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ON THE FULLNESS OF SURJECTIVE MAPS OF AN INTERVAL

BY

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ABSTRACT. Let $I = [0, 1]$, $\mathfrak{B}$ = Lebesgue measurable subsets of $[0, 1]$, and let λ denote the Lebesgue measure on $(I, \mathfrak{B})$. Let $\tau: I \to I$ be measurable and surjective. We say τ is full, if for all $A \in \mathfrak{B}$, $\lambda(A) > 0$, $\tau(A), \tau^2(A), \ldots,$ measurable, the condition

(1) $$\lim_{n\to\infty} \lambda(\tau^n(A)) = 1$$

holds. We say τ is interval full if (1) holds for any interval $A \subset I$. In this note, we give an example of $\tau: I \to I$ which is continuous and interval full, but not full. We also show that for a class of transformations τ satisfying Renyi's condition, interval fullness implies fullness. Finally, we show that fullness is not preserved under limits on the surjections.

1. Introduction. Let I be $[0, 1]$, $\mathfrak{B}$ be the set of Lebesgue measurable subsets of $[0, 1]$ and let μ be a measure on $(I, \mathfrak{B})$. We say $\tau: I \to I$ is an endomorphism if it is surjective (onto) and μ is invariant under τ, i.e., $\mu(\tau^{-1}(A)) = \mu(A)$, $A \in \mathfrak{B}$. An endomorphism τ is (μ-exact) *exact* [**1**] if and only if for every set $A \in \mathfrak{B}$ of μ-positive measure with measurable images $\tau(A), \tau^2(A), \ldots,$ the relationship

(1) $$\lim_{n\to\infty} \mu(\tau^n(A)) = 1$$

holds. The exactness of an endomorphism τ has interesting and important consequences, among them the property that τ is mixing of all degrees [**1**]. Among examples of exact endomorphisms are the Renyi maps [**1**] of the form

$$\tau(x) = p(x) \mod(1),$$

where $p(x)$ is continuous, strictly monotonic and satisfies certain slope and end-point conditions. Other examples are given in [**2**].

The notion of the images $\{\tau^n(A)\}$ expanding to the entire space can be stated without requiring the existence of an invariant measure μ.

DEFINITION 1. Let λ denote the Lebesgue measure on $(I, \mathfrak{B})$ and let $\tau: I \to I$ be measurable and surjective. We shall say τ is *full*, if whenever $A \in \mathfrak{B}$, $\lambda(A) > 0$, and $\tau(A), \tau^2(A), \ldots,$ are measurable, the condition

(2) $$\lim_{n\to\infty} \lambda(\tau^n(A)) = 1$$

holds; and it is *interval full* if (2) holds for any interval $A \subset I$.

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In [3] it was shown that certain Markov maps, not necessarily piecewise linear, are interval full, and the full measure is attained in a finite number of iterations. By a Markov map we mean a map which is piecewise continuous on a partition

$$a_0 < a_1 < \cdots < a_{n-1} < a_n$$

of [0, 1] such that for $i = 0, 1, \ldots, n-1$, $\tau_i = \tau|_{(a_{i-1}, a_i)}$ is a homeomorphism onto some interval $(a_{j(i)}, a_{k(i)})$.

An obvious question that arises from the results in [3] is: are these interval full maps also full? We shall prove that this is true for maps satisfying the Renyi condition (§3). However, this is not true in general. In §2 we construct a continuous transformation which is interval full but not full. From this it follows that, in general, μ-exactness is not a consequence of interval μ-exactness, where by interval μ-exactness we mean that (1) holds for intervals $A \in \mathfrak{B}$.

Finally, in §4, we present an example in which τ_n, $n \geqslant 1$, is full, and exact but $\tau(x) = \lim_{n\to\infty} \tau_n(x)$ is neither full nor exact.

2. Interval fullness does not imply fullness. First, we construct a Cantor set C_δ of measure $1 - \delta$ by systematically removing open intervals in the same way as the ordinary Cantor set C_1 is constructed. Let

E_1 = open interval of length $\delta/3$ centered at $1/2$.

E_2 = 2 open intervals each of length $\delta/3^2$ centered at the respective midpoints of the complement E_1^c of E_1.

