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PICARD'S THEOREM

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ABSTRACT. This paper deals with the numerical content of Picard's Theorem. Two classically equivalent versions of this theorem are proved which are distinct from a computational point of view. The proofs are elementary, and constructive in the sense of Bishop. A Brouwerian counterexample is given to the original version of the theorem.

1. Introduction. A century ago, Picard [8] showed that, in any neighbourhood of an essential singularity, a complex function attains every complex value, with at most one exception, an infinite number of times. Picard's proof was nonelementary, in that it made use of the theory of modular functions. The first elementary proof was given twenty-five years later by Schottky [9], and was based on a theorem which now bears his name. Subsequently, Montel [7] used Schottky's Theorem and the notion of a normal family to give what has become the standard elementary proof of Picard's Theorem.

In this paper, we shall discuss the numerical content of Picard's Theorem. We shall prove the following two versions of the theorem:

(A) *Let f be a complex-valued analytic function on the annulus* $\{z \in \mathbf{C}\colon 0 < |z - \zeta| < r\}$, *and suppose that f omits the values* 0 *and* 1. *Then we can compute the (finite) order of f at ζ.*

(B) *Let f be a complex-valued analytic function on the annulus* $\{z \in \mathbf{C}\colon 0 < |z - \zeta| < r\}$, *with an essential singularity at ζ, and let ξ, ξ' be distinct complex numbers. Then, in any neighbourhood of ζ, we can compute z such that either $f(z) = \xi$ or $f(z) = \xi'$.*

The reader can easily see that these two statements are quite distinct from a computational point of view. We prove them by elementary methods, based on Schottky's Theorem and constructive in the sense of Bishop [1]. We also give a Brouwerian counterexample to the original version of Picard's Theorem.

2. The Schottky theory. Classically, there are at least three approaches to Schottky's Theorem. The most sophisticated of these uses the theory of modular functions [5, Chapter V]. A second approach, via Bloch's Theorem, is described in §§1–3 of Chapter XII of [3]. However, we prefer to follow an argument of

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Titchmarsh [**10**, 8.85] which appears to be based on the original one of Schottky and which, in many respects, seems the most natural way to arrive at Schottky's Theorem.

The main part of our proof of Schottky's Theorem follows very closely the lines of Titchmarsh's proof. However, what Titchmarsh describes as "Schottky's Theorem" is not the full form of the theorem as understood by other authors, and we are obliged to supplement his argument in order to reach that full form.

It is convenient at this stage to introduce some notation and definitions. First, we denote by $B(\zeta, r)$ (resp. $\bar{B}(\zeta, r)$) the open (resp. closed) ball in $\mathbf{C}$ of centre ζ and radius r, and by $A(\zeta, r, s)$ the open annulus $\{z \in \mathbf{C}: r < |z - \zeta| < s\}$, where $0 \leqslant r < s$. We also write $A(\zeta, s)$ instead of $A(\zeta, 0, s)$. Next, if $0 \leqslant r < r' \leqslant s' < s$ and f is analytic in $A(0, r, s)$, we define

$$\left.\begin{aligned}\lambda(f, r', s') &= \inf\\ \mu(f, r', s') &= \sup\end{aligned}\right\}\{|f(z)|: |z| = r' \text{ or } |z| = s'\}.$$

When $r' = s'$, we write $\lambda(f, r')$ (resp. $\mu(f, r')$) instead of $\lambda(f, r', r')$ (resp. $\mu(f, r', r')$).

2.1. *Let f be analytic and not identically zero in $A(0, r, s)$, where $0 < r < s$. Then $\lambda(f, r') > 0$ for all but finitely many r' with $r + \frac{1}{4}(s - r) \leqslant r' \leqslant r + \frac{3}{4}(s - r)$.*

