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AN ALTERNATING SUM FORMULA FOR MULTIPLICITIES IN $L^2(\Gamma \setminus G)$

BY

ROBERTO J. MIATELLO[1]

ABSTRACT. We prove an alternating sum formula for multiplicities in $L^2(\Gamma \setminus G)$, where G is a semisimple Lie group of split rank one with finite center and Γ is a discrete cocompact torsion free subgroup.

Introduction. Let G be a connected semisimple Lie group with finite center. Let $G = K \cdot A \cdot N$ be an Iwasawa decomposition of G and $g = k \oplus a \oplus n$ the corresponding decomposition at the algebra level. Let M denote the centralizer of A in K. We will assume throughout this paper that G has rank one, that is, $\dim(a) = 1$. We will not assume that G is linear. Let $\mathcal{E}(G)$ denote the set of equivalence classes of irreducible unitary representations of G. Given a homomorphism $\mu\colon \mathfrak{Z} \to \mathbf{C}$, where $\mathfrak{Z}$ is the center of $\mathfrak{U}(g)$ (the universal enveloping algebra of G), let $\mathcal{E}_\mu(G)$ denote the (finite) subset of $\mathcal{E}(G)$ of those classes with infinitesimal character μ. If $\mathcal{S}$ is any subset of $\mathcal{E}(G)$ we write $\mathcal{S}_\mu = \mathcal{S} \cap \mathcal{E}_\mu(G)$.

Let Γ be a discrete, torsion free subgroup of G such that $\Gamma \setminus G$ is compact. The right regular representation decomposes into a direct sum $\pi_\Gamma = \sum_{\omega \in \mathcal{E}(G)} n_\Gamma(\omega) \cdot \omega$ and $n_\Gamma(\omega) < \infty$, for any $\omega \in \mathcal{E}(G)$. The purpose of this paper is to prove an alternating sum formula connecting the multiplicities $n_\Gamma(\omega)$ of representations having the same infinitesimal character. We now describe our result.

Let K and M be as above. Let $i^*\colon R(K) \to R(M)$ denote the restriction map from the representation ring of K to that of M. If $\tau \in \mathcal{E}(K)$ and $\omega \in \mathcal{E}(G)$, then $[\tau : \omega]$ stands for the multiplicity of τ in the restriction of ω to K. If $\eta \in R(K)$ we abuse notation by writing $[\eta : \omega] = \sum m_j \cdot [\tau_j : \omega]$, where $\eta = \sum m_j \cdot \tau_j$.

Let $\mathcal{E}_c(G)$ denote those classes in $\mathcal{E}(G)$ corresponding to the irreducible unitary principal series or the complementary series and let $\mathcal{E}_2(G)$ denote the discrete series of G. Let $\alpha\colon \mathcal{E}(G) \to \mathbf{Z}$ be defined as follows: $\alpha(\omega) = 0$, if $\omega \in \mathcal{E}_c(G)$; $\alpha(\omega) = n_\Gamma(\omega) - d(\omega) \cdot \operatorname{vol}(\Gamma \setminus G)$, if $\omega \in \mathcal{E}_2(G)$; and $\alpha(\omega) = n_\Gamma(\omega)$ if $\omega \in \mathcal{E}(G) - (\mathcal{E}_c(G) \cup \mathcal{E}_2(G))$. Here $d(\omega)$ denotes the formal degree of ω. (We observe that $d(\omega)$ and $\operatorname{vol}(\Gamma \setminus G)$ depend on the choice of the Haar measure on G, but their product does not). Finally let Λ be a fixed infinitesimal character of G. We prove (Theorem 1.2) that if $\eta \in \ker(i^*)$, then

$$(1) \qquad \sum_{\omega \in \mathcal{E}_\Lambda(G)} \alpha(\omega) \cdot [\eta : \omega] = 0.$$

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From the definition of $\alpha(\omega)$ we see that the formula relates multiplicities of unitarizable Langlands quotients with multiplicities of nonintegrable discrete series (resp. limits of discrete series) when Λ is regular (resp. singular). These connections were already observed by Wallach in [**9**, §9], in the case of $SU(2, 1)$, and our formula can be regarded as a generalization of the family of equations that he obtains. In fact, when (1) is specialized to $SU(2, 1)$, it yields precisely Wallach's formulas.

The outline is as follows. In §1 we introduce preparatory material to prove our result (Theorem 1.2), while in §2 we discuss in detail the examples of $Sl(2, \mathbf{R})$ and $SU(2, 1)$, making formula (1) very explicit for these groups. We wish to thank D. De George, N. Wallach and C. Rader for useful comments on the contents of this paper.

