TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 270, Number 1, March 1982

A HYPERSURFACE DEFECT RELATION FOR A CLASS OF MEROMORPHIC MAPS[1]

BY

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ABSTRACT. Let $D_1, \ldots, D_q$ be hypersurfaces of degree p in $\mathbf{P}_n$ with normal crossings. We prove for a certain class of meromorphic maps $f\colon \mathbf{C}^m \to \mathbf{P}_n$ a defect relation $\delta_f(D_1) + \cdots + \delta_f(D_q) \leqslant (n+1)/p$ conjectured by Ph. Griffiths and B. Shiffman.

Introduction. Let $f\colon \mathbf{C}^m \to \mathbf{P}_n$ be a meromorphic map. Let $D_1, \ldots, D_q$ be hypersurfaces of degree p in $\mathbf{P}_n$ such that $f(\mathbf{C}^m) \not\subseteq D_j$ for $j = 1, \ldots, q$. The Nevanlinna defect $\delta_f(D_j)$ of f for D_j is defined for $j = 1, \ldots, q$. When does the defect relation

$$\sum_{j=1}^{q} \delta_f(D_j) \leqslant (n+1)/p \tag{1}$$

hold? We shall provide a partial answer to this question.

If $p = 1$, this is the classical defect relation (Nevanlinna [**8**], H. Cartan [**4**], Ahlfors [**1**], Weyl [**13**], Stoll [**11**], Vitter [**12**]). Here we are concerned with the case $p \geqslant 2$. The Carlson-Griffiths-King theory [**3**, **6**] implies (1) if $D_1, \ldots, D_q$ have normal crossings and if $m \geqslant n = \operatorname{rank} f$. Hopefully this rank condition can be replaced by a more natural assumption which permits $m \leqslant n$.

P. Griffiths [**5**] conjectured that (1) holds if the image of f is not contained in any hypersurface of degree p and if $D_1, \ldots, D_q$ have normal crossings. We provide a counterexample (§5). If $f(\mathbf{C}^m)$ is not contained in any hypersurface, the conjecture (Shiffman [**9**]) remains unresolved, even if $m = 1$, despite many attempts.

In 1979 B. Shiffman [**10**] investigated a particular class $\mathfrak{S}$ of meromorphic maps of finite order. He considers $D_1, \ldots, D_q$ distinct hypersurfaces of degree p such that no point of $\mathbf{P}_n$ is contained in $n+1$ distinct D_j. If $f \in \mathfrak{S}$ and $f(\mathbf{C}^m) \not\subseteq D_j$ for $j = 1, \ldots, q$ he shows

$$\sum_{j=1}^{q} \delta_f(D_j) \leqslant 2n.$$

Under these assumptions, $2n$ is the best possible bound.

We prove (1) for a more general class $\mathfrak{R}$ of meromorphic maps, but, naturally we impose the stricter condition of normal crossings on the divisors. The maps of our

Received by the editors May 12, 1980. Paper presented at the 784th A.M.S. Meeting, Notre Dame, March 20–21, 1981.

1980 *Mathematics Subject Classification*. Primary 32A22, 32H25, 32H30; Secondary 30D35.

Key words and phrases. Defect relation, Second Main Theorem.

[1]This research was partially supported by a grant of the Consiglio Nazionale delle Ricerche (Italy).

class $\mathfrak{R}$ have infinite order or finite integral order and are "projections" of meromorphic maps with maximal linear Nevanlinna deficiencies.

In §1 and §2, we assemble the basic notions. In §3, we investigate the properties of normal crossings. The defect relation (1) is proved in §4. In §5 we provide a counterexample to (1), if the image of f is not contained in any hypersurface of degree p, but is contained in a higher dimensional hypersurface. In §6 we investigate the relation of our class of maps with Shiffman class.

I wish to thank my thesis advisor, Professor W. Stoll, who introduced me to value distribution theory and placed all his knowledge at my disposal.

1. Preliminaries. Let $f\colon \mathbf{C}^m \to \mathbf{P}_n$ be a meromorphic map. A holomorphic vector function $\mathfrak{v}\colon \mathbf{C}^m \to \mathbf{C}^{n+1}$ is said to be a *representation* for f if $\mathfrak{v}^{-1}(0) \neq \mathbf{C}^m$ and $\mathbf{P} \circ \mathfrak{v} = f$ on $\mathbf{C}^m - \mathfrak{v}^{-1}(0)$. The representation is said to be *reduced* if $\dim \mathfrak{v}^{-1}(0) \leqslant m - 2$.

A meromorphic map $f\colon \mathbf{C}^m \to \mathbf{P}_n$ is said to be *nondegenerate of degree p* if for any hypersurface D of degree p in $\mathbf{P}_n$ we have $f(\mathbf{C}^m) \not\subseteq \operatorname{supp} D$, otherwise f is said to be *degenerate of degree p*. If a meromorphic map is nondegenerate (or degenerate) of degree 1 we say that f is nondegenerate (or degenerate).

2. Nevanlinna theory. If $z = (z_1, \ldots, z_m) \in \mathbf{C}^m$, define

$$|z| = \left(|z_1|^2 + \cdots + |z_m|^2\right)^{1/2} \quad \text{and} \quad \tau(z) = |z|^2.$$

If $r > 0$, set $\mathbf{C}^m[r] = \{z \in \mathbf{C}^m \mid |z| \leqslant r\}$ and $\mathbf{C}^m\langle r\rangle = \partial \mathbf{C}^m[r]$. We define

$$\begin{aligned} &\upsilon = dd^c\tau \quad \text{on } \mathbf{C}^m, \\ &\sigma = \sigma_m = d^c \log \tau \wedge \left(dd^c \log \tau\right)^{m-1} \quad \text{on } \mathbf{C}^m - \{0\}, \end{aligned}$$

where $d^c = (i/4\pi)(\bar\partial - \partial)$.

Let ν be a divisor on $\mathbf{C}^m$. Abbreviate $S[r] = \mathbf{C}^m[r] \cap \operatorname{supp} \nu$, if $0 < r \in \mathbf{R}$, the *counting function* of ν is defined by

$$n_\nu(r) = \begin{cases} r^{2-2m} \int_{S[r]} \nu \upsilon^{m-1} & \text{if } m > 1, \\ \sum_{z \in S[r]} \nu(z) & \text{if } m = 1. \end{cases}$$

For all $0 < r_0 < r$ the *valence function* of ν is defined by

$$N_\nu(r, r_0) = \int_{r_0}^{r} n_\nu(t) t^{-1}\, dt.$$

Let $f\colon \mathbf{C}^m \to \mathbf{P}_n$ be a meromorphic map. Let ω be the Fubini-Kaehler form on $\mathbf{P}_n$. For $r > r_0 > 0$, define the *characteristic function* of f

$$T_f(r, r_0) = \int_{r_0}^{r} t^{1-2m}\, dt \int_{\mathbf{C}^m[r]} f^*(\omega) \wedge \upsilon^{m-1}$$

and for any hypersurface of degree p in $\mathbf{P}_n$ we define the valence function of f for D

$$N_f(r, r_0, D) = N_{\nu_f^D}(r, r_0) \geqslant 0,$$

where $\nu_f^D = f^*(D)$ is the pull-back divisor.