Proof. Let K be the compact set $\{z \in \mathbf{C}: r + \frac{1}{4}(s - r) \leqslant |z| \leqslant r + \frac{3}{4}(s - r)\}$. By [**2**, Theorem 4], either $\inf\{|f(z)|: z \in K\} > 0$ or, as we may suppose, there exist finitely many points $z_1, \ldots, z_n$ in $A(0, r, s)$ and an analytic function g on $A(0, r, s)$ such that $\inf\{|g(z)|: z \in K\} > 0$ and

$$f(z) = (z - z_1) \ldots (z - z_n)g(z) \qquad (z \in A(0, r, s)).$$

For any r' such that $r + (s - r)/4 \leqslant r' \leqslant r + 3(s - r)/4$ and $r' \neq |z_k|$ for each k, we now have $\lambda(f, r') > 0$. $\square$

Let U be an open subset of $\mathbf{C}$, f a mapping of U into $\mathbf{C}$ and ζ a complex number. We say that f *omits the value* ζ if $f(z) \neq \zeta$ for all z in U; and that f is a *Picard function* (*on* U) if f is analytic and omits the values 0 and 1.

Before going any further, we introduce the continuous mapping

$$C: (\alpha, r) \to \exp\left(2^{56}\alpha^8 / (1 - \surd(2r - r^2))^4\right)$$

of $\mathbf{R}^+ \times [0, 1)$ into $\mathbf{R}^+$. For fixed r in $[0, 1)$, the mapping $\alpha \to C(\alpha, r)$ is strictly increasing; also, $C(\alpha, r) > \alpha$ for each $\alpha > 0$.

2.2. Schottky's Theorem. *Let $\alpha > 0$, and f a Picard function on $B(0, 1)$ such that $|f(0)| \leqslant \alpha$. Then $|f(z)| \leqslant C(\alpha, r)$ whenever $|z| \leqslant r < 1$.*

Proof. Without loss of generality, we can assume that $\alpha > 3$ and that $|1 - f(0)| < \alpha$. Either $1/3 \leqslant \min(|f(0)|, |1 - f(0)|)$ or $\min(|f(0)|, |1 - f(0)|) < 1/2$. In the former case, minor constructive modifications of an argument of Titchmarsh [**10**, 8.85] (as corrected on pp. 150–154 of [**6**]) enable us to show that, for $0 \leqslant r < 1$, $\mu(f, r) \leqslant K(\alpha, r)$, where $K(\alpha, r) = \exp(2^{56}\alpha^8/(1 - r)^4)$.

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This leaves us with the case $\min(|f(0)|, |1 - f(0)|) < 1/2$. Replacing f by $1 - f$ if necessary, we may assume that $|f(0)| < 1/2$. Now, we know from the maximum principle that

$$\lambda(f, r') = 1/\mu(1/f, r') < 1/2 \qquad (0 \leqslant r' < 1).$$

Let $0 \leqslant r < 1$. Either $\mu(f, r) < 3$ or, as we may assume, $2 < \mu(f, r)$. Applying the Intermediate Value Theorem to the function $t \to |f(re^{it})|$ on $[0, 2\pi]$, we compute ζ with $|\zeta| = r$ and $4/3 < |f(\zeta)| < 2$. Then $\tau(z) = (z - \zeta)/(\bar{\zeta}z - 1)$ defines a conformal mapping of $\bar{B}(0, 1)$ onto itself with $\tau(0) = \zeta$. Moreover, as a straightforward computation shows, $\tau(\bar{B}(0, \sqrt{(2r - r^2)})) \supset \bar{B}(0, r)$. Now, $F = f \circ \tau$ is a Picard function on $\bar{B}(0, 1)$ such that

$$\max(|F(0)|, |1 - F(0)|) < \alpha \quad \text{and} \quad \min(|F(0)|, |1 - F(0)|) \geqslant 1/3.$$

Thus

$$\begin{aligned}\mu(f, r) &\leqslant \sup\left\{|f(z)|: z \in \tau\left(\bar{B}\left(0, \sqrt{(2r - r^2)}\right)\right)\right\} \\ &= \mu\left(F, \sqrt{(2r - r^2)}\right) \leqslant K\left(\alpha, \sqrt{(2r - r^2)}\right).\end{aligned}$$

We note that $K(\alpha, t)$ is an increasing function of t in $[0, 1)$; so that

$$\max\left(3, K(\alpha, r), K\left(\alpha, \sqrt{(2r - r^2)}\right)\right) = K\left(\alpha, \sqrt{(2r - r^2)}\right) = C(\alpha, r).$$

