A NOTE ON COMPLETE INTERSECTIONS

BY

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Abstract. Let $R$ be a regular local ring and let $R[T]$ be a polynomial algebra in one variable over $R$. In this paper the author proves that every maximal ideal of $R[T]$ is complete intersection in each of the following cases: (1) $R$ is a local ring of an affine algebra over an infinite perfect field, (2) $R$ is a power series ring over a field.

Introduction. Let $R$ be a regular local ring. Let $R[T]$ be a polynomial algebra in one variable over $R$. In [D-G] the following question has been asked.

Question. Is every maximal ideal of $R[T]$ complete intersection?

In this paper we prove that the answer to the above question is affirmative in each of the following cases:

1. $R$ is a local ring of an affine algebra over an infinite perfect field.
2. $R$ is a power series ring over a field.

This paper is divided into three sections. In §1 we fix notations and state a theorem without proof which is used in §§2 and 3. In §2 we prove some lemmas and propositions which are used in proving the result when $R$ is a local ring of an affine algebra. §3 deals with the power series case.

1. Throughout this paper we consider commutative noetherian rings with 1. For a ring $R$, $\dim R$ denotes its Krull dimension which we always assume to be finite. If $R$ is a local ring then $\mathfrak{m}(R)$ will always denote its unique maximal ideal. If $M$ is a finitely generated $R$-module then $\mu(M)$ will denote the minimal number of generators of $M$. For an ideal $I$ of $R$ $\text{ht}(I)$ denotes the height of $I$.

Definition. Let $I$ be an unmixed ideal of $R$ of height $r$. Then $I$ is said to be complete intersection in $R$ if $I = \sum_{i=1}^{r} Ra_i$, where $a_1, a_2, \ldots, a_r$ is a regular $R$-sequence.

Remark. If $R$ is Cohen-Macaulay then $I$ is complete intersection if and only if $\mu(I) = \text{ht}(I)$.

Let $R$ and $S$ be two local rings.

Definition. $R$ is said to be a local extension of $S$ if $S$ is a subring of $R$ and $\mathfrak{m}(S) = \mathfrak{m}(R) \cap S$. $R$ is said to be unramified over $S$ if $\mathfrak{m}(S)R = \mathfrak{m}(R)$ and $R/\mathfrak{m}(R)$ is separable over $S/\mathfrak{m}(S)$.

Let $L/K$ be a finite separable extension of $K$. Then $L$ is a simple extension of $K$. By a minimal polynomial of $L$ over $K$ we always mean an irreducible monic polynomial over $K$ satisfied by a generator of $L$ over $K$.

Now we state a theorem which has been proved in [D-G, Theorem 3].

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Theorem. Let $R$ be a regular ring. Let $A = R[X, Y]$ be a polynomial algebra in two variables over $R$. Then every maximal ideal of $A$ is complete intersection.

In subsequent sections this theorem will always be referred to as the D-G theorem.

2. In this section we prove the following theorem.

Theorem 2.1. Let $k$ be an infinite perfect field. Let $C$ be an affine $k$-algebra. Let $\mathfrak{p}$ be a prime ideal of $C$ such that $C_{\mathfrak{p}} = R$ is regular. Let $M$ be a maximal ideal of $R[T]$. Then $M$ is complete intersection.

For the proof of this theorem we need some lemmas and propositions.

Lemma 2.2. Let $A$ be an affine domain of dim 1 over a field $K$. Let $\mathfrak{m}$ be a nonregular maximal ideal of $A$ such that $A/\mathfrak{m}$ is a finite separable (therefore simple) extension of $K$. Then there exist $y_1, y_2, \ldots, y_r \in A$ such that

1. $A$ is integral over $K[y_1]$,
2. the inclusion map $K[y_1]/\mathfrak{m} \cap K[y_1] \to A/\mathfrak{m}$ is an isomorphism,
3. $\mathfrak{m} = (f(y_1), y_2, \ldots, y_r)$ where $r = \mu(\mathfrak{m}/\mathfrak{m}^2)$ and $f$ is a minimal polynomial of $A_{\mathfrak{p}}$ over $K$.

