WEAK P-POINTS IN COMPACT CCC F-SPACES
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Abstract. Using a technique due to van Mill we show that each compact ccc F-space of weight greater than $2^\omega$ contains a weak P-point, i.e. a point $x \in X$ such that $x \notin F$ for each countable $F \subseteq X - \{x\}$. We show that, assuming $BF(c)$, each nowhere separable compact F-space has a weak P-point. We show the existence of points which are not limit points of any countable nowhere dense set in compact F-spaces of weight $\aleph_1$. We also discuss remote points and points not the limit point of any countable discrete set.

Introduction. All spaces considered are completely regular and $X^*$ denotes $\beta X - X$. A space $X$ is an F-space if each cozero set is $C^*$-embedded. A ccc F-space is easily seen to be extremally disconnected; the closure of each open set is open. A point $x \in X$ is a weak P-point if $x \notin \overline{F}$ for each countable $F \subseteq X - \{x\}$. Kunen [K] has shown that $\omega^*$ has a dense set of weak P-points. Jan van Mill then showed that each compact infinite F-space of weight $2^\omega$ in which nonempty $G_\delta$s have nonempty interior has weak P-points [vM]. He also showed that if there is a ccc nonseparable growth of $\omega$ then he could remove the weight restriction. Subsequently Murray Bell [B] constructed such a growth of $\omega$. Then in [DvM] the author and van Mill extended van Mill’s result to “each compact nowhere ccc F-space has weak P-points”. It is easy to see that a separable space cannot have weak P-points. In §5 we give an example of a nonseparable F-space in which there are no weak P-points. We see, therefore, that we need to assume nowhere separable rather than nonseparable. However in the case of ccc spaces we can consider all nonseparable spaces because such a space contains a nowhere separable open set. We address the open question “do all compact nowhere separable F-spaces have weak P-points?”.

We are able to show that for compact ccc F-spaces of weight greater than $c$ the answer is yes and that assuming $BF(c)$ it is also true for nonseparable compact ccc F-spaces of weight $\leq c$. We are also able to show that for spaces of weight $\aleph_1$ we do not need to assume $BF(c)$, that is, there are points not the limit point of any countable nowhere dense set.

The point $x \in X^*$ is called a remote point of $X$ if $x \notin \text{cl}_{\beta X} A$ for each nowhere dense subset of $A$ of $X$. It is known that if $X$ is a nonpseudocompact space with countable $\Pi$-weight then $X$ has a remote point [vD, 1.5], [CS]. In [vDvM] the authors show that not every nonpseudocompact space has remote points but ask if $\omega \times 2^\omega$ has remote points if CH fails. Our methods enable us to show that under $BF(c)$ each nonpseudocompact ccc space of weight $\leq c$ has a remote point.

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Finally we also show that under some further assumptions on the \( \Pi \)-character of a ccc \( F \)-space there are points which are not the limit point of any countable discrete set.

1. **Nice filters.** Let \( X \) be a normal space. A **filter base** on \( X \) is a collection of closed subsets of \( X \) which is closed under finite intersections and which does not contain the empty set. The **filter** generated by the filterbase \( \mathcal{F} \) is the collection of \( \{ A \subseteq X : A = \overline{A} \text{ and } \exists F \in \mathcal{F} \text{ with } F \subseteq A \} \). Let \( X \) be the topological sum of countably many nonempty compact spaces, say \( X_n \) (\( n < \omega \)). In [vM] van Mill defines a filter \( \mathcal{F} \) to be **nice** provided that for each \( F \in \mathcal{F} \), the set \( \{ n < \omega : F \cap X_n = \emptyset \} \) is finite while, in addition, \( \cap \mathcal{F} = \emptyset \). We will say that a nice filter \( \mathcal{F} \) on \( X \) avoids countable sets if for each countable \( D \subseteq X \) there is an \( F \in \mathcal{F} \) with \( D \cap F = \emptyset \). In [vM], Jan van Mill has done most of the work of constructing weak \( P \)-points.

**Lemma 1.1.** Let \( K \) be a compact ccc \( F \)-space and let \( \{ Z_n : n \in \omega \} \) be disjoint nonempty clopen subsets of \( K \). Then if \( \mathcal{F} \) is a nice filter on \( X = \bigcup Z_n \), there is a point \( x \in \bigcap \{ \overline{F} : F \in \mathcal{F} \} \) which is not in the closure of any countable subset of \( K \setminus (X \cup \{ x \}) \). In particular, if \( \mathcal{F} \) avoids countable subsets of \( X \) then \( x \) is a weak \( P \)-point of \( K \).