1. We keep the notation from the introduction. If $\sigma \in \mathcal{E}(M)$ and $\nu \in a_{\mathbf{C}}^{*}$, let $\pi_{\sigma,\nu}$ denote the principal series of G, parametrized as in [**1**, §3]. In this parametrization $\pi_{\sigma,\nu}$ is unitary if and only if $i\nu \in a^{*}$, and $\pi_{\sigma,\nu}$ is reducible only if $\nu = 0$ [**5**]. In this case $\pi_{\sigma,\nu}$ splits as a direct sum of two inequivalent irreducible representations. We denote by $\mathcal{E}_r(G)$ the classes or irreducible constituents of reducible unitary principal series. It is a well-known fact that $\mathcal{E}_r(G) \cap \mathcal{E}_c(G) = \varnothing$. We will also set

$$\mathcal{E}_L(G) = \mathcal{E}(G) - (\mathcal{E}_c(G) \cup \mathcal{E}_2(G) \cup \mathcal{E}_r(G)).$$

If $\omega \in \mathcal{E}(G)$ let Λ_ω denote the infinitesimal character of ω. If Ω is the Casimir element of G we write $\Lambda_\omega(\Omega) = \lambda_\omega$. Also, the eigenvalue of Ω for the (nonunitary) principal series $\pi_{\sigma,\nu}$ will be denoted by $\lambda_{\sigma,\nu}$.

For fixed $(\tau, V_\tau) \in \mathcal{E}(K)$ we form the homogeneous vector bundle over G/K:

$$E_\tau = G \times_{\rho_\tau} \operatorname{End}(V_\tau)$$

where $\rho_\tau(k)(A) = A \cdot \tau(k^{-1})$, if $A \in \operatorname{End}(V_\tau)$. The Casimir element of G defines an elliptic, second order differential operator D on E_τ. Under the standard identification of cross sections with vector-valued functions $f\colon G \to \operatorname{End}(V_\tau)$ transforming according to ρ_τ, and of these with scalar-valued functions that are invariant under convolution with $d_\tau \cdot \chi_\tau$, the fundamental solution of the heat equation $D - \partial/\partial s = 0$ on E_τ is given by convolution with a scalar multiple of $g_{\tau,s}$, where

$$g_{\tau,s}(x) = \int_{\mathcal{E}(G)} d_\tau^{-1} \cdot \phi_{\tau,\omega}(x^{-1}) \cdot e^{s\lambda_\omega}\, d\mu(\omega)$$

for $x \in G$ and $s > 0$. Here $\mu(\omega)$ denotes the Plancherel measure and $\phi_{\tau,\omega}$ is the τ-spherical trace function of ω. That is,

$$\phi_{\tau,\omega}(x) = \operatorname{tr}(E_\tau \cdot \pi_\omega(x) \cdot E_\tau)$$

where E_τ is the orthogonal projection onto the space of vectors transforming according to τ. In the next theorem we collect the properties of $g_{\tau,s}$ that we will need. If $\omega \in \mathcal{E}(G)$, θ_ω will denote the distributional character of ω.

1.1 THEOREM. *Let G be a Lie group of split rank one with finite center. Fix $\tau \in \mathcal{E}(K)$ and let $\psi_\tau = d_\tau \cdot \bar{\chi}_\tau$. Then*

(i) $\psi_\tau * g_{\tau,s} * \psi_\tau = g_{\tau,s}$.

(ii) $g_{\tau,s} \in \mathcal{C}^p(G)$, *for any* $p > 0$.

(iii) $\theta_\omega(g_{\tau,s}) = [\tau : \omega] \cdot e^{s\lambda_\omega}$, *for any* $\omega \in \mathcal{E}(G)$.

Here $\mathcal{C}^p(G)$ denotes (if $0 < p \leqslant 2$) the p-Schwartz space of G (see **[1**, §4], *for instance).*