In general,

$E_n = 2^{n-1}$ open intervals each of length $\delta/3^n$ centered at the midpoints of the 2^{n-1} components of $(E_1 \cup E_2 \cup \ldots \cup E_{n-1})^c$. Then, it can be easily shown that $C_\delta = \bigcap_{n=1}^\infty E_n^c$ has Lebesgue measure $1 - \delta$.

We now introduce the following notation for the individual deleted intervals. Let

$E_1 = I(1)$,

$E_2 = I(0, 1) \cup I(2, 1)$,

$\vdots$

$E_n = \bigcup I(a_1, a_2, \ldots, a_{n-1}, 1)$ where the union is over all $(n-1)$-tuples with $a_i \in \{0, 2\}$.

In effect, we are setting up a 1-1 correspondence (which can be extended to a homeomorphism) of C_δ and C_1 (see Lemma 1) between the deleted intervals of C_δ and those of C_1: if $\delta = 1$, $I(a_1, a_2, \ldots, a_{n-1}, 1) = \{x \in [0, 1]: .a_1a_2 \cdots a_{n-1}1 < x < .a_1a_2 \cdots a_{n-1}2\}$ where the expansions are base 3-expansions. For $0 < \delta < 1$, the above labelling of intervals means:

(1) if π is an n-tuple, then $I(\pi) \subset E_n$, i.e., $I(\pi)$ is removed at the nth step;

(2) if $\pi = (a_1, a_2, \ldots, a_{n-1}, 1)$, $\sigma = (b_1, b_2, \ldots, b_{m-1}, 1)$, then $I(\pi) < I(\sigma)$ (i.e., $\forall\, x \in I(\pi), \forall\, y \in I(\sigma), x < y$) if and only if

$$.a_1a_2 \cdots a_{n-1}1 < .b_1b_2 \cdots b_{m-1}1 \quad \text{(base 3)}.$$

We now define a transformation τ_δ: $[0, 1] \to [0, 1]$ in two steps.

On $I(1)$, define τ_δ to be linear on $[(3 - \delta/6), 1/2]$ satisfying $\tau_\delta((3 - \delta)/6) = 1$ and $\tau_\delta(1/2) = 0$; and on $[1/2, (3 + \delta)/6]$, τ is linear, satisfying $\tau_\delta(1/2) = 0$ and

$\tau_\delta((3+\delta)/6) = 1$. On the deleted interval $I(\pi)$, define τ_δ to be linear and such that:

(i) $\tau_\delta(I(0, a_2, a_3, \ldots, a_{n-1}, 1)) = I(a_2, a_3, \ldots, a_{n-1}, 1)$ with the slope positive,

(ii) $\tau_\delta(I(2, a_2, a_3, \ldots, a_{n-1}, 1)) = I(2 - a_2, 2 - a_3, \ldots, 2 - a_{n-1}, 1)$ with the slope negative.

Now τ_δ is defined on $\bigcup_{n=1}^{\infty} E_n$ and is continuous since this set is open and τ_δ is locally linear; moreover $\tau_\delta(1 - x) = \tau_\delta(x)$.

It is easy to see that there is a unique extension of τ_δ to a continuous function (which we also call τ_δ) on $[0, 1]$ satisfying $\tau_\delta(1 - x) = \tau_\delta(x)$. Let $\alpha = \frac{1}{2} - \frac{\delta}{6}$ and $D = (\bigcup_{n=1}^{\infty} E_n) \cap [0, \alpha]$. Since τ_δ is strictly increasing on $[0, \alpha]$, $\tau_\delta|_{[0,\alpha]}$ is a homeomorphism.

THEOREM 1. (1) $\tau_\delta(C_\delta) \subset C_\delta$.

(2) *For any open interval J, there exists an integer n such that $\tau_\delta^n(J) = I$ (i.e., interval fullness in a finite number of iterations).*

PROOF. (1) This follows from the fact that $\tau_\delta|_{[0,\alpha]}$ is a homeomorphism, $\tau_\delta(D) = [0, 1] - C_\delta$, and the symmetry of C_δ and τ_δ about $x = \frac{1}{2}$.