Proof. Since $A$ is one dimensional and $\mathfrak{m}$ nonregular we have $\mu(\mathfrak{m}/\mathfrak{m}^2) = r \geq 2 = \dim A + 1$. Therefore by [Mo, Corollary 3] it follows that $\mu(\mathfrak{m}) = \mu(\mathfrak{m}/\mathfrak{m}^2)$.

Let $A/\mathfrak{m} = K[\alpha]$. Let $f(X)$ be the minimal polynomial of $\alpha$ over $K$. Let $b \in A$ be such that $\alpha = b \mod \mathfrak{m}$. Then $\alpha$ is separable over $K$ and $f(X)$ is its minimal polynomial imply that $f(b) \in \mathfrak{m}$ and $\frac{\partial f(b)}{\partial X} \notin \mathfrak{m}$. If $f(b) \in \mathfrak{m}^2$ then replacing $b$ by $b + x$ for some $x \in \mathfrak{m} - \mathfrak{m}^2$ we get $f(b) \notin \mathfrak{m}^2$. This in particular implies that $b$ is not algebraic over $K$.

Since $A$ is an one dimensional affine, by the normalization theorem [Z-S, p. 200] there exists $y \in A$ such that $A$ is integral over $K[y]$. Let $\mathfrak{m} \cap K[y] = (h(y))$. Let $y_1 = b + h(y)^l$ where $l$ is a positive integer. Then by taking sufficiently large $l \geq 2$ one can see that $K[y, b] = K[y_1]$ is integral over $K[y_1]$. Moreover

$$f(y_1) = f(b) + (\frac{\partial f}{\partial X})(b)h(y)^l + ch(y)^{2l}, \quad c \in K[y, b].$$

Since $f(b) \notin \mathfrak{m}^2$, $h(y) \in \mathfrak{m}$ and $l \geq 2$ we get $f(y_1) \in \mathfrak{m} - \mathfrak{m}^2$. Since $A$ is integral over $K[y, b]$, $A$ is integral over $K[y_1]$ and $\mathfrak{m} \cap K[y_1] = (f(y_1))$. Therefore the inclusion map $K[y_1]/\mathfrak{m} \cap K[y_1] \to A/\mathfrak{m}$ is an isomorphism.

Let $A' = A/(f(y_1))$, $\mathfrak{m}' = \mathfrak{m}/(f(y_1))$. Then $A'$ is zero dimensional and $\mu(\mathfrak{m}'/\mathfrak{m}'^2) = \mu(\mathfrak{m}/\mathfrak{m}^2) - 1 = r - 1 \geq 1$. Therefore by [Mo, Corollary 3] there exist $y'_2, y'_3, \ldots, y'_r \in A'$ such that $\mathfrak{m}' = (y'_2, y'_3, \ldots, y'_r)$. Let $y_i$ be a pull back of $y'_i$ in $A$ for $2 \leq i \leq r$. Then $\mathfrak{m} = (f(y_1), y_2, y_3, \ldots, y_r)$.

This completes the proof of Lemma 2.2. Now we state a lemma the proof of which is easy and can be found in [L, Lemma 2].

Lemma 2.3. Let $k$ be a perfect field. Let $C$ be an affine $k$-algebra. Let $\mathfrak{p}$ be a prime ideal of $C$ such that $C_{\mathfrak{p}} = R$ is regular. Then there exists a field extension $K/k$ and regular affine $K$-domain $B$ contained in $R$ such that

1. $R = B_{\mathfrak{p}}$ for some maximal ideal $\mathfrak{p}$ of $B$,
2. $B/\mathfrak{p} = R/\mathfrak{p}(R)$ is a finite separable extension of $K$. 


The following two propositions are very crucial for the proof of Theorem 2.1.