To complete the proof, it only remains to refer to the maximum principle. □

2.3. Corollary. *Let f be analytic in $B(0, 1)$, $0 < r < 1$ and $\mu(f, r) > C(|f(0)|, r)$. Then f attains at least one of the values 0, 1 in $B(0, 1)$.*

Proof. The proof is similar to, but simpler than, that of 2.4 below, and is left to the reader. □

We now introduce a mapping which will play a major role in our discussion of Picard's Theorem. For each $\alpha > 0$, define $\delta_0(\alpha) = 2\alpha$,

$$\delta_{n+1}(\alpha) = C(\delta_n(\alpha), 1/8) \qquad (n = 0, 1, \ldots, 25),$$

and let δ be the increasing, continuous mapping $\alpha \to \delta_{26}(\alpha)$ of $\mathbf{R}^+$ into $\mathbf{R}^+$. Note that $\delta(\alpha) > \alpha$ for each $\alpha > 0$.

2.4. Corollary. *Let f be analytic on $A(0, 1)$, $0 < r \leqslant \frac{1}{2}$, $0 < \lambda(f, r)$ and $\delta(\lambda(f, r)) < \mu(f, r)$. Then f attains at least one of the values 0, 1 in the annulus $\{z \in \mathbf{C}: r/2 \leqslant |z| \leqslant 3r/2\}$.*

Proof. Consider first the case where $r = \frac{1}{2}$. As $\mu(f, r) > \delta(\lambda(f, r)) > \lambda(f, r)$, f is nonconstant. Hence, by 2.1, there exist r_1, r_2 such that $1/4 < r_1 < 3/8$, $5/8 < r_2 < 3/4$, $0 < \lambda(f, r_1, r_2)$ and $0 < \lambda(f - 1, r_1, r_2)$. By [**2**, Theorem 1], either f attains at least one of the values 0, 1 in $\{z \in \mathbf{C}: r_1 \leqslant |z| \leqslant r_2\}$, or

$$(*) \qquad 0 < \inf\{\min(|f(z)|, |f(z) - 1|): r_1 \leqslant |z| \leqslant r_2\}.$$

We now show that $(*)$ cannot happen. To do so, suppose that $(*)$ obtains, choose ζ so that $|\zeta| = \frac{1}{2}$ and $|f(\zeta)| < 2\lambda(f, \frac{1}{2})$, and define $z_n = \zeta \exp(n\pi i/26)$ $(n = 0, 1, \ldots, 26)$. Then the balls $\bar{B}(z_n, 1/8)$ $(n = 0, \ldots, 26)$ cover the semicircle

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$\{\zeta \exp(i\phi)\colon 0 \leqslant \phi \leqslant \pi\}$. As each of these balls lies in the annulus $\{z\colon r_1 \leqslant |z| \leqslant r_2\}$, the restriction of f to each $B(z_n, 1/8)$ is a Picard function. Moreover,

$$|z_{n+1} - z_n| < \pi/26 < 1/8 \qquad (n = 0, 1, \ldots, 25).$$

A simple induction argument using Schottky's Theorem now shows that $|f(z)| \leqslant \delta_n(\lambda(f, \frac{1}{2}))$ whenever $|z - z_n| < 1/8$ and $n = 0, \ldots, 26$. Hence $|f(z)| \leqslant \delta_{26}(\lambda(f, \frac{1}{2})) = \delta(\lambda(f, \frac{1}{2}))$ whenever $z = \zeta \exp(i\phi)$ and $0 \leqslant \phi \leqslant \pi$. Similar considerations involving the points $\zeta \exp(-n\pi i/26)$ $(n = 0, \ldots, 26)$ enable us to show that $|f(z)| \leqslant \delta(\lambda(f, \frac{1}{2}))$ whenever $|z| = \frac{1}{2}$. As this contradicts our hypotheses, we conclude that $(*)$ cannot hold.