(2) Note that $|d\tau_\delta/dx| \geqslant 3$ on C_δ^c. Let J be any open interval.

Case (a). $J \cap C_\delta \neq \varnothing$. This implies the existence of an n-tuple π such that $I(\pi) \subset J$, since $J \cap C_\delta$ contains infinitely many points and any two points of C_δ are separated by some $I(\pi)$. Then $\tau_\delta^{n-1}(J) \supset I(1)$, and $\tau_\delta^n(J) = [0, 1]$.

Case (b). $J \cap C_\delta = \varnothing$, i.e., $J \subset I(\pi)$ for some n-tuple π. Since $\lambda(\tau_\delta(J)) \geqslant 3\lambda(J)$, where λ is the Lebesgue measure, it follows that the sequence $J, \tau_\delta(J), \tau_\delta^2(J), \ldots$ keeps expanding until case (a) is satisfied, i.e., for some m, $\tau_\delta^m(J) \cap C_\delta \neq \varnothing$, and the result follows. Q.E.D.

We note that

$$\tau_1(x) = \begin{cases} 3x, & 0 \leqslant x \leqslant \frac{1}{3}, \\ 3 - 6x, & \frac{1}{3} < x \leqslant \frac{1}{2}, \\ \tau_1(1 - x), & \frac{1}{2} < x \leqslant 1. \end{cases}$$

We now construct a family of endomorphisms which are interval μ-exact but not exact.

LEMMA 1. *For each $0 < \delta \leqslant 1$ there exists a homeomorphism h_δ: $[0, 1] \to [0, 1]$ such that $h_\delta(C_1) = C_\delta$. Moreover,*

$$h_\delta \circ \tau_1 = \tau_\delta \circ h_\delta.$$

PROOF. We define h_δ by requiring that it map each deleted interval in $[0, 1] - C_1$ linearly onto the corresponding interval in $[0, 1] - C_\delta$. Clearly there is a unique continuous extension to all of $[0, 1]$ which is strictly increasing, and hence a homeomorphism which maps C_1 onto C_δ. By construction, $h_\delta \circ \tau_1 = \tau_\delta \circ h_\delta$ on each deleted interval, and hence on $[0, 1]$. Q.E.D.

LEMMA 2. *For each* $0 < \delta \leqslant 1$ *there exist measures* η_δ *and* υ_δ, *invariant under* τ_δ, *and supported on* C_δ *and* $[0, 1] - C_\delta$, *respectively. Moreover,* υ_1 *can be chosen to be absolutely continuous with respect to Lebesgue measure.*

PROOF. We will construct η_1 and υ_1 invariant under τ_1 and then define the induced measures η_δ and υ_δ by:

$$\eta_\delta(A) = \eta_1\big(h_\delta^{-1}(A)\big) \quad \text{and} \quad \upsilon_\delta(A) = \upsilon_1\big(h_\delta^{-1}(A)\big).$$

The invariance of η_δ and υ_δ under τ_δ follows from the invariance of η_1 and υ_1 under τ_1 and Lemma 1:

$$\mu_\delta\big(\tau_\delta^{-1}A\big) = \mu_1\big(h_\delta^{-1}\tau_\delta^{-1}A\big) = \mu_1\big(\tau_1^{-1}h_\delta^{-1}A\big) = \mu_1\big(h_\delta^{-1}A\big) = \mu_\delta(A),$$

where μ_δ represents η_δ or υ_δ.

Let us define $\eta_1([0, x]) = \psi(x)$, where ψ is the Cantor function. This defines η_1 uniquely on the Lebesgue measurable subsets of $[0, 1]$, and if $A \subset [0, 1] - C_1$, $\eta_1(A) = 0$. To show that η_1 is τ_1-invariant, it is enough to show that

$$\eta_1\big(\tau_1^{-1}[0, x]\big) = \eta_1\big([0, x]\big) = \psi(x).$$

But $\tau_1^{-1}([0, x]) = [0, \frac{1}{3}x] \cup [1 - \frac{1}{3}x, 1] \cup B$, where $B \subset [\frac{1}{3}, \frac{2}{3}]$. Since $\eta_1[1 - \frac{1}{3}x, 1] = \eta_1[0, \frac{1}{3}x] = \psi(\frac{1}{3}x)$ and $\eta_1(B) = 0$, we have

