A CLASS OF L¹-CONVERGENCE

BY
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ABSTRACT. It is proved that if the Fourier coefficients \( \{a_n\} \) of \( f \in L¹(0, \pi) \) satisfy
\[ n^{-1} \sum_{k=n}^{\infty} k^p |a_k| = o(1), \]
for some \( 1 < p < 2 \), then \( \|s_n - f\| = o(1) \), if and only if \( a_n \lg n = o(1) \). For cosine trigonometric series with coefficients of bounded variation and satisfying (*) it is proved that a necessary and sufficient condition for the series to be a Fourier series is \( \{a_n\} \in \mathcal{C} \), where \( \mathcal{C} \) is the Garrett-Stanojević [4] class.

1. Introduction. Let \( f \) be a \( 2\pi \)-periodic and even function in \( L¹(0, \pi) \), and let \( \{a_k\} \) be the sequence of its Fourier coefficients. Denote by \( \mathcal{F} \) the class of sequences of Fourier coefficients of all such functions. It is well known that, in general, it does not follow from \( \{a_k\} \in \mathcal{F} \) that
\[ s_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx \]
converges to \( f \) in the \( L¹(0, \pi) \)-norm, i.e. it does not follow that \( \|s_n - f\| = o(1) \), \( n \to \infty \), where \( \|\cdot\| \) is the \( L¹(0, \pi) \)-norm.

However, there are examples of subclasses of \( \mathcal{F} \) for which \( a_n \lg n = o(1) \), \( n \to \infty \), is a necessary and sufficient condition for \( \|s_n - f\| = o(1) \), \( n \to \infty \).

DEFINITION 1.1. A subclass \( \mathcal{K} \) of \( \mathcal{F} \) is called a class of \( L¹ \)-convergence if
\[ \|s_n - f\| = o(1), n \to \infty, \]
is equivalent with \( a_n \lg n = o(1) \), \( n \to \infty \).

There are three classical examples of classes of \( L¹ \)-convergence. The first one is due to Young [1] and \( \mathcal{K} \) is defined to be the class of all convex sequences \( \{a_k\} \) \( (\Delta^2 a_k > 0) \). The second one is the class of all sequences \( \{a_k\} \) such that \( \sum_{k=1}^{\infty} k|\Delta^2 a_k| < \infty \), introduced by Kolmogorov [2]. The latter one is an obvious extension of the first one, and both are subclasses of \( \mathcal{B} \mathcal{V} \), the class of all null-sequences of bounded variation.

The third example is the Telyakovskii [3] class \( \mathcal{S} \). A sequence \( \{a_k\} \) belongs to \( \mathcal{S} \) if there exists a monotone sequence \( \{A_k\} \) such that \( \sum_{k=1}^{\infty} A_k < \infty \) and \( |\Delta a_k| < A_k \), for all \( k \). Clearly \( \mathcal{S} \subset \mathcal{B} \mathcal{V} \).

Garrett and Stanojević [4] introduced the following class \( \mathcal{C} \). A null-sequence \( \{a_k\} \) belongs to the class \( \mathcal{C} \) if for every \( \epsilon > 0 \) there exists \( \delta > 0 \), independent of \( n \), and such that
\[ C_n(\delta) = \frac{1}{\pi} \int_{0}^{\delta} \left| \sum_{k=n}^{\infty} \Delta a_k D_k(x) \right| \, dx < \epsilon, \]
for all \( n \), where \( D_k \) is the Dirichlet kernel. As a corollary to their main result in [4], they proved that \( \mathcal{C} \cap \mathcal{B} \mathcal{V} \) is a class of \( L^1 \)-convergence. Later, Garrett, Rees and Stanojević [5] proved that \( \mathcal{S} \subset \mathcal{C} \cap \mathcal{B} \mathcal{V} \).