We shall need the following well-known results. Let D be a hypersurface of degree p on $\mathbf{P}_n$ such that $f(\mathbf{C}^m) \not\subseteq \operatorname{supp} D$. Let α be a homogeneous polynomial of degree p on $\mathbf{C}^{n+1}$ such that $\mathbf{P}(\alpha^{-1}(0)) = \operatorname{supp} D$. Then for $r > r_0 > 0$ we have

(2.1) FIRST MAIN THEOREM.

$$pT_f(r, r_0) + O(1) \geqslant N_f(r, r_0, D).$$

(2.2) JENSEN'S FORMULA.

$$N_f(r, r_0, D) = \int_{\mathbf{C}^m\langle r\rangle} \log|\alpha \circ \mathfrak{v}|\,\sigma - \int_{\mathbf{C}^m\langle r_0\rangle} \log|\alpha \circ \mathfrak{v}|\,\sigma.$$

$$T_f(r, r_0) = \int_{\mathbf{C}^m\langle r\rangle} \log|\mathfrak{v}|\,\sigma - \int_{\mathbf{C}^m\langle r_0\rangle} \log|\mathfrak{v}|\,\sigma, \tag{2.3}$$

where $\mathfrak{v}$ denotes a reduced representation of f.

Let D be a hypersurface of degree p and assume that $f(\mathbf{C}^m) \not\subseteq \operatorname{supp} D$. The *defect* of f for the hypersurface D is defined by

$$\delta_f(D) = \liminf_{r\to\infty}\left(1 - \frac{N_f(r, r_0, D)}{pT_f(r, r_0)}\right).$$

By (2.1) we have

$$0 \leqslant \delta_f \leqslant 1.$$

We shall use from now on the following notation. Let g and h be real valued functions on $\mathbf{R}(r_0, \infty)$. We write $g(r) \underset{\cdot}{\leqslant} h(r)$ if there exists a subset E of $\mathbf{R}(r_0, \infty)$ with finite Lebesgue measure such that $g(r) \leqslant h(r)$ for all $r \in \mathbf{R}(r_0, \infty) - E$.

Now we can state

SECOND MAIN THEOREM (STOLL [**11**]). *Let f: $\mathbf{C}^m \to \mathbf{P}_n$ be a nondegenerate, meromorphic map. Let $H_1, \ldots, H_q$ be hyperplanes in $\mathbf{P}_n$ in general position with $q \geqslant n + 1$. Then*

$$(q - n - 1)T_f(r, r_0) \underset{\cdot}{\leqslant} \sum_{j=1}^{q} N_f(r, r_0, H_j) + O\big(\log rT_f(r, r_0)\big). \tag{2.4}$$

3. Normal crossings. Let D be a hypersurface of degree p in $\mathbf{P}_n$. Let α be a homogeneous polynomial of degree p on $\mathbf{C}^{n+1}$ such that $\mathbf{P}(\alpha^{-1}(0)) = \operatorname{supp} D$. Then we set $D = D[\alpha]$.

Let $D_j = D[\alpha_j]$ be hypersurfaces of degree p on $\mathbf{P}_n$, for $j = 1, \ldots, q$. For $y = \mathbf{P}(\mathfrak{y}) \in \mathbf{P}_n$, let

$$d = d(y) = \#\{j \in \mathbf{N}[1, q] \mid \alpha_j(\mathfrak{y}) = 0\}.$$

We have $0 \leqslant d \leqslant q$. Call $d = d(y)$ the *crossings number* of $D_1, \ldots, D_q$ at y. If $d(y) \geqslant 1$ then there exists one and only one $\kappa = \kappa(y)$: $\mathbf{Z}[1, d] \to \mathbf{Z}[1, q]$ such that $\alpha_{\kappa(j)}(\mathfrak{y}) = 0$ for every $j = 1, \ldots, d$. Call κ the *crossings selector* of $D_1, \ldots, D_q$ at y and

$$J(y) = J(D_1, \ldots, D_q, y) = d\alpha_{\kappa(1)}(\mathfrak{y}) \wedge \cdots \wedge d\alpha_{\kappa(d)}(\mathfrak{y})$$

the *crossings Jacobian* of $D_1,\dots,D_q$ at y (here $d\alpha_j(\mathfrak{y})$ is considered as an element of $(\mathbf{C}^{n+1})^*$).

We say that $D_1,\dots,D_q$ have *normal crossings* at y if and only if $d(y) \geqslant 1$ and $J(y) \neq 0$. Also we say that $D_1,\dots,D_q$ have *normal crossings* if and only if $D_1,\dots,D_q$ have normal crossings at every $y \in \bigcup_{j=1}^{q} \operatorname{supp} D_j$.

REMARK 3.1. (a) If $D_1,\dots,D_q$ have normal crossings at y then $d(y) \leqslant n$.

(b) If $D_1,\dots,D_q$ have normal crossings then D_j is smooth for $j = 1,\dots,q$.

LEMMA 3.2. *Let φ: $\mathbf{C}^{N+1} \to \mathbf{C}^{n+1}$ be a surjective linear map. Take $\mathfrak{y} \in \mathbf{C}^{N+1} - \ker\varphi$. Let $D_j = D[\alpha_j]$ be hypersurfaces of degree p in $\mathbf{P}_n$, for $j = 1,\dots,q$. Set $\tilde{D}_j = D[\alpha_j \circ \varphi]$. Then $D_1,\dots,D_q$ have normal crossings at $\mathbf{P}(\varphi(\mathfrak{y}))$ with crossings number d and crossings selector κ if and only if $\tilde{D}_1,\dots,\tilde{D}_q$ have normal crossings at $\mathbf{P}(\mathfrak{y})$ with crossings number d and crossings selector κ.*

PROOF. By definition the crossings number and the crossings selector are the same. Set $\beta_j = \alpha_j \circ \varphi$. We have

$$d\beta_j(\mathfrak{y}) = d\alpha_j(\varphi(\mathfrak{y})) \circ d\varphi(\mathfrak{y}) = d\alpha_j(\varphi(\mathfrak{y})) \circ \varphi.$$

So if $y = \mathbf{P}(\mathfrak{y})$ and $x = \mathbf{P}(\varphi(\mathfrak{y}))$

$$\begin{aligned} J\big(\tilde{D}_1,\dots,\tilde{D}_q, y\big) &= d\beta_{\kappa(1)}(\mathfrak{y}) \wedge \cdots \wedge d\beta_{\kappa(d)}(\mathfrak{y}) \\ &= d\alpha_{\kappa(1)}(\varphi(\mathfrak{y})) \circ \varphi \wedge \cdots \wedge d\alpha_{\kappa(d)}(\varphi(\mathfrak{y})) \circ \varphi \\ &= \varphi^*\big(d\alpha_{\kappa(1)}(\varphi(\mathfrak{y})) \wedge \cdots \wedge d\alpha_{\kappa(d)}(\varphi(\mathfrak{y}))\big) \\ &= \varphi^*\big(J(D_1,\dots,D_q, x)\big). \end{aligned}$$

Since φ is surjective, φ^* is injective. Hence $J(\tilde{D}_1,\dots,\tilde{D}_q, y) \neq 0$ if and only if $J(D_1,\dots,D_q, x) \neq 0$. Q.E.D.