$$\eta_1\big(\tau_1^{-1}[0, x]\big) = 2\psi\big(\tfrac{1}{3}x\big) = \psi(x).$$

To define υ_1, let $U = \bigcup_{n=1}^{\infty} E_n = [0, 1] - C_1$. For each deleted interval $I \subset U$, let $V_I = \tau_1^{-1}(I) \cap (\frac{1}{3}, \frac{2}{3})$. Clearly, $V_I \cap V_J = \varnothing$ if $I \cap J = \varnothing$. Now for each I except E_1 let

$$g_I(x) = \frac{2^{-2n+1}}{\lambda(I)}\chi_I(x),$$

where χ_I is the characteristic function of the set I and $n \geqslant 2$ is uniquely determined by $I \subset E_n$. For each I (including E_1) let

$$g_{V_I}(x) = 2^{-2n}\chi_{V_I}(x)/\lambda(V_I).$$

Then

$$g(x) = \sum_{\substack{I \subset U \\ I \neq E_1}} g_I(x) + \sum_{\text{all } I} g_{V_I}(x)$$

is well defined and measurable. For any given x at most one term in each sum will be nonzero. Furthermore,

$$\int_0^{1/3} g(x)\,dx = \int_{2/3}^1 g(x)\,dx = \sum_{n=2}^{\infty} 2^{n-2}2^{-2n+1} = \frac{1}{4}$$

and

$$\int_{1/3}^{2/3} g(x)\,dx = \sum_I \int g_{V_I}(x)\,dx = \sum_{n=1}^{\infty} 2^{n-1}2^{-2n} = \frac{1}{2}.$$

Therefore, $g(x)$ is a density function. Let

$$\upsilon_1(A) = \int_A g(x)\,dx.$$

Then the support of $v_1 \subset U$; and if $I \neq E_1$, $v_1(I) = 2/4^n$, where $I \subset E_n$ determines an $n \geqslant 2$. Since $\tau_1^{-1}(I)$ consists of V_I together with two intervals of E_{n+1},

$$v_1(\tau_1^{-1}(I)) = \frac{2}{4^{n+1}} + \frac{2}{4^{n+1}} + \frac{1}{4^n} = \frac{2}{4^n}.$$

This last equality holds also for $n = 1$, i.e., for E_1, while

$$v_1(E_1) = \int_{1/3}^{2/3} g(x)\,dx = \frac{1}{2}.$$

Thus $v_1(\tau_1^{-1}(I)) = v_1(I)$ for every I.

Now let A be a measurable subset of U. To show that $v_1(\tau_1^{-1}A) = v_1(A)$, it is sufficient to show that $v_1(\tau_1^{-1}(A \cap I)) = v_1(A \cap I)$ for each I. Thus we may assume $A \subset I$. If $I \neq E_1$, the result follows from the facts that (i) $v_1(\tau_1^{-1}(I)) = v_1(I)$, (ii) g is constant and τ_1 is linear on each component of $\tau_1^{-1}(I)$. If $I = E_1$, let $A = A_1 \cup A_0$, where $A_1 = A \cap \bigcup_I V_I$ and

$$A_0 = A - A_1 \subset E_1 - \bigcup_I V_I = E_1 \cap \tau_1^{-1}C_1.$$

We note that $v_1(A_0) = 0$ since g vanishes on A_0. Moreover, $\tau_1^{-1}(A_0)$ has Lebesgue measure 0 and hence v_1-measure 0.

We may therefore assume $A = A_1$, i.e., $A \subset \bigcup_I V_I$. Since (i) $v_1(\tau_1^{-1}E_1) = v_1(E_1)$, (ii) g is constant and τ_1 is linear on each component of $\tau_1^{-1}(E_1) \cap [0, \frac{1}{2}]$ and $\tau_1^{-1}(E_1) \cap [\frac{1}{2}, 1]$, and (iii) τ_1 is symmetric about $x = \frac{1}{2}$, the result follows. Q.E.D.