Due to the length and complexity we will not include this proof. The reader is referred to the proof of Theorem 0.3 in [vM]. One need only observe that the restriction of \( \mathcal{F} \) to any infinite subcollection of the \( Z_n \)'s is again a nice filter on this union and that Bell [B] has shown the existence of a ccc nowhere separable growth of \( \omega \). We then follow the proof of Theorem 0.3 in [vM] verbatim. \( \Box \)

Our investigation therefore turns to constructing nice filters on topological sums of compact ccc \( F \)-spaces which avoid countable sets.

2. **Large \( F \)-spaces.** We will first investigate compact ccc \( F \)-spaces with weight greater than \( c \). We begin by stating some results which we will require.

**Theorem 2.1 [BF].** Every infinite complete boolean algebra \( B \) contains a free subalgebra \( A \) with \( | A | = | B | \).

The clopen subsets of a compact ccc \( F \)-space form a complete boolean algebra so by the above result and Stone’s duality theorem [W] we obtain

**Theorem 2.2.** Each compact ccc \( F \)-space of weight greater than \( c \) contains weak \( P \)-points.

Proof. It follows easily from 2.2 that we can choose countably many disjoint clopen subsets \( \{ Z_n : n \in \omega \} \) of \( X \) each of weight greater than \( c \). From Lemma 1.1 we need only construct a nice filter \( \mathcal{F} \) on \( \bigcup Z_n \) which avoids countable subsets of \( \bigcup Z_n \). Our technique is very similar to that used in [vM]. By Theorem 2.2, we can let \( g_n \) be a continuous surjection from \( Z_n \) to \( 2^{c^+} \). Note that \( 2^{c^+} \) is ccc nowhere separable. For
each countable $D \subseteq \bigcup Z_n$, let $D' = \bigcup g_n[D \cap Z_n]$. Choose a cellular family of clopen subsets $\{A^D_k : k \in \omega\}$ of $2^\omega$ whose union is dense and $A^D_k \cap D' = \emptyset$ for each $k$. This can be done because $D'$ is nowhere dense in $2^\omega$. We will let $\mathcal{F}$ be the filter generated by $\{\bigcup_{n \in \omega} \bigcup_{k \leq n} g_n[A^D_k] : D \in [\bigcup Z_n]^\omega\}$.

It is a simple matter to check that $\mathcal{F}$ is a nice filter on $\bigcup Z_n$ and obviously $\mathcal{F}$ avoids countable sets. Let $\{D_j : 1 \leq j \leq m\}$ be countable subsets of $\bigcup Z_n$. Then for each of the cellular families $\{A^D_k : k \in \omega\}$, $\bigcup_{k \in \omega} A^D_k$ is dense in $2^\omega$. Hence we can recursively choose $k_j$, $1 \leq j \leq m$, so that $\bigcap_{j=1}^m A^{A_{k_j}} \neq \emptyset$. Let $N = \max\{k_j : 1 \leq j \leq m\}$ and let $n > N$. Hence

$$Z_n \cap \left( \bigcup_{k \leq n} g_n[A^D_k] \right) \cap \cdots \cap \left( \bigcup_{k \leq n} g_n[A^D_k] \right)$$

contains $g_{n'}[\bigcap_{j=1}^m A^{A_{k_j}}]$ and is therefore not empty. This proves that $\mathcal{F}$ is a nice filter and completes the proof of the theorem. □

Remark. It is worth noting that any ccc nowhere separable space $Y$ could have taken the place of $2^\omega$ in the above proof so long as one has surjections from each $Z_n$ to $Y$. This fact can be used to conclude that many well-known $F$-spaces have weak $P$-points, for instance any ccc nowhere separable space which has countably many disjoint clopen sets which are pairwise homeomorphic.

3. Small $F$-spaces. In the case of small $F$-spaces, that is, spaces of weight less than or equal to $c$, we cannot use the above method because $2^\kappa$ for $\kappa \leq c$ is separable. The method we use is an attempt to capture within the space the essential idea behind the above method. We only managed to succeed with the aid of the set-theoretic principle $BF(c)$. Let $F$ be the set of all functions from $\omega$ into $\omega$. If $f$ and $g$ belong to $F$, define $g < f$ provided $\{n \in \omega : g(n) > f(n)\}$ is finite. A subset $G$ of $F$ is bounded if there is an $f \in F$ such that for each $g \in G$, $g \leq f$. $BF(c)$ is equivalent to the statement: each subset of $F$ of cardinality less than $c$ is bounded. $BF(c)$ is known to be consistent with the usual axioms of set theory and follows from $MA$ or even $P(c)$ [R, pp. 82, 88].

