AN ALTERNATING SUM FORMULA
FOR MULTIPLICITIES IN $L^2(T \setminus G)$

BY

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Abstract. We prove an alternating sum formula for multiplicities in $L^2(T \setminus G)$, where $G$ is a semisimple Lie group of split rank one with finite center and $T$ is a discrete cocompact torsion free subgroup.

Introduction. Let $G$ be a connected semisimple Lie group with finite center. Let $G = K \cdot A \cdot N$ be an Iwasawa decomposition of $G$ and $g = k \oplus a \oplus n$ the corresponding decomposition at the algebra level. Let $M$ denote the centralizer of $A$ in $K$. We will assume throughout this paper that $G$ has rank one, that is, $\dim(a) = 1$. We will not assume that $G$ is linear. Let $\mathcal{E}(G)$ denote the set of equivalence classes of irreducible unitary representations of $G$. Given a homomorphism $\mu : \mathcal{L} \rightarrow \mathbb{C}$, where $\mathcal{L}$ is the center of $\mathfrak{u}(g)$ (the universal enveloping algebra of $G$), let $\mathcal{E}_\mu(G)$ denote the (finite) subset of $\mathcal{E}(G)$ of those classes with infinitesimal character $\mu$. If $\mathcal{S}$ is any subset of $\mathcal{E}(G)$ we write $\mathcal{S}_\mu = \mathcal{S} \cap \mathcal{E}_\mu(G)$.

Let $T$ be a discrete, torsion free subgroup of $G$ such that $T \setminus G$ is compact. The right regular representation decomposes into a direct sum $\tau = \bigoplus_{\omega \in \mathcal{E}(G)} \omega$, and $\mathcal{N}(\omega) < \infty$, for any $\omega \in \mathcal{E}(G)$. The purpose of this paper is to prove an alternating sum formula connecting the multiplicities $\mathcal{N}(\omega)$ of representations having the same infinitesimal character. We now describe our result.

Let $K$ and $M$ be as above. Let $i^* : R(K) \rightarrow R(M)$ denote the restriction map from the representation ring of $K$ to that of $M$. If $\tau \in \mathcal{E}(K)$ and $\omega \in \mathcal{E}(G)$, then $[\tau : \omega]$ stands for the multiplicity of $\tau$ in the restriction of $\omega$ to $K$. If $\eta \in R(K)$ we abuse notation by writing $[\eta : \omega] = \sum \mathcal{M}_\eta \cdot [\tau : \omega]$, where $\mathcal{M}_\eta = \sum \mathcal{M}_\eta \cdot \tau$.

Let $\mathcal{E}_c(G)$ denote those classes in $\mathcal{E}(G)$ corresponding to the irreducible unitary principal series or the complementary series and let $\mathcal{E}_d(G)$ denote the discrete series of $G$. Let $\alpha : \mathcal{E}(G) \rightarrow \mathbb{Z}$ be defined as follows: $\alpha(\omega) = 0$, if $\omega \in \mathcal{E}_c(G)$; $\alpha(\omega) = n_\tau(\omega) - d(\omega) \cdot \text{vol}(T \setminus G)$, if $\omega \in \mathcal{E}_2(G)$; and $\alpha(\omega) = n_\tau(\omega)$ if $\omega \in \mathcal{E}(G) - (\mathcal{E}_c(G) \cup \mathcal{E}_2(G))$. Here $d(\omega)$ denotes the formal degree of $\omega$. (We observe that $d(\omega)$ and vol$(T \setminus G)$ depend on the choice of the Haar measure on $G$, but their product does not). Finally let $\Lambda$ be a fixed infinitesimal character of $G$. We prove (Theorem 1.2) that if $\eta \in \ker(i^*)$, then

$$\sum_{\omega \in \mathcal{E}_\Lambda(G)} \alpha(\omega) \cdot [\eta : \omega] = 0.$$
From the definition of $\alpha(\omega)$ we see that the formula relates multiplicities of unitarizable Langlands quotients with multiplicities of nonintegrable discrete series (resp. limits of discrete series) when $\Lambda$ is regular (resp. singular). These connections were already observed by Wallach in [9, §9], in the case of $SU(2, 1)$, and our formula can be regarded as a generalization of the family of equations that he obtains. In fact, when (1) is specialized to $SU(2, 1)$, it yields precisely Wallach’s formulas.

The outline is as follows. In §1 we introduce preparatory material to prove our result (Theorem 1.2), while in §2 we discuss in detail the examples of $SL(2, \mathbb{R})$ and $SU(2, 1)$, making formula (1) very explicit for these groups. We wish to thank D. De George, N. Wallach and C. Rader for useful comments on the contents of this paper.

1. We keep the notation from the introduction. If $\sigma \in \mathcal{S}(M)$ and $\nu \in a^*_\mathbb{C}$, let $\pi_{\sigma,\nu}$ denote the principal series of $G$, parametrized as in [1, §3]. In this parametrization $\pi_{\sigma,\nu}$ is unitary if and only if $\nu \in a^*$, and $\pi_{\sigma,\nu}$ is reducible only if $\nu = 0$ [5]. In this case $\pi_{\sigma,\nu}$ splits as a direct sum of two inequivalent irreducible representations. We denote by $\mathcal{S}_r(G)$ the classes or irreducible constituents of reducible unitary principal series. It is a well-known fact that $\mathcal{S}_r(G) \cap \mathcal{S}_c(G) = \emptyset$. We will also set

$$\mathcal{S}_L(G) = \mathcal{S}(G) - (\mathcal{S}_r(G) \cup \mathcal{S}_2(G) \cup \mathcal{S}_c(G)).$$

If $\omega \in \mathcal{S}(G)$ let $\Lambda_{\omega}$ denote the infinitesimal character of $\omega$. If $\Omega$ is the Casimir element of $G$ we write $\Lambda_{\omega}(\Omega) = \lambda_{\omega}$. Also, the eigenvalue of $\Omega$ for the (nonunitary) principal series $\pi_{\sigma,\nu}$ will be denoted by $\lambda_{\sigma,\nu}$.

For fixed $(\tau, V_\tau) \in \mathcal{S}(K)$ we form the homogeneous vector bundle over $G/K$:

$$E_\tau = G \times_{\rho_\tau} \text{End}(V_\tau)$$

where $\rho_\tau(k)(A) = A \cdot \tau(k^{-1})$, if $A \in \text{End}(V_\tau)$. The Casimir element of $G$ defines an elliptic, second order differential operator $D$ on $E_\tau$. Under the standard identification of cross sections with vector-valued functions $f: G \to \text{End}(V_\tau)$ transforming according to $\rho_\tau$, and of these with scalar-valued functions that are invariant under convolution with $d_\tau \cdot \chi_\tau$, the fundamental solution of the heat equation $D = \partial/\partial s = 0$ on $E_\tau$ is given by convolution with a scalar multiple of $g_{\tau,s}$, where

$$g_{\tau,s}(x) = \int_{\mathcal{S}_r(G)} d_{\tau}^{-1} \cdot \phi_{\tau,\omega}(x^{-1}) \cdot e^{s\lambda_{\omega}} \mu(\omega)$$

for $x \in G$ and $s > 0$. Here $\mu(\omega)$ denotes the Plancherel measure and $\phi_{\tau,\omega}$ is the $\tau$-spherical trace function of $\omega$. That is,

$$\phi_{\tau,\omega}(x) = \text{tr}(E_\tau \cdot \pi_\omega(x) \cdot E_\tau)$$

where $E_\tau$ is the orthogonal projection onto the space of vectors transforming according to $\tau$. In the next theorem we collect the properties of $g_{\tau,s}$ that we will need. If $\omega \in \mathcal{S}(G)$, $\theta_\omega$ will denote the distributional character of $\omega$. 

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1.1 Theorem. Let $G$ be a Lie group of split rank one with finite center. Fix $	au \in \mathcal{S}(K)$ and let $\psi_\tau = d_\tau \cdot \tilde{\chi}_\tau$. Then

(i) $\psi_\tau \ast g_{r,s} \ast \psi_\tau = g_{r,s}$.
(ii) $g_{r,s} \in \mathcal{O}^p(G)$, for any $p > 0$.
(iii) $\theta_\omega(g_{r,s}) = [\tau : \omega] \cdot e^{\lambda_\omega}$, for any $\omega \in \mathcal{S}(G)$.

Here $\mathcal{O}^p(G)$ denotes (if $0 < p < 2$) the $p$-Schwartz space of $G$ (see [1, §4], for instance).

Proof. Assertion (i) follows from the definition of $g_{r,s}$. In [7, Theorem 4.11], (ii) was proved under the assumption that $\tau$ restrict simply to $M$ ($g_{r,s}$ coincides with the function $h_{r,s}$ defined there). Recently De George and Wallach (unpublished) have shown that (ii) is true for semisimple groups of arbitrary rank. We now prove (iii). From the definition,

$$g_{r,s}(x) = \sum_{\sigma \in \mathcal{S}(M)} d_{r,s}^{-1} \int_{a^*} \mathrm{tr}(E_{r} \pi_{\sigma,s}(x) E_{r}) e^{i\lambda_\sigma \cdot \mu_\omega(v)} dv + g_{r,s}^0(x),$$

where $g_{r,s}^0$ is a linear combination of matrix entries of discrete series representations. By the Plancherel theorem [3] the operator $\pi_{\sigma,s}(g_{r,s}) = d_{r,s}^{-1} \cdot e^{i\lambda_\sigma} \cdot E_{r}$, if $\nu \in a^*$. Since $g_{r,s} \in \mathcal{O}^p(G)$ for any $p > 0$, by analytic continuation we see that this holds for $\nu \in a^*_e$. Since any irreducible unitary representation $(\pi, V)$ of $G$ is infinitesimally equivalent to a subquotient of $(H_{p,s}, \pi_{0,s})$ for some $\sigma \in \mathcal{S}(M)$, $\nu \in a^*_e$, the above clearly implies that $\pi(g_{r,s}) = d_{r,s}^{-1} \cdot e^{i\lambda_\sigma} \cdot E_{r,\sigma}$, where $E_{r,\sigma}$ denotes the orthogonal projection onto the subspace of $V$ of vectors transforming according to $\tau$. Hence $\theta_\omega(g_{r,s}) = \mathrm{tr} \pi(g_{r,s}) = [\tau : \pi] \cdot e^{i\lambda_\sigma}$.

Let $\alpha : \mathcal{S}(G) \to \mathbb{Z}$ be as defined in the introduction and let $i^* : R(K) \to R(M)$ be the ring homomorphism induced by the inclusion $i : M \to K$.

1.2 Theorem. Let $G$ be a semisimple Lie group of split rank one with finite center and let $\Gamma \subset G$ be a discrete, cocompact, torsion free subgroup. Fix $\Lambda$ an infinitesimal character of $G$. If $\eta \in \ker(i^*)$, then

$$\sum_{\omega \in \mathcal{S}_{\Lambda}(G)} \alpha(\omega) \cdot [\eta : \omega] = 0.$$

Proof. Let $\eta = \sum_{j=1}^k m_j \cdot \tau_j$. For fixed $\tau$ the set

$$\mathcal{S}(\tau) = \{ \omega \in \mathcal{S}(G) - \mathcal{S}_e(G) | [\tau : \omega] \neq 0 \}$$

is finite (see for instance [1, §4]). Set $\mathcal{S} = \bigcup_{\tau} \mathcal{S}(\tau)$ and let $\mathcal{F} = \{ \Lambda_1, \ldots, \Lambda_m \}$ be the distinct infinitesimal characters of elements of $\mathcal{S}$. If $\Lambda \notin \mathcal{F}$ and $\omega \in \mathcal{S}_{\Lambda}(G)$ then either $[\eta : \omega] = 0$ or $\omega \in \mathcal{S}_e(G)$ and in this case $\alpha(\omega) = 0$. Hence if $\Lambda \notin \mathcal{F}$, the left-hand side of (1) equals zero. We thus assume that $\Lambda \in \mathcal{F}$. Then there exists $z \in \mathcal{Z}$ such that $\Lambda(z) = 1$ and $\Lambda'(z) = 0$, if $\Lambda' \in \mathcal{F}, \Lambda' \neq \Lambda$. Indeed, if $\Lambda = \Lambda_1$, let $u \in \mathcal{Z}$ be such that $\Lambda_1(u) \neq \Lambda_j(u)$ (if $i \neq j$). We may then take

$$z = \prod_{j>1} \frac{u - \Lambda_j(u)}{\Lambda_1(u) - \Lambda_j(u)}.$$
Let us set \( g_{\eta,z} = \sum m_j \cdot g_{\eta,z} \). By 1.1, \( \theta_\omega(g_{\eta,z}) = [\eta : \omega] \cdot e^{i\lambda_\omega} \) is valid for any \( \omega \in \mathcal{E}(G) \). We now apply the Selberg trace formula to \( z \cdot g_{\eta,z} \). This is legitimate since \( z \cdot g_{\eta,z} \in C^l(G) \) is \( K \)-finite (see [7, Corollary 2.16]). Moreover, by the polynomial character of \( \Lambda_{\eta,z}(z) \) it follows, as in the proof of Theorem 5.1 in [7], that

\[
\text{tr} \pi_\Gamma(z \cdot g_{\eta,z}) \sim \text{vol}(\Gamma \setminus G) \cdot (z \cdot g_{\eta,z})(1), \quad \text{as } s \to 0^+.
\]

(If \( \phi, \psi \in C^\infty(R^+) \) we write \( \phi(s) \sim \psi(s) \) if \( \phi(s) - \psi(s) = o(s^n) \) as \( s \to 0^+ \), for all \( n \in \mathbb{N} \).)

Hence by 1.1 we have, as \( s \to 0^+ \),

\[
\sum_{\eta \in \mathcal{E}(G)} n_\Gamma(\eta) \cdot [\eta : \omega] \cdot \Lambda_\omega(z) \cdot e^{i\lambda_\omega} \\
\sim \text{vol}(\Gamma \setminus G) \left( \sum_{\sigma \in \mathcal{E}(M)} [\sigma : \eta] \cdot \int_{a^*} \Lambda_{\eta,z}(z) \cdot e^{i\lambda_\omega} \cdot \mu_\nu(v) \, dv \right) \\
+ \text{vol}(\Gamma \setminus G) \left( \sum_{\omega \in \mathcal{E}(G)} d(\omega) \cdot [\eta : \omega] \cdot \Lambda_\omega(z) \cdot e^{i\lambda_\omega} \right).
\]

Since by assumption \( i^*(\eta) = 0 \), the sum corresponding to \( \mathcal{E}(G) \) in the left-hand side drops out and so does the first summand in the right-hand side. On the other hand, by the choice of \( z \), we need only sum over \( \mathcal{E}_\Lambda(G) \). Hence, as \( s \to 0^+ \),

\[
\sum_{\eta \in \mathcal{E}_\Lambda(G) - \mathcal{E}_\omega(G)} n_\Gamma(\eta) \cdot [\eta : \omega] \cdot e^{i\lambda_\omega} \sim \sum_{\omega \in \mathcal{E}_\omega(G)} \text{vol}(\Gamma \setminus G) \cdot d(\omega) \cdot [\eta : \omega] \cdot e^{i\lambda_\omega},
\]

and then both sides are equal since all sums are finite. This proves the theorem.

**Remark.** It is well known that if \( \omega \in \mathcal{E}_\omega(G) \) is integrable then \( n_\Gamma(\omega) = d(\omega) \cdot \text{vol}(\Gamma \setminus G) \) (in fact, this equality holds for certain families of nonintegrable classes as well (see [4, 8])). Hence, in formula (1), only nonintegrable classes do contribute to the sum.

2. We now analyze the cases of \( SL(2, \mathbb{R}) \) and \( SU(2, 1) \). We describe a basis over \( \mathbb{Z} \) of \( \ker(i^*) \) for each group and apply formula (1) to \( \eta \), an element of the basis. In this manner we retrieve Langlands’ formulas for the multiplicities of the discrete series for \( SL(2, \mathbb{R}) \). In the case of \( SU(2, 1) \), when \( \eta \) runs over a basis of \( \ker(i^*) \), we obtain (aside from some trivial equations) exactly Wallach’s formulas of [9, §9].

Let \( G = SL(2, \mathbb{R}) \). Then \( K = SO(2) \) and \( M = \{ \pm I \} \). If

\[
k(\theta) = \begin{vmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{vmatrix}
\]

let \( \tau_\eta(k(\theta)) = e^{i\eta \theta} \). We have that \( \mathcal{E}(K) = \{ \tau_n | n \in \mathbb{Z} \} \) and \( \mathcal{E}(M) = \{ 1, e \} \), where \( e \) stands for the signum representation. The following lemma is straightforward.

**2.1 Lemma.** A basis of \( \ker(i^*) \) is given by

\[
\mathcal{B} = \{ \eta_j = \tau_j - \tau_{j-2} | j \in \mathbb{Z} \}.
\]

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We recall some well-known facts on $\mathcal{D}(G)$ (see [6]). We have that $\mathcal{D}_2(G) = \{\pi_k|k > 1\}$ and $\mathcal{D}_2(G) = \{\pi^\pm\}$ where

$$\pi^\pm_k = \sum_{j=0}^{\infty} \pi^\pm(2j+k) \quad \text{and} \quad \pi^\pm = \sum_{j=0}^{\infty} \pi^\pm(2j+1).$$

Moreover $\pi^\pm_j$ is integrable if and only if $j > 2$ and $\mathcal{D}_2(G) = \{1\}$.

Formula (1) when applied to $\eta_k = \pi_k - \pi_{k-2}$ gives ($k > 1$)

$$a(\pi_k) \cdot [\eta_k : \pi_k] = 0 \quad \text{or} \quad n_1(\pi_k) = d(\pi_k) \cdot \text{vol}(\Gamma \setminus G).$$

If $k = 2$, $\eta_2 = \pi_2 - 1$ and then $[\eta_2, \omega] = 1$ if $\omega = \pi_2$, $[\eta_2, \omega] = -1$ if $\omega = 1$ and $[\eta_2, \omega] = 0$ for any other representation $\omega \in \mathcal{D}(G) - \mathcal{D}_2(G)$. Since $n_1(1) = 1$ one thus obtains Langlands' formula

$$n_1(\pi_2) = d(\pi_2) \cdot \text{vol}(\Gamma \setminus G) + 1.$$  

Similar equations to (2) and (3) are obtained for $\pi_{-k-2}$ and $\pi_{-2}$, by using $\eta_{-k}$ and $\eta_0$. Finally, if $\eta_1 = \pi_1 - \pi_{-1}$, we get

$$n_1(\pi^+) = n_1(\pi^-).$$

Note that formula (1) gives no more information on $n_1(\pi^\pm)$.

We now discuss the case of $G = SU(2, 1)$, i.e.

$$SU(2, 1) = \{A \in Gl(3, C)|H(A(z)) = H(z)\}$$

where $z = (z_1, z_2, z_3) \in C^3$ and $H(z) = |z_1|^2 + |z_2|^2 - |z_3|^2$. Then

$$K = \begin{cases} A & 0 \\ 0 & \det A^{-1} \end{cases} |A \in U(2)$$

is a maximal compact subgroup and we may take $a \subset p$ so that $M = \{\text{diag}(a, b, a)|a^2 \cdot b = 1\}$. Here $\text{diag}(z_1, z_2, z_3)$ denotes the diagonal matrix in $Gl(3, C)$ with entries $z_1, z_2, z_3$. Let $l(\theta) = \text{diag}(e^{i\theta}, e^{i\theta}, e^{-2i\theta})$ and $m(\theta) = \text{diag}(e^{i\theta}, e^{-2i\theta}, e^{i\theta})$. Given $l \in Z, p \in Z$ such that $p > 0, p \equiv l \pmod{2}$ there exists a unique irreducible representation of $K, \pi_p$, such that $\dim \pi_p = p + 1$ and $\pi_p(l(\theta)) = e^{-ip\theta} \cdot I$. Furthermore, the representations $\pi_p$ exhaust $\mathcal{D}(K)$.

If $k \in Z$, let $\sigma(k)$ denote the irreducible representation of $M$ such that $\sigma(k)(m(\theta)) = e^{ik\theta} \cdot I$. When restricted to $M, \pi_p$ decomposes as follows:

$$i^*(\pi_p) = \sum_{j=0}^{p} \sigma(r - 3j)$$

where $r = (l + 3p)/2$. We may thus associate to $\pi_p \in \mathcal{D}(K)$ an interval $[r - 3p, r]$ in $r + 3Z$. Conversely, given an interval $[a, b]$ in $c + 3Z (c = 0, 1$ or $2$) there is a unique $K$-type $\pi_p$ such that the associated interval is $[a, b]$ (in fact $p = (b - a)/3$ and $l = a + b$). In the light of this correspondence we may construct elements in $\ker(i^*)$ in a simple manner. Let $I_1, \ldots, I_k$ be disjoint intervals in $c + 3Z$ such that their union is an interval $I$. Then, if $\pi(I)$ denotes the $K$-type associated to $I$, clearly $\eta = \pi(I) - \sum j \pi(I_j) \in \ker(i^*)$. In particular, any interval can be decomposed as a union of singletons. We will set

$$\eta(I) = \pi(I) - \sum_{j=0}^{b} \tau(j) \quad \text{where} \quad \tau(j) = \pi([j, j]).$$

The following lemma is not difficult.
2.2 Lemma. The family $\mathcal{B} = \{ \eta(I) | I an interval in c + 3\mathbb{Z}, c = 0, 1 or 2 \}$ is a basis of $\ker(i^*)$.

In order to apply formula (1) to elements in $\mathcal{B}$ we need to recall several facts from the representation theory of $SU(2, 1)$. We will recall them very briefly. We refer the reader to [9, §7] for details.

Let $\Delta^+$ be the system of positive roots as defined in [9, p. 181] and let $\delta = \frac{1}{2}(\sum_\alpha \alpha)$. Also, let $\alpha_1, \alpha_2$ be the simple roots and $\Lambda_1, \Lambda_2$ the basic highest weights for this system. Finally, let $s_1, s_2$ be the reflections with respect to $\alpha_1$ and $\alpha_2$, respectively.

In the notation of [9, §7], if $\omega \in \mathcal{E}(G) - \mathcal{E}_c(G)$, then $\omega$ is one of:

$$D_{s_1 s_2 s_1}(\Lambda + \delta) - \delta, \quad D_{s_2 s_1 s_2}(\Lambda + \delta) - \delta, \quad \Lambda = k_1 \Lambda_1 + k_2 \Lambda_2, \quad k_i \in \mathbb{Z}, \quad k_0 > 0;$$

$$T_k^\pm, \quad k \in \mathbb{Z}, \quad k \geq -1;$$

$$\pi_{\Lambda}^\pm, \quad \text{where } \Lambda = k\Lambda_1 + (-2 - k)\Lambda_2 \text{ or } \Lambda = (-2 - k)\Lambda_1 + k\Lambda_2, \quad k > 0; \text{ or}$$

$$\omega = 1, \text{ the trivial representation.}$$

If $\Lambda = k_1 \Lambda_1 + k_2 \Lambda_2 \in \mathfrak{h}^*$ let $\chi_\Lambda$ denote the homomorphism $\chi_\Lambda: \delta \rightarrow \mathbb{C}$ as defined by Harish-Chandra [2]. The infinitesimal characters of the above representations are as follows:

$$D_{s_1 s_2 s_1}(\Lambda + \delta) - \delta, \quad D_{s_2 s_1 s_2}(\Lambda + \delta) - \delta \text{ have infinitesimal character } \chi_{\Lambda + \delta};$$

$$T_k^+ \text{ (resp. } T_k^-) \text{ has infinitesimal character } \chi_{\Lambda_1 + \delta} \text{ (resp. } \chi_{\Lambda_2 + \delta});$$

$$\pi_{\Lambda}^\pm \text{ have infinitesimal character } \chi_{\Lambda + \delta}.$$

From now on we will write $D_{\Lambda}^+, D_{\Lambda}^- \text{ and } D_{\Lambda}$ instead of $D_{s_1 s_2 s_1}(\Lambda + \delta) - \delta, \quad D_{s_2 s_1 s_2}(\Lambda + \delta) - \delta \text{ and } D_{s_1 s_2 s_1}(\Lambda + \delta) - \delta$, respectively. Also $\pi_{\Lambda}^\pm$ stand for $\pi_{\Lambda}^\pm$, if $\Lambda = k\Lambda_1 + (-2 - k)\Lambda_2$, and similarly $\pi_{\Lambda k}^\pm = \pi_{\Lambda}^\pm$, if $\Lambda = (-2 - k)\Lambda_1 + k\Lambda_2$. Finally, if $\mathcal{S} \subset \mathcal{E}(G)$ and $\Lambda \in \mathfrak{h}^*$, we abbreviate $\mathcal{S}_\Lambda = \mathcal{S}_{\chi_{\Lambda + \delta}}$. With this notation in mind the classification of $\mathcal{E}(G) - \mathcal{E}_c(G)$ according to the infinitesimal character is:

$$\mathcal{E}_{d,\Lambda} = \{ D_{\Lambda}^+, \quad D_{\Lambda}^-, \quad D_{\Lambda} \}, \quad \text{if } \Lambda = k_1 \Lambda_1 + k_2 \Lambda_2, \quad k_i > 0,$$

$$\mathcal{E}_{r,\Lambda} = \{ T_k^+ \}, \quad \text{if } \Lambda = k\Lambda_1, \quad k > 0 \text{ or } k = -1,$$

$$\mathcal{E}_{r,\Lambda} = \{ T_k^- \}, \quad \text{if } \Lambda = k\Lambda_2, \quad k > 0 \text{ or } k = -1,$$

$$\mathcal{E}_{r,\Lambda} = \{ T_0^+, \quad T_0^-, \quad 1 \}, \quad \text{if } \Lambda = 0,$$

$$\mathcal{E}_{r,\Lambda} = \{ \pi_{\Lambda k}^+ \}, \quad \text{if } \Lambda = k\Lambda_1 + (-2 - k)\Lambda_2, \quad k > 0,$$

$$\mathcal{E}_{r,\Lambda} = \{ \pi_{\Lambda k}^- \}, \quad \text{if } \Lambda = (-2 - k)\Lambda_1 + k\Lambda_2, \quad k > 0.$$