REMARK. The foregoing construction will not work for C_δ, $\delta < 1$, because if $A = E_1 \cap \tau_\delta^{-1}C_\delta$, $v_\delta(A) = 0$, but $\tau_\delta^{-1}A$ consists of eight copies of C_δ, and consequently is a set of positive Lebesgue measure in the support of g. The invariant measure v_δ is not absolutely continuous because the homeomorphism h_δ is not an absolutely continuous function.

THEOREM 2. *Let* $0 < \alpha < 1$, $0 < \delta \leqslant 1$ *and define*

$$\mu_{\alpha,\delta} = \alpha\eta_\delta + (1 - \alpha)v_\delta.$$

Then τ_δ is an endomorphism which is interval $\mu_{\alpha,\delta}$-exact, but not $\mu_{\alpha,\delta}$-exact.

PROOF. Since $0 < \mu_{\alpha,\delta}(C_\delta) < 1$ and $\tau_\delta(C_\delta) \subset C_\delta$, τ_δ is not $\mu_{\alpha,\delta}$-exact. However, since for any interval J, $\tau_\delta^n(J) = [0, 1]$ for some n, τ_δ is interval $\mu_{\alpha,\delta}$-exact. Q.E.D.

3. A class of transformations for which interval fullness implies fullness. We shall need the following result which appears as Theorem 16A in [7].

LEMMA 3. *Let A be a measurable subset of* $[0, 1]$ *with* $\lambda(A) > 0$. *Given* $\varepsilon > 0$, *there exists an open interval U such that* $\lambda(A \cap U) > (1 - \varepsilon)\lambda(U)$.

Let τ: $I \to I$ be piecewise C^1 and Markov, i.e., if $a_0 < a_1 < \cdots < a_p$ is the partition of $[0, 1]$ such that τ is C^1 on (a_i, a_{i+1}), $0 \leqslant i < p$, then τ maps (a_i, a_{i+1}) homeomorphically onto $(a_{j(i)}, a_{k(i)})$. For each $n > 1$, τ^n: $I \to I$ determines a partition $a_0^{(n)} < a_1^{(n)} < \cdots < a_{p(n)}^{(n)}$. Let $I_{n,i} = (a_{i-1}^{(n)}, a_i^{(n)})$ and $\tau_{n,i} = \tau^{(n)}|_{I_{n,i}}$. Put

$$C_{n,i} = \frac{\sup_{t \in I_{n,i}} |\tau'_{n,i}(t)|}{\inf_{t \in I_{n,i}} |\tau'_{n,i}(t)|}, \quad C_n = \max_{1 \leqslant i \leqslant p(n)} C_{n,i},$$

and

$$C = \sup_n C_n.$$

The condition $C = C(\tau) < \infty$ is referred to as Renyi's condition. See [4] for a different but equivalent definition of Renyi's condition. Now, let $\mathcal{C}$ denote the class of transformations which are piecewise C^1, Markov, interval full, and satisfy Renyi's condition. Examples of subclasses of $\mathcal{C}$ are the Renyi transformations discussed in [4], as well as maps satisfying:

$$\delta_1 \delta_2 \ldots \delta_{p(n)}(1 + c) > 1, \tag{4}$$

where

$$\delta_i = \frac{\inf_{t \in I_i}|\tau_i'(t)|}{\sup_{t \in I_i}|\tau_i'(t)|} = \frac{1}{C_{1,i}}; \qquad \tau_i = \tau|_{I_i}, I_i = (a_{i-1}, a_i),$$

$$c = \frac{\min_{i=1,\ldots,p} \lambda(I_i)}{\max_{i=1,\ldots,p} \lambda(I_i)},$$

and the condition that each interval I_i maps onto I eventually. A proof of a more general result than that stated is given in [3]. Clearly, if τ is piecewise linear, $\delta_i = 1$, $i = 1, \ldots, p(n)$, and condition (4) reduces to $(1 + c) > 1$, which is always satisfied.