In the case of general r in $(0, 1/2]$, we need only apply the foregoing to the Picard function $z \to f(2rz)$ to complete the proof. □

3. **Picard's Theorem.** Let f be analytic in $A(\zeta, r)$, with Laurent expansion $\sum_{n=-\infty}^{\infty} a_n(z - \zeta)^n$, and let ν be an integer. We say that *f has a pole of order at most ν at ζ* if $a_{-n} = 0$ for all $n > \nu$; if also $a_{-\nu} \neq 0$, then we say that *f has a pole of determinate order at ζ*, and that the *order* of this pole is ν. (In standard terminology, a pole of nonpositive order is a removable singularity; while a pole of negative order ν is a zero of order $-\nu$.) A necessary and sufficient condition that f have a pole of order at most ν at ζ is that $(z - \zeta)^\nu f(z)$ be bounded in some neighbourhood of ζ [**4**, 9.15.2].

3.1. *Let f be an analytic function in $A(\zeta, r)$ which omits a complex value α and has a pole at ζ. Then either f has a removable singularity at ζ or the order of pole of f at ζ is positive and determinate.*

Proof. Let $\sum_{n=-\infty}^{\infty} a_n(z - \zeta)^n$ be the Laurent expansion of f in $A(\zeta, r)$, and ν a positive integer such that $a_{-n} = 0$ for all $n > \nu$. It clearly suffices to prove that either $a_{-\nu} = 0$ or $a_{-\nu} \neq 0$. Now, $z \to (z - \zeta)^\nu(f(z) - \alpha)$ extends to an analytic function g on $B(0, r)$. Since $g(\zeta + r/2) \neq 0$, it follows from [**2**, Theorem 4] that either $\inf\{|g(z)|\colon |z - \zeta| \leqslant r/2\} > 0$—in which case $a_{-\nu} = g(\zeta) \neq 0$—or there exists z' in $B(\zeta, r)$ with $g(z') = 0$. In the latter case, were $z' \neq \zeta$, $f(z')$ would be defined and we would have $f(z') = (z' - \zeta)^{-\nu}g(z') + \alpha = \alpha$, a contradiction. Hence $z' = \zeta$ and $a_{-\nu} = g(\zeta) = 0$. □

3.2. *Let f be analytic and nonvanishing on $A(\zeta, r)$ with a pole at ζ. Then the order of this pole is determinate.*

Proof. Let $f(z) = \sum_{-\nu}^{\infty} a_n(z - \zeta)^n$ be the Laurent expansion of f in $A(\zeta, r)$, where ν is an integer. In view of 3.1, we can assume that $\nu \leqslant 0$ and that f has been extended to an analytic function on $B(\zeta, r)$. By [**2**, Theorem 4], either $\inf\{|f(z)|\colon |z - \zeta| \leqslant r/2\} > 0$, in which case f has a pole of order 0 at ζ; or there exist a positive integer m, points $z_1, \ldots, z_m$ of $B(\zeta, r)$, and an analytic function g on $B(\zeta, r)$ such that $f(z) = (z - z_1) \ldots (z - z_m)g(z)$ for each z in $B(\zeta, r)$, and $\inf\{|g(z)|\colon |z - \zeta| \leqslant r/2\} > 0$. In the latter case, as f is nonvanishing in $A(\zeta, r)$, we must have $z_1 = \cdots = z_m = \zeta$; whence f has a pole of order $-m$ at ζ. □

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Now let f be a Picard function on $A(\zeta, r)$. Then $1/f$ is analytic [2, Theorem 5, Corollary], and is therefore a Picard function. For $0 < s < 1$, let $\gamma(s)$ be the path $t \to \zeta + se^{it}$ $(0 \leqslant t \leqslant 2\pi)$ in $A(\zeta, r)$. It follows from [1, Chapter 5, Proposition 3] that

$$\nu_0(f) = (2\pi i)^{-1} \int_{\gamma(s)} f'(z)\, dz/f(z)$$

is an integer whose value is independent of s. These observations will be used in the proof of

3.3. Picard's Theorem (first form). *Let f be a Picard function on $A(\zeta, r)$. Then f has a pole of determinate order at ζ.*