To prove our result for small nowhere separable compact ccc $F$-spaces we will first prove a lemma in greater generality than is needed for weak $P$-points.

**Lemma 3.1.** Assume $BF(c)$. Let $\{Z_n : n \in \omega\}$ be compact ccc spaces of weight less than or equal to $c$ and let $X$ be the topological sum of $\{Z_n : n \in \omega\}$. There is a nice filter $\mathcal{F}$ on $X$ which avoids all nowhere dense subsets of $X$. (Such a filter has been called a remote filter [vM].)

**Proof.** Since $X$ is ccc and of weight less than or equal to $c$ there are only $c$ maximal cellular families of regular closed sets. Let $\langle \{A^\alpha_{n,m} : n, m \in \omega\} : \alpha < c \rangle$ list all maximal cellular families of regular closed sets such that for each $\alpha < c$ and $n \in \omega$, $A^\alpha_{n,m} \subset Z_n$ for all $m \in \omega$. Let $\mathcal{D}$ be the set of nowhere dense subsets of $X$. Notice that for each $D \in \mathcal{D}$ there is an $\alpha < c$ such that $D \cap (\bigcup_{n,m} A^\alpha_{n,m}) = \emptyset$.

Our plan is to select, for each $\alpha < c$, a function $h_\alpha$ from $\omega$ into $\omega$. We will define our filter $\mathcal{G}$ to be generated by the set of closed sets $\{\bigcup_{n \in \omega} \bigcup_{j < h_\alpha(n)} A^\alpha_{n,j} : \alpha < c\}$.
So the idea is to select the $h_a$’s to ensure that this filter is nice. This procedure is actually a simple recursion using $BF(c)$.

Let $h_0(n) = n$ for each $n \in \omega$. Suppose we have defined $h_\gamma$ for $\gamma < \alpha < c$ such that for any finite sequence $\gamma_1 < \gamma_2 < \cdots < \gamma_k < \alpha$ there is an $N \in \omega$ such that for $n \geq N$,

$$Z_n \cap \left( \bigcup_{j < h_{\gamma_j(n)}} A_{n,j}^{\gamma_j} \right) \cap \cdots \cap \left( \bigcup_{j < h_{\gamma_j(n)}} A_{n,j}^{\alpha_k} \right) \neq \emptyset.$$  

(The $Z_n$ is here only for emphasis.) This is the condition we require to ensure we get a nice filter.

Let us select $h_\alpha$. For each $E \in [\alpha]^{<\omega}$ we define a function $g_E$ as follows. Let $E$ be the sequence $\gamma_1 < \gamma_2 < \cdots < \gamma_k$. Let $g_E(n) = 0$ if $\bigcap_{i=1}^k \left( \bigcup_{j < h_{\gamma_j(n)}} A_{n,j}^{\gamma_i} \right) = \emptyset$. Otherwise let $g_E(n)$ be the smallest integer $p$ such that

$$A_{n,p} \cap \bigcap_{i=1}^k \left( \bigcup_{j < h_{\gamma_j(n)}} A_{n,j}^{\gamma_i} \right) \neq \emptyset.$$  

Since $| \{ g_E : E \in [\alpha]^{<\omega} \} | < | [\alpha]^{<\omega} | < c$, the set $\{ g_E : E \in [\alpha]^{<\omega} \}$ is a bounded family if we assume $BF(c)$. Hence we can choose a function $h_\alpha$ so that for each $E \in [\alpha]^{<\omega}$ the set $\{ n : g_E(n) > h_\alpha(n) \}$ is finite.

To see that we have preserved our induction assumption, let $E = \{ \gamma_i : 1 \leq i \leq k, \gamma_i < \alpha \}$. By assumption, there is an integer $N$ such that for $n \geq N$,

$$Z_n \cap \bigcap_{i=1}^k \left( \bigcup_{j < h_{\gamma_j(n)}} A_{n,j}^{\gamma_i} \right) \neq \emptyset.$$  

Therefore, for $n \geq N$,

$$A_{n,g_E(n)}^\alpha \cap \bigcap_{i=1}^k \left( \bigcup_{j < h_{\gamma_j(n)}} A_{n,j}^{\gamma_i} \right) \neq \emptyset.$$  

By the definition of $h_\alpha$, there is an integer $N_1$, so that for $n \geq N_1$, $h_\alpha(n) \geq g_E(n)$. Therefore, for $n \geq \max( N, N_1 )$,

$$Z_n \cap \bigcap_{i=1}^k \left( \bigcup_{j < h_{\gamma_j(n)}} A_{n,j}^{\gamma_i} \right) \cap \bigcap_{j < h_\alpha(n)} A_{n,j}^\alpha \neq \emptyset.$$  

This completes the induction.