PROOF. Assertion (i) follows from the definition of $g_{\tau,s}$. In [7, Theorem 4.11], (ii) was proved under the assumption that τ restrict simply to M ($g_{\tau,s}$ coincides with the function $h^1_{\tau,s}$ defined there). Recently De George and Wallach (unpublished) have shown that (ii) is true for semisimple groups of arbitrary rank. We now prove (iii). From the definition,

$$g_{\tau,s}(x) = \sum_{\sigma \in \mathcal{E}(M)} d_\tau^{-1} \int_{a^*} \operatorname{tr}\left(E_\tau \pi_{\sigma,\nu}(x) E_\tau\right) e^{s\lambda_{\sigma,\nu}} \mu_\sigma(\nu)\, d\nu + g^0_{\tau,s}(x),$$

where $g^0_{\tau,s}$ is a linear combination of matrix entries of discrete series representations. By the Plancherel theorem [3] the operator $\pi_{\sigma,\nu}(g_{\tau,s}) = d_\tau^{-1} \cdot e^{s\lambda_{\sigma,\nu}} \cdot E_\tau$, if $\nu \in a^*$. Since $g_{\tau,s} \in \mathcal{C}^p(G)$ for any $p > 0$, by analytic continuation we see that this holds for $\nu \in a^*_{\mathbf{C}}$. Since any irreducible unitary representation (π, V) of G is infinitesimally equivalent to a subquotient of $(H_{\sigma,\nu}, \pi_{\sigma,\nu})$ for some $\sigma \in \mathcal{E}(M)$, $\nu \in a^*_{\mathbf{C}}$, the above clearly implies that $\pi(g_{\tau,s}) = d_\tau^{-1} \cdot e^{s\lambda_\pi} \cdot E_{\tau,\pi}$, where $E_{\tau,\pi}$ denotes the orthogonal projection onto the subspace of V of vectors transforming according to τ. Hence $\theta_\pi(g_{\tau,s}) = \operatorname{tr} \pi(g_{\tau,s}) = [\tau : \pi] \cdot e^{s\lambda_\pi}$.

Let α: $\mathcal{E}(G) \to \mathbf{Z}$ be as defined in the introduction and let i^*: $R(K) \to R(M)$ be the ring homomorphism induced by the inclusion i: $M \to K$.

1.2 THEOREM. *Let G be a semisimple Lie group of split rank one with finite center and let $\Gamma \subset G$ be a discrete, cocompact, torsion free subgroup. Fix Λ an infinitesimal character of G. If $\eta \in \ker(i^*)$, then*

$$\sum_{\omega \in \mathcal{E}_\Lambda(G)} \alpha(\omega) \cdot [\eta : \omega] = 0. \tag{1}$$

PROOF. Let $\eta = \sum_{j=1}^k m_j \cdot \tau_j$. For fixed τ the set

$$\mathcal{S}(\tau) = \{\omega \in \mathcal{E}(G) - \mathcal{E}_c(G) | [\tau : \omega] \neq 0\}$$

is finite (see for instance [1, §4]). Set $\mathcal{S} = \bigcup_1^k \mathcal{S}(\tau_j)$ and let $\mathcal{F} = \{\Lambda_1, \ldots, \Lambda_m\}$ be the distinct infinitesimal characters of elements of $\mathcal{S}$. If $\Lambda \notin \mathcal{F}$ and $\omega \in \mathcal{E}_\Lambda(G)$ then either $[\eta : \omega] = 0$ or $\omega \in \mathcal{E}_c(G)$ and in this case $\alpha(\omega) = 0$. Hence if $\Lambda \notin \mathcal{F}$, the left-hand side of (1) equals zero. We thus assume that $\Lambda \in \mathcal{F}$. Then there exists $z \in \mathfrak{Z}$ such that $\Lambda(z) = 1$ and $\Lambda'(z) = 0$, if $\Lambda' \in \mathcal{F}$, $\Lambda' \neq \Lambda$. Indeed, if $\Lambda = \Lambda_1$, let $u \in \mathfrak{Z}$ be such that $\Lambda_i(u) \neq \Lambda_j(u)$ (if $i \neq j$). We may then take

$$z = \prod_{j>1} \frac{u - \Lambda_j(u)}{\Lambda_1(u) - \Lambda_j(u)}.$$

Let us set $g_{\eta,s} = \Sigma m_j \cdot g_{\tau_j,s}$. By 1.1, $\theta_\omega(g_{\eta,s}) = [\eta : \omega] \cdot e^{s\lambda_\omega}$ is valid for any $\omega \in \mathcal{E}(G)$. We now apply the Selberg trace formula to $z \cdot g_{\eta,s}$. This is legitimate since $z \cdot g_{\eta,s} \in \mathcal{C}^1(G)$ is K-finite (see [7, Corollary 2.16]). Moreover, by the polynomial character of $\Lambda_{\sigma,\nu}(z)$ it follows, as in the proof of Theorem 5.1 in [7], that

$$\operatorname{tr} \pi_\Gamma(z \cdot g_{\eta,s}) \sim \operatorname{vol}(\Gamma \setminus G) \cdot (z \cdot g_{\eta,s})(1), \quad \text{as } s \to 0^+.$$

(If $\phi, \psi \in C^\infty(\mathbf{R}^+)$ we write $\phi(s) \sim \psi(s)$ if $\phi(s) - \psi(s) = o(s^n)$ as $s \to 0^+$, for all $n \in \mathbf{N}$.)