Proof. We lose no generality in taking $\zeta = 0$, $r = 1$. Let $\Sigma_{k=-\infty}^{\infty} a_k z^k$ be the Laurent expansion of f in $A(0, 1)$. Let n be a positive integer, and suppose that $a_{-n} \neq 0$. For each s with $0 < s < 1$, we have

$$|a_{-n}| = \left|(2\pi i)^{-1} \int_{\gamma(s)} z^{n-1} f(z)\, dz\right| \leqslant s^n \delta(\lambda(f, s)).$$

Hence $\delta(\lambda(f, s)) \geqslant s^{-n}|a_{-n}| \to \infty$ as $s \to 0$. As δ is increasing, it follows that $\lambda(f, s) \to \infty$ as $s \to 0$. Thus $1/f$ is bounded in the neighbourhood of 0, and so has a removable singularity at 0. Using 3.2, we see that $1/f$, and therefore f, has a pole of determinate order at 0. Computation of the integral defining $\nu_0(f)$ now shows that the order of the pole of f is $-\nu_0(f)$; whence $n \leqslant -\nu_0(f)$.

It now follows that $a_{-n} = 0$ for all $n > -\nu_0(f)$; whence (3.2) f has a pole of determinate order at 0. $\square$

Let f be analytic, with Laurent expansion $\Sigma_{-\infty}^{\infty} a_n(z - \zeta)^n$, in the annulus $A(\zeta, r)$. We say that ζ is an *essential singularity* of f if there exists a strictly increasing, infinite sequence $(n(k))_{k \geqslant 1}$ of positive integers such that $a_{-n(k)} \neq 0$ for each k. Note that we may be unable to tell whether ζ is a pole or an essential singularity of f.

In order to discuss the behaviour of an analytic function in the neighbourhood of an essential singularity, we need some information about entire functions. We say that an entire function g is of *infinite degree* if the function $z \to g(1/z)$ has an essential singularity at 0. This is equivalent to the condition that, for any polynomial p, the function $g - p$ takes a nonzero value.

3.4. *Let g be a nonconstant entire function, p a polynomial function. Then there exists $\xi \in \mathbf{C}$ such that $g(\xi) \neq 0$ and $1/g(\xi) \neq p(\xi)$.*

Proof. Let $p(z) = \Sigma_{n=0}^{\nu} p_n z^n$, where ν is a positive integer, and choose $r > 0$ so that $\lambda(g, r) > 0$. Either $\mu(p, r) < \mu(1/g, r)$, in which case we need only choose ξ so that $|\xi| = r$ and $\mu(p, r) < 1/g(\xi)$; or $0 < \mu(p, r)$. In the latter case, there exists m with $p_m \neq 0$. If $m = 0$, we choose ξ so that $g(\xi) \neq 0$ and $g(\xi) \neq 1/p_0$. Then either $|\Sigma_{n=1}^{\nu} p_n \xi^n| < |1/g(\xi) - p_0|$, and $1/g(\xi) \neq p(\xi)$; or $0 < |\Sigma_{n=1}^{\nu} p_n \xi^n|$. It follows that we can take $m \geqslant 1$.

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Now let x be any root of p. If $0 < \inf\{|g(z)|: |z - x| \leqslant 1\}$, then $g(x) \neq 0$, $(1/g(x) - p(x))g(x) = 1$, $1/g(x) \neq p(x)$ and we can take $\xi = x$. On the other hand, if

$$\inf\{|g(z)|: |z - x| \leqslant 1\} < 1/\sup\{|p(z)|: |z - x| \leqslant 1\},$$

we need only choose $\xi \in B(x, 1)$ so that $0 < g(\xi) < 1/\sup\{|g(z)|: |z - x| \leqslant 1\}$. □

3.5. *Let g be an entire function of infinite degree, and R, ε positive numbers. Then there exists $r > R$ such that either $\lambda(g, r) < \varepsilon$ or g has a zero z with $R < |z| < r$.*

