi_n\underline{Y}$ is a monomorphism; hence, it is an isomorphism. By Lemma (3.3.2) of

[**SI₁**] and the argument given above at the limit ordinals, the induced map $\pi_n \underline{X}/\Gamma_\alpha \pi_n \underline{X} \to \pi_n \underline{Y}/\Gamma_\alpha \pi_n \underline{Y}$ is an isomorphism for any ordinal α.

Observe that for cardinality reasons, there exists an ordinal α such that $\Gamma\pi_n\underline{X} = \Gamma_\alpha \pi_n \underline{X}$. Consider a diagram of pro-groups whose rows are $0 \to \Gamma\pi_n\underline{X} \to \pi_n\underline{X} \to \pi_n\underline{X}/\Gamma\pi_n\underline{X} \to 0$ and consider a similar one for $\underline{Y}$. The vertical maps of this diagram are induced by $\underline{f}$. By the "Five Lemma" (see (3.6.6)) it follows that $\pi_n \underline{f}$ is a monomorphism; hence, $\pi_n \underline{f}$ is a bimorphism. This proves $\pi_n \underline{f}$ is an isomorphism; see [**MA**]. □

(3.6) *The setting for a "Five Lemma"*. Given the following diagram

$$(3.6.1)\qquad \begin{array}{ccccccccc} 0 & \to & \underline{H} & \xrightarrow{\underline{j}} & \underline{G} & \xrightarrow{\underline{i}} & \underline{K} & \xrightarrow{\underline{f}} & \underline{L} & \to & 0 \\ & & \downarrow\underline{\alpha} & & \downarrow\underline{\beta} & & \downarrow\underline{\gamma} & & \downarrow\underline{\delta} & & \\ 0 & \to & \underline{H}' & \xrightarrow{\underline{j}'} & \underline{G}' & \xrightarrow{\underline{i}'} & \underline{K}' & \xrightarrow{\underline{f}'} & \underline{L}' & \to & 0 \end{array}$$

such that for each λ in Λ we have a commutative diagram

$$(3.6.2)\qquad \begin{array}{ccccccccc} 0 & \to & H_\lambda & \xrightarrow{j_\lambda} & G_\lambda & \xrightarrow{i_\lambda} & K_\lambda & \xrightarrow{f_\lambda} & L_\lambda & \to & 0 \\ & & \downarrow\alpha_\lambda & & \downarrow\beta_\lambda & & \downarrow\gamma_\lambda & & \downarrow\delta_\lambda & & \\ 0 & \to & H'_\lambda & \xrightarrow{j'_\lambda} & G'_\lambda & \xrightarrow{i'_\lambda} & K'_\lambda & \xrightarrow{f'_\lambda} & L'_\lambda & \to & 0 \end{array}$$

such that each row of (3.6.2) is exact; furthermore, the maps $\underline{\alpha}$, $\underline{\beta}$, and $\underline{\gamma}$ are maps of pro-groups in pro-$\mathcal{G}$, and $\underline{\delta}$ is a map of pro-(pointed sets). More specifically, $L_\lambda = K_\lambda / i_\lambda G_\lambda$ and $L'_\lambda = K'_\lambda / i'_\lambda G'_\lambda$ for each λ in Λ; and the maps f_λ and f'_λ are the quotient maps. Notation:

$$\begin{aligned} &\underline{G} = (G_\lambda, p_{\lambda\lambda'}, \Lambda), \quad \underline{G}' = (G'_\lambda, p'_{\lambda\lambda'}, \Lambda), \quad \underline{K} = (K_\lambda, q_{\lambda\lambda'}, \Lambda), \\ &\underline{K}' = (K'_\lambda, q'_{\lambda\lambda'}, \Lambda), \quad \underline{H} = (H_\lambda, p_{\lambda\lambda'}, \Lambda), \quad \underline{H}' = (H'_\lambda, p'_{\lambda\lambda'}, \Lambda), \\ &\underline{L} = (L_\lambda, r_{\lambda\lambda'}, \Lambda) \quad \text{and} \quad \underline{L}' = (L'_\lambda, r'_{\lambda\lambda'}, \Lambda). \end{aligned}$$