Let ζ_j: $\mathbf{C}^{N+1} \to \mathbf{C}$ be the jth coordinate function, i.e. $\zeta_j(z_0,\dots,z_N) = z_j$. Set

$$e_j = e_j^{(N)} = \Big(\underbrace{0,\dots,0}_{j},1,0,\dots,0\Big) \in \mathbf{C}^{N+1}.$$

If J: $\mathbf{Z}[0, N] \to \mathbf{Z}[0, p]$ let $J(k) = j_k$ and $J = (j_0,\dots,j_N)$. Let $K = K(p)$ be the set of all J: $\mathbf{Z}[0, N] \to \mathbf{Z}[0, p]$ such that $j_0 + \cdots + j_N = p$. If $J \in K$ then $\zeta^J = \zeta_0^{j_0} \cdots \zeta_N^{j_N}$: $\mathbf{C}^{N+1} \to \mathbf{C}$ is a homogeneous polynomial of degree p. Then if we denote by $\mathbf{C}_{(p)}^{N+1}$ all the homogeneous polynomials of degree p on $\mathbf{C}^{N+1}$ we have that $\{\zeta^J \mid J \in K\}$ is a base for $\mathbf{C}_{(p)}^{N+1}$ over $\mathbf{C}$. Therefore if $\alpha \in \mathbf{C}_{(p)}^{N+1}$ then $\alpha = \sum_{J \in K} \alpha_J \zeta^J$ where the coefficients $\alpha_J \in \mathbf{C}$ are unique. We set $B = B(p) = \{J \in K(p) \mid j_k \geqslant p - 1$ for some $k \in \mathbf{Z}[0, N]\}$. Consider the map $\delta = \delta_p$: $(\mathbf{Z}[0, N])^2 \to B$, for $p \geqslant 2$ defined as follows:

$$\delta(h, k)(j) = \begin{cases} 0 & \text{if } h \neq j \neq k, \\ 1 & \text{if } h = j \neq k, \\ p - 1 & \text{if } h \neq j = k, \\ p & \text{if } h = j = k. \end{cases}$$

Define $\varepsilon = \varepsilon_p\colon (\mathbf{Z}[0, N])^2 \to \{1,2\}$ by $\varepsilon(h, k) = \#\delta^{-1}(\delta(h, k))$.Then we have $\varepsilon(h, k) \equiv 1$ if $p > 2$ and for $p = 2$

$$\varepsilon(h, k) = \begin{cases} 1 & \text{if } h = k, \\ 2 & \text{if } h \neq k. \end{cases}$$

If $\alpha = \Sigma_{J \in K} \alpha_J \zeta^J \in \mathbf{C}_{(p)}^{N+1}$, we set

$$\rho(\alpha) = \sum_{J \in B} \alpha_J \zeta^J = \sum_{h,k=0}^{N} \alpha_{hk} \zeta_h \zeta_k^{p-1} \in \mathbf{C}_{(p)}^{N+1},$$

$$\rho_k(\alpha) = \sum_{h=0}^{N} \alpha_{hk} \zeta_h \in \mathbf{C}_{(1)}^{N+1} \quad \text{for } k = 0, \ldots, N,$$

where $\alpha_{hk} = \varepsilon(h, k)^{-1} \alpha_{\delta(h,k)}$. We have

$$\rho(\alpha) = \sum_{k=0}^{N} \rho_k(\alpha) \zeta_k^{p-1}. \tag{3.1}$$

LEMMA 3.3. *Let* $\alpha \in \mathbf{C}_{(p)}^{N+1}$. *Then, for* $k \in \mathbf{Z}[0, N]$

(i) $\rho(\alpha)(e_k) = \alpha(e_k) = \rho_k(\alpha)(e_k)$.

(ii) $(d\rho(\alpha))(e_k) = d\alpha(e_k)$.

(iii)

$$d\alpha(e_k) = \begin{cases} 2\rho_k(\alpha) & \text{if } p = 2, \\ \rho_k(\alpha) & \text{if } p > 2 \text{ and } \alpha(e_k) = 0. \end{cases}$$

PROOF. (i) Set $\chi = \alpha - \rho(\alpha)$. We have $\chi(e_k) = 0$, $d\chi(e_k) = 0$ and $\rho(\chi) = \rho_k(\chi) = 0$ for $k = 0, \ldots, N$. If $\rho(\alpha) = \Sigma_{h,k=0}^{N} \alpha_{hk} \zeta_h \zeta_k^{p-1}$ then $\alpha(e_k) = \rho(\alpha)(e_k) = \alpha_{kk} = \rho_k(\alpha)(e_k)$.

(ii) It is a direct consequence of $d\chi(e_k) = 0$.

(iii) We have

$$(d\rho(\alpha))(e_k) = \begin{cases} p\alpha_{kk}\zeta_k + \Sigma_{h \neq k} \alpha_{hk} \zeta_k & \text{if } p > 2, \\ 2\rho_k(\alpha) & \text{if } p = 2. \end{cases}$$

Therefore by (ii), if $p = 2$ we have $d\alpha(e_k) = 2\rho_k(\alpha)$, if $p > 2$ and $\alpha(e_k) = \alpha_{kk} = 0$ then $d\alpha(e_k) = \rho^k(\alpha)$. Q.E.D.

Let $D_j = D[\alpha_j]$ be hypersurfaces of degree p for $j = 1, \ldots, q$. Set $R_j = D[\rho(\alpha_j)]$ and $R_{kj} = D[\rho_k(\alpha_j)]$. Let $\varphi\colon \mathbf{C}^{N+1} \to \mathbf{C}^{n+1}$ be a surjective linear map. Set $\tilde{D}_j = D[\alpha_j \circ \varphi]$, $\tilde{R}_j = D[\rho(\alpha_j \circ \varphi)]$ and $\tilde{R}_{kj} = D[\rho_k(\alpha_j \circ \varphi)]$ for $j = 1, \ldots, q$. Then as a consequence of Lemma 3.2 and Lemma 3.3 we get

COROLLARY 3.4. (a) $\{D_j\}$, $\{R_j\}$ *and* $\{R_{kj}\}$ *have the same crossings number and crossings selector at* $\mathbf{P}(e_k)$. *Moreover if one of them has normal crossings at* $\mathbf{P}(e_k)$ *then all of them have normal crossings at* $\mathbf{P}(e_k)$.