**Proposition 2.4.** Let \( k, C, \mathfrak{X}, R, K, B, \mathfrak{M} \) be as in Lemma 2.3. Let \( p \) be a prime ideal of \( R \) such that \( R/p \) is one dimensional and nonregular. Then \( R \) contains a local domain \( S \) such that

1. \( S \) is a localization of a polynomial algebra \( C' \) over \( K \) at some maximal ideal \( \eta \) of \( C' \),

2. there exists \( h \in p \cap S \) such that the inclusion of \( S \) in \( R \) gives rise to an inclusion of \( S/hS \) in \( R/hR \) which is an isomorphism, i.e. \( S/hS = R/hR \).

**Proof.** Since \( R = B_{\mathfrak{M}} \) there exists a prime ideal \( q \) of \( B \) such that \( qB_{\mathfrak{M}} = p \). Then \( B/q \) is one dimensional and \( \mathfrak{M}/q \) is a nonregular maximal ideal of \( B/q \).

Let \( A = B/q, \mathfrak{M} = \mathfrak{M}/q \). Then by Lemma 2.2 there exist \( y_1, y_2, \ldots, y_r \in A \) satisfying properties 1, 2 and 3 of Lemma 2.2. Let \( \phi: B \to A (=B/q) \) be the canonical map. Let \( x_i \in B \) be such that \( \phi(x_i) = y_i \) for \( 1 \leq i \leq r \). Then \( q + (f(x_1), x_2, \ldots, x_r) = \mathfrak{M} \) and \( f(x_1), x_2, \ldots, x_r \) generate \( \mathfrak{M} \) mod \( \mathfrak{M}^2 + q \) where \( r = \dim_{B/\mathfrak{M}}(\mathfrak{M}/\mathfrak{M}^2 + q) \). Let \( \dim_{B/\mathfrak{M}}(\mathfrak{M}/\mathfrak{M}^2) = \mu(\mathfrak{M}/\mathfrak{M}^2) = 1 = \dim R = n \). Then since we have the following exact sequence

\[
0 \to q/q \cap \mathfrak{M}^2 \to \mathfrak{M}/\mathfrak{M}^2 \to \mathfrak{M}/\mathfrak{M}^2 + q \to 0
\]

we get \( \dim_{B/\mathfrak{M}}(q/q \cap \mathfrak{M}^2) = n - r \). Let \( x_{r+1}, x_{r+2}, \ldots, x_n \in q \) be such that \( (x_{r+1}, x_{r+2}, \ldots, x_n) + q \cap \mathfrak{M}^2 = q \). Then it is easy to see that \( (f(x_1), x_2, \ldots, x_r, x_{r+1}, x_{r+2}, \ldots, x_n) + \mathfrak{M} = \mathfrak{M} \). Since \( R = B_{\mathfrak{M}} \) is regular of dim \( n \) it follows that \( (f(x_1), x_2, \ldots, x_n)R = \mathfrak{M}(R) \) and \( f(x_1), x_2, \ldots, x_n \) are algebraically independent over \( K \). Therefore \( x_1, x_2, \ldots, x_n \) are also algebraically independent over \( K \) and hence \( C' = K[x_1, x_2, \ldots, x_n] \) is a polynomial algebra over \( K \) contained in \( B \).

Let \( \eta = C' \cap \mathfrak{M} \). Then \( \eta = (f(x_1), x_2, \ldots, x_n) \) is a maximal ideal of \( C' \) and the inclusion map \( C'/\eta \to B/\mathfrak{M} \) is an isomorphism. Moreover \( A (=B/q) \) is integral over \( C'/\mathfrak{M} \), where \( \mathfrak{M} \) is the only maximal ideal of \( A \) lying over the maximal ideal \( \eta/\mathfrak{M} \) of \( C'/\mathfrak{M} \).

Let \( L = \text{quotient field of } B, L' = \text{quotient field of } C' \). Then since \( B \) and \( C' \) are affine \( K \)-domains of dim \( n \), \( L \) is a finite algebraic extension of \( L' \). Let \( B' \) be the integral closure of \( C' \) in \( L \). Then \( B' \) is a finitely generated \( C' \)-module contained in \( B \).