Hence by 1.1 we have, as $s \to 0^+$,

$$\sum_{\omega \in \mathcal{E}(G)} n_\Gamma(\omega) \cdot [\eta : \omega] \cdot \Lambda_\omega(z) \cdot e^{s\lambda_\omega}$$

$$\sim \operatorname{vol}(\Gamma \setminus G) \left(\sum_{\sigma \in \mathcal{E}(M)} [\sigma : \eta] \cdot \int_{a^*} \Lambda_{\sigma,\nu}(z) \cdot e^{s\lambda_{\sigma,\nu}} \cdot \mu_\sigma(\nu)\, d\nu \right)$$

$$+ \operatorname{vol}(\Gamma \setminus G) \left(\sum_{\omega \in \mathcal{E}_2(G)} d(\omega) \cdot [\eta : \omega] \cdot \Lambda_\omega(z) \cdot e^{s\lambda_\omega} \right).$$

Since by assumption $i^*(\eta) = 0$, the sum corresponding to $\mathcal{E}_c(G)$ in the left-hand side drops out and so does the first summand in the right-hand side. On the other hand, by the choice of z, we need only sum over $\mathcal{E}_\Lambda(G)$. Hence, as $s \to 0^+$,

$$\sum_{\omega \in \mathcal{E}_\Lambda(G) - \mathcal{E}_c(G)} n_\Gamma(\omega) \cdot [\eta : \omega] \cdot e^{s\lambda_\omega} \sim \sum_{\omega \in \mathcal{E}_{2,\Lambda}(G)} \operatorname{vol}(\Gamma \setminus G)\, d(\omega) \cdot [\eta : \omega] \cdot e^{s\lambda_\omega},$$

and then both sides are equal since all sums are finite. This proves the theorem.

REMARK. It is well known that if $\omega \in \mathcal{E}_2(G)$ is integrable then $n_\Gamma(\omega) = d(\omega) \cdot \operatorname{vol}(\Gamma \setminus G)$ (in fact, this equality holds for certain families of nonintegrable classes as well (see [**4**, **8**])). Hence, in formula (1), only nonintegrable classes do contribute to the sum.

2. We now analyze the cases of $Sl(2, \mathbf{R})$ and $SU(2, 1)$. We describe a basis over $\mathbf{Z}$ of $\ker(i^*)$ for each group and apply formula (1) to η, an element of the basis. In this manner we retrieve Langlands' formulas for the multiplicities of the discrete series for $Sl(2, \mathbf{R})$. In the case of $SU(2, 1)$, when η runs over a basis of $\ker(i^*)$, we obtain (aside from some trivial equations) exactly Wallach's formulas of [**9**, §9].

Let $G = Sl(2, \mathbf{R})$. Then $K = SO(2)$ and $M = \{\pm I\}$. If

$$k(\theta) = \begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix}$$

let $\tau_n(k(\theta)) = e^{in\theta}$. We have that $\mathcal{E}(K) = \{\tau_n | n \in \mathbf{Z}\}$ and $\mathcal{E}(M) = \{1, \varepsilon\}$, where ε stands for the signum representation. The following lemma is straightforward.

2.1 LEMMA. *A basis of* $\ker(i^*)$ *is given by*

$$\mathcal{B} = \{\eta_j = \tau_j - \tau_{j-2} | j \in \mathbf{Z}\}.$$

We recall some well-known facts on $\mathcal{E}(G)$ (see [6]). We have that $\mathcal{E}_2(G) = \{\pi_{\pm k} | k > 1\}$ and $\mathcal{E}_r(G) = \{\pi^{\pm}\}$ where

$$\pi_{\pm k|K} = \sum_{j=0}^{\infty} \tau_{\pm(2j+k)} \quad \text{and} \quad \pi^{\pm}_{|K} = \sum_{0}^{\infty} \tau_{\pm(2i+1)}.$$

Moreover $\pi_{\pm j}$ is integrable if and only if $j > 2$ and $\mathcal{E}_L(G) = \{1\}$.