(b) *If* $\{D_j\}$ *has normal crossings at* $\mathbf{P}(\varphi(e_k))$ *with crossings number* d *and crossings selector* κ, *then* $\{\tilde{D}_j\}$, $\{\tilde{R}_j\}$ *and* $\{\tilde{R}_{kj}\}$ *have normal crossings at* $\mathbf{P}(e_k)$ *with crossings number* d *and crossings selector* κ.

Let η_0: $\mathbf{C} \to \{0, 1\}$ be defined by $\eta_0(a) = 1$ if $a \neq 0$ and $\eta_0(0) = 0$. If $\alpha = \sum_{J \in K} \alpha_J \zeta^J \in \mathbf{C}_{(p)}^{N+1}$ we define

$$\eta_p(\alpha) = \sum_{J \in K} \eta_0(\alpha_J) \,|\, \zeta^J \,|\,.$$

PROPOSITION 3.5. *Let* φ: $\mathbf{C}^{N+1} \to \mathbf{C}^{n+1}$ *be a surjective linear map. Let* $D_j = D[\alpha_j]$ *be hypersurfaces of degree* p *in* $\mathbf{P}_n$, *for* $j = 1, \ldots, q$. *Assume that* $D_1, \ldots, D_q$ *have normal crossings. Then there exist a constant* $c > 0$ *and an injective map* τ: $\mathbf{Z}[0, n] \to \mathbf{Z}[0, N]$ *such that*

$$\prod_{j=1}^{q} \eta_p(\alpha_j \circ \varphi) \geqslant c|\zeta|^{qp-n-1} \prod_{h=0}^{n} |\zeta_{\tau(h)}|, \tag{3.2}$$

where $|\,\zeta\,| = (\sum_{h=0}^{N} |\,\zeta_h\,|^2)^{1/2}$.

PROOF. We set $\beta_j = \alpha_j \circ \varphi$ for $j = 1, \ldots, q$, $\gamma = \eta_p \circ \rho$ and $\gamma_k = \eta_1 \circ \rho_k$. Then by (3.1) we have

$$\gamma(\beta_j) = \sum_{k=0}^{N} \gamma_k(\beta_j) \,|\, \zeta_k \,|^{p-1} \quad \text{for } j = 1, \ldots, q$$

and

$$\eta_p(\beta_j) \geqslant \gamma(\beta_j) = \eta_p(\rho(\beta_j)).$$

So

$$\begin{aligned} \prod_{j=1}^{q} \eta_p(\beta_j) &\geqslant \prod_{j=1}^{q} \gamma(\beta_j) = \prod_{j=1}^{q} \sum_{h=0}^{N} \gamma_h(\beta_j) |\zeta_h|^{p-1} \\ &\geqslant \sum_{h=0}^{N} \left(\prod_{j=1}^{q} \gamma_h(\beta_j) \right) |\zeta_h|^{qp-q}. \end{aligned} \tag{3.3}$$

Since φ is surjective, there exists τ: $\mathbf{Z}[0, n] \to \mathbf{Z}[0, N]$ such that $\{\varphi(e_{\tau(0)}), \ldots, \varphi(e_{\tau(n)})\}$ is a base for $\mathbf{C}^{n+1}$. Set $T = \tau(\mathbf{Z}[0, n]) \subseteq \mathbf{Z}[0, N]$ and $T_h = T - \{h\}$.

CLAIM 1. Let $d = d_h$ be the crossings number of $D_1, \ldots, D_q$ at $\mathbf{P}(\varphi(e_h))$ then there exists $\varepsilon = \varepsilon_h$: $\mathbf{Z}[1, d_h] \to T_h$ such that

$$\prod_{j=1}^{q} \gamma(\beta_j) \geqslant |\zeta_{\varepsilon(1)}| \cdots |\zeta_{\varepsilon(d)}| \, |\zeta_h|^{q-d}. \tag{3.4}$$

PROOF OF CLAIM 1. By definition we have $\gamma_h(\beta_j) = \eta_1(\rho_h(\beta_j))$. Let $\kappa = \kappa_h$ be the crossings selector of $D_1, \ldots, D_q$ at $\mathbf{P}(\varphi(e_h))$. Then by Corollary 3.4(b), $\tilde{R}_{hj} = D[\rho_h(\beta_j)]$ for $j = 1, \ldots, q$ have normal crossings at $\mathbf{P}(e_h)$ with crossings number d and crossings selector κ. Set $\chi_j = \rho_h(\beta_{\kappa(j)})$. Then we have

$$\prod_{i=1}^{q} \gamma_h(\beta_i) = \left(\prod_{j=1}^{d} \eta_1(\chi_j) \right) |\zeta_h|^{q-d}.$$

Now $\chi_1 \wedge \cdots \wedge \chi_d \neq 0$, $\chi_j(e_h) = 0$ for $j = 1, \ldots, d$, $\{\varphi(e_{\tau(0)}), \ldots, \varphi(e_{\tau(n)})\}$ a base for $\mathbf{C}^{n+1}$ and an easy argument of linear algebra show that there exists $\varepsilon = \varepsilon_h$: $\mathbf{Z}[1, d] \to T_h$ injective such that

$$\prod_{j=1}^{d} \eta_1(\chi_j) \geqslant |\zeta_{\varepsilon(1)}| \cdots |\zeta_{\varepsilon(d)}|.$$

Therefore we get (3.4).

From (3.3) and (3.4) we obtain

$$\prod_{j=1}^{q} \eta_p(\beta_j) \geqslant \sum_{h=0}^{N} |\zeta_{\varepsilon_h(1)}| \cdots |\zeta_{\varepsilon_h(d_h)}| \, |\zeta_h|^{qp-d_h}. \tag{3.5}$$

Abbreviate

$$f_h = |\zeta_{\varepsilon_h(1)}| \cdots |\zeta_{\varepsilon_h(d_h)}| \, |\zeta_h|^{qp-d_h}, \quad g_h = \left(\prod_{j=0}^{n} |\zeta_{\tau(j)}| \right) |\zeta_h|^{qp-n-1}$$

and

$$f = \sum_{h=0}^{N} f_h, \quad g = \sum_{h=0}^{N} g_h.$$

CLAIM 2.