Recently Fomin [6] extended the Telyakovskii class \( \mathcal{S} \) in the following way. A sequence \( \{a_k\} \) belongs to the class \( \mathcal{G}_p \), if for some \( 1 < p < 2 \),

\[
\sum_{n=1}^{\infty} \left( \frac{\sum_{k=n}^{\infty} |\Delta a_k|^p}{n} \right)^{1/p} < \infty.
\]

Fomin [6] proved that \( \mathcal{G}_p \) is a class of \( L^1 \)-convergence and that \( \mathcal{S} \subset \mathcal{G}_p \).

On the other hand Stanojević [7] proved that \( \mathcal{G}_p \subset \mathcal{C} \cap \mathcal{B} \mathcal{V} \), and obtained several new classes of \( L^1 \)-convergence. A sequence \( \{a_k\} \) of Fourier coefficients belongs to the class \( \mathcal{C}_p \) if, for some \( 1 < p < 2 \), \( \sum_{k=1}^{\infty} |\Delta a_k|^p < \infty \), and

\[
\left( \frac{\sum_{k=n}^{\infty} |\Delta a_k|^p}{n} \right)^{1/p} = o(1), \quad n \to \infty.
\]

Clearly \( \{a_k\} \in \mathcal{G}_p \) implies that \( \{a_k\} \in \mathcal{C}_p \). Hence the result of Stanojević [7] is stronger than the one of Fomin [6].

In [7], the class \( \mathcal{B} \mathcal{V} \) is extended in the following manner.

**Definition 1.2.** A null-sequence \( \{a_k\} \) belongs to the class \( \mathcal{G} \) if

\[
\frac{1}{n} \sum_{k=1}^{n} k|\Delta a_k| = o(1), \quad n \to \infty.
\]

It is proved in [7] that if \( \{a_k\} \in \mathcal{G} \cap \mathcal{G}_p \) and if \( n\Delta a_n = O(1), n \to \infty \), then (1.1) holds. The proof is based on the estimate

\[
\|g_n - a_n\| = O\left(\frac{1}{n} \sum_{k=1}^{n} k|\Delta a_k|\right), \quad n \to \infty,
\]

where \( g_n(x) = s_n(x) - a_{n+1}D_n(x) \), and \( s_n \) is the Fejér's sum of \( s_n \).

We have a threefold objective in this paper. First, we shall, within a subclass of \( \mathcal{G} \), remove the condition \( n\Delta a_n = O(1), n \to \infty \). The subclass \( \mathcal{G}_p \) of \( \mathcal{G} \) is defined as follows.

**Definition 1.3.** A sequence \( \{a_k\} \) belongs to the class \( \mathcal{G}_p \) if, for some \( 1 < p < 2 \),

\[
\frac{1}{n} \sum_{k=1}^{n} k^p|\Delta a_k|^p = o(1), \quad n \to \infty.
\]

Secondly we shall estimate \( \|s_n - f\| = \|a_n\| \|D_n\| \) directly without using the estimate (1.3).

Finally we shall obtain a necessary and sufficient condition for \( a_0/2 + \sum_{k=1}^{\infty} a_k \cos kx, \{a_n\} \in \mathcal{G}_p \cap \mathcal{B} \mathcal{V} \), to be a Fourier series.

**2. Lemmas.** In the proof of our main theorem, instead of the condition (1.4) we shall use an equivalent one

\[
\frac{1}{n} \sum_{k=n}^{2n-1} k^p|\Delta a_k|^p = o(1), \quad n \to \infty.
\]

The following lemma establishes that equivalence.
Lemma 2.1. Let \( \{c_k\} \) be a sequence of nonnegative numbers. Then
\[
\alpha_n = \frac{1}{n} \sum_{k=1}^{n} c_k = o(1), \quad n \to \infty,
\]
if and only if
\[
\beta_n = \frac{1}{n} \sum_{k=n}^{2n} c_k = o(1), \quad n \to \infty.
\]