Let $\mathcal{F}$ be the filter generated by the set $\{ \bigcup_{n < \omega} \bigcup_{j < h_\alpha(n)} A_{n,j}^\alpha : \alpha < c \}$. Since each $A_{n,j}^\alpha$ is closed in $Z_n$ and we are only taking a finite union in each $Z_n$, $\mathcal{F}$ is indeed a filter of closed sets. It follows easily from the induction that $\mathcal{F}$ is nice. Finally, for each $D \in \mathcal{D}$, as pointed out above, there is an $\alpha < c$ such that

$$D \cap \left( \bigcup_{n,m} A_{n,m}^\alpha \right) = \emptyset.$$  

This completes the proof. $\square$

We can now state the main result of this section.

**Theorem 3.2.** Assume $BF(c)$. Each compact ccc nowhere separable $F$-space of weight less than or equal to $c$ contains weak $P$-points.
Proof. Let $K$ be such an $F$-space and let $\{Z_n: n \in \omega\}$ be nonempty disjoint clopen subsets of $K$ (recall that $K$ is extremally disconnected). By Lemma 1.1, we need only construct a nice filter $\mathcal{F}$ on $X = \bigcup_n Z_n$ which avoids countable sets. From Lemma 3.1, we have a nice filter $\mathcal{F}$ on $X$ which avoids all nowhere dense sets. Since $X$ is nowhere separable we are done. $\square$

Our theorem concerning remote points also follows from Lemma 3.1.

**Theorem 3.3.** Assume $BF(c)$. Each nonpseudocompact ccc space of weight less than or equal to $c$ has remote points.

**Proof.** Let $X$ be such a space. Since $X$ is not pseudocompact there is a nonempty zero set $Z$ of $\beta X$ contained in $X^*$. Let $Y = \beta X \setminus Z$. Since $X$ is ccc and of weight less than or equal to $c$, so is $Y$. Let $\{Z_n: n \in \omega\}$ be a locally finite collection of disjoint nonempty regular closed subsets of $Y$. Let $\mathcal{F}$ be a remote filter on $\bigcup Z_n$ as constructed in Lemma 3.1. Then any $p \in \bigcap_{F \in \mathcal{F}} \text{cl}_{\beta X} F$ is a remote point of $Y$ and hence of $X$. To see this, first observe that since each $Z_n$ is a regular closed subset of $Y$, there is an $F \in \mathcal{F}$ such that $F \subseteq \text{int}_{Y}(\bigcup Z_n)$. Since $Y$ is normal, $\text{cl}_{\beta X} F \cap \text{cl}_{\beta X}(Y \setminus \bigcup Z_n) = \emptyset$. Therefore the only nowhere dense subsets of $X$ we have to worry about are those contained in $\bigcup Z_n$, but $\mathcal{F}$ avoids all such sets. $\square$

4. Another technique. In this section we develop another technique of constructing nice filters, which is an adaptation of Eric van Douwen's construction of remote points [vD]; the construction in [CS] is also similar. The $\Pi$-weight, $\Pi w X$, of a space $X$ is the least cardinality of a $\Pi$-base. A $\Pi$-base is a collection of nonempty open sets such that every nonempty open set of the space contains one from the collection. We construct nice filters in compact ccc $F$-spaces of $\Pi$-weight $\aleph_1$ which avoid all countable nowhere dense sets. We cannot hope to avoid all countable sets because it is consistent that all such spaces are separable [T]. We use the same method to construct nice filters which avoid countable discrete sets in compact ccc $F$-spaces with further $\Pi$-weight assumptions.

We first need a result of Efimov.

**Theorem 4.1** [E, p. 260]. A compact ccc $F$-space $X$ with no isolated points can be represented uniquely as $X = \bigcup_{n \in \mathbb{N}} U_n$, where the $U_n$ are disjoint open and closed sets of homogeneous $\Pi$-weight: that is, $\Pi w V = \Pi w U_n$ for any open $V \subseteq U_n$, and if $n \neq m$ then $\kappa_n \neq \kappa_m$.