$$f \leqslant g/(N+1). \tag{3.6}$$

PROOF OF CLAIM 2. W.l.o.g. we may assume $\tau(j) = j$ and therefore $\varepsilon_h(\mathbf{Z}[1, d_h]) \subseteq \mathbf{Z}[0, n]$. We set $x_j = |\zeta_j(z)| \in \mathbf{R}_+$ for $z \in \mathbf{C}^{N+1}$ and $j = 0, \ldots, N$. Let $j_0, \ldots, j_N$ be a permutation of $\{0, \ldots, N\}$ such that $x_{j_0} \geqslant \cdots \geqslant x_{j_N}$. Abbreviate $k = j_0$. Then

$$\begin{aligned}(N+1)^{-1} g(z) &\leqslant \operatorname*{Max}_{0 \leqslant j \leqslant N} g_j(z) = x_0 \cdots x_n x_k^{qp-n-1} \\ &= x_0 \cdots x_n x_k^{d_k - n - 1} x_k^{qp-d_k} \\ &\leqslant x_{\varepsilon_k(1)} \cdots x_{\varepsilon_k(d_k)} x_k^{qp-d_k} = f_k(z) \leqslant f(z).\end{aligned}$$

Using (3.5), (3.6) and the inequalities

$$(N+1)^{(n+2-qp)} \left(\sum_{j=0}^{N} |\zeta_j|^{qp-n-1} \right) \leqslant \sum_{j=0}^{N} |\zeta_j|^{qp-n-1} \leqslant \sum_{j=0}^{N} |\zeta_j|^{qp-n-1}$$

we get (3.2) where $c = (N+1)^{n+1-qp}$. Q.E.D.

4. Defect relation. Let ζ_j: $\mathbf{C}^{N+1} \to \mathbf{C}$ be the jth coordinate function. Set $H_j = \mathbf{P}(\zeta_j^{-1}(0)) \subseteq \mathbf{P}_N$. If φ: $\mathbf{C}^{N+1} \to \mathbf{C}^{n+1}$ is a linear map we denote by $\mathbf{P}(\varphi)$ the projective map induced by φ. Abbreviate

$$e_j = (\underbrace{0, \ldots, 0}_{j}, 1, 0, \ldots, 0) \quad \text{for } j = 0, \ldots, N.$$

DEFINITION 4.1. Let $f\colon \mathbf{C}^m \to \mathbf{P}_n$ be a meromorphic map. We say that $f \in \mathfrak{D}$, if there exist a meromorphic map $g\colon \mathbf{C}^m \to \mathbf{P}_N$ and a linear map $\varphi\colon \mathbf{C}^{N+1} \to \mathbf{C}^{n+1}$ such that

1. $f = \mathbf{P}(\varphi) \circ g$.
2. $\delta_g(H_j) = 1$ or $N_g(r, r_0, H_j) = o(T_g(r, r_0))$, $j = 0, \ldots, N$.
3. If $\mathfrak{v}$ and $\mathfrak{g}$ are reduced representations of f and g respectively, let $u\colon \mathbf{C}^m \to \mathbf{C}$ be a holomorphic function such that $u\mathfrak{v} = \varphi \circ \mathfrak{g}$, then we require that

$$N_u(r, r_0, 0) = o\big(T_g(r, r_0)\big).$$

We say that (g, φ) satisfying conditions 1, 2 and 3 is a *decomposition* of f.

REMARK 4.2. A meromorphic map g satisfying condition 2 in Definition 4.1 is transcendental (see Mori [7]).

PROPOSITION 4.3. *Let (g, φ) be a decomposition of $f \in \mathfrak{D}$. Then*

$$T_f(r, r_0) \leqslant T_g(r, r_0) + O(1). \tag{4.1}$$

If, in addition, g is nondegenerate and $\varphi(e_j) \neq 0$ for $j = 0, \ldots, N$, then

$$T_g(r, r_0) \mathrel{\underset{\cdot}{\leqslant}} T_f(r, r_0) + o\big(T_f(r, r_0)\big). \tag{4.2}$$

PROOF. Let $\mathfrak{v}$ and $\mathfrak{g}$ be reduced representations of f and g respectively and u a holomorphic function such that $u\mathfrak{v} = \varphi \circ \mathfrak{g}$. Put $\mathfrak{w} = u\mathfrak{v} = \varphi \circ \mathfrak{g}$. Take $0 \neq \beta$: $\mathbf{C}^{n+1} \to \mathbf{C}$ linear function with $|\beta| = 1$. Define $\chi = \beta \circ \varphi\colon \mathbf{C}^{N+1} \to \mathbf{C}$ and $F = u(\beta \circ \mathfrak{v}) = \beta \circ \mathfrak{w} = \chi \circ \mathfrak{g}$. Schwarz's inequality implies

$$|F| \leqslant |\beta|\,|\mathfrak{w}| = |\mathfrak{w}| = |\varphi \circ \mathfrak{g}| \leqslant |\varphi|\,|\mathfrak{g}|$$

and

$$\begin{aligned}\int_{\mathbf{C}^m\langle r\rangle} \log|F|\sigma &\leqslant \int_{\mathbf{C}^m\langle r\rangle} \log|\mathfrak{w}|\sigma = \int_{\mathbf{C}^m\langle r\rangle} \log|u|\sigma + \int_{\mathbf{C}^m\langle r\rangle} \log|\mathfrak{v}|\sigma \\ &\leqslant \int_{\mathbf{C}^m\langle r\rangle} \log|\mathfrak{g}|\sigma + O(1).\end{aligned}$$

Therefore we get

$$\begin{aligned}\int_{\mathbf{C}^m\langle r\rangle} \log|F|\sigma &\leqslant N_u(r, r_0, 0) + T_f(r, r_0) + O(1) \\ &\leqslant T_g(r, r_0) + O(1)\end{aligned} \tag{4.3}$$

which implies (4.1).

In order to prove (4.2), consider β in such a way that $\chi(e_j) \neq 0$ for $j = 0, \ldots, N$.

Set $H_{N+1} = \mathbf{P}(\operatorname{Ker} \chi) \subseteq \mathbf{P}_N$. Then $H_0, \ldots, H_{N+1}$ are in general position and by (2.4) we have

$$T_g(r, r_0) \mathrel{\underset{\cdot}{\leqslant}} \sum_{j=0}^{N+1} N_g(r, r_0, H_j) + O\big(\log r T_g(r, r_0)\big).$$

But

$$N_g(r, r_0, H_j) = o\big(T_g(r, r_0)\big) \quad \text{for } j = 0, \ldots, N$$

and

$$N_g(r, r_0, H_{N+1}) = \int_{\mathbf{C}^m\langle r\rangle} \log|\chi \circ \mathfrak{g}|\sigma + O(1) = \int_{\mathbf{C}^m\langle r\rangle} \log|F|\sigma + O(1)$$
$$\leqslant T_f(r, r_0) + N_u(r, r_0, 0) + O(1) = T_f(r, r_0) + o\big(T_g(r, r_0)\big).$$

Therefore by Remark 4.2 we get

$$T_g(r, r_0) \underset{\cdot}{\leqslant} T_f(r, r_0) + o\big(T_g(r, r_0)\big).$$

Now, for every $0 < \varepsilon < 1$, we have

$$(1 - \varepsilon)T_g(r, r_0) \underset{\cdot}{\leqslant} T_f(r, r_0)$$

or

$$o\big(T_g(r, r_0)\big) \underset{\cdot}{\leqslant} o\big(T_f(r, r_0)\big).$$

Hence we get (4.2). Q.E.D.