Formula (1) when applied to $\eta_k = \tau_k - \tau_{k-2}$ gives $(k > 1)$

$$\alpha(\pi_k) \cdot [\eta_k : \pi_k] = 0 \quad \text{or} \quad n_\Gamma(\pi_k) = d(\pi_k) \cdot \mathrm{vol}(\Gamma \setminus G). \tag{2}$$

If $k = 2$, $\eta_2 = \tau_2 - 1$ and then $[\eta_2, \omega] = 1$ if $\omega = \pi_2$, $[\eta_2, \omega] = -1$ if $\omega = 1$ and $[\eta_2, \omega] = 0$ for any other representation $\omega \in \mathcal{E}(G) - \mathcal{E}_c(G)$. Since $n_\Gamma(1) = 1$ one thus obtains Langlands' formula

$$n_\Gamma(\pi_2) = d(\pi_2) \cdot \mathrm{vol}(\Gamma \setminus G) + 1. \tag{3}$$

Similar equations to (2) and (3) are obtained for π_{-k-2} and π_{-2}, by using η_{-k} and η_0. Finally, if $\eta_1 = \tau_1 - \tau_{-1}$, we get

$$n_\Gamma(\pi^+) = n_\Gamma(\pi^-). \tag{4}$$

Note that formula (1) gives no more information on $n_\Gamma(\pi^{\pm})$.

We now discuss the case of $G = SU(2, 1)$, i.e.

$$SU(2,1) = \{A \in Gl(3, \mathbf{C}) | H(A(z)) = H(z)\}$$

where $z = (z_1, z_2, z_3) \in \mathbf{C}^3$ and $H(z) = |z_1|^2 + |z_2|^2 - |z_3|^2$. Then

$$K = \left\{ \begin{vmatrix} A & 0 \\ 0 & \det A^{-1} \end{vmatrix} \,\middle|\, A \in U(2) \right\}$$

is a maximal compact subgroup and we may take $\mathfrak{a} \subset \mathfrak{p}$ so that $M = \{\mathrm{diag}(a, b, a) | a^2 \cdot b = 1\}$. Here $\mathrm{diag}(z_1, z_2, z_3)$ denotes the diagonal matrix in $Gl(3, \mathbf{C})$ with entries z_1, z_2, z_3. Let $t(\theta) = \mathrm{diag}(e^{i\theta}, e^{i\theta}, e^{-2i\theta})$ and $m(\theta) = \mathrm{diag}(e^{i\theta}, e^{-2i\theta}, e^{i\theta})$. Given $l \in \mathbf{Z}, p \in \mathbf{Z}$ such that $p \geqslant 0, p \equiv l \pmod 2$ there exists a unique irreducible representation of K, τ_p^l, such that $\dim \tau_p^l = p + 1$ and $\tau_p^l(t(\theta)) = e^{-il\theta} \cdot I$. Furthermore, the representations τ_p^l exhaust $\mathcal{E}(K)$.

If $k \in \mathbf{Z}$, let $\sigma(k)$ denote the irreducible representation of M such that $\sigma(k)(m(\theta)) = e^{ik\theta} \cdot I$. When restricted to M, τ_p^l decomposes as follows:

$$i^*(\tau_p^l) = \sum_{j=0}^{p} \sigma(r - 3j)$$

where $r = (l + 3p)/2$. We may thus associate to $\tau_p^l \in \mathcal{E}(K)$ an interval $[r - 3p, r]$ in $r + 3\mathbf{Z}$. Conversely, given an interval $[a, b]$ in $c + 3\mathbf{Z}$ ($c = 0$, 1 or 2) there is a unique K-type τ_p^l such that the associated interval is $[a, b]$ (in fact $p = (b - a)/3$ and $l = a + b$). In the light of this correspondence we may construct elements in $\ker(i^*)$ in a simple manner. Let $I_1, \ldots, I_k$ be disjoint intervals in $c + 3\mathbf{Z}$ such that their union is an interval I. Then, if $\tau(I)$ denotes the K-type associated to I, clearly $\eta = \tau(I) - \sum_j \tau(I_j) \in \ker(i^*)$. In particular, any interval can be decomposed as a union of singletons. We will set

$$\eta(I) = \tau(I) - \sum_{a}^{b} \tau(j)$$

where $\tau(j) = \tau([j, j])$. The following lemma is not difficult.

2.2 LEMMA. *The family* $\mathfrak{B} = \{\eta(I) | I$ *an interval in* $c + 3\mathbf{Z}$, $c = 0, 1$ *or* $2\}$ *is a basis of* $\ker(i^*)$.

In order to apply formula (1) to elements in $\mathfrak{B}$ we need to recall several facts from the representation theory of $SU(2, 1)$. We will recall them very briefly. We refer the reader to [**9**, §7] for details.