Let \( \mathfrak{M}' = \mathfrak{M} \cap B' \), \( B'_{\mathfrak{M}} = R' \), \( C'_{\mathfrak{M}} = S \). Then we get a tower of local extensions \( S \to R' \to R \). Since \( S/\mathfrak{M}(S) = C'/\mathfrak{M} \to B/\mathfrak{M} = R/\mathfrak{M}(R) \) and \( R \) is unramified over \( S \), \( R \) is also unramified over \( R' \) and \( R'/\mathfrak{M}(R') \to R/\mathfrak{M}(R) \). But since \( R' \) and \( R \) have the same quotient field \( L \) and \( R' \) is normal, by Zariski's main theorem \([BI, p. 93] \) we have \( R' = R \).

Let \( q' = q \cap B' \). Then we get a tower of integral extensions \( C'/\mathfrak{M}' \to B'/\mathfrak{M}' \to B/q (=A) \). Since \( \mathfrak{M}' (=\mathfrak{M}/q) \) is the only maximal ideal of \( A \) lying over \( \eta/\mathfrak{M} \), \( \mathfrak{M}'/q' \) will be the only maximal ideal of \( B'/\mathfrak{M}' \) lying over \( \eta/\mathfrak{M} \). Therefore \( \eta B' + q' \) is \( \mathfrak{M}' \)-primary. Since \( B'_{\mathfrak{M}} = R' = R \) and \( \eta R = \mathfrak{M}'(R) \) we have \( \eta B' + \mathfrak{M}'^2 = \mathfrak{M}' \). But this implies that \( \eta B' + \mathfrak{M}'^l = \mathfrak{M}' \) for every positive integer \( l \). Since \( \eta B' + q' \) is \( \mathfrak{M}' \)-primary, there exists a positive integer, say \( l_0 \), such that \( \mathfrak{M}'^{l_0} \subset \eta B' + q' \). Therefore \( \eta B' + q' = \mathfrak{M}' \). Moreover \( \eta B' + \mathfrak{M}'^2 = \mathfrak{M}' \) implies that \( \mathfrak{M}'/\eta B' \) is an
idempotent and therefore principal ideal of $B'/\eta B'$. Hence there exists $t \in q'$ such that $tB' + \eta B' = \mathfrak{M}'$.

Let $B'' = C[t], \mathfrak{M}'' = \mathfrak{M}' \cap B'', q'' = q' \cap B''$. It is obvious that $\mathfrak{M}''B' = \mathfrak{M}'$ and $B''/\mathfrak{M}'' \rightarrow B'/\mathfrak{M}'$. Since $B'$ is a finitely generated $B''$-module we have $B''_{\mathfrak{M}'} = B'_{\mathfrak{M}'} = R$ and $q'' R = p$.

Since $B''$ is a simple integral extension of $C'$ and $C'$ is a unique factorization domain we get $B'' = C'[T]/(g(T))$ where $g(T)$ is a monic irreducible polynomial in $T$.

Let $\psi: C'[T] \rightarrow B'' = C'[t]$ be the canonical map. Let $M = \psi^{-1}(\mathfrak{M}'')$. Since $\psi(T) = t \in \mathfrak{M}''$ we have $T \in M$. Also $\mathfrak{M}'' \cap C' = \eta$ implies $M \cap C' = \eta$. Therefore $M = TC'[T] + \eta C'[T]$.

Let $g(T) = T + a_{-1}T^{-1} + \cdots + a_0$. Then $g(t) = 0$ and $t \in q''$ implies $a_0 \in q_1 = q'' \cap C'$. Since $B''_{\mathfrak{M}'} = R$ and $\eta R = \mathfrak{M}(R)$ it follows that $a_1 \notin \eta$, and therefore $tR = hR$ where $h = a_0$. Therefore the map $S/hS \rightarrow R/hR$ is an isomorphism. Thus the proof of Proposition 2.4 is complete.