DEFINITION 4.4. Let (g, φ) be a decomposition of $f \in \mathfrak{D}$. We say that (g, φ) is a *reduced decomposition* if g is nondegenerate and $\varphi(e_j) \neq 0$ for $j = 0, \ldots, N$. Denote by $\mathfrak{R}$ the class of all meromorphic maps $f \in \mathfrak{D}$ such that f admits a reduced decomposition.

We shall use the following general assumptions.

(A1) Let f: $\mathbf{C}^m \to \mathbf{P}_n$ be a meromorphic map. Assume $f \in \mathfrak{R}$ with (g, φ) as a reduced decomposition.

(A2) Let $\mathfrak{v}$ and $\mathfrak{g}$ be reduced representations of f and g respectively.

(A3) Let u: $\mathbf{C}^m \to \mathbf{C}$ be a holomorphic function such that $u\mathfrak{v} = \varphi \circ \mathfrak{g}$ and $N_u(r, r_0, 0) = o(T_g(r, r_0))$.

LEMMA 4.5. *Assume that* (A1)–(A3) *holds. Let $D = D[\alpha]$ be a hypersurface of degree p. Then if g is nondegenerate of degree p, we have*

$$o\big(T_f(r, r_0)\big) + N_f(r, r_0, D) \underset{\cdot}{\geqslant} \int_{\mathbf{C}^m\langle r\rangle} \log\big(\eta_p(\alpha \circ \varphi) \circ \mathfrak{g}\big)\sigma. \tag{4.4}$$

PROOF. Set $\beta = \alpha \circ \varphi$. If $\beta \equiv 0$ then (4.4) is trivially true. Suppose $\beta \not\equiv 0$. We have $u^p\alpha \circ \mathfrak{v} = \beta \circ \mathfrak{g}$. Set $\tilde{D} = D[\beta]$. Then (2.2) implies

$$N_f(r, r_0, D) + pN_u(r, r_0, 0) = N_g\big(r, r_0, \tilde{D}\big).$$

Therefore

$$o\big(T_g(r, r_0)\big) + N_f(r, r_0, D) = N_g\big(r, r_0, \tilde{D}\big).$$

Let $\beta = \sum_{J\in K}\beta_J\zeta^J$ then $\beta \circ \mathfrak{g} = \sum_{J\in K}\beta_J\mathfrak{g}^J$ where $\mathfrak{g}^J = \prod_{h=0}^N \mathfrak{g}_h^{j_h}$. Denote by D^J the hypersurface of degree p such that supp $D^J = \mathbf{P}((\zeta^J)^{-1}(0))$. Then by (2.2), we get

$$N_g\big(r, r_0, D^J\big) = \sum_{k=0}^{N} j_k\left(\int_{\mathbf{C}^m\langle r\rangle} \log|\mathfrak{g}_k|\sigma - \int_{\mathbf{C}^m\langle r\rangle} \log|\mathfrak{g}_k|\sigma\right)$$
$$= \sum_{k=0}^{N} j_k N_g(r, r_0, H_k) = o\big(T_g(r, r_0)\big)$$

for every $J \in K$.

If $K(\beta) = \{J \in K \mid \beta_J \neq 0\}$ let $K(\beta) = \{J_0,\ldots,J_t\}$ then $\beta \circ \mathfrak{g} = \Sigma_{s=0}^{t} \beta_{J_s} g^{J_s}$. Set $\tilde{\mathfrak{h}} = (g^{J_0},\ldots,g^{J_t})\colon \mathbf{C}^m \to \mathbf{C}^{t+1}$ and $h = \mathbf{P}(\tilde{\mathfrak{h}})$. Since g is nondegenerate of degree p then h is nondegenerate. Let $\mathfrak{h} = (h_0,\ldots,h_t)$ be a reduced representation of h and v a holomorphic function such that $\tilde{\mathfrak{h}} = v\mathfrak{h}$. We have $g^{J_s} = vh_s$ for $s = 0,\ldots,t$. Let $\tilde{H}_0,\ldots,\tilde{H}_t$ be the coordinate hyperplanes in $\mathbf{P}_t$. Then we have

$$N_g(r, r_0, D^{J_s}) = N_h(r, r_0, \tilde{H}_s) + N_v(r, r_0, 0) \quad \text{for } s = 0,\ldots,t,$$

$$N_g(r, r_0, \tilde{D}) = N_h(r, r_0, \tilde{H}_{t+1}) + N_v(r, r_0, 0),$$

where H_{t+1} is the hyperplane in $\mathbf{P}_t$ given by $\Sigma_{s=0}^{t} \beta_{J_s} z_s = 0$. If $t = 0$ then

$$\int_{\mathbf{C}^m\langle r\rangle} \log(\eta_\rho(\alpha \circ \varphi) \circ \mathfrak{g})\sigma \underset{\cdot}{\leqslant} o(T_f(r, r_0))$$

and (4.4) is true. Let $t \geqslant 1$. Then proceeding as in the proof of (4.2) of Proposition 4.3 we get

$$T_h(r, r_0) \underset{\cdot}{\leqslant} \sum_{s=0}^{t+1} N_h(r, r_0, H_s) + O(\log rT_h(r, r_0)).$$

Since g is transcendental and

$$T_h(r, r_0) + N_v(r, r_0, 0) = \int_{\mathbf{C}^m\langle r\rangle} \log|\tilde{\mathfrak{h}}|\sigma + O(1) \leqslant T_g(r, r_0)$$

we obtain

$$\begin{aligned}\int_{\mathbf{C}^m\langle r\rangle} \log|\tilde{\mathfrak{h}}|\sigma &\underset{\cdot}{\leqslant} N_g(r, r_0, \tilde{D}) - (t+1)N_v(r, r_0, 0) + o(T_g(r, r_0)) \\ &\leqslant N_g(r, r_0, \tilde{D}) + o(T_g(r, r_0)).\end{aligned}$$

Finally from Proposition 4.3 and since

$$|\tilde{\mathfrak{h}}| = \left(\sum_{J \in K(\beta)} |g^J|^2\right)^{1/2} \geqslant c\eta_p(\beta) \circ \mathfrak{g}$$

for some positive constant c, we get (4.4). Q.E.D.