Proof. The “only if” part is obvious. For the “if” part notice that
\[
2^n \beta_{2^n} = 2^{n+1} \alpha_{2^{n+1}} - 2^n \alpha_{2^n}
\]
or
\[
\alpha_{2^n} = \frac{\alpha_1}{2^m} + \frac{1}{2^m} \sum_{k=0}^{m-1} 2^k \beta_{2^k}.
\]

For \( 2^m < n < 2^{m+1} \), we have
\[
\alpha_n < \frac{2^{m+1}}{n} \alpha_{2^{m+1}} < 2 \alpha_{2^{m+1}}.
\]

Hence from (2.2) and (2.3) it follows that \( \beta_n \to 0, \quad n \to \infty \), implies that \( \alpha_n \to 0, \quad n \to \infty \).

The next lemma provides an identity that we need in order to avoid using (1.3) explicitly in our proof.

Lemma 2.2. Let
\[
s_n(x) = \frac{1}{2} a_0 + \sum_{k=1}^{n} a_k \cos kx
\]
and let \( \{\sigma_n(x)\} \) be the sequence of Fejér sums of the sequence \( \{s_n(x)\} \). Then
\[
s_n(x) - f(x) = 2(\sigma_{2n-1}(x) - f(x)) - (\sigma_{n-1}(x) - f(x))
\]
\[
- \frac{1}{n} \sum_{k=n+1}^{2n} (2n - k) a_k \cos kx.
\]

The core of the proof is supplied by the following lemma.

Lemma 2.3. Let \( \{c_k\} \) be a sequence of real numbers. Then for any \( 1 < p < 2 \) and \( n > 1 \)
\[
\frac{1}{n} \int_0^{\pi} \left| \sum_{k=n}^{2n-1} c_k D_k(x) \right| dx \leq A_p \left( \frac{1}{n} \sum_{k=n}^{2n-1} |c_k|^p \right)^{1/p},
\]
where \( A_p \) is an absolute constant.
Proof. We write
\[
\frac{1}{n} \int_{0}^{\pi} \left| \sum_{k=n}^{2n-1} c_k D_k(x) \right| \, dx = \frac{1}{n} \int_{0}^{\pi} \left| \sum_{k=n}^{2n-1} c_k D_k(x) \right| \, dx \\
+ \frac{1}{n} \int_{\pi/n}^{\pi} \left| \sum_{k=n}^{2n-1} c_k D_k(x) \right| \, dx \\
= I_1 + I_2.
\] (2.5)

Since \( |D_k| \leq k + 1 \), for the first integral in (2.5) we have \( I_1 \leq n^{-1} \sum_{k=n}^{2n-1} |c_k| \), and by Hölder’s inequality
\[
I_1 \leq \left( \frac{1}{n} \sum_{k=n}^{2n-1} |c_k|^p \right)^{1/p}.
\]

Let \( 1/p + 1/q = 1, p > 1 \). Then by applying the Hölder inequality to the second integral in (2.5) we get
\[
I_2 = \frac{1}{n} \int_{\pi/n}^{\pi} \frac{1}{2 \sin(x/2)} \left| \sum_{k=n}^{2n-1} c_k \sin(k + \frac{1}{2})x \right| \, dx \\
\leq \frac{1}{2n} \left( \int_{\pi/n}^{\pi} \frac{dx}{(2 \sin(x/2))^p} \right)^{1/p} \left( \int_{\pi/n}^{\pi} \left| \sum_{k=n}^{2n-1} c_k \sin(k + \frac{1}{2})x \right|^q \, dx \right)^{1/q}.
\]

Since
\[
\int_{\pi/n}^{\pi} \frac{dx}{(2 \sin(x/2))^p} \leq \pi^p \frac{dx}{x^p} \leq \frac{\pi}{p - 1} n^{p - 1},
\]

it follows that
\[
I_2 \leq \frac{1}{2} \left( \frac{\pi}{p - 1} \right)^{1/p} n^{-1/p} \left( \int_{\pi/n}^{\pi} \left| \sum_{k=n}^{2n-1} c_k \sin(k + \frac{1}{2})x \right|^q \, dx \right)^{1/q}.
\]