Remark. Under the assumptions of Proposition 2.4 Lindel [L, Proposition 2] also has shown the existence of $S$ and $h$. Our proof is a variation of his proof because of the requirement that $h$ should belong to $p$.

**Proposition 2.5.** Let $K$ be an infinite field. Let $D = K[X_1, X_2, \ldots, X_n]$ be a polynomial algebra over $K$. Let $\mathfrak{M} = (f(X_1), X_2, \ldots, X_n)$ be a maximal ideal of $D$. Let $p$ be a prime ideal of dim 1 contained in $\mathfrak{M}$. If $n \geq 3$ then $D$ contains a $K$-algebra $D'$ of dim $n - 1$ such that

1. $D = D'[Y]$,
2. $p + \mathfrak{M}'D$ is $\mathfrak{M}$-primary where $\mathfrak{M}' = \mathfrak{M} \cap D'$.

**Proof.** If $p$ contains one of the generators $f(X_1), X_2, \ldots, X_n$, say $f(X_1)$, then $p + (X_2, \ldots, X_n) = \mathfrak{M}$. Therefore by taking $D' = K[X_2, \ldots, X_n]$ we get the required result.

Now we assume that $X_i \notin p$ for $2 \leq i \leq n$ and $f(X_1) \notin p$. Then $p + (X_n) = I$ is a zero dimensional ideal of $D$ and hence contained in only finitely many maximal ideals of $D$. Let $T = \{\mathfrak{M} = \mathfrak{M}_1, \mathfrak{M}_2, \ldots, \mathfrak{M}_t\}$ be a finite set of maximal ideals of $D$ containing $I$.

For every $i$, $2 \leq i \leq t$, let $V_i$ denote a subspace of $K^n$ consisting of $n$-tuples $(\lambda_1, \ldots, \lambda_n)$ such that $\lambda_1 f(X_1) + \lambda_2 X_2 + \cdots + \lambda_n X_n \in \mathfrak{M}_i$. Then $V_i \neq K^n$ for $2 \leq i \leq t$. Since $K$ is infinite we have $\bigcup_{2 \leq i \leq t} V_i \neq K^n$. Let $(\beta_1, \beta_2, \ldots, \beta_n)$ be such that $(\beta_1, \beta_2, \ldots, \beta_n) \notin V_i$ for every $i$, $2 \leq i \leq t$. Let $Z = \beta_1 f(X_1) + \beta_2 X_2 + \cdots + \beta_n X_n$. Since $X_n \in \mathfrak{M}_i$ for every $i$, $2 \leq i \leq t$, we have $\beta_i \neq 0$ for some $l$, $1 \leq l \leq n - 1$.

If $\beta_1 = 0$ then taking $D' = D[X_1, X_2, \ldots, X_n]$ we get $X_n, Z \in \mathfrak{M}' = \mathfrak{M} \cap D'$, and the ideal $p + (X_n, Z)D$ is $\mathfrak{M}$-primary. Therefore $p + \mathfrak{M}'D$ is $\mathfrak{M}$-primary. Since $D = D'[X_2]$ we get the required result.

If $\beta_1 \neq 0$ then obviously $D = K[X_1, Z, X_3, \ldots, X_n]$. Taking $D' = K[Z, X_3, \ldots, X_n]$ we get $X_n, Z \in \mathfrak{M}' = \mathfrak{M} \cap D'$. Therefore as before we see that $\mathfrak{M}'D + p$ is $\mathfrak{M}$-primary. Since $D = D'[X_1]$ we get the required result.
Proof of Theorem 2.1. Let \( p = M \cap R \). Then \( \dim R/p \leq 1 \). If \( R/p \) is regular then since \( \text{ht}(M/pR[T]) = 1 \), \( M/pR[T] \) is a principal ideal of \( R/p[T] \). Therefore

\[
\mu(M) \leq 1 + \mu(pR[T]) = 1 + \mu(p) = 1 + \text{ht}(p) = \text{ht}(M).
\]

Since we always have \( \text{ht}(M) \leq \mu(M) \) we get the equality \( \mu(M) = \text{ht}(M) \) which shows that \( M \) is complete intersection.