THEOREM 4.6 (SECOND MAIN THEOREM AND DEFECT RELATION). *Assume that* (A1)–(A3) *holds, with g nondegenerate of degree p and* φ *surjective. Let* $D_j = D[\alpha_j]$ *be hypersurfaces of degree p for* $j = 1,\ldots,q$, $q \geqslant n+1$. *Assume* $D_1,\ldots,D_q$ *have normal crossings. Then*

$$(4.5) \qquad (qp - n - 1)T_f(r, r_0) \underset{\cdot}{\leqslant} \sum_{j=1}^{q} N_f(r, r_0, D_j) + o(T_f(r, r_0)),$$

$$(4.6) \qquad \sum_{j=1}^{q} \delta_f(D_j) \leqslant (n+1)/p.$$

PROOF. Lemma 4.5 and Proposition 3.5 imply

$$o\big(T_f(r, r_0)\big) + \sum_{j=1}^{q} N_f\big(r, r_0, D_j\big) \geqslant \int_{\mathbf{C}^m\langle r\rangle} \log\left(\prod_{j=1}^{q} \eta_p\big(\alpha_j \circ \varphi\big) \circ \mathfrak{g}\right)\sigma$$

$$\geqslant \int_{\mathbf{C}^m\langle r\rangle} \log\Big(|g_{\tau(0)}| \cdots |g_{\tau(n)}|\, |\mathfrak{g}|^{qp-n-1}\Big)\sigma + O(1)$$

$$= \sum_{j=0}^{n} N_g\big(r, r_0, H_{\tau(j)}\big) + (qp - n - 1)T_g(r, r_0) + O(1).$$

Since $N_g(r, r_0, H_{\tau(j)}) = o(T_g(r, r_0))$ for $j = 0,\ldots,n$, then Proposition 4.3 implies (4.5).

From (4.5) we get

$$\sum_{j=1}^{q} \delta_f(a_j) = \sum_{j=1}^{q} \liminf_{r\to\infty}\left(1 - \frac{N_f\big(r, r_0, D_j\big)}{pT_f(r, r_0)}\right)$$

$$\leqslant \liminf_{r\to\infty} \sum_{j=1}^{q}\left(1 - \frac{N_f\big(r, r_0, D_j\big)}{pT_f(r, r_0)}\right)$$

$$\leqslant (n+1)/p. \quad \text{Q.E.D.}$$

REMARK 4.7. In **[2]** Theorem 4.6 is proved with g and φ satisfying weaker conditions, but the proof is much more complicated. So here, also suggested by the referee, is given a weaker version which has a much simpler proof.

5. An example. Let g: $\mathbf{C} \to \mathbf{P}_3$ be the holomorphic map defined by the reduced representation $\mathfrak{g} = (1, e^t, e^{2t}, e^{3t})$: $\mathbf{C} \to \mathbf{C}^4$. We have that g is nondegenerate but is degenerate of degree 2. Let φ: $\mathbf{C}^4 \to \mathbf{C}^3$ be a surjective linear map defined by the matrix

$$A = \begin{pmatrix} 1 & -2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{pmatrix}$$

and let $f = \mathbf{P}(\varphi) \circ g$. We have that f is nondegenerate of degree 2, but an easy computation shows that f is degenerate of degree 3. Consider $p = 2$. Let $D_j = D[\alpha_j]$ be conics in $\mathbf{P}_2$ for $j = 1, 2, 3$ defined by

$$\alpha_1 = 9\zeta_0^2 + 36\zeta_0\zeta_1 + 8\zeta_0\zeta_2 + 16\zeta_1^2,$$

$$\alpha_2 = 8\zeta_0\zeta_2 + 16\zeta_1^2 + 36\zeta_1\zeta_2 + 9\zeta_2^2,$$

$$\alpha_3 = \zeta_0\zeta_1 + \zeta_1\zeta_2 + \zeta_0\zeta_2.$$

We have that D_1, D_2, D_3 have normal crossings. By **[12**, pp. 94–106**]** we have $T_f(r, r_0) = (3r)/\pi + O(1)$ and since $\alpha_1 \circ \varphi \circ \mathfrak{g} = 9$, $\alpha_2 \circ \varphi \circ \mathfrak{g} = 9e^{6t}$ and $\alpha_3 \circ \varphi \circ \mathfrak{g} = e^t(1 - 3e^t + e^{2t} - 3e^{3t} + e^{4t})$ then

$$N_f\big(r, r_0, D_j\big) = O(1) \quad \text{for } j = 1, 2$$

and

$$N_f\big(r, r_0, D_3\big) = 4r/\pi + O(1).$$

Hence

$$\delta_f(D_1) = \delta_f(D_2) = 1 \quad \text{and} \quad \delta_f(D_3) = \tfrac{1}{3}$$

so

$$\sum_{j=1}^{3} \delta_f(D_j) = \tfrac{7}{3} > \tfrac{3}{2}.$$

Therefore the condition "g nondegenerate of degree p" in Theorem 4.6 cannot be substituted by the weaker condition "f nondegenerate of degree p".

6. Appendix. Recall the class of meromorphic maps from $\mathbf{C}^m$ to $\mathbf{P}_n$ defined by B. Shiffman in [**10**]. Take $\lambda \in \mathbf{N}$. Denote by $\mathfrak{E}_\lambda(\mathbf{C}^m)$ the ring of holomorphic function g on $\mathbf{C}^m$ of the form

$$g = \sum_{k=1}^{q} \Phi_k \exp P_k,$$

where P_k are polynomials on $\mathbf{C}^m$ of degree at most λ and Φ_k are meromorphic functions such that $T_{\Phi_k}(r, r_0) = o(r^\lambda)$. We shall use the following notation. Let f and g be real valued functions on $\mathbf{R}[r_0, \infty]$. We write $f(r) \sim g(r)$ if $f(r)g(r)^{-1}$ is bounded above and below by positive finite constants for r sufficiently large.

Definition 6.1. A meromorphic map $f\colon \mathbf{C}^m \to \mathbf{P}_n$ is of *special exponential type of order* λ if $T_f(r, r_0) \sim r^\lambda$ and it has a reduced representation $\mathfrak{v} \in (\mathfrak{E}_\lambda(\mathbf{C}^m))^{n+1}$.

Denote by $\mathfrak{S}_\lambda$ the class of all meromorphic maps from $\mathbf{C}^m$ to $\mathbf{P}_n$ of special exponential type of order λ and $\mathfrak{S} = \bigcup_{\lambda=1}^{\infty} \mathfrak{S}_\lambda$.

Proposition 6.2. *With the same notation as above we have* $\mathfrak{S} \subseteq \mathfrak{R}$.

Proof. Take $f \in \mathfrak{S}_\lambda$, then there exists a reduced representation $\mathfrak{v} \in (\mathfrak{E}_\lambda(\mathbf{C}^m))^{n+1}$ of f such that if $\mathfrak{v} = (f_0, \ldots, f_n)$ then

$$f_j = \sum_{k=1}^{q_j} \Phi_{jk} \exp P_{jk} \quad \text{for } j = 0, \ldots, n.$$

Denote by $\mathfrak{M}$ the complex vector space of all meromorphic functions on $\mathbf{C}^m$. Let $\mathfrak{T} \subseteq \mathfrak{M}$ be the subspace spanned by

$$T = \{\Phi_{jk} \exp P_{jk} \mid j = 0, \ldots, n \text{ and } k = 1, \ldots, q_j\}.$$

Let $\mathfrak{T}' \subseteq \mathfrak{T}$ be the subspace spanned by $f_0, \ldots, f_n$. Consider $\{g_0', \ldots, g_t'\} \subseteq T$ a base for $\mathfrak{T}$. Let $0 \leq j_0 < \cdots < j_N \leq t$ be such that $g_{j_0}', \ldots, g_{j_N}'$ is a minimal system of generators for $\mathfrak{T}'$, i.e. for any $h = \sum_{j=0}^{t} a_j g_j' \in \mathfrak{T}'$ we have $a_j = 0$ for $j \in \mathbf{Z}[0, t] - \{j_0, \ldots, j_N\}$. Set $\tilde{g}_k = g_{j_k}'$, $k = 0, \ldots, N$.

By definition we have $\tilde{g}_j = \Phi_j \exp P_j$ where $T_{\Phi_j}(r, r_0) = o(r^\lambda)$ and P_j is a polynomial of degree at most λ. Since Φ_j is a meromorphic function on $\mathbf{C}^m$ there exist k_j and h_j holomorphic functions on $\mathbf{C}^m$ such that $\Phi_j = k_j/h_j$. Let $s_j = k_j \exp P_j$. Then we can choose k_j and h_j such that s_j and h_j are coprime. Let ν_j be the 0-divisor of h_j. Consider the divisor ν such that (i) $\operatorname{supp} \nu = \bigcup_{j=0}^{N} \operatorname{supp} \nu_j$; (ii) the multiplicity of every branch B in ν is equal to the maximum of the multiplicity of B in ν_j for

$j = 0, \ldots, N$. Let u be a holomorphic function such that u has a 0-divisor ν. Then there exist coprime holomorphic functions $u_0, \ldots, u_N$ such that $u = u_j h_j$ for $j = 0, \ldots, N$. In addition we have that a holomorphic function w exists such that

$$h_0 \cdots h_N = wu. \tag{6.1}$$

Then $g_j = u\tilde{g}_j = u_j s_j$ is a holomorphic function for $j = 0, \ldots, N$. Define $\mathfrak{g} = (g_0, \ldots, g_N)$ and $g = \mathbf{P}(\mathfrak{g})$. Since $f_k \in \mathfrak{X}'$ for $k = 0, \ldots, n$, we have

$$uf_k = \sum_{j=0}^{N} a_k^j g_j, \quad \text{where } a_k^j \in \mathbf{C}.$$

Let $\varphi\colon \mathbf{C}^{N+1} \to \mathbf{C}^{n+1}$ be a linear map defined by the matrix (a_k^j). Then we have

$$u\mathfrak{v} = \varphi \circ \mathfrak{g}.$$

In order to prove $f \in \mathfrak{R}$ it remains to check the following.

(6.2) g is nondegenerate.

(6.3) $\mathfrak{g}$ is a reduced representation of g.

(6.4) $\varphi(e_j) \neq 0$ for $j = 0, \ldots, N$.

(6.5) $N_g(r, r_0, H_j) = o(T_g(r, r_0))$ for $j = 0, \ldots, N$.

(6.6) $N_u(r, r_0, 0) = o(T_g(r, r_0))$.

First we note that g is nondegenerate for the choice of $\tilde{g}_0, \ldots, \tilde{g}_N$. In order to prove (6.3) we claim that

(6.7) $s_0, \ldots, s_N$ are coprime.

Suppose $s_0, \ldots, s_N$ are not coprime. Then there exist holomorphic functions μ, $\tilde{s}_0, \ldots, \tilde{s}_N$ such that $s_j = \mu \tilde{s}_j$ for $j = 0, \ldots, N$ and $\tilde{s}_0, \ldots, \tilde{s}_N$ are coprime. Let B be an irreducible branch of the 0-divisor of μ and denote by μ_1 a holomorphic function which has B as 0-divisor. Then $\tilde{\mu} = \mu\mu_1^{-1}$ is a holomorphic function and

$$uf_k = \sum_{j=0}^{N} a_k^j u_j s_j = \mu_1 \tilde{f}_k \quad \text{for } k = 0, \ldots, n,$$

where $\tilde{f}_k = \tilde{\mu}\sum_{j=0}^{N} a_k^j u_j \tilde{s}_j$. Since s_j and h_j are coprime, for $j = 0, \ldots, N$, then μ_1 does not divide h_j for $j = 0, \ldots, N$ and so μ_1 does not divide u. Hence μ_1 must divide f_k for $k = 0, \ldots, n$. This implies that $\mathfrak{v}$ is not a reduced representation of f. Contradiction. Therefore we have (6.7).

PROOF OF (6.3). Suppose $g_0, \ldots, g_N$ are not coprime. Then there exist holomorphic functions η, $\hat{g}_0, \ldots, \hat{g}_N$ such that $g_j = \eta \hat{g}_j$ for $j = 0, \ldots, N$ and $\hat{g}_0, \ldots, \hat{g}_N$ are coprime. Let B be an irreducible branch of the 0-divisor of η and denote by η_1 a holomorphic function which has B as 0-divisor. Then $\tilde{\eta} = \eta\eta_1^{-1}$ is a holomorphic function. Since η_1 divides $g_j = u_j s_j$ for $j = 0, \ldots, N$, we have that η_1 must divide either u_j or s_j. But $u_0, \ldots, u_N$ and $s_0, \ldots, s_N$ are respectively coprime, so there exist $j_0, j_1 \in \mathbf{Z}[0, N]$ such that η_1 does not divide u_{j_0} and s_{j_1}. Hence η_1 must divide s_{j_0} and u_{j_1}. Since η_1 divides u_{j_1} but not u_{j_0} then η_1 divides u and h_{j_0}, which implies that s_{j_0} and h_{j_0} are not coprime. Contradiction. Hence $\mathfrak{g}$ is reduced.

Since $\varphi(e_j) = (a_0^j, \ldots, a_n^j)$ then $\varphi(e_j) = 0$ implies that $\{\tilde{g}_0, \ldots, (\tilde{g}_j)\check{}, \ldots, \tilde{g}_N\}$ is a system of generators for $\mathfrak{X}'$, which is in contradiction with the minimality of $\{\tilde{g}_0, \ldots, \tilde{g}_N\}$. Therefore we have (6.4).

Since $T_g(r, r_0) \sim r^\lambda$ and $T_{\Phi_j}(r, r_0) = o(r^\lambda)$, then (2.1) and (6.1) imply

$$N_u(r, r_0, 0) = o(r^\lambda) = o\big(T_g(r, r_0)\big)$$

and

$$\begin{aligned} N_{g_j}(r, r_0, 0) &= N_{u_j}(r, r_0, 0) + N_{k_j}(r, r_0, 0) \\ &\leqslant N_u(r, r_0, 0) + T_{\Phi_j}(r, r_0) = o\big(T_g(r, r_0)\big) \end{aligned}$$

for $j = 0, \ldots, N$. Hence (6.5) and (6.6) are proved. Q.E.D.

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