Proof. By [2, Theorem 4], we can find an integer $\nu \geqslant 0$, a polynomial p of degree ν, and an analytic function g_1 on $B(0, R + 2)$ such that all the roots of p lie in $B(0, R + 2)$, $\inf\{|g_1(z)|: |z| \leqslant R + 1\} > 0$ and $g(z) = p(z)g_1(z)$ for each z in $B(0, R + 2)$. It is easy to see that g_1 extends to an entire function of infinite degree such that $g = pg_1$ everywhere. By [2, Theorem 5, Corollary], $1/g_1$ is analytic on $B(0, R + 1)$. Let $\sum_{n=0}^{\infty} b_n z^n$ be the Taylor expansion of $1/g_1$ about 0 in $B(0, R + 1)$. According to 3.4, we can compute ξ so that $g_1(\xi) \neq 0$ and $1/g_1(\xi) \neq \sum_{n=0}^{\nu} b_n \xi^n$; whence there exists an integer $m > \nu$ with $b_m \neq 0$. Using 2.1, we now choose $r > R$ so that $r^m|b_m| > \mu(p, r)/\varepsilon$ and $\lambda(g_1, r) > 0$. If $0 < \inf\{|g_1(z)|: |z| \leqslant r\}$, then, for all s in $(0, r)$,

$$\begin{aligned} |b_m| &= \left|(2\pi i)^{-1} \int_{\gamma(s)} dz/z^{m+1}g_1(z)\right| \\ &\leqslant \mu(1/g_1, s)/s^m = 1/s^m\lambda(g_1, s), \end{aligned}$$

and so, by continuity, $|b_m| \leqslant 1/r^m\lambda(g_1, r)$. Thus

$$\lambda(g, r) \leqslant \mu(p, r)\lambda(g_1, r) \leqslant \mu(p, r)/r^m|b_m| < \varepsilon.$$

On the other hand, if $\inf\{|g_1(z)|: |z| \leqslant r\} < \lambda(g_1, r)$, then g_1 has a zero z in $B(0, r)$ [1, Chapter 5, Theorem 7]. Clearly, $R < |z| < r$ and z is a zero of g. □

3.6. The Casorati-Weierstrass Theorem. *Let f be analytic in $A(\zeta, r)$, with an essential singularity at ζ. Let $\xi \in \mathbf{C}$, $0 < r' < r$, and $0 < \varepsilon$. Then there exists z with $0 < |z - \zeta| < r'$ and $|f(z) - \xi| < \varepsilon$.*

Proof. Without loss of generality we can take $\zeta = 0$, $r = 1$ and $\xi = 0$. Let $\sum_{n=-\infty}^{\infty} a_n z^n$ be the Laurent expansion of f about 0, and choose $R > 1/r'$ so that $|\sum_{n=1}^{\infty} a_n z^n| < \varepsilon/2$ for all z in $B(0, 1/R)$. As the power series $\sum_{n=0}^{\infty} a_{-n} z^n$ is convergent for $|z| > 1$, there is an entire function g such that $g(z) = \sum_{n=0}^{\infty} a_{-n} z^n$ for all complex z. By 3.5, there exists z such that $|1/z| > R$ and $|g(1/z)| < \varepsilon/2$. Thus $|z| < 1/R < r'$ and $|f(z)| \leqslant |\sum_{n=1}^{\infty} a_n z^n| + |g(1/z)| < \varepsilon$. □

Our next theorem constitutes a considerable strengthening of the Casorati-Weierstrass Theorem.

3.7. Picard's Theorem (second version). *Let f be analytic on $A(\zeta, r)$, with an essential singularity at ζ, and let ξ, ξ' be distinct complex numbers. Then, in any neighbourhood of ζ, f attains at least one of the values ξ, ξ'.*

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PROOF. It is easy to see that we may take $\zeta = 0$, $r = 1$, and that it will then suffice to show that f attains one of the values 0, 1 in $A(0, 1)$. Let $\sum_{n=-\infty}^{\infty} a_n z^n$ be the Laurent expansion of f in $A(0, 1)$, and choose $\nu \geqslant 1$ with $a_{-\nu} \neq 0$. Compute r' so that $0 < r' \leqslant 1/2$ and $0 < \alpha = \min(\lambda(f, r'), \lambda(f - 1, r'))$. By 3.6, there exists ρ such that $0 < \rho < \min(r', (\delta(\alpha)^{-1}|a_{-\nu}|)^{1/\nu})$ and $\lambda(f, \rho) < \alpha$. For such ρ we have

$$|a_{-\nu}| = \left|(2\pi i)^{-1}\int_{\gamma(\rho)} z^{\nu-1}f(z)\, dz\right| \leqslant \rho^{\nu}\mu(f, \rho)$$

and therefore $\mu(f, \rho) > \rho^{-\nu}|a_{-\nu}| > \delta(\alpha) \geqslant \delta(\lambda(f, \rho))$. It follows from 2.4 that f attains at least one of the values 0, 1 in the annulus $\{z: \rho/2 \leqslant |z| \leqslant 3\rho/2\}$. □