Now we suppose that \( R/p \) is not regular. Then \( \dim R/p = 1 \), \( \text{ht}(M) = \text{ht}(p) + 1 = \dim R \) and \( \dim R \geq 2 \).

Case 1. \( \dim R = 2 \). Then \( \dim R/p = 1 \) implies \( \text{ht}(p) = 1 \). Therefore we have \( \text{ht}(M) = \text{ht}(p) + 1 = 2 \). Since \( R[T] \) is regular, \( M \) is locally generated by a regular sequence of length 2. Therefore \( \text{hd}_{R[T]}(M) = 1 \) where \( \text{hd}_{R[T]}(M) \) denotes the homological dimension of the \( R[T] \)-module \( M \). Since

\[
\Ext^1_{R[T]}(M, R[T]) \cong \Ext^2_{R[T]}(R[T]/M, R[T]) \cong R[T]/M,
\]

we get \( \Ext^1_{R[T]}(M, R[T]) \) to be a cyclic \( R[T] \)-module. Therefore by [S, p. 8] there is an exact sequence \( 0 \rightarrow R[T] \rightarrow P \rightarrow M \rightarrow 0 \) with \( P \) finitely generated projective \( R[T] \)-module of rank 2. But by [Mu, Theorem] \( P \) is free. Therefore \( \mu(P) = 2 \). Since \( M \) is an epimorphic image of \( P \) we have

\[
\mu(M) \leq \mu(P) = 2 = \text{ht}(M) \leq \mu(M).
\]

Hence \( M \) is complete intersection.

Case 2. \( \dim R = n \geq 3 \). By Lemma 2.3 and Proposition 2.4 there exist a field extension \( K/k \) and a local domain \( S \) contained in \( R \) such that

1. \( S = K[X_1, \ldots, X_n]_\eta \) where \( \eta \) is a maximal ideal of \( K[X_1, \ldots, X_n] \) generated by \( f(X_1), X_2, \ldots, X_n \) for some irreducible monic polynomial \( f(X) \) over \( K \).

2. There exists \( \eta \in S \) such that \( S/hS = R/hR \) and therefore \( S[T]/hS[T] = R[T]/hR[T] \).

Let \( \tilde{M} = M \cap S[T] \). Since \( h \in \tilde{M} \), \( \tilde{M} \) is a maximal ideal of \( S[T] \). Moreover \( \tilde{M}R[T] = M \) and \( \text{ht}(\tilde{M}) = \text{ht}(M) \). Therefore it is enough to prove that \( \tilde{M} \) is a complete intersection ideal of \( S[T] \).

Let \( q = S \cap p = \tilde{M} \cap S \). Then \( h \in q \) and hence \( S/q = R/p \). Therefore \( \dim S/q = 1 \). Let \( D = K[X_1, \ldots, X_n] \), \( \tilde{M}' = \tilde{M} \cap D[T] \), \( q' = q \cap D = \tilde{M}' \cap D \). Then since \( D_\eta = S \) we have \( \tilde{M}'S[T] = \tilde{M}, \text{ht}(\tilde{M}') = \text{ht}(\tilde{M}) = n = \dim D \) and \( \text{ht}(q') = \text{ht}(q) = n - 1 \). Therefore \( \dim q' = \dim D/q' = 1 \).

Since \( n \geq 3 \) by Proposition 2.5 there exists a subalgebra \( D' \) of \( D \) of \( \dim n - 1 \) such that

1. \( D = D'[Y] \),
2. \( \eta'D + q' \) is \( \eta' \)-primary where \( \eta' = \eta \cap D' \).