Moreover, for each λ in Λ, the homomorphisms j_λ and j'_λ are inclusion; and, therefore, we have again denoted the restrictions of the homomorphisms $p_{\lambda\lambda'}$ and $p'_{\lambda\lambda'}$ by $p_{\lambda\lambda'}$ and $p'_{\lambda\lambda'}$.

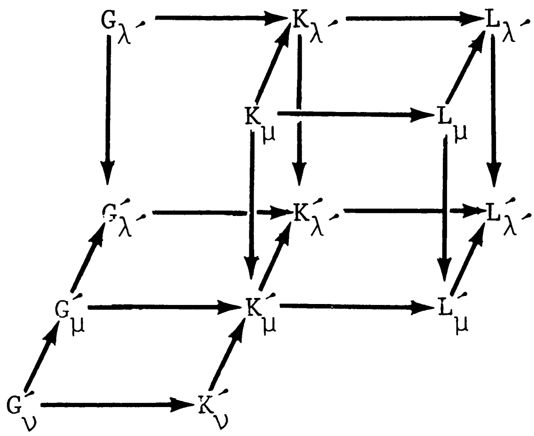
(3.6.3) LEMMA ("A FIVE LEMMA"). *In the setting: If $\underline{\alpha}$ is an epimorphism, $\underline{\gamma}$ is an epimorphism, and $\underline{\delta}$ is a monomorphism, then $\underline{\beta}$ is an epimorphism.*

PROOF. Given an index λ in Λ. (I) Since $\underline{\alpha}$ is an epimorphism, there exists $\lambda' \geqslant \lambda$ such that the image $\operatorname{Im}(H'_{\lambda'} \to H'_\lambda)$ is contained in the image $\operatorname{Im}(H_\lambda \to H'_\lambda)$. (II) Given $\lambda' \geqslant \lambda$ as asserted, choose $\mu \geqslant \lambda'$ such that the induced map between the kernels, $[\operatorname{Ker}(L_\mu \to L'_\mu) \to \operatorname{Ker}(L_{\lambda'} \to L'_{\lambda'})]$, is the zero map; this follows since $\underline{\delta}$ is a monomorphism. (III) Choose $\nu \geqslant \mu$ such that $\operatorname{Im}(K'_\nu \to K'_\mu)$ is contained in $\operatorname{Im}(K_\mu \to K'_\mu)$.

Claim. The image $\operatorname{Im}(G'_\nu \to G'_\lambda)$ is contained in the image $\operatorname{Im}(G_\lambda \to G'_\lambda)$.

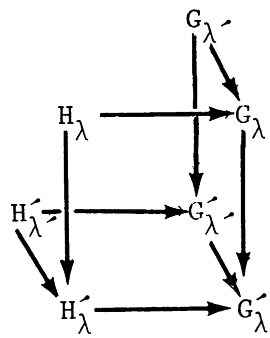
We next prove this claim by considering the following commutative diagrams:

(3.6.4)



and

(3.6.5)