THEOREM 3. *If $\tau \in \mathcal{C}$, then it is full.*

PROOF. Let $A \subset I$ be measurable with $\lambda(A) > 0$. Let $C = C(\tau)$ be the Renyi bound for τ as defined above, and let

$$N = \min\{j: \tau^j([a_i, a_{i+1}]) = [0, 1], 0 \leqslant i \leqslant p\},$$
$$m = \max\{|\tau'(x)|: x \in [0, 1] - \{a_0, a_1, \ldots, a_p\}\},$$
$$L = \max\{a_{i+1} - a_i: 0 \leqslant i < p\}.$$

Choose $\varepsilon > 0$ and let

$$\eta \leqslant \min\left\{\frac{\varepsilon}{4LCm^N}, \frac{1}{4}\right\}.$$

By Lemma 3 there is an open interval $U = (a, b) \subset [0, 1]$ such that

$$\lambda(U \cap A) > (1 - \eta)\lambda(U).$$

Since τ is interval full, we can choose n such that $\tau^n(U) = [0, 1]$ and $x_1, \ldots, x_l \in U$ with $a < x_1 < \ldots < x_l < b$ so that

(i) τ^n maps (x_i, x_{i+1}) homeomorphically onto some (a_j, a_{j+1}) for each $1 \leqslant i < l$,
(ii) $x_1 - a < \eta(b - a)/4$ and $b - x_l < \eta(b - a)/4$.

This can be done, for example, by first choosing $\gamma, \rho \in U$ so that

$$\gamma - a < \eta(b - a)/2 \quad \text{and} \quad b - \rho < \eta(b - a)/2,$$

then choosing n_1, n_2, n_3 to satisfy $\tau^{n_1}((a, \gamma)) = \tau^{n_2}((\rho, b)) = \tau^{n_3}((\gamma, \rho)) = [0, 1]$. Letting $n = \max\{n_1, n_2, n_3\}$ and $\{x_1, x_2, \ldots, x_l\} = \tau^{-n}(\{a_0, \ldots, a_p\}) \cap U$, (i) and (ii) are clearly satisfied. Now let $V = (x_1, x_l)$. We note that $\lambda(U) < 2\lambda(V)$, and for each $1 \leqslant k < l$, $(x_k, x_{k+1}) \subset I_{n,i}$ for some i. Then

$$\begin{aligned}\lambda(V \cap A) &= \lambda(U \cap A) - \lambda((U - V) \cap A)\\ &> (1 - \eta)\lambda(V) - \lambda(U - V)\\ &> (1 - \eta)\lambda(V) - \eta\lambda(V) = (1 - 2\eta)\lambda(V).\end{aligned}$$

Thus, for some k, $1 \leqslant k < l$,

$$\lambda((x_k, x_{k+1}) \cap A) > (1 - 2\eta)\lambda((x_k, x_{k+1})),$$

since

$$\begin{aligned}\lambda(V \cap A) &= \sum_{k=1}^{l-1} \lambda((x_k, x_{k+1}) \cap A)\\ &> (1 - 2\eta) \sum_{k=1}^{l-1} \lambda((x_k, x_{k+1})).\end{aligned}$$

Let $E = (x_k, x_{k+1}) \cap A$ and $F = (x_k, x_{k+1}) - A$. Then since $\tau^n|_{E \cup F} = \tau_{n,i}$ is a differentiable homeomorphism it follows that

$$\frac{\lambda(\tau^n(F))}{\lambda(\tau^n(E \cup F))} \leqslant \frac{\lambda(\tau^n(F))}{\lambda(\tau^n(E))} \leqslant C\frac{\lambda(F)}{\lambda(E)} < \frac{2\eta}{1 - 2\eta} C \leqslant 4C\eta.$$

Therefore,

$$\lambda(\tau^n(F)) < 4C\eta\lambda(\tau^n(E \cup F)) \leqslant 4CL\eta.$$

Now,

$$\tau^N(\tau^n(E) \cup \tau^n(F)) = [0, 1] = \tau^{n+N}(E) \cup \tau^{n+N}(F).$$

Thus, we have $[0, 1] - \tau^{n+N}(E) \subset \tau^{n+N}(F)$, and

$$\lambda\big([0, 1] - \tau^{n+N}(E)\big) \leqslant \lambda(\tau^{n+N}(F)) \leqslant m^N\lambda(\tau^n(F)) < 4CLm^N\eta \leqslant \varepsilon.$$

Since $A \supset E$, it follows that $\lambda(\tau^{n+N}(A)) > 1 - \varepsilon$. Q.E.D.