Consider the following commutative diagram

\[
\begin{array}{ccc}
D' & \hookrightarrow & D'[Y] = D \\
\downarrow & & \downarrow \\
D[T] & \hookrightarrow & S[T]
\end{array}
\]

\( \tilde{M}' \) is a prime ideal of \( D[T] \) of height \( n = \dim D[T] - 1 \). Therefore every prime ideal of \( D[T] \) which contains \( \tilde{M}' \) properly is a maximal ideal of \( D[T] \). Let \( M_1 \) be one.
such maximal ideal. Then since $D', D, D[T]$ all are affine rings, $N_1 = M_1 \cap D'$ will be a maximal ideal of $D'$. If $\eta' = N_1$ then since $\tilde{M}' \subset M_1$ we have $\eta'D + \eta' \subset M_1 \cap D$. But $\eta'D + \eta'$ is $\eta$-primary and $\eta$ is maximal; therefore $\eta = M_1 \cap D$. Since $S = D_\eta$, $\eta = M_1 \cap D$ implies that $M_1S[T]$ is a prime ideal of $S[T]$ which contains $\tilde{M}'S[T] = \tilde{M}$ properly which contradicts the fact that $\tilde{M}$ is maximal. Therefore $N_1 \neq \eta'$.

The above discussion shows that no prime ideal of $D[T]$ which contains $\tilde{M}'$ properly can lie over a prime ideal of $D'$ contained in $\eta'$. Therefore $\tilde{M}'S'[T]$ becomes a maximal of $S'[T]$ of height $= ht(\tilde{M}')$ where $S' = D'[Y]$. Then by the D-G theorem $\tilde{M}'S'[T]$ is complete intersection. Now we have the following tower of rings:

$$D'[Y, T] = D[T] \subset S'[T] \subset S[T].$$

Since $\tilde{M}'S'[T]$ is complete intersection, $\tilde{M}'S[T] = \tilde{M}$ and $ht(\tilde{M}'S'[T]) = ht(\tilde{M}') = ht(\tilde{M})$, it follows that $\tilde{M}$ is also complete intersection. Thus the proof of Theorem 2.1 is complete.

3. We begin this section with the following theorem.

**Theorem 3.1.** Let $k$ be a field. Let $R = k[[X_1, X_2, \ldots, X_n]]$ be a power series ring in $n$ variables over $k$. Let $M$ be a maximal ideal of $R[T]$. Then $M$ is complete intersection.

**Proof.** Let $p = R \cap M$. If $p = 0$ then $ht(M) = ht(p) + 1 = 1$.

Since $R[T]$ is a unique factorization domain, $M$ will be a principal ideal and hence complete intersection.

If $p \neq 0$ then let $f$ be a nonzero element of $p$. It is easy to see that there exist $Y_1, Y_2, \ldots, Y_n \in R$ such that $R = k[[Y_1, Y_2, \ldots, Y_n]]$ and $f$ as a power series in $Y_1, Y_2, \ldots, Y_n$ is regular in $Y_n$. Therefore without loss of generality we can assume that $f = f(X_1, \ldots, X_n)$ is regular in $X_n$. Then by the Weierstrass preparation theorem [Z-S, p. 139] there exists a unit $u(X_1, \ldots, X_n)$ in $R$ such that

$$u(X_1, \ldots, X_n)f(X_1, \ldots, X_n) = f'(X_1, \ldots, X_n) = X_n^r + g_1X_n^{r-1} + \cdots + g_r$$

where $g_i \in k[[X_1, \ldots, X_{n-1}]]$ and $g_i(0, 0, \ldots, 0) = 0$ for $1 \leq i \leq r$. Let $S = k[[X_1, \ldots, X_{n-1}]]X_n \subset R$. Then it also follows from the above-mentioned theorem that $S/f'S = R/f'R$. Therefore $S[T]/f'S[T] = R[T]/f'R[T]$.

Let $\tilde{M} = M \cap S[T]$ Then since $f' \in p \cap S \subset \tilde{M}$ it follows that $\tilde{M}$ is a maximal ideal of $S[T]$, $\tilde{M}R[T] = M$ and $ht(\tilde{M}) = ht(M)$. Since $S[T] = k[[X_1, \ldots, X_{n-1}]]X_n, T]$ by the D-G theorem $\tilde{M}$ is complete intersection. Hence $M$ is also complete intersection.