Suppose an element g'_ν of G'_ν goes to an element $g'_{\lambda'}$ in $G'_{\lambda'}$ in (3.6.4). By a chase of (3.6.4) along with utilizing (II) and (III), we conclude that there exists an element $g_{\lambda'}$ in $G_{\lambda'}$ whose image $x'_{\lambda'}$ in $G'_{\lambda'}$ has the property that $(x'_{\lambda'})^{-1}g_{\lambda'}$ belongs to $H'_{\lambda'}$. Now chase the diagram (3.6.4) and use (I) to conclude that there exists an element h_λ in H_λ whose image g'_λ in G'_λ equals the image $(x'_\lambda)^{-1}g'_\lambda$ in G'_λ of $(x'_{\lambda'})^{-1}g'_{\lambda'}$. Hence, the elements x'_λ and $(x'_\lambda)^{-1}g'_\lambda$ are in $\operatorname{Im}(G_\lambda \to G'_\lambda)$. This means $g'_\lambda = x'_\lambda(x'_\lambda)^{-1}g'_\lambda$ is in $\operatorname{Im}(G_\lambda \to G'_\lambda)$. This proves our claim and suffices to prove that $\underline{\beta}$ is an epimorphism (cf. [**MA**]). □

(3.6.6) *Remarks on the "Five Lemma"*. The category of pro-(abelian groups) is abelian [**AM**]; hence, one may assume the "Five Lemma" while working in this category. In pro-$\mathcal{G}$, the various versions of this lemma require some proof: Although we have not studied all the possible versions, we have proved (for lack of better reference concerning these matters) a "Weak Five Lemma" in pro-$\mathcal{G}$ (see [**SI$_1$**]), along with the result of (3.6) which suffices for our applications.

For convenience of reference, we shall summarize the main results of this section in the following section.

4. A summary of the main results.

(4.1) *The setting*. Throughout the following we let $f\colon \underline{X} \to \underline{Y}$ denote a map in $\mathcal{C}$. We say $\underline{f}$ is a *weak pro-homotopy equivalence* if $\underline{f}$ induces isomorphisms of all homotopy pro-groups of $\underline{X}$ and $\underline{Y}$. All homology is considered with coefficients in $\mathfrak{Z}$. The map $\underline{f}$ is called a *pro-homology equivalence* if $\underline{f}$ induces isomorphisms of all the homology pro-groups of $\underline{X}$ and $\underline{Y}$. We have proved the following results.

(4.1.1) THEOREM. *Given a map* $\underline{f}\colon \underline{X} \to \underline{Y}$ *in* $\mathcal{C}$. *Suppose* (a) $\pi_i\underline{f}\colon \pi_i\underline{X} \to \pi_i\underline{Y}$ *is an isomorphism for* $0 \leqslant i \leqslant (n-1)$, (b) *the induced map* $H_n\underline{f}\colon H_n\underline{X} \to H_n\underline{Y}$ *of the homology pro-groups is an isomorphism, and* (c) *the induced map* $H_{n+1}\underline{f}\colon H_{n+1}\underline{X} \to H_{n+1}\underline{Y}$ *is an epimorphism. Then the induced map* $\hat{\Gamma}_\omega\pi_n\underline{f}\colon \hat{\Gamma}_\omega\pi_n\underline{X} \to \hat{\Gamma}_\omega\pi_n\underline{Y}$ *is an isomorphism.*

(4.1.2) THEOREM. *Suppose* $\underline{f}\colon \underline{X} \to \underline{Y}$ *satisfies the hypotheses of Theorem* (4.1.1), *and suppose* $\underline{f}$ *satisfies the following additional hypotheses*:

(a) $\Gamma_\omega\pi_n\underline{f}\colon \Gamma_\omega\pi_n\underline{X} \to \Gamma_\omega\pi_n\underline{Y}$ *is an epimorphism*,

(b) $\Gamma'_\omega\pi_n\underline{f}\colon \Gamma'_\omega\pi_n\underline{X} \to \Gamma'_\omega\pi_n\underline{Y}$ *is a monomorphism, and*

(c) $\Gamma\pi_n\underline{f}\colon \Gamma\pi_n\underline{X} \to \Gamma\pi_n\underline{Y}$ *is a monomorphism.*

Then $\pi_n\underline{f}\colon \pi_n\underline{X} \to \pi_n\underline{Y}$ *is an isomorphism.*