From the above theorem it follows that when we are considering compact ccc $F$-spaces we may assume they have homogeneous $\Pi$-weight. For the remainder of this section all hypothesized spaces will be compact ccc $F$-spaces with homogeneous $\Pi$-weight. The $\Pi$-character of a point $x$ of $X$, denoted $\Pi X(x, X)$, is the least cardinal of a local $\Pi$-base at $x$.

**Lemma 4.2.** Let $X$ have $\Pi$-weight $\aleph_1$. Then

(i) $\Pi X(x, X) = \aleph_1$ for all $x \in X$, and

(ii) $\Pi X(D, X) = \aleph_1$ for any countable nowhere dense set $D$, i.e. given countably many open subsets of $X$ there is a neighborhood of $D$ containing none of them.
Proof. Recall that we are assuming that the \( \Pi \)-weight of every open set is \( S \).

(i) Let \( x \in X \) and let \( \{A_n : n \in \omega \} \) be nonempty clopen subsets of \( X \). It suffices to find a neighborhood of \( x \) which does not contain any of the \( A_n \)'s. Since \( \Pi w(A_n) = S \), for each \( n \in \omega \), we can begin by finding a clopen set \( B_0 \) such that \( B_0 \subseteq A_0 \) and \( A_n \setminus B_0 \neq \emptyset \) for all \( n \in \omega \). Similarly the set \( \{A_n \setminus B_0 : n \in \omega \} \) is not a \( \Pi \)-base for any open set, in particular, not for \( A_0 \setminus B_0 \). We can therefore choose a clopen set \( B_1 \) such that \( B_1 \subseteq A_0 \setminus B_0 \) and \( (A_n \setminus B_0) \setminus B_1 \neq \emptyset \) for all \( n \in \omega \). Similarly, recursively select clopen sets \( B_{2n} \) and \( B_{2n+1} \) such that

1. \( B_i \cap B_j = \emptyset \) for \( i < j \leq 2n + 1 \),
2. \( B_{2n} \cup B_{2n+1} \subseteq A_n \), and
3. \( A_k \setminus \left( \bigcup_{j=1}^{2n+1} B_j \right) \neq \emptyset \) for all \( k \in \omega \).

Therefore both \( \bigcup_{j \in \omega} B_{2j} \) and \( V = \bigcup_{j \in \omega} B_{2j+1} \) intersect each \( A_n \) by (2) and \( U \cap V = \emptyset \) because \( X \) is extremally disconnected. Hence either \( U \) or \( X \setminus U \) is the required neighborhood of \( x \) because neither set contains any of the \( A_n \)'s.

(ii) Let \( D = \{d_n : n \in \omega \} \) be nowhere dense in \( X \) and let \( \{A_n : n \in \omega \} \) be clopen subsets of \( X \). We again wish to find a neighborhood of \( D \) which does not contain any \( A_n \). One recursively selects nonempty clopen sets \( B_k \) and \( C_k \) for \( k \in \omega \) such that:

1. \( d_k \in B_k \), \( B_k \cap \left( \bigcup_{j<k} C_j \right) = \emptyset \) and \( A_n \setminus \left( B_k \cup \bigcup_{j<k} (B_j \cup C_j) \right) \neq \emptyset \) for \( n \in \omega \); and
2. \( C_k \cap \bar{D} = \emptyset \), \( C_k \subseteq A_k \setminus \left( \bigcup_{j<k} B_j \right) \) and \( A_n \setminus \left( \bigcup_{j<k} (B_j \cup C_j) \right) \neq \emptyset \) for \( n \in \omega \).

This recursion can be carried out because \( \Pi w(d_k, X) = S_1 \) for each \( k \in \omega \), \( \Pi w(A_n \setminus \left( \bigcup_{j<k} (B_j \cup C_j) \right)) = S_1 \) and because \( \bar{D} \) is nowhere dense in \( X \). This completes the proof.

\[ \Box \]

Theorem 4.3. Let \( X \) have \( \Pi \)-weight \( S_1 \). There are points in \( X \) which are not in the closure of any countable nowhere dense sets.