Let \( 1 < p < 2 \). Then using the Hausdorff-Young inequality we get
\[
\left( \frac{1}{n} \int_{0}^{\pi} \left| \sum_{k=n}^{2n-1} c_k \sin(k + \frac{1}{2})x \right|^q \, dx \right)^{1/q} \leq \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \sum_{k=n}^{2n-1} c_k e^{ikx} \right|^q \, dx \right)^{1/q} \leq \left( \sum_{k=n}^{2n-1} |c_k|^p \right)^{1/p}.
\]

Thus
\[
I_2 \leq (\pi/2)(p - 1)^{-1/p}(n^{-1} \sum_{k=n}^{2n-1} |c_k|^p)^{1/p}.
\]

Combining all estimates we get
\[
I_1 + I_2 \leq \pi (2 + \frac{1}{2} (p - 1)^{-1/p}) \left( \frac{1}{n} \sum_{k=n}^{2n-1} |c_k|^p \right)^{1/p},
\]

and the proof of Lemma 2.3 is completed.

3. Main result. In our main theorem we shall prove that for some \( 1 < p < 2 \), the class \( \mathcal{V}_p \cap \mathcal{F} \) is an \( L^1 \)-convergence class. To make use of our lemmas we shall reformulate that result in the following manner.
Theorem 3.1. Let \( f \in L^1(0, \pi) \) be an even and \( 2\pi \)-periodic function and for some \( 1 < p < 2 \) let the sequence \( \{a_k\} \) of its Fourier coefficients satisfy

\[
\left(2.1\right) \quad \frac{1}{n} \sum_{k=n}^{2n} k^p |\Delta a_k|^p = o(1), \quad n \to \infty.
\]

Then (1.1) holds.

Proof. Since

\[
\sum_{k=n+1}^{2n} (2n - k) a_k \cos kx = \sum_{k=n+1}^{2n} (2n - k) a_k (D_k(x) - D_{k-1}(x))
\]

\[
= -na_nD_n(x) + \sum_{k=n}^{2n-1} ((2n - k)a_k - (2n - k - 1)a_{k+1}) D_k(x),
\]

from (2.4) we get

\[
s_n(x) - f(x) = a_nD_n(x) + 2(\sigma_{2n-1}(x) - f(x))
\]

\[
- (\sigma_n(x) - f(x)) - \frac{1}{n} \sum_{k=n}^{2n-1} a_{k+1} D_k(x)
\]

\[
+ \frac{1}{n} \sum_{k=n}^{2n-1} (2n - k - 1) \Delta a_k D_k(x);
\]

hence

\[
\|s_n - f\| - |a_n| \|D_n\| \leq 2\|\sigma_{2n-1} - f\| + \|\sigma_n - f\|
\]

\[
+ \frac{1}{n} \int_0^\pi \left| \sum_{k=n}^{2n-1} a_k D_k(x) \right| \, dx
\]

\[
+ \frac{1}{n} \int_0^\pi \left| \sum_{k=n}^{2n-1} (2n - k - 1) \Delta a_k D_k(x) \right| \, dx.
\]

Applying Lemma 2.3 to the last two integrals we get

\[
\frac{1}{n} \int_0^\pi \left| \sum_{k=n}^{2n-1} a_k D_k(x) \right| \, dx \leq A_p \left( \frac{1}{n} \sum_{k=n}^{2n-1} |a_k|^p \right)^{1/p}
\]

and

\[
\frac{1}{n} \int_0^\pi \left| \sum_{k=n}^{2n-1} (2n - k - 1) \Delta a_k D_k(x) \right| \, dx \leq A_p \left( \frac{1}{n} \sum_{k=n}^{2n-1} k^p |\Delta a_k|^p \right)^{1/p}.
\]