(4.1.3) THEOREM. *Suppose* $\underline{f}\colon \underline{X} \to \underline{Y}$ *is a pro-homology equivalence*; *see* (4.1) *for terminology. Then the following are equivalent*:

(a) $\underline{f}$ *is a weak pro-homotopy equivalence.*

(b) *For all* i, $1 \leqslant i < \infty$, $\Gamma_\omega\pi_i\underline{f}$ *is an epimorphism*, $\Gamma'_\omega\pi_i\underline{f}$ *is a monomorphism, and* $\Gamma\pi_i\underline{f}$ *is a monomorphism.*

(4.1.4) REMARK. These theorems are the exact analogues of Dror's results (see [**DR**, Theorem (3.1) and Proposition (3.2)]).

(4.2) *Some calculations*. Suppose $\underline{X}$ is an object of $\mathcal{C}$. It is easy to see that for any i, $1 \leqslant i < \infty$, we have $\Gamma_\omega\pi_i\underline{X} \approx \Gamma'_\omega\pi_i\underline{X} \approx \Gamma\pi_i\underline{X} \approx 0$ in any of the following cases:

(4.2.1) $\pi_1\underline{X} \approx 0$;

(4.2.2) $\underline{X}$ is simple (see (3.2.3) for a definition);

(4.2.3) $\underline{X}$ is nilpotent (see (3.2.3) for a definition);

(4.2.4) $\underline{X}$ is complete (see (3.2.4) for a definition).

(4.3) A COROLLARY OF THEOREM (4.1.3). *Given a map* $\underline{f}\colon \underline{X} \to \underline{Y}$ *in* $\mathcal{C}$. *Suppose any one of the following holds*: $\pi_1\underline{X} \approx \pi_1\underline{Y} \approx 0$; $\underline{X}$ *and* $\underline{Y}$ *are simple*; $\underline{X}$ *and* $\underline{Y}$ *are nilpotent*; *or* $\underline{X}$ *and* $\underline{Y}$ *are complete. Then the following are equivalent*:

(i) $\underline{f}$ *is a pro-homology equivalence.*

(ii) $\underline{f}$ *is a weak pro-homotopy equivalence.*

(4.4) $\underline{H}$*-structures in* $\mathcal{C}$. The notion of an H-space or a space with an H-structure is well known in homotopy theory; moreover, Eckmann and Hilton [**EH**] have discussed an analogous notion of $\underline{H}$-structure on objects of suitable categories. In our earlier work [**SI$_2$**], we have concretized the notion of $\underline{H}$-structure in suitable pro-homotopy categories; indeed, it follows from Theorem (4.2.1) of [**SI$_2$**] that an object $\underline{X}$ of pro-$\mathcal{H}\mathcal{C}\mathcal{W}_0$ with an $\underline{H}$-structure is simple.

(4.5) *Shape theoretic considerations.* A pointed topological space X is called *s-simple* ("shape simple"), s-nilpotent, or s-complete if there exists an object $\underline{X}$ of pro-$\mathcal{H}\mathcal{C}\mathcal{W}_0$ associated with X in the sense of Morita (cf. **[MA]**, **[DS]**) such that $\underline{X}$ is simple, nilpotent, or complete; see (3.2). We shall be brief; see **[SI$_2$]** for a discussion of many related matters. There are many versions of the Whitehead theorem in shape theory, for instance, **[MA]**, **[MO]**, **[MR]**; moreover, the books **[BO]**, **[DS]**, **[ED]** contain many other references and related discussions. As a sample, we shall next state an analogue of the Dror-Whitehead theorem in shape theory.