This completes the proof of Theorem 3.1.

Let $R$ be an equicharacteristic regular local ring. Let $\hat{R}$ be the completion of $R$ with respect to $\mathfrak{M}(R)$-adic topology. Then $\hat{R} = k[[X_1, \ldots, X_n]]$ where $k$ is the residue field of $R$ and $n = \dim R$.

Now we state a proposition which is a generalization of Theorem 3.1.

**Proposition 3.2.** Let $R$ be an equicharacteristic regular local ring. Let $\hat{R}$ be its completion with respect to $\mathfrak{M}(R)$-adic topology. Let $M$ be a maximal ideal of $R[T]$. Let $I = M\hat{R}[T]$. Then $ht(I) = ht(M)$ and $I$ is complete intersection.
Proof. Let \( \hat{R} = k[[X_1, \ldots, X_n]] \) where \( k = R/\mathfrak{m}(R) \). Since \( M \) is locally generated by a regular sequence of length \( = \text{ht}(M) \) and \( \hat{R}[T] \) is a faithfully flat extension of \( R[T] \) it follows that \( \text{ht}(M) = \text{ht}(I) \). If \( \text{ht}(M) = 1 \) then \( M \) itself is complete intersection and therefore \( I \) is also complete intersection. Now we assume that \( \text{ht}(M) \geq 2 \).

Let \( J = I \cap \hat{R} \). Then \( \text{ht}(I) = \text{ht}(M) \geq 2 \) implies that \( J \neq 0 \). Then as in Theorem 3.1 we can assume that \( J \) contains an element \( f \) such that \( f \in S \), \( S/fS = R/fR \) where \( S = k[[X_1, \ldots, X_{n-1}]]X_n] \). Moreover we can assume that \( f \) is monic in \( X_n \).

Let \( I' = I \cap S[T] \). Since \( f \in I' \) we have \( \mu(I'/I'^2) = \mu(I/I^2) \) and \( I'\hat{R}[T] = I \). But \( \hat{R}[T] \) is faithfully flat over \( R[T] \), \( M\hat{R}[T] = I \) and \( M \) is a maximal ideal of \( R[T] \). Therefore \( \mu(I/I^2) = \mu(M/M^2) = \text{ht}(M) = \text{ht}(I) \).

Since \( S[T] = k[[X_1, \ldots, X_{n-1}]]T[X_n] \) and \( f \in I' \), \( I' \) contains a monic polynomial in \( X_n \) with coefficients in \( k[[X_1, \ldots, X_{n-1}]][T] \). Since \( \mu(I'/I'^2) = \mu(I/I^2) = \text{ht}(I) \geq 2 \) and \( \dim S[T]/I' = \dim R[T]/I = 0 \) (this is easy to check) by [Mo, Theorem 5] there exists a finitely generated projective \( S[T] \)-module \( P \) of rank \( = \mu(I'/I'^2) \) and a surjective homomorphism \( \psi: P \to I' \). But by [L-L, Theorem 2] \( P \) is free and therefore \( \mu(P) = \text{rank}(P) = \mu(I'/I'^2) \). This implies that \( \mu(I') \leq \mu(I'/I'^2) = \mu(I/I^2) = \text{ht}(I) \). Since \( I'\hat{R}[T] = I \), we have \( \mu(I) \leq \mu(I') \leq \text{ht}(I) \leq \mu(I) \). Therefore \( I \) is complete intersection.

This completes the proof of Proposition 3.2.

Remark. In view of known results regarding projective modules over \( R[T] \) when \( R \) is regular local, one can obtain the results of §§2 and 3 in one stroke if one can prove the following theorem.

Theorem. Let \( R \) be a regular local ring. Let \( M \) be a maximal ideal of \( R[T] \). Then there exists a projective \( R[T] \)-module \( P \) of rank \( = \text{ht}(M) \) and a surjective homomorphism \( \psi: P \to M \).

References


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