Since \( \|D_n\| = (4/\pi^2)\lg n + O(1), \ n \to \infty, \) and since \( f \in L^1(0, \pi) \) implies that \( \|s_n - f\| = o(1), n \to \infty, \) we finally get

\[
\|s_n - f\| - |a_n| \lg n = O\left( \left( \frac{1}{n} \sum_{k=n}^{2n} k^p |\Delta a_k|^p \right)^{1/p} \right), \quad n \to \infty.
\]

In view of (2.1) this completes the proof of Theorem 3.1. We have now a corollary to this theorem.
Corollary 3.1. The class $C_p \cap \mathbb{Y}$ is a $L^1$-convergence class.

Proof. Since $n^{p-1} \sum_{k=n}^{2n-1} |A_k|^p = o(1)$, $n \to \infty$, implies

$$\frac{1}{n} \sum_{k=n}^{2n-1} k^p |A_k|^p = o(1), \quad n \to \infty,$$

it follows that $C_p \subset \mathcal{V}_p$.

4. Necessary and sufficient integrability conditions. Let $\{a_k\} \in \mathbb{V}$. Then

$$a_0 + \sum_{k=1}^{\infty} a_k \cos kx$$

converges pointwise to some $f$ in $(0, \pi]$. In order to prove that (4.1) is a Fourier series it suffices to show that $f \in L^1(0, \pi)$.

Theorem 4.1. Let $\{a_k\} \in \mathcal{V}_p \cap \mathbb{V}$. Then (4.1) is a Fourier series if and only if $\{a_k\} \in \mathcal{C}$.

Proof. The “if” part. Garrett and Stanojević [4] proved that if $\{a_k\} \in \mathbb{V}$, then $\|g_n - f\| = o(1)$, $n \to \infty$, if and only if $\{a_k\} \in \mathcal{C}$. Hence $\{a_k\} \in \mathcal{C}$ is always a sufficient condition for $f \in L^1(0, \pi)$ whenever $\{a_k\} \in \mathbb{V}$.

The “only if” part. The identity (3.1) can be rewritten as

$$s_n(x) - a_{n+1}D_n(x) - f(x) = g_n(x) - f(x)$$

$$= 2(\sigma_{2n-1}(x) - f(x)) - (\sigma_{n-1}(x) - f(x))$$

$$- \frac{1}{n} \sum_{k=n}^{2n-1} a_{k+1}D_k(x) - \frac{1}{n} \sum_{k=n}^{2n-1} A_kD_k(x) + \frac{\Delta_n D_n(x)}{n}.$$

Assuming that $f \in L^1(0, \pi)$ and applying Lemma 2.3 we get

$$\|g_n - f\| = O\left((\frac{1}{n} \sum_{k=n}^{2n} k^p |A_k|^p)^{1/p}\right), \quad n \to \infty.$$

Since $\{a_k\} \in \mathcal{V}_p \cap \mathbb{V}$ we have that $\|g_n - f\| = o(1)$, $n \to \infty$, and finally that $\{a_k\} \in \mathcal{C}$. This completes the proof of Theorem 4.1.

Corollary 4.1. Let $\{a_k\} \in C_p \cap \mathbb{V}$. Then (4.1) is a Fourier series if and only if $\{a_k\} \in \mathcal{C}$.

Proof. $C_p \subset \mathcal{V}_p$.

Let $\mathcal{M}$ be a class of all monotone null-sequences $\{a_k\}$. As a consequence to Corollary 4.1 we obtain a partial answer to the classical outstanding question: Let $\{a_k\} \in \mathcal{M}$. What are necessary and sufficient conditions for (4.1) to be a Fourier series?

Corollary 4.2. Let $\{a_k\} \in \mathcal{V}_p \cap \mathcal{M}$. Then (4.1) is a Fourier series if and only if $\{a_k\} \in \mathcal{C}$.

Theorem 4.1, Corollary 4.1 and Corollary 4.2 extend the integrability classes found by Stanojević [7].

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