The next lemma gives the $K$-type structure of these representations. It is a consequence of Lemma 7.9 in [9] and the relations

$$s_1 s_2(\Lambda + \delta) - \delta = -(k_1 + k_2 + 3)\Lambda_1 + k_1 \Lambda_2,$$

$$s_2 s_1(\Lambda + \delta) - \delta = k_2 \Lambda_1 - (k_1 + k_2 + 3)\Lambda_2,$$

$$s_1 s_2(\Lambda + \delta) - \delta = -(k_2 + 2)\Lambda_1 - (k_1 + 2)\Lambda_2.$$
2.3 Lemma. If $\omega \in \tilde{\mathcal{E}}(G)$, let $j^*(\omega)$ denote the restriction of $\omega$ to $K$. Then

$$j^*(D^+_{\Lambda}) = \sum_{0 < q < k_1, 0 < p < k_2} \tau^{4k_1 + 2k_2 + 6 + 3(p - q)}, \quad j^*(D^-_{\Lambda}) = \sum_{0 < p < k_1, 0 < q} \tau^{4k_2 - 2k_1 - 6 + 3(p - q)},$$

$$j^*(D_{\Lambda}) = \sum_{k_1 < p, k_2 < q} \tau^{2k_2 - 2k_1 + 3(p - q)},$$

$$j^*(T^+_{k}) = \sum_{k < p} \tau^{-2k + 3p}, \quad j^*(T^-_{k}) = \sum_{k < q} \tau^{2k - 3q} \quad (k > -1),$$

$$j^*(\pi^+_{1,k}) = \sum_{0 < p < k, 0 < q} \tau^{-4k - 4k + 3(p - q)}, \quad j^*(\pi^-_{1,k}) = \sum_{k < p, 0 < q} \tau^{-4k + 4k + 3(p - q)},$$

$$j^*(\pi^+_{2,k}) = \sum_{0 < p < k, 0 < q < k} \tau^{4k + 4k + 3(p - q)}, \quad j^*(\pi^-_{2,k}) = \sum_{0 < p, k < q} \tau^{4k + 4k + 3(p - q)}.$$ 

It will be convenient to write this information on the $K$-types in terms of intervals in $r + 3 \cdot \mathbb{Z}$. Recall that $\tau_p^I$ is associated to $[r - 3p, r]$ where $r = (l + 3p)/2$. We then have the following table of representations and corresponding intervals (or $K$-types).

- $D^+_{\Lambda} : [2k_1 + 2k_2 + 3 - 3q, 2k_1 + 2k_2 + 3 + 3p], \quad p > 0, \quad k_1 > q > 0,$
- $D^-_{\Lambda} : [-2k_2 - k_1 - 3 - 3q, -2k_2 - k_1 - 3 + 3p], \quad q > 0, \quad k_2 > p > 0,$
- $D_{\Lambda} : [k_2 - k_1 - 3q, k_2 - k_1 + 3p], \quad p > k_1, \quad q > k_2,$
- $T^+_{k} : [-k, -k + 3p], \quad p > k,$
- $T^-_{k} : [k - 3q, k], \quad q > k,$
- $\pi^+_{1,k} : [-2 - 2k - 3q, -2 - 2k + 3p], \quad k > p > 0, \quad q > 0,$
- $\pi^-_{1,k} : [-2 - 2k - 3q, -2 - 2k + 3p], \quad p > k, \quad q > 0,$
- $\pi^+_{2,k} : [2 + 2k - 3q, 2 + 2k + 3p], \quad p > 0, \quad k > q > 0,$
- $\pi^-_{2,k} : [2 + 2k - 3q, 2 + 2k + 3p], \quad p > 0, \quad q > k,$
- $1 : [0, 0].$

It is now a simple matter to apply formula (1) of Theorem 1.2 to $K$-types of the form $\eta(I) = \tau(I) - \sum_{a} \tau(j)$ where $I = [a, b]$. By Lemma 2.2 any other $\eta \in \ker(i^*)$ yields an equation which is a combination of equations of this type. We assume first that $\Lambda = k_1 \Lambda_1 + k_2 \Lambda_2, k_i > 1$. Then $\tilde{\mathcal{E}}_{\Lambda}(G) = \{D^+_{\Lambda}, D^-_{\Lambda}, D_{\Lambda}\}$. It is easy to check that if $\tau(I)$ is a $K$-type of $D_{\Lambda}$, $\eta(I)$ yields the equation

$$n_1(D^+_{\Lambda}) + n_1(D^-_{\Lambda}) - n_1(D_{\Lambda}) = d(D_{\Lambda}) \cdot \text{vol}(\Gamma \setminus G).$$

(Note that $d(D^+_{\Lambda}) = d(D^-_{\Lambda}) = d(D_{\Lambda})$.) On the other hand, if $\tau(I)$ is a $K$-type of $D^+_{\Lambda}$ or $D^-_{\Lambda}$ we get the trivial equation. If we use $\eta(I_1), \eta(I_2)$ where $I_1 = [k_2 - k_1, 2k_1 + k_2 + 3], I_2 = [-2k_2 - k_1 - 3, k_2 - k_1]$, we obtain

$$n_1(D^+_{\Lambda}) = d(D^+_{\Lambda}) \cdot \text{vol}(\Gamma \setminus G),$$

$$n_1(D^-_{\Lambda}) = d(D^-_{\Lambda}) \cdot \text{vol}(\Gamma \setminus G);$$
hence we have also

\begin{equation}
    n_{t}(D_{A}) = d(D_{A}) \cdot \text{vol}(\Gamma \setminus G). \tag{8}
\end{equation}

We recall from \cite[Lemma 7.7]{9} that if \(k_{1} \geq 2, k_{2} = 1\) then \(D_{A}\) and \(D_{A}^{+}\) are not integrable. If \(k_{1} = 1, k_{2} \geq 1\), then \(D_{A}\) and \(D_{A}^{-}\) are not integrable.

When \(k_{2} = 0\) and \(k_{1} = k > 0\), (5) and (7) are still true and (6) changes into

\begin{equation}
    n_{t}(D_{A}^{+}) - d(D_{A}^{+}) \cdot \text{vol}(\Gamma \setminus G) = n_{t}(T_{k}^{+}) \tag{9}
\end{equation}

since \(\tau([-k, 2k + 3])\) is a \(K\)-type of \(T_{k}^{+}\). Instead of (8),

\begin{equation}
    n_{t}(D_{A}) = n_{t}(D_{A}^{+}). \tag{10}
\end{equation}

Similarly, if \(k_{1} = 0, k_{2} = k > 0\), (5) and (6) are valid and, moreover,

\begin{equation}
    n_{t}(D_{A}^{-}) - d(T_{k}^{-}) \cdot \text{vol}(\Gamma \setminus G) = n_{t}(T_{k}^{-}), \tag{11}
\end{equation}

\begin{equation}
    n_{t}(D_{A}) = n_{t}(D_{A}^{-}). \tag{12}
\end{equation}

If \(\Lambda = 0\), because of the presence of the trivial representation, (5), (9) and (11) change into

\begin{equation}
    n_{t}(D_{0}^{+}) + n_{t}(D_{0}^{-}) - n_{t}(D_{0}) = d(D_{0}) \cdot \text{vol}(\Gamma \setminus G) - 1, \tag{13}
\end{equation}

\begin{equation}
    n_{t}(D_{0}^{+}) - d(D_{0}^{+}) \cdot \text{vol}(\Gamma \setminus G) = n_{t}(T_{0}^{+}) - 1, \tag{14}
\end{equation}

\begin{equation}
    n_{t}(D_{0}^{-}) - d(D_{0}^{-}) \cdot \text{vol}(\Gamma \setminus G) = n_{t}(T_{0}^{-}) - 1. \tag{15}
\end{equation}

If \(\Lambda = k\Lambda_{1} + (-2 - k)\Lambda_{2}\) then

\begin{equation}
    n_{t}(\pi_{1,k}^{+}) = n_{t}(\pi_{1,k}^{+}) \quad \text{if} \quad k > 0, \quad \text{and} \quad n_{t}(\pi_{1,0}^{-}) - n_{t}(\pi_{1,0}^{+}) = n_{t}(T_{0}^{+}). \tag{16}
\end{equation}

If \(\Lambda = (-2 - k)\Lambda_{1} + k\Lambda_{2}\) then

\begin{equation}
    n_{t}(\pi_{2,k}^{+}) = n_{t}(\pi_{2,k}^{+}) \quad \text{if} \quad k > 0, \quad \text{and} \quad n_{t}(\pi_{2,0}^{-}) - n_{t}(\pi_{2,0}^{+}) = n_{t}(T_{0}^{-}). \tag{17}
\end{equation}

It is not difficult to verify that this is all the information one can get by using the \(\eta(I)\)'s. Equations (5) through (17) are the equations obtained by Wallach in \cite[§9]{9}.

\textbf{References}


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