Let Δ^+ be the system of positive roots as defined in [**9**, p. 181] and let $\delta = \frac{1}{2}(\Sigma_{\Delta^+} \alpha)$. Also, let α_1, α_2 be the simple roots and Λ_1, Λ_2 the basic highest weights for this system. Finally, let s_1, s_2 be the reflections with respect to α_1 and α_2, respectively.

In the notation of [**9**, §7], if $\omega \in \mathcal{E}(G) - \mathcal{E}_c(G)$, then ω is one of:
$D^+_{s_1s_2(\Lambda+\delta)-\delta}$, $D^-_{s_2s_1(\Lambda+\delta)-\delta}$, $D_{s_1s_2s_1(\Lambda+\delta)-\delta}$, $\Lambda = k_1\Lambda_1 + k_2\Lambda_2$, $k_i \in Z$, $k_i \geq 0$;
$T_k^{\pm}$, $k \in Z$, $k \geq -1$;
$\pi_\Lambda^{\pm}$, where $\Lambda = k\Lambda_1 + (-2 - k)\Lambda_2$ or $\Lambda = (-2 - k)\Lambda_1 + k\Lambda_2$, $k \geq 0$; or
$\omega = 1$, the trivial representation.

If $\Lambda = k_1\Lambda_1 + k_2\Lambda_2 \in \mathfrak{h}^*$ let χ_Λ denote the homomorphism $\chi_\Lambda\colon \mathfrak{z} \to \mathbf{C}$ as defined by Harish-Chandra [**2**]. The infinitesimal characters of the above representations are as follows:
$D^+_{s_1s_2(\Lambda+\delta)-\delta}$, $D^-_{s_2s_1(\Lambda+\delta)-\delta}$, $D_{s_1s_2s_1(\Lambda+\delta)-\delta}$ have infinitesimal character $\chi_{\Lambda+\delta}$;
T_k^+ (resp. T_k^-) has infinitesimal character $\chi_{k\Lambda_1+\delta}$ (resp. $\chi_{k\Lambda_2+\delta}$); and
$\pi_\Lambda^{\pm}$ have infinitesimal character $\chi_{\Lambda+\delta}$.

From now on we will write D_Λ^+, D_Λ^- and D_Λ instead of $D^+_{s_1s_2(\Lambda+\delta)-\delta}$, $D^-_{s_2s_1(\Lambda+\delta)-\delta}$ and $D_{s_1s_2s_1(\Lambda+\delta)-\delta}$, respectively. Also $\pi_{1,k}^{\pm}$ stand for $\pi_\Lambda^{\pm}$, if $\Lambda = k\Lambda_1 + (-2 - k)\Lambda_2$, and similarly $\pi_{2,k}^{\pm} = \pi_\Lambda^{\pm}$, if $\Lambda = (-2 - k)\Lambda_1 + k\Lambda_2$. Finally, if $\mathcal{S} \subset \mathcal{E}(G)$ and $\Lambda \in \mathfrak{h}^*$, we abbreviate $\mathcal{S}_\Lambda = \mathcal{S}_{\chi_{\Lambda+\delta}}$. With this notation in mind the classification of $\mathcal{E}(G) - \mathcal{E}_c(G)$ according to the infinitesimal character is:

$$\begin{aligned}
\mathcal{E}_{d,\Lambda} &= \{D_\Lambda^+, D_\Lambda^-, D_\Lambda\}, \quad \text{if } \Lambda = k_1\Lambda_1 + k_2\Lambda_2, \qquad k_i \geq 0,\\
\mathcal{E}_{L,\Lambda} &= \{T_k^+\}, \quad \text{if } \Lambda = k\Lambda_1, \qquad k > 0 \text{ or } k = -1,\\
\mathcal{E}_{L,\Lambda} &= \{T_k^-\}, \quad \text{if } \Lambda = k\Lambda_2, \qquad k > 0 \text{ or } k = -1,\\
\mathcal{E}_{L,\Lambda} &= \{T_0^+, T_0^-, 1\}, \quad \text{if } \Lambda = 0,\\
\mathcal{E}_{r,\Lambda} &= \{\pi_{1,k}^{\pm}\}, \quad \text{if } \Lambda = k\Lambda_1 + (-2 - k)\Lambda_2, \qquad k \geq 0,\\
\mathcal{E}_{r,\Lambda} &= \{\pi_{2,k}^{\pm}\}, \quad \text{if } \Lambda = (-2 - k)\Lambda_1 + k\Lambda_2, \qquad k \geq 0.
\end{aligned}$$