(4.5.1) THEOREM. *Suppose $\underline{f}$: $X \to Y$ is a shape map (or a shaping) of pointed continua of finite fundamental dimension and suppose X and Y are s-simple or, more generally, s-nilpotent. Then the following are equivalent:*

(a) *$\underline{f}$ is a shape equivalence.*

(b) *$\underline{f}$ induces isomorphisms of all the homotopy pro-groups.*

(c) *$\underline{f}$ induces isomorphisms of the homology pro-groups with coefficients in $\mathfrak{Z}$.*

PROOF. The equivalence of (a) and (b) is well known (cf. **[DS]**). It follows from Corollary (4.3) that (c) implies (b). This proves the theorem. □

REFERENCES

[AM] M. Artin and B. Mazur, *Étale homotopy*, Lecture Notes in Math., vol. 100, Springer, Berlin and New York, 1969.

[BK] A. K. Bousfield and D. M. Kan, *Homotopy limits, completions and localizations*, Lecture Notes in Math., vol. 304, Springer, New York, 1972.

[BO] K. Borsuk, *Theory of shape*, Monografie Mat., no. 59, PWN, Warsaw, 1975.

[DR] E. Dror, *A generalization of the Whitehead theorem*, Lecture Notes in Math., vol. 249, Springer, Berlin and New York, 1971, pp. 13–22.

[DS] J. Dydak and J. Segal, *Shape theory*, Lecture Notes in Math., vol. 688, Springer, Berlin and New York, 1978.

[ED] D. A. Edwards and H. M. Hastings, *Čech and Steenrod homotopy theories with applications to geometric topology*, Lecture Notes in Math., vol. 542, Springer, New York, 1976.

[EH] B. Eckmann and P. Hilton, *Group-like structures in general cattegories*. I: *Multiplications and comultiplications*, Math. Ann. **145** (1962), 227–255.

[EW] D. A. Edwards, *Étale homotopy theory and shape*, Algebraic and Geometric Methods in Topology (edited by L. McAuley), Lecture Notes in Math., vol. 428, Springer, Berlin and New York, 1974, pp. 58–107.

[GR] A. Grothendieck, *Technique de decene* . . . , Sém. Bourbaki, 12 année, 1959–60, Exp. 195, pp. 1–22.

[HI] P. Hilton, G. Mislin and J. Roitberg, *Localization of nilpotent groups and spaces*, North-Holland Math. Studies, North-Holland, Amsterdam; American Elsevier, New York, 1975.

[HU] S. T. Hu, *Homotopy theory*, Academic Press, New York, 1959.

[MA] S. Mardešić, *On the Whitehead theorem in shape theory*. I, II, Fund. Math. **91** (1976), 51–64; 93–103.

[ML] S. Mac Lane, *Homology*, Springer, New York, 1975.

[MO] M. Moszyńska, *The Whitehead theorem in the theory of shape*, Fund. Math. **80** (1973), 221–263.

[MR] K. Morita, *The Hurewicz and Whitehead theorems in shape theory*, Sci. Rep. Tokyo Kyoiku Daigaku Sect. A **12** (1974), 246–258.

[PO] M. M. Postnikov, *Localization of topological spaces*, Uspehi Mat. Nauk **32** (6) (1977), 117–181 = Russian Math. Surveys **32** (6) (1977), 121–184.

[RA] M. Raussen, *Hurewicz and Whitehead Theorems in pro-categories*, Arch. Math. **30** (1978), 153–164.

[SI$_1$] S. Singh, *Pro-homology and isomorphisms of pro-groups*, J. Pure Appl. Algebra (to appear).

[SI$_2$] ______, *Mittag-Leffler for homotopy pro-groups of movable continua*, Glas. Mat. Ser. III (to appear).

[SP] E. H. Spanier, *Algebraic topology*, McGraw-Hill, New York, 1966.

[ST] J. R. Stallings, *Homology and central series of groups*, J. Algebra **2** (1965), 170–181.

[WH] G. W. Whitehead, *Elements of homotopy theory*, Graduate Texts in Math., vol. 61, Springer, New York, 1978.

DEPARTMENT OF MATHEMATICS, THE PENNSYLVANIA STATE UNIVERSITY, ALTOONA, PENNSYLVANIA 16603