3.8. COROLLARY. *Let f be an analytic function on $A(\zeta, r)$ which has an essential singularity at ζ and omits the value* 0. *Then f attains every nonzero complex value.* □

The following is a Brouwerian counterexample to the classical Picard Theorem that, under the conditions of 3.7, one of the values 0, 1 is attained infinitely often by f in $A(0, 1)$. Let $(a_n)_{n\geqslant 1}$ be an increasing sequence in $\{0, 1\}$, and define entire functions g_n by

$$g_n(z) = \begin{cases} \left(1 + \sum_{k=0}^{n} z^k/k!\right)e^{z/n} & \text{if } a_n = 0, \\ \left(1 + \sum_{k=0}^{\nu} z^k/k!\right)e^{z/\nu} & \text{if } a_n = 1, \\ \text{where } \nu \text{ is the smallest positive integer } k \text{ with } a_k = 1. \end{cases}$$

Then g: $z \to \lim_{n\to\infty} g_n(z)$ is an entire function of infinite degree; so that f: $z \to g(1/z)$ is analytic everywhere except at 0, where it has an essential singularity. If $f(z) = 0$ for infinitely many distinct z, then we can prove the statement, "for each positive integer n, $a_n = 0$"; while if $f(z) = 1$ for infinitely many distinct z, then we can prove the negation of that statement.

Turning now to the Little Picard Theorem [3, Chapter XII, 2.3], we first observe that our constructive version of that theorem is not an immediate consequence of 3.7: this is because there is no constructive procedure for showing that a given entire function is either a polynomial or of infinite degree (cf. [3, Chapter XII, introductory remarks]). However, the proofs of the following results are similar to, but simpler than, those of their counterparts above, and are left to the reader.

3.9. LITTLE PICARD THEOREM. *Let f be a nonconstant entire function, and ξ, ξ' distinct complex numbers. Then f attains at least one of the values ξ, ξ'.* □

3.10. COROLLARY. *Let f be a nonconstant entire function which omits the value* 0. *Then f attains every nonzero complex value.* □

REFERENCES

1. Errett Bishop, *Foundations of constructive analysis*, McGraw-Hill, New York, 1967.

2. D. S. Bridges, *On the isolation of zeroes of an analytic function*, Pacific J. Math. (to appear).

3. J. B. Conway, *Functions of one complex variable*, 2nd ed., Graduate Texts in Math., no. 11, Springer-Verlag, Berlin, 1975.

4. J. Dieudonné, *Foundations of modern analysis*, Academic Press, New York, 1960.

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5. W. H. J. Fuchs, *Topics in the theory of functions of one complex variable*, Van Nostrand, Princeton, N. J., 1967.

6. A. S. B. Holland, *Introduction to the theory of entire functions*, Pure and Appl. Math., vol. 56, Academic Press, New York, 1973.

7. P. Montel, *Sur les familles de fonctions analytiques qui admettent des valeurs exceptionnelles dans un domaine*, Ann. Ecole Norm. Sup. (3) **23** (1912), 487–535.

8. E. Picard, *Sur les fonctions analytiques uniformes dans le voisinage d'un point singulier essentiel*, C. R. Acad. Sci. **89** (1879), 745–747.

9. F. Schottky, *Über den Picard'schen Satz und die Borel'schen Ungleichungen*, Sitzung. der K. Preussischen Akad. der Wissenschaften, Berlin, 1904, pp. 1244–1262.

10. E. C. Titchmarsh, *The theory of functions*, 2nd ed. (corrected), Clarendon Press, Oxford, 1968.

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