The next lemma gives the K-type structure of these representations. It is a consequence of Lemma 7.9 in [**9**] and the relations

$$\begin{aligned}
s_1s_2(\Lambda + \delta) - \delta &= -(k_1 + k_2 + 3)\Lambda_1 + k_1\Lambda_2,\\
s_2s_1(\Lambda + \delta) - \delta &= k_2\Lambda_1 - (k_1 + k_2 + 3)\Lambda_2,\\
s_1s_2s_1(\Lambda + \delta) - \delta &= -(k_2 + 2)\Lambda_1 - (k_1 + 2)\Lambda_2.
\end{aligned}$$

2.3 LEMMA. *If $\omega \in \mathfrak{E}(G)$, let $j^*(\omega)$ denote the restriction of ω to K. Then*

$$j^*(D_\Lambda^+) = \sum_{\substack{0 \leq q \leq k_1 \\ 0 \leq p}} \tau_{p+q}^{4k_1+2k_2+6+3(p-q)}, \qquad j^*(D_\Lambda^-) = \sum_{\substack{0 \leq p \leq k_2 \\ 0 \leq q}} \tau_{p+q}^{-4k_2-2k_1-6+3(p-q)},$$

$$j^*(D_\Lambda) = \sum_{\substack{k_1 < p \\ k_2 < q}} \tau_{p+q}^{2k_2-2k_1+3(p-q)},$$

$$j^*(T_k^+) = \sum_{k<p} \tau_p^{-2k+3p}, \qquad j^*(T_k^-) = \sum_{k<q} \tau_q^{2k-3q} \quad (k \geq -1),$$

$$j^*(\pi_{1,k}^+) = \sum_{\substack{0 \leq p \leq k \\ 0 \leq q}} \tau_{p+q}^{-4-4k+3(p-q)}, \qquad j^*(\pi_{1,k}^-) = \sum_{\substack{k < p \\ 0 \leq q}} \tau_{p+q}^{-4-4k+3(p-q)},$$

$$j^*(\pi_{2,k}^+) = \sum_{\substack{0 \leq p \\ 0 \leq q \leq k}} \tau_{p+q}^{4+4k+3(p-q)}, \qquad j^*(\pi_{2,k}^-) = \sum_{\substack{0 \leq p \\ k < q}} \tau_{p+q}^{4+4k+3(p-q)}.$$

It will be convenient to write this information on the K-types in terms of intervals in $r + 3 \cdot \mathbf{Z}$. Recall that τ_p^l is associated to $[r - 3p, r]$ where $r = (l + 3p)/2$. We then have the following table of representations and corresponding intervals (or K-types).

$$\begin{array}{rll}
D_\Lambda^+ : [2k_1 + 2k_2 + 3 - 3q, 2k_1 + 2k_2 + 3 + 3p], & p \geq 0, & k_1 \geq q \geq 0, \\
D_\Lambda^- : [-2k_2 - k_1 - 3 - 3q, -2k_2 - k_1 - 3 + 3p], & q \geq 0, & k_2 \geq p \geq 0, \\
D_\Lambda : [k_2 - k_1 - 3q, k_2 - k_1 + 3p], & p > k_1, & q > k_2, \\
T_k^+ : [-k, -k + 3p], & p > k, & \\
T_k^- : [k - 3q, k], & q > k, & \\
\pi_{1,k}^+ : [-2 - 2k - 3q, -2 - 2k + 3p], & k \geq p \geq 0, & q \geq 0, \\
\pi_{1,k}^- : [-2 - 2k - 3q, -2 - 2k + 3p], & p > k, & q \geq 0, \\
\pi_{2,k}^+ : [2 + 2k - 3q, 2 + 2k + 3p], & p \geq 0, & k \geq q \geq 0, \\
\pi_{2,k}^- : [2 + 2k - 3q, 2 + 2k + 3p], & p \geq 0, & q > k, \\
1 : [0, 0]. & &
\end{array}$$

It is now a simple matter to apply formula (1) of Theorem 1.2 to K-types of the form $\eta(I) = \tau(I) - \sum_a^b \tau(j)$ where $I = [a, b]$. By Lemma 2.2 any other $\eta \in \ker(i^*)$ yields an equation which is a combination of equations of this type. We assume first that $\Lambda = k_1\Lambda_1 + k_2\Lambda_2$, $k_i \geq 1$. Then $\mathfrak{E}_\Lambda(G) = \{D_\Lambda^+, D_\Lambda^-, D_\Lambda\}$. It is easy to check that if $\tau(I)$ is a K-type of D_Λ, $\eta(I)$ yields the equation

$$(5) \qquad n_\Gamma(D_\Lambda^+) + n_\Gamma(D_\Lambda^-) - n_\Gamma(D_\Lambda) = d(D_\Lambda) \cdot \mathrm{vol}(\Gamma \setminus G).$$