COROLLARY 1. *Let $\tau \in \mathcal{C}$ admit an absolutely continuous invariant measure μ. Then τ is μ-exact.*

4. Fullness is not preserved under limits on the surjections. In this section, we shall construct a sequence of functions which are piecewise-linear, Markov and full, but such that their limit function is not full.

Let

$$F_n = \bigcup_{i=1}^{n} E_i \quad \text{and} \quad G_n = [0, 1] - F_n.$$

We note that G_n consists of 2^n closed intervals, each of length $2^{-n}(1 - \delta) + 3^{-n}\delta$ and that $C_\delta = \bigcap_{n=1}^{\infty} G_n$. Let us now label the intervals of G_n in a manner compatible with the notation used for the deleted intervals. A typical interval in F_n is

$$I(a_1, \ldots, a_{m-1}, 1),$$

where $1 \leqslant m \leqslant n$. We denote by $J^{(n)}(a_1, \ldots, a_{m-1}, 1)$ the interval of G_n which lies immediately to the left of $I(a_1, \ldots, a_{m-1}, 1)$ and by $J^{(n)}(2, 2, \ldots, 2)$ the extreme right hand interval of G_n. The explicit dependence on n is clearly required, because the intervals in G_n and G_{n+1} adjacent to some fixed $I(a_1, \ldots, a_{n-1}, 1)$ are not the same. We define $\tau_{n,\delta}$ on

$$\bigcup_{m=1}^{n} \left\{ I(a_1, \ldots, a_{m-1}, 1) \cup J^{(n)}(a_1, \ldots, a_{m-1}, 1) \colon a_1 = 0 \right\},$$

i.e., on $[0, (3 - \delta)/6)$ as follows:

$$\tau_{n,\delta}(I(0, \ldots, a_{m-1}, 1)) = \tau_\delta(I(0, \ldots, a_{m-1}, 1)),$$

$$\tau_{n,\delta}\big(J^{(n)}(0, a_2, \ldots, a_{m-1}, 1)\big) = J^{(n-1)}(a_2, \ldots, a_{m-1}, 1)$$

and

$$\tau_{n,\delta}(J^{(n)}(1)) = J^{(n-1)}(2, 2, \ldots, 2),$$

where $\tau_{n,\delta}$ is linear on each of the specified intervals and has positive slope and τ_δ is as defined in §2. On $[(3 - \delta)/6, 1/2]$ define $\tau_{n,\delta}(x) = \tau_\delta(x)$, and extend to all of $[0, 1]$ by $\tau_{n,\delta}(x) = \tau_{n,\delta}(1 - x)$. The map $\tau_{n,\delta}$ is piecewise linear, continuous, and Markov. We observe that $|d\tau_{n,\delta}/dx| \geqslant 3$ on F_n and

$$\left| \frac{d\tau_n}{dx} \right| = \frac{2^{-n+1}(1 - \delta) + 3^{-n+1}\delta}{2^{-n}(1 - \delta) + 3^{-n}\delta} = 3 - \frac{1 - \delta}{1 - \delta + \left(\frac{2}{3}\right)^n \delta} > 2$$

on G_n, where we have used the fact that

$$\lambda\big(J^{(n)}(a_1, \ldots, a_{m-1}, 1)\big) = 2^{-n}(1 - \delta) + 3^{-n}\delta.$$

Since $\tau_{n,\delta}$ maps each interval of F_n or G_n onto $[0, 1]$ in a finite number of iterations, $\tau_{n,\delta}$ is interval full [**3**] and hence full by Theorem 3. Finally,

$$|\tau_{n,\delta}(x) - \tau_\delta(x)| \leqslant \lambda\big(J^{(n-1)}(a_1, \ldots, a_{m-1}, 1)\big) = 2^{-n+1}(1 - \delta) + 3^{-n+1}\delta.$$

Thus, $\tau_{n,\delta}$ converges uniformly to τ_δ, but τ_δ is not full.

REMARK. If $\delta = 1$, $|d\tau_{n,\delta}/dx| = 3$ except on $I(1)$. In this case C_δ is C_1 and $\tau_{n,\delta}(x) = \tau_1(x)$, defined in §2.

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