Proof. Let \( \{Z_n : n \in \omega \} \) be disjoint nonempty clopen subsets of \( X \). By Lemma 1.1 we must construct a nice filter on \( \bigcup Z_n \) which avoids all countable nowhere dense subsets of \( \bigcup Z_n \). For each \( n \in \omega \), let \( \{B_{n, a} : \alpha < \omega_1 \} \) be nonempty clopen subsets of \( Z_n \) which form a \( \Pi \)-base for \( Z_n \). Let \( \mathcal{D} = \{D \subseteq \bigcup Z_n : D \) is countable and nowhere dense\}. We will construct a family \( \{F_{D,n} : D \in \mathcal{D}, n \in \omega \} \) of clopen sets satisfying:

1. \( F_{D,n} \cap \bar{D} = \emptyset \);
2. for any \( \mathcal{D}_1 \subseteq \mathcal{D} \) with \( 1 \leq |\mathcal{D}_1| \leq n \), \( \bigcap_{D \in \mathcal{D}_1} F_{D,n} \neq \emptyset \); and
3. \( F_{D,n} \subseteq Z_n \); and then define \( F_D = \bigcup_n F_{D,n} \).

Our nice filter will be \( \mathcal{F} \) which has \( \{F_D : D \in \mathcal{D} \} \) as a filter base.

For each \( D \in \mathcal{D} \) and \( n \in \omega \) we define \( \alpha(D, n, 0) = \min\{\alpha : B_{n, \alpha} \cap \bar{D} = \emptyset \} \). Note that \( \alpha(D, n, 0) \) exists because \( \bar{D} \) is nowhere dense. By Lemma 4.2(ii) there is a clopen neighborhood \( U(D, n, 1) \) of \( D \) such that \( B_{n, \gamma} \setminus U(D, n, 1) \neq \emptyset \) for each \( \gamma < \alpha(D, n, 0) \). Let \( \alpha(D, n, 1) = \min\{\alpha : \) for each \( \gamma \leq \alpha(D, n, 0) \) there is a \( \xi < \alpha \) with \( B_{n, \xi} \subseteq B_{n, \xi} \cap U(D, n, 1) \}).

Then let \( K(D, n, 1) = \{\xi < \alpha(D, n, 1) : B_{n, \xi} \cap U(D, n, 1) = \emptyset \} \). Note that \( \bigcup \{B_{n, \xi} : \xi \in K(D, n, 1) \} \cap \bar{D} = \emptyset \) and that, for each \( \gamma < \alpha(D, n, 0) \), there is a \( \xi \in K(D, n, 1) \) with \( B_{n, \xi} \subseteq B_{n, \gamma} \). Recursively construct for \( m < n \), \( U(D, n, m + 1) \), clopen neighborhoods of \( D \) such that for each \( \gamma < \alpha(D, n, m) \),

\[ B_{n, \gamma} \setminus U(D, n, m + 1) \neq \emptyset, \]
ordinals $\alpha(D, n, m + 1) = \min \{ \xi < \alpha: \text{for each } \gamma \leq \alpha(D, n, m) \text{ there is a } \xi < \alpha \text{ with } B_{n, \xi} \subset B_{n, \gamma} \setminus U(D, n, m + 1) \}$, and sets $K(D, n, m) = \{ \xi < \alpha(D, n, m + 1): B_{n, \xi} \cap U(D, n, m + 1) = \emptyset \}$. Define

$$F_{D,n} = B_{n, \alpha(D,n,0)} \cup \bigcup_{m=1}^{n} \left[ \bigcup \{ B_{n,\xi}; \xi \in K(D,n,m) \} \right].$$

Let us check that this satisfies (2). Let $S$ be a subfamily of $\mathcal{O}$ with $1 \leq |S| \leq n$. Let $|S| = e$. With recursion on $j$ pick $E_j \in \mathcal{O} - \{ E_i: 0 < i \text{ and } i < j \}$, for $0 \leq j < e$ in such a way that

(4) $\alpha(E_j, n, j) \leq \alpha(E, n, j)$ for all $E \in \mathcal{O} - \{ E_i: 0 < i \text{ and } i < j \}$.

Next define $s(j) \in \omega_1$, for $0 \leq j < e$ by $s(0) = \alpha(E_0, n, 0)$;

$$s(j + 1) = \min \{ \xi \in K(E_{j+1}, n, j + 1): B_{n, \xi} \subseteq B_{n,s(j)} \}. $$

This is possible by the definition of $K(E_j, n, j + 1)$ and the fact that $s(j) < \alpha(E_j, n, j + 1)$ for each $0 \leq j < e$ by (4). Since each $B_{n,s(j)} \subseteq F_{E_j,n}$ for $0 \leq j < e$, it follows that $\bigcap_{j<e} F_{E_j,n} \supseteq B_{n,s(e)} \neq \emptyset$. This completes the proof. $\square$

**Corollary 4.4.** Let $X$ have $\Pi$-weight $\aleph_1$, and be nowhere separable. Then $X$ contains weak $P$-points.

**Remark.** An easy adaptation of the proof of Theorem 4.3 can be used to show the following. Let $X$ be a nonpseudocompact space of $\Pi$-weight at most $\aleph_1$. Then there is a point $x \in X^*$ such that $x \notin F$ for any countable nowhere dense $F \subset X$ of $\Pi$-character $\aleph_1$.

**Theorem 4.5.** Let $\Pi w(X) = \kappa$ be a regular cardinal and suppose that $\Pi \chi(x, X) = \kappa$ for each $x \in X$. Then $X$ contains points which are not the limit point of any countable discrete set.

**Proof.** If $\Pi w(X) = \omega$, this is shown in [vM]. For larger cardinals $\kappa$, it is simple to show that $\Pi \chi(D, X) = \kappa$ for each countable discrete set $D$ in $X$. Then simply replace $\aleph_1$ by $\kappa$ in the proof of 4.3.

We do not know if it is necessarily true that $\Pi \chi(x, X) = \Pi w(X)$ for each $x \in X$ in a compact ccc $F$-space of homogeneous $\Pi$-weight. We showed that it was true for $\Pi$-weight $\aleph_1$ in 4.2 and in the following theorem we show it for separable compact $F$-spaces assuming Martin's axiom for $\sigma$-centred posets, denoted MAS.

**Theorem 4.6 (MAS).** Let $X$ be a compact separable $F$-space. Then $\Pi \chi(x, X) = \Pi w(X)$ for each $x \in X$.

**Proof.** Let $\Pi w(X) = \kappa$ and note that $\kappa \leq c$ since $X$ is separable. Let $S$ be a countable dense subset of $X$ and suppose there is an $x \in X$ with $\Pi \chi(x, X) < \kappa$. Let us first prove the following fact.

**Fact 1.** Suppose that $x$ is the limit point of a discrete set $D$ and that $\Pi \chi(d, X) > \lambda$ for each $d \in D$. Then $\Pi \chi(x, X) > \lambda$. Indeed, suppose $\{ A_\alpha: \alpha < \lambda \}$ is a collection of clopen subsets of $X$. To show that it is not a $\Pi$-base at $x$ we will find a clopen neighborhood of $x$ not containing any $A_\alpha$. Since $D$ is discrete, let $D = \{ d_n: n \in \omega \}$ and $\{ V_n: n \in \omega \}$ be disjoint neighborhoods of the $d_n$'s. Since $\Pi \chi(d_n, X) > \lambda$ for
each \( n \), we can choose a neighborhood \( U_n \) of \( d_n \) such that \( U_n \subset V_n \) and, if \( A_\alpha \cap V_n \neq \emptyset \), then \( (A_\alpha \cap V_n) \setminus U_n \neq \emptyset \) for \( \alpha < \lambda \). Since \( X \) is extremely disconnected and \( x \in U = \bigcup U_n \), \( U \) is a neighborhood of \( x \). Let \( \alpha < \lambda \) and suppose that \( A_\alpha \subset U \). Then there is an \( n < \omega \) such that \( A_\alpha \cap U_n \neq \emptyset \) and therefore \( A_\alpha \cap V_n \neq \emptyset \). This is a contradiction because \( (A_\alpha \cap V_n) \setminus U_n \neq \emptyset \) by assumption and \( (U \cap U_n) \cap (A_\alpha \cap V_n) = \emptyset \). Therefore Fact 1 is true.

Now suppose that \( x \in X \) and \( \Pi X(x, X) = \lambda < \kappa \). Note that \( \{ s \in S : \Pi X(s, X) \leq \lambda \} \) is nowhere dense in \( X \) because the union of local \( \Pi \)-bases of a dense set is a \( \Pi \)-base for \( X \). Therefore we assume \( \Pi X(s, X) > \lambda \), for each \( s \in S \). Our plan now is to find a discrete subset of \( S \) which has \( x \) as a limit point, which will complete the proof.

Let \( \{ A_\alpha : \alpha < \lambda \} \) be a \( \Pi \)-base at \( x \) consisting of clopen subsets of \( X \). Also let \( \{ B_\alpha : \alpha < \kappa \} \) be clopen subsets of \( X \) which form a \( \Pi \)-base for \( X \). Recall that we are assuming that \( X \) has homogeneous \( \Pi \)-weight. We can inductively define \( B'_\alpha \) so that \( B'_\alpha \subset B_\alpha \) and for each \( \xi < \lambda \) and \( \gamma_1 < \cdots < \gamma_n < \alpha \), \( B'_\alpha \neq A_\xi \setminus \bigcup_{i=1}^n B'_{\gamma_i} \). Therefore \( \{ B'_\alpha : \alpha < \kappa \} \) is a \( \Pi \)-base such that no finite union of its members contains any \( A_\xi \).

Let \( (P, \leq) \) be the poset whose members are \( \{ (F, V) : F \in [S]^{\omega_1}, V \) is a finite union of sets from the \( \Pi \)-base; \( \Pi C V \} \). We will define \( (G, W) \leq (F, V) \) if \( F \subset G \), \( V \subset W \) and \( G \setminus F \cap V = \emptyset \). To see that \( (P, \leq) \) is \( \sigma \)-centred, we simply let \( P_\alpha = \{ (F, V) : (F, V) \in P \} \) for each \( F \in [S]^{\omega_1} \). Let \( E_\alpha = \{ (F, V) : F \cap A_\alpha \neq \emptyset \} \) for each \( \alpha < \lambda \). It is easy to check that \( E_\alpha \) is dense in \( (P, \leq) \). Therefore MAS allows us to choose a generic filter \( \mathcal{G} \) such that for each \( \alpha < \lambda \) there is an \( (F_\alpha, V_\alpha) \in \mathcal{G} \) with \( (F_\alpha, V_\alpha) \) also in \( E_\alpha \). Define \( D = \bigcup \{ F : \exists V \text{ with } (F, V) \in \mathcal{G} \} \).

First note that \( D \cap A_\alpha \neq \emptyset \) for each \( \alpha < \lambda \) and so \( x \in \bar{D} \). All we have to show is that \( D \) is discrete. Let \( d \in D \) and find \( (F_d, V_d) \in \mathcal{G} \) with \( d \in F_d \). By the definition of the partial ordering we see that \( D \cap V_D = F_D \). It follows that \( D \) is discrete. \( \square \)

We remark that MA implies that if \( X \) is compact, ccc and \( \Pi w(X) < \omega_1 \) then \( X \) is separable \([T]\).

**Corollary 4.7 (MAS).** Let \( X \) be a compact separable F-space with \( \Pi w(X) \) regular; then \( X \) contains points not the limit point of any countable discrete set.

**Proof.** Theorems 4.5 and 4.6.

**5. Example and remarks.**

**Example.** We now give the promised example of a compact nonseparable F-space which does not have any weak P-points. The author is grateful to the referee and also to Jan van Mill for suggesting this example. Let \( \omega_1 + 1 \) be the ordinals less than or equal to \( \omega_1 \) endowed with the order topology. Let \( E \) be the projective cover of \( \omega_1 + 1 \) and let \( k : \omega \times E \rightarrow \omega_1 + 1 \) be the canonical map. Let \( Z = k^{-1}([\omega_1]) \) and let \( Y = \beta(\omega \times E) \). Observe that \( Y \setminus \text{cl}_Y(\omega \times Z) \) is locally separable. Let \( \Pi : \omega \times Z \rightarrow \omega \) be the projection map and let \( \beta \Pi : \beta(\omega \times Z) \rightarrow \beta \omega \) be its Stone extension. Note that \( \beta(\omega \times Z) = \text{cl}_Y(\omega \times Z) \), as \( Y \) is an F-space and \( \omega \times Z \) is \( \sigma \)-compact. We can form the adjunction space \( M = Y \cup_{\beta \Pi} \beta \omega \) (see \([W, \text{Chapter 10}]\)). Since \( \text{cl}_Y(\omega \times Z) \) is a P-set of \( Y \) it is easy to show that \( M \) is an F-space. Clearly \( M \) is not separable and has no weak P-points.
Remarks. One would naturally conjecture that all compact nowhere separable $F$-spaces contain weak $P$-points. The remaining problem is to remove special set-theoretic assumptions for the case of compact ccc $F$-spaces of weight $\leq c$. Murray Bell has observed that if such a space has a $\sigma$-$n$-linked base for each $n$ then it contains weak $P$-points. One would also like to show that all compact $F$-spaces contain points which are not the limit point of any countable discrete set. Is it true that all compact ccc $F$-spaces of homogeneous $\Pi$-weight also have homogeneous $\Pi$-character?

References


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