(Note that $d(D_\Lambda^+) = d(D_\Lambda^-) = d(D_\Lambda)$.) On the other hand, if $\tau(I)$ is a K-type of D_Λ^+ or D_Λ^- we get the trivial equation. If we use $\eta(I_1)$, $\eta(I_2)$ where $I_1 = [k_2 - k_1, 2k_1 + k_2 + 3]$, $I_2 = [-2k_2 - k_1 - 3, k_2 - k_1]$, we obtain

$$(6) \qquad n_\Gamma(D_\Lambda^+) = d(D_\Lambda^+) \cdot \mathrm{vol}(\Gamma \setminus G),$$

$$(7) \qquad n_\Gamma(D_\Lambda^-) = d(D_\Lambda^-) \cdot \mathrm{vol}(\Gamma \setminus G);$$

hence we have also

$$n_\Gamma(D_\Lambda) = d(D_\Lambda) \cdot \operatorname{vol}(\Gamma \setminus G). \tag{8}$$

We recall from [9, Lemma 7.7] that if $k_1 \geqslant 2$, $k_2 = 1$ then D_Λ and D_Λ^+ are not integrable. If $k_1 = 1$, $k_2 \geqslant 1$, then D_Λ and D_Λ^- are not integrable.

When $k_2 = 0$ and $k_1 = k > 0$, (5) and (7) are still true and (6) changes into

$$n_\Gamma(D_\Lambda^+) - d(D_\Lambda^+) \cdot \operatorname{vol}(\Gamma \setminus G) = n_\Gamma(T_k^+) \tag{9}$$

since $\tau([-k, 2k+3])$ is a K-type of T_k^+. Instead of (8),

$$n_\Gamma(D_\Lambda) = n_\Gamma(D_\Lambda^+). \tag{10}$$

Similarly, if $k_1 = 0$, $k_2 = k > 0$, (5) and (6) are valid and, moreover,

$$n_\Gamma(D_\Lambda^-) - d(T_\Lambda^-) \cdot \operatorname{vol}(\Gamma \setminus G) = n_\Gamma(T_k^-), \tag{11}$$

$$n_\Gamma(D_\Lambda) = n_\Gamma(D_\Lambda^-). \tag{12}$$

If $\Lambda = 0$, because of the presence of the trivial representation, (5), (9) and (11) change into

$$n_\Gamma(D_0^+) + n_\Gamma(D_0^-) - n_\Gamma(D_0) = d(D_0) \cdot \operatorname{vol}(\Gamma \setminus G) - 1, \tag{13}$$

$$n_\Gamma(D_0^-) - d(D_0^-) \cdot \operatorname{vol}(\Gamma \setminus G) = n_\Gamma(T_0^-) - 1, \tag{14}$$

$$n_\Gamma(D_0^+) - d(D_0^+) \cdot \operatorname{vol}(\Gamma \setminus G) = n_\Gamma(T_0^+) - 1. \tag{15}$$

If $\Lambda = k\Lambda_1 + (-2-k)\Lambda_2$ then

$$n_\Gamma(\pi_{1,k}^-) = n_\Gamma(\pi_{1,k}^+) \quad \text{if } k > 0, \quad \text{and} \quad n_\Gamma(\pi_{1,0}^-) - n_\Gamma(\pi_{1,0}^+) = n_\Gamma(T_{-1}^+). \tag{16}$$

If $\Lambda = (-2-k)\Lambda_1 + k\Lambda_2$ then

$$n_\Gamma(\pi_{2,k}^-) = n_\Gamma(\pi_{2,k}^+) \quad \text{if } k > 0, \quad \text{and} \quad n_\Gamma(\pi_{2,k}^-) - n_\Gamma(\pi_{2,k}^+) = n_\Gamma(T_{-1}^-). \tag{17}$$

It is not difficult to verify that this is all the information one can get by using the $\eta(I)$'s. Equations (5) through (17) are the equations obtained by Wallach in [9, §9].

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Departamento de Matemática, Universidade Federal de Pernambuco, 50000 Recife, Pe., Brasil

School of Mathematics, Institute for Advanced Study, Princeton, New Jersey 08540

Current address: Department of Mathematics, Rutgers University, New Brunswick, New Jersey 08903