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FUNDAMENTAL GROUPS OF TOPOLOGICAL R-MODULES

BY

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ABSTRACT. The main result of this paper is that if R is a countable, Noetherian ring, then the underlying abelian group of every R-module is isomorphic to the fundamental group of some topological R-module. As a corollary, it is shown that for certain varieties V (e.g., varieties of finite type) every abelian group in V is isomorphic to the fundamental group of some arcwise connected topological algebra in V.

In this paper, we examine the class of abelian groups isomorphic to fundamental groups of topological algebras in a variety V. Our main result (Theorem 3.1) is that if R is a countable, Noetherian ring, then the underlying abelian group of every R-module is isomorphic to the fundamental group of some topological R-module. Given any variety V, one can construct a unital ring R_V such that the abelian groups in V are precisely the underlying abelian groups of R_V-modules [see §4]. Hence, we have as a corollary to Theorem 3.1 that if R_V is a countable, Noetherian ring (e.g., V has finite type), every abelian group in V is isomorphic to the fundamental group of some arcwise connected topological algebra in V.

In [**6**], Milnor proves that every countable abelian group is isomorphic to the fundamental group of some topological group. As another consequence of Theorem 3.1, we have that Milnor's result is true for every abelian group regardless of its cardinality.

§1 is devoted to preliminaries, with special emphasis on CW complexes. In §2 we construct, for every unital ring R and every countable R-module A, a topological R-module $T(A)$ with $\pi_1(T(A))$ isomorphic to A. We will construct $T(A)$ as a countable CW complex. If we try to repeat this construction with an uncountable R-module, we run into the usual problem that the product topology on $T(A) \times T(A)$ does not coincide with the CW topology and $T(A)$ is not a topological R-module. Milnor's method in [**6**] has the same limitation. A common procedure for dealing with this problem with products is to change the topology on the product by working in a "convenient category". Such a procedure would work to extend the results of this section to all R-modules in the category of compactly generated spaces, but Theorem 3.1 shows that we need not resort to a "convenient category" when R is a countable, Noetherian ring.

In §3, we prove Theorem 3.1 by using structure theorems for modules over a countable, Noetherian ring. If R is a countable, Noetherian ring and A is an R-module, we first construct a topological R-module T with $\pi_1(T)$ containing the

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underlying abelain group of A as a subgroup. T is contained in a product P of countable CW complexes and we topologize P with the box topology in order for T to be a locally simply connected space. We complete the proof of Theorem 3.1 with a covering space argument.

In §4, we examine the implications of Theorem 3.1 for varieties more general than varieties of R-modules.

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1. Preliminaries and notation. We first present the basic concepts connected with CW complexes.

Let R stand for the set of real numbers and R^n for real n-dimensional space. Throughout this paper I will stand for the unit interval and $I^n \subseteq R^n$ for the n-dimensional cube. By a *closed Euclidean n-cell*, denoted by E^n, we mean a homeomorphic image of I^n. $\dot{E}^n$ will stand for the boundary of E^n as a subset of R^n.

DEFINITION 1.1. A *cell structure* on a set X is a pair (X, Φ), where Φ is a collection of maps of closed Euclidean cells into X satisfying:

(i) if $\varphi \in \Phi$ and φ has domain E^n, then φ is injective on $E^n - \dot{E}^n$;

(ii) the images $\{\varphi(E^n - \dot{E}^n) \mid \varphi \in \Phi\}$ partition X;

(iii) if $\varphi \in \Phi$ has domain E^n, then $\varphi(\dot{E}^n) \subseteq \bigcup\{\psi(E^k - \dot{E}^k) \mid \psi \in \Phi$ has domain E^k and $k \leqslant n-1\}$.

If $\varphi \in \Phi$ has domain E^n, then $\varphi(E^n) = \sigma^n$ is called an *n-cell* and we say that φ is a *characteristic map* for σ^n.

DEFINITION 1.2. A cell structure is called *closure finite* if each n-cell σ^n meets only finitely many open cells $\sigma^p - \dot{\sigma}^p$ with $p < n$.

A cell structure is topologized in the following way.

DEFINITION 1.3. Let (X, Φ) be a cell structure. We define the *weak topology* on X with respect to Φ by:

(i) each n-cell σ has the quotient topology induced by φ_σ where $\varphi_\sigma(E^n) = \sigma$;

(ii) a set $F \subseteq X$ is closed if and only if $F \cap \sigma$ is closed in σ for each cell σ of X.

DEFINITION 1.4. A space X is a *CW complex* if and only if there is a sequence of closed subspaces $X_0 \subseteq X_1 \subseteq \cdots \subseteq X$ such that $X = \bigcup X_n$ and

(i) the set X_0 is discrete;

(ii) for each n, X_n is obtained from X_{n-1} by attaching n-cells;

(iii) the space X has the weak topology with respect to the closed sets X_n.

We conclude with some miscellaneous notation. Let X be a topological space and $w, v\colon I \to X$ loops based at a. By $w \simeq_H v$, we mean that w is homotopic to v relative to a by a homotopy H. By $[w]$ we mean the homotopy equivalence class of loops based at a containing w and by $w * v$ we mean the loop defined by

$$\begin{aligned}(w * v)(t) &= v(2t) \quad \text{if } 0 \leqslant t \leqslant \tfrac{1}{2},\\ &= w(2t-1) \quad \text{if } \tfrac{1}{2} \leqslant t \leqslant 1.\end{aligned}$$

All rings will be unital and all R-modules will be left R-modules. If A is an R-module and S a submodule of A, for any $a, b \in A$, $a \equiv_S b$ means $a - b \in S$. If T is a topological R-module, $\pi_1(T)$ will stand for the fundamental group of T with basepoint 0—i.e., $\pi_1(T) = \pi_1(T, 0)$.

2. Countable R-modules. In this section we prove

THEOREM 2.1. *For any countable R-module A there exists an arcwise connected topological R-module $T(A)$ such that $A \simeq \pi_1(T(A))$ as R-modules. Moreover, we can construct $T(A)$ such that*

(i) *for every $a \in A$ there is a loop w_a: $I \to T(A)$ such that for every $r \in R$ and $t \in I$, $rw_a(t) = w_{ra}(t)$ and the correspondence $a \to [w_a]$ is an isomorphism of A onto $\pi_1(T(A))$;*

(ii) *$T(A)$ is a CW complex.*

PROOF. Let $T(A)$ be the R-module consisting of all A-valued functions on the unit interval with finite support and vanishing at the endpoints. Let $a \otimes t$ be the function on the unit interval with value a at t and 0 elsewhere. Set $a \otimes 0 = a \otimes 1 = 0$. Then, clearly,

LEMMA 2.1. *Every element f of $T(A)$ can be uniquely written in the form*

$$f = a_1 \otimes t_1 + \cdots + a_n \otimes t_n,$$

for some nonzero elements $a_1, \ldots, a_n$ of A and $t_1, \ldots, t_n \in I$ with $0 < t_1 < \cdots < t_n < 1$.

We will call $a_1 \otimes t_1 + \cdots + a_n \otimes t_n$ as in Lemma 2.1 the canonical representation of f.

We now define a cell structure on $T(A)$. We will have an n-cell σ for every sequence $(a_1, \ldots, a_n)$ of elements of A and σ will consist of all elements of the form

$$a_1 \otimes t_1 + \cdots + a_n \otimes t_n$$

where each t_i varies over I subject to the condition: $0 \leqslant t_1 \leqslant t_2 \leqslant \cdots \leqslant t_n \leqslant 1$. We will use the expression "a cell σ is given by $(a_1, \ldots, a_n)$" to mean σ consists of the above elements. We define a closed Euclidean cell $E_\sigma \subseteq [0, 1]^n$ by

$$E_\sigma = \{(t_1, \ldots, t_n) \in I : 0 \leqslant t_1 \quad \cdots \leqslant t_n \leqslant 1\},$$

and a characteristic map φ_σ: $E_\sigma \to \sigma$ by

$$\varphi_\sigma(t_1, \ldots, t_n) = a_1 \otimes t_1 + \cdots + a_n \otimes t_n.$$

Let $\Phi^n = \{\varphi_\sigma : \sigma \text{ is a cell given by } (a_1, \ldots, a_n)\}$, $\Phi^0 = \{\varphi : 0 \to 0\}$ and $\Phi = \bigcup_{n=0}^{\infty} \Phi^n$.

LEMMA 2.2. *$(T(A), \Phi)$ is a cell structure.*

PROOF. We must show (i)–(iii) of Definition 1.1. Each φ_σ is injective on $E_\sigma - \dot{E}_\sigma$ by Lemma 1.2. Likewise, $\{\psi_\sigma(E_\sigma - \dot{E}_\sigma) : \sigma \text{ varies over the cells of } T(A)\}$ partitions $T(A)$. If $f \in \varphi(\dot{E}_\sigma)$, then $f = a_1 \otimes t_1 + \cdots + a_n \otimes t_n$ where some of the t_i's are possibly 0 or 1 or equal. In any case, when we reduce f to its canonical representation, f belongs to a cell of a lower dimension.

Let $(T(A), \Phi)$ have the weak topology with respect to Φ. To show that $(T(A), \Phi)$ is a CW complex we need the following two lemmas.

LEMMA 2.3. *$(T(A), \Phi)$ is closure finite.*

PROOF. We must show that each n-cell σ^n meets only finitely many open m-cells $\sigma^m - \dot{\sigma}^m$ for $m < n$. Suppose

$$\sigma^n \cap (\sigma^m - \dot{\sigma}^m) \neq \varnothing, \qquad m < n,$$

and σ^n consists of all elements of the form

$$a_1 \otimes t_1 + \cdots + a_n \otimes t_n, \tag{1}$$

where $0 \leqslant t_1 \ldots \leqslant t_n \leqslant 1$. Since there are only finitely many ways (1) can be reduced in length, there are only finitely many possibilities for σ^m.

LEMMA 2.4. *The cells of $(T(A), \Phi)$ are closed subsets of $T(A)$ with the weak topology.*

PROOF. Let σ be a cell given by $(a_1, \ldots, a_n)$ and let $f \in \dot{\sigma}$. Let $x = (s_1, \ldots, s_n) \in \dot{E}_\sigma$ with $\varphi_\sigma(x) = f$. We define a set $G_x = \{(t_1, \ldots, t_n) \in \dot{E}_\sigma$: if $s_i = 0$ or 1, then $t_i = s_i$ and if $s_i = s_{i+1}$, then $t_i = t_{i+1}\}$. Note that each G_x is a closed subset of E_σ and, as x varies over $\dot{E}_\sigma$, there are only finitely many distinct sets G_x. We claim that $\varphi_\sigma(G_x) = \tau$, where τ is the cell with $f \in \tau - \dot{\tau}$. Assume τ is given by $(b_1, \ldots, b_m)$. Then we may reduce

$$f = a_1 \otimes s_1 + \cdots + a_n \otimes s_n \tag{2}$$

to its canonical representation

$$b_1 \otimes r_1 + \cdots + b_m \otimes r_m \tag{3}$$

where $(r_1, \ldots, r_m) \in E_\tau - \dot{E}_\tau$. Now we reduce (2) to its canonical representation by deleting all $a_i \otimes s_i = 0$ and replacing all $a_i \otimes s_i + a_{i+1} \otimes s_{i+1}$ by $(a_i + a_{i+1}) \otimes s_i$ if $s_i = s_{i+1}$. Hence the sequence of group elements $(b_1, \ldots, b_m)$ depends only on those i for which either $s_i = 0$ or 1 or $s_i = s_{i+1}$, and not on the particular value of s_i. Thus $\varphi_\sigma(G_x) \subseteq \tau$. To prove the reverse inclusion, we first observe that by the reduction process $(r_1, \ldots, r_m)$ is a subsequence $(s_{i_1}, \ldots, s_{i_m})$ of $(s_1, \ldots, s_n)$ with $s_{i_k} \neq s_{i_j}$ for $k \neq j$. If $g \in \tau$, $g = b_1 \otimes t'_1 + \cdots + b_m \otimes t'_m$ for some $(t'_1, \ldots, t'_m) \in E_\tau$. We may embed $(t'_1, \ldots, t'_m)$ into a sequence $(t_1, \ldots, t_n) \in G_x$ with $t_{i_k} = t'_k$. Then, $\varphi_\sigma(t_1, \ldots, t_n) = g$ and $\varphi_\sigma(G_x) = \tau$.

Let σ be any cell of $(T(A), \Phi)$. To show σ is closed we must show that $\sigma \cap \sigma'$ is closed in σ' for all cells σ'. If $\sigma \cap \sigma' = \varnothing$ or σ', we are done. Let $f \in \sigma \cap \sigma'$. If $f \in (\sigma - \dot{\sigma}) \cap (\sigma' - \dot{\sigma}')$, then $\sigma = \sigma'$ and $\sigma \cap \sigma' = \sigma'$. If $f \in \dot{\sigma} \cap (\sigma' - \dot{\sigma}')$ and $\varphi_\sigma(x) = f$, then $\varphi_\sigma(G_x) = \sigma' \subseteq \dot{\sigma}$ and again $\sigma \cap \sigma' = \sigma'$. Assume $f \in \dot{\sigma}'$ and let $\varphi_{\sigma'}(y) = f$, $y \in \dot{E}_{\sigma'}$. Then $\varphi_{\sigma'}(G_y) = \tau$, where $f \in \tau - \dot{\tau}$. We look at two cases. If $f \in \sigma - \dot{\sigma}$, then $\sigma = \tau$ and $\varphi_{\sigma'}(G_y) = \sigma \cap \sigma'$. If $f \in \dot{\sigma}$, let $\varphi_\sigma(x) = f$. Then $\varphi_\sigma(G_x) = \varphi_{\sigma'}(G_y) = \tau$ and $\varphi_{\sigma'}(G_y) \subseteq \sigma \cap \sigma'$. Hence, in both cases $\varphi_{\sigma'}(G_y) \subseteq \sigma \cap \sigma'$ and so $\varphi_{\sigma'}^{-1}(\sigma \cap \sigma') = \bigcup G_y$, where the union is taken over all y such that $\varphi_{\sigma'}(y) \in \sigma \cap \sigma'$. Since there are only finitely many sets G_y and each G_y is closed in $E_{\sigma'}$, $\varphi^{-1}(\sigma \cap \sigma')$ is a closed subset of $E_{\sigma'}$. Since σ' has the quotient topology with respect to $\varphi_{\sigma'}$, $\sigma \cap \sigma'$ is closed in σ'.

LEMMA 2.5. *$(T(A), \Phi)$ is a CW complex.*

PROOF. In Definition 1.4, take each $X_n =$ the set of cells of $T(A)$ of dimension $\leqslant n$. It follows from Lemmas 2.3 and 2.4 and from standard arguments that each X_n is closed. Conditions (i)–(iii) are obvious. For (i), note that $X_0 = \{0\}$.

LEMMA 2.6. *$T(A)$ is a topological R-module.*

PROOF. We must show that addition and scalar multiplication are continuous. Since A is by assumption a countable R-module, $T(A)$ is a countable CW complex, i.e., $T(A)$ has only countably many cells. Hence by Theorem II.5.2 in [5], the product space $T(A) \times T(A)$ is a CW complex with cells $\sigma \times \sigma'$, where σ and σ' are cells of $T(A)$. By Proposition II.1.3 [5], it suffices to show that $+$ restricted to $\sigma \times \sigma'$ is continuous for all cells σ, σ' of $T(A)$. It will be convenient to use the symbol h to stand for the binary operation $+$—that is, h: $T(A) \times T(A) \to T(A)$ is defined by $h(a, b) = a + b$.

Let σ be an n-cell given by $(a_1,\ldots,a_n)$ and σ' an m-cell given by $(a_{n+1},\ldots,a_{n+m})$. Then $h(\sigma, \sigma') = \sigma + \sigma'$ is the set of all elements of the form

$$a_1 \otimes t_1 + \cdots + a_n \otimes t_n + \cdots + a_{n+m} \otimes t_{n+m},$$

where $0 \leqslant t_1 \leqslant \cdots \leqslant t_n \leqslant 1$ and $0 \leqslant t_{n+1} \leqslant \cdots \leqslant t_{n+m} \leqslant 1$.

Let $E = E_\sigma \times E_{\sigma'}$. For each permutation π of $\{1,\ldots,m+n\}$ such that $\pi(j) = i$ and $\pi(l) = k$, where $1 \leqslant i \leqslant k \leqslant n$ or $n+1 \leqslant i \leqslant k \leqslant n+m$, implies $j \leqslant l$, we define a set

$$H_\pi = \{(x_1,\ldots,x_{m+n}) \in E\colon 0 \leqslant x_{\pi(1)} \leqslant \cdots \leqslant x_{\pi(n+m)} \leqslant 1\}.$$

Note that H_π is a closed subset of E, there are only finitely many sets H_π and $E = \bigcup H_\pi$. Let σ_π be the cell given by $(a_{\pi(1)},\ldots,a_{\pi(n+m)})$. Then H_π is homeomorphic to E_{σ_π} by a homeomorphism λ: $E_{\sigma_\pi} \to H_\pi$. If φ: $E \to \sigma \times \sigma'$ is the characteristic map for $\sigma \times \sigma'$, $\varphi_{\sigma_\pi} = (h\varphi)\lambda$. Since φ_{σ_π} is continuous and λ is a homeomorphism, $h\varphi \restriction H_\pi$ is continuous for each π.

Let F be a closed subset of $T(A)$. We will show that $h^{-1}(F)$ is closed. Now $\varphi^{-1}h^{-1}(F) = \bigcup(\varphi^{-1}h^{-1}(F) \cap H_\pi)$ and each $\varphi^{-1}h^{-1}(F) \cap H_\pi$ is closed in H_π, and hence in E, since $h\varphi \restriction H_\pi$ is continuous. Since there are only finitely many sets H_π, $\varphi^{-1}h^{-1}(F)$ is closed in E. Since $\sigma \times \sigma'$ has the quotient topology with respect to φ, $h^{-1}(F)$ is closed in $\sigma \times \sigma'$.

To show scalar multiplication is continuous, first note that for each $r \in R$ and each cell σ given by $(a_1,\ldots,a_n)$, if $(a_{i_1},\ldots,a_{i_k})$ is the largest subsequence of $(a_1,\ldots,a_n)$ such that for each $j = 1,\ldots,k$, $ra_{i_j} \neq 0$, then $r\sigma = \{rf\colon f \in \sigma\}$ is the k-cell given by $(ra_{i_1},\ldots,ra_{i_k})$. We have the commutative diagram

$$\begin{array}{ccc} E_\sigma & \xrightarrow{p} & E_{r\sigma} \\ \varphi_\sigma \downarrow & & \downarrow \varphi_{r\sigma} \\ \sigma & \xrightarrow{\bar{r}} & r\sigma \end{array}$$

where p is the projection $p(t_1,\ldots,t_n) = (t_{i_1},\ldots,t_{i_k}) \in E_{\sigma_r}$ and $\bar{r}(f) = rf, f \in \sigma$. If H is a closed subset of $r\sigma$, then $\varphi_\sigma^{-1}\bar{r}^{-1}(H) = p^{-1}\varphi_{r\sigma}^{-1}(H)$ is closed in E_σ since p and

$\varphi_{r\sigma}$ are continuous. Since σ has the quotient topology with respect to φ_σ, $\bar{r}^{-1}(H)$ is closed in σ. Hence scalar multiplication restricted to each cell σ is continuous, whence by Proposition II.1.3 [5] it is continuous on $T(A)$.

From Lemmas 2.5 and 2.6 we know that $T(A)$ is a CW complex and a topological R-module. Hence we need only prove (i) of Theorem 2.1 and that $A \simeq \pi_1(T(A))$ as R-modules. Associate with each $a \in A$ the loop w_a in $T(A)$ defined by $w_a(t) = a \otimes t$ for each t in the unit interval. Since $w_a = \varphi_\sigma$ where σ is given by (a), it is continuous. Clearly, for every $r \in R$ and $t \in I$, $rw_a(t) = w_{ra}(t)$ in $T(A)$ and we will show that the correspondence $a \to [w_a]$ is a group isomorphism of A onto $\pi_1(T(A))$. Since $T(A)$ is a CW complex, the 1-cells of $(T(A), \Phi)$ give the generators of $\pi_1(T(A))$ and the 2-cells give the relations. Since each 1-cell σ' is given by (a) for some $a \in A$, $\pi_1(T(A))$ consists of the homotopy classes $[w_a]$, $a \in A$. Each 2-cell σ^2 is given by some (a, b), $a, b \in A$, and consists of elements of the form

$$a \otimes t_1 \oplus b \otimes t_2$$

where $0 \leqslant t_1 \leqslant t_2 \leqslant 1$. Note that E_{σ^2} is a triangle with edges $e_1 = I \times \{1\}$, $e_2 = \{0\} \times I$ and $e_3 = \{(t, t)\colon t \in I\}$. $\varphi_{\sigma^2} \restriction e_1$ traces out w_a, $\varphi_{\sigma^2} \restriction e_2$ traces out w_b and $\varphi_\sigma \restriction e_3$ traces out $w_a + w_b$. But since each $a \otimes t + b \otimes t = (a + b) \otimes t$ for all $t \in I$, $\varphi_{\sigma^2} \restriction e_3$ traces out $w_{a+b}(t)$ and the relations of $\pi_1(T(A), 0)$ are

$$[w_a] + [w_b] = [w_{a+b}],$$

whence $a \to [w_a]$ is an isomorphism of A onto $\pi_1(T(A))$.

We can define a scalar multiplication on $\pi_1(T(A), 0)$ by $r[w_a] = [rw_a] = [w_{ra}]$. This scalar multiplication makes $\pi_1(T(A), 0)$ into an R-module and A and $\pi_1(T(A))$ are isomorphic as R-modules.

3. Modules over a countable, Noetherian ring. We use Theorem 2.1 to prove

THEOREM 3.1. *If R is a countable, Noetherian ring, then the underlying abelian group A_G of every R-module A is isomorphic to $\pi_1(T(A))$ for some arcwise connected topological R-module $T(A)$.*

The link between Theorems 2.1 and 3.1 is the following lemma concerning the structure of modules over a countable, Noetherian ring.

LEMMA 3.1. *If R is a countable, Noetherian ring, then every R-module can be embedded into a direct sum of countable R-modules.*

PROOF. Assume R is a countable, Noetherian ring. Every R-module can be embedded into an injective R-module [**3**, p. 80] and since R is Noetherian, every injective R-module is isomorphic to a direct sum of uniform injective R-modules [**3**, p. 103]. (An R-module is uniform if and only if each of its submodules is indecomposable.) Hence, we need only show that every uniform injective R-module is countable. Now a uniform injective R-module is isomorphic to the injective hull of R/L for some meet irreducible left ideal L of R [**3**, p. 100]. Since R is countable, so is R/L. Let $A_0 = R/L$ and let E be the injective hull of A_0. We will show that E is countable.

Since E is injective, given any left ideal L' of R and any homomorphism α: $L' \to A_0$, α has an extension to a homomorphism α': $R \to E$. For each such α, fix an extension α'. Let

$$A_1 = A_0 + \left(\sum_{L'_1 \text{ an ideal of } R,\ \alpha'\colon L' \to A_0} \alpha'(R) \right),$$

where $+$ and Σ indicate the sum, not necessarily direct, of the submodule A_0 and the submodules $\alpha'(R)$. Then, A_1 is a submodule of E such that every homomorphism α: $L' \to A_0$ has an extension to a homomorphism α': $R \to A_1$. Since R is a countable Noetherian ring and A_0 is a countable R-module,

(i) each $\alpha'(R)$ is countable,

(ii) the number of left ideals L' of R is countable, and

(iii) for each left ideal L, the number of homomorphisms α: $L' \to A_0$ is countable.

Hence, A_1 is a countable submodule of E.

Repeating the construction of A_1 from A_0, each time using A_i in place of A_0 to construct A_{i+1}, we generate a nest of countable submodules of E

$$A_0 \subseteq A_1 \subseteq \cdots \subseteq A_i \subseteq A_{i+1} \cdots$$

such that for each left ideal L' and each homomorphism α: $L' \to A_i$, α has an extension to a homomorphism α': $R \to A_{i+1}$. Then, $A = \bigcup A_i$ is a countable submodule of E. We claim A is injective. Let α: $L' \to A$ be a homomorphism and L' a left ideal of R. Since R is Noetherian, L' is finitely generated, whence $\alpha(L') \subseteq A_i$ for some i. But then α has an extension to a homomorphism α': $R \to A_{i+1} \subseteq A$. That A is injective follows from the Injective Test Lemma [**3**, p. 73]. By the minimality of the injective hull $E \supseteq A \supseteq A_0$, $E = A$ and so E is countable.

For the rest of this section R will stand for a countable, Noetherian ring.

PROOF OF THEOREM 3.1. Let A be an arbitrary R-module. Then, by Lemma 3.1, there exist countable R-modules A_λ, $\lambda \in \Lambda$, such that A can be embedded into $\oplus A_\lambda$ as a submodule. For each $\lambda \in \Lambda$, let $T(A_\lambda)$ be the topological R-module of Theorem 2.1. In particular, the underlying abelian group of A_λ is isomorphic to $\pi_1(T(A_\lambda))$.

We look at the product $\prod T(A_\lambda)$ with the box topology—i.e., sets of the form $\prod G_\lambda$, G_λ open in $T(A_\lambda)$, form a basis for the topology. We will use the notation $\prod_B T(A_\lambda)$ to indicate the box topology. We will write elements of a product $\prod X_\lambda$ as $\langle x_\lambda \rangle$, $x_\lambda \in X_\lambda$ for each λ. We will use the symbol ρ_λ to stand for the projection of $\prod X_\lambda$ onto X_λ. If f: $X \to \prod X_\lambda$, $f_\lambda = \rho_\lambda \circ f$. If f_λ: $X \to X_\lambda$, $\langle f_\lambda \rangle$: $X \to \prod X_\lambda$ is the function defined by $\langle f_\lambda \rangle_\lambda = f_\lambda$. Given an element $\langle x_\lambda \rangle \in \prod X_\lambda$, by the expression "$x_\lambda$ has property P almost everywhere" (abbreviated a.e.), we mean x_λ has this property for all but finitely many λ. Similarly, if f: $X \to \prod X_\lambda$, for "f has a property P a.e." we mean f_λ has this property for all but finitely many λ. We will use the following remarks about the box topology.

REMARK 3.1. Each projection ρ_λ: $\prod_B X_\lambda \to X_\lambda$ is continuous, and hence if f: $X \to \prod_B X_\lambda$ is continuous, so is each f_λ: $X \to X_\lambda$.

REMARK 3.2. If f: $X \to \prod_B X_\lambda$ is such that f_λ is identically constant a.e. and each f_λ is continuous, then f is continuous.

PROOF. If $\prod_B G_\lambda$ is a basic open set, $f^{-1}(\prod_B G_\lambda) = \bigcap f_\lambda^{-1}(G_\lambda)$, which is open since each $f_\lambda^{-1}(G_\lambda)$ is open and $f_\lambda^{-1}(G_\lambda) = X$ or $\varnothing$ a.e.

LEMMA 3.2. *$\prod_B T(A_\lambda)$ is a topological R-module.*

PROOF. Immediate from the fact that each $T(A_\lambda)$ is a topological R-module.

We restrict our attention to the direct sum $\bigoplus_B T(A_\lambda)$, topologized as a subspace of $\prod_B T(A_\lambda)$. Since $\prod_B T(A_\lambda)$ is a topological R-module and $\bigoplus_B T(A_\lambda)$ is a submodule of $\prod T(A_\lambda)$, $\bigoplus_B T(A_\lambda)$ is also a topological R-module.

Since A can be embedded into $\bigoplus A_\lambda \subseteq \prod A_\lambda$, we will express each $a \in A$ as a sequence $\langle a_\lambda \rangle$, $a_\lambda \in A_\lambda$ with $a_\lambda = 0$ a.e.. Let $w_{a_\lambda}\colon I \to T(A_\lambda)$ be the loop representing a_λ in Theorem I (ii). If $a_\lambda = 0$, $w_{a_\lambda} \equiv 0$. Let $w_a = \langle w_{a_\lambda} \rangle$. Then, since $w_{a_\lambda} \equiv 0$ a.e., $w_a\colon I \to \bigoplus_B T(A_\lambda)$ and, by Remark 3.2, w_a is continuous. Let

$$\pi_0 = \{[w_a]\colon a \in A\} \subseteq \pi_1\Big(\bigoplus_B T(A_\lambda)\Big).$$

Let $\varphi\colon A \to \pi_0$ be defined by $\varphi(a) = [w_a]$. Then,

LEMMA 3.3. *π_0 is a subgroup of $\pi_1(\bigoplus_B T(A_\lambda))$ and $\varphi\colon A_G \to \pi_0$ is an isomorphism of A_G onto π_0, where A_G is the underlying abelian group of the R-module A.*

PROOF. Since $\varphi(A) = \pi_0$, it suffices to show that φ is a monomorphism. Let $a = \langle a_\lambda \rangle$ and $b = \langle b_\lambda \rangle$. Then $\varphi(a) * \varphi(b) = [\langle w_{a_\lambda} \rangle] * [\langle w_{b_\lambda} \rangle] = [\langle w_{a_\lambda} * w_{b_\lambda} \rangle]$. By Theorem 2.1, for each λ, $w_{a_\lambda} * w_{b_\lambda}$ is homotopic to $w_{a_\lambda + b_\lambda}$ in $T(A_\lambda)$ by a homotopy H_λ. Since $w_{a_\lambda} \equiv w_{b_\lambda} \equiv 0$ a.e., H_λ may be taken to be identically 0 a.e. Hence by Remark 3.2, $\langle H_\lambda \rangle$ is continuous and $[\langle w_{a_\lambda} * w_{b_\lambda} \rangle] = [\langle w_{a_\lambda + b_\lambda} \rangle]$. That is, $\varphi(a) * \varphi(b) = \varphi(a + b)$, or φ is a homomorphism. If $\varphi(a) = \varphi(b)$, then $\langle w_{a_\lambda} \rangle$ is homotopic to $\langle w_{b_\lambda} \rangle$ by a homotopy H. By Remark 3.1, each $H_\lambda\colon I \times I \to T(A_\lambda)$ is continuous, whence, for each λ, w_{a_λ} is homotopic to w_{b_λ}, $a_\lambda = b_\lambda$ and so $a = b$.

To construct a covering space $T(A)$ of $\bigoplus_B T(A_\lambda)$ with fundamental group isomorphic to π_0 we need that $\bigoplus_B T(A_\lambda)$ is an arcwise connected, locally arcwise connected, and locally simply connected space. Since each $T(A_\lambda)$ is arcwise connected, it follows from Remark 3.2 that $\bigoplus_B T(A_\lambda)$ is arcwise connected. We prove that $\bigoplus_B T(A_\lambda)$ is locally arcwise connected and locally simply connected by proving the stronger result in Lemma 3.5. For the proof of Lemma 3.5, we need

LEMMA 3.4. *For every vertex a of a CW complex X and for every open set V with $a \in V$, there exists an open set U, $a \in U \subseteq V$, and a strong deformation retraction p of U onto a such that for any open set G, $a \in G \subseteq U$, there exists an open set H, $a \in H \subseteq G$, and $p(H \times I) \subseteq H$.*

PROOF. We review the proof of Theorem II.6.1. [**5**, p. 63], in which an open set $U \subseteq V$ and a strong deformation retraction p of U onto a are constructed. The existence of an open set H with $a \in H \subseteq G \subseteq U$ and $p(H \times I) \subseteq H$ follows immediately from an examination of this proof.

For each n-cell σ of X, fix a characteristic map $\varphi_\sigma\colon D^n \to \sigma$ where D^n is the Euclidean n-disc. X^n will denote the n-dimensional skeleton of X and C^n will denote $C \cap X^n$, where $C \subseteq X$.

U and p are constructed by recursively constructing each U^n and a strong deformation retraction p^n of U^n onto a. For $n = 0$, let $U^0 = a$ and define p^0 by $p^0(a, t) = a$, $0 \leq t \leq 1$. Suppose for each $k \leq n$ we have constructed U^k and a strong deformation retraction p^k of U^k onto a such that

1. U^k is open in X^k, $\overline{U}^k \subseteq V^k$, where $\overline{U}^k$ is the closure of U^k, and $U^k \cap X^{k-1} = U^{k-1}$, and

2. $p^k(x, t) = x$ for $0 \leq t \leq 2^{-k}$ and $p^k \restriction U^{k-1} \times I = p^{k-1}$.

We construct U^n and p^n with these same properties.

For each n-cell σ whose interior meets V, we look at a characteristic map φ_σ: $D^n \to \sigma$. Note that $\varphi_\sigma^{-1}(U^{n-1}) \subseteq \dot{D}^n$, where $\dot{D}^n$ is the boundary of D^n. Let δ equal the distance between $\varphi_\sigma^{-1}(U^{n-1})$ and $D^n - \varphi_\sigma^{-1}(V)$ and let

$$W_\sigma^n = \{(1 - s)z : z \in \varphi_\sigma^{-1}(U^{n-1}) \text{ and } 0 \leq s \leq \delta/4\}.$$

We have $\overline{\varphi_\sigma(W_\sigma^n)} \subseteq V^n$. Let $U_\sigma^n = \varphi_\sigma(W_\sigma^n)$ and $U^n = U^{n-1} \cup (\bigcup_\sigma U_\sigma^n)$. Then $U^n \subseteq \overline{U}^n \subseteq V^n$ and one may easily verify that U^n is open in X^n.

Let $D^n - \{0\}$ stand for the n-disc with the center removed. Define r^n: $(D^n - \{0\}) \times [0, 2^{-n+1}] \to D^n \times \{0\}$ by

$$r^n(x, t) = x \quad \text{for } 0 \leq t \leq 2^{-n},$$

$$r^n(x, t) = (2 - 2^n t)x + (2^n t - 1)\left(\frac{x}{\|x\|}\right) \quad \text{for } 2^{-n} \leq t \leq 2^{-n+1}.$$

That is, r^n is a radial deformation retraction of $D^n - \{0\}$ onto $\dot{D}^n$. If σ is a n-cell, $W_\sigma^n \subseteq D^n - \{0\}$ and $r^n(W_\sigma^n \times [0, 2^{-n+1}]) \subseteq W_\sigma^n$. Let $r_\sigma^n = r^n \restriction W_\sigma^n \times [0, 2^{-n+1}]$. Note that r_σ^n is a radial deformation retraction of W_σ^n onto $\varphi^{-1}(U^{n-1})$.

We define a strong deformation retraction p^n of U^n onto a by

$$p^n(x, t) = \varphi_\sigma r_\sigma^n(\varphi_\sigma^{-1}(x), t) \quad \text{for } x \in \sigma, \qquad 0 \leq t \leq 2^{-n+1},$$

$$p^n(x, t) = p^{n-1}(p^n(x, 2^{-n+1}), t) \quad \text{for } 2^{-n+1} \leq t \leq 1.$$

Then, p^n has the properties in (ii) above and our induction is completed. We take $p = \bigcup p^n$ and $U = \bigcup U^n$.

Given an open set G with $a \in G \subseteq U$, we may replace V by G in the above construction to obtain an open set H, $a \in H \subseteq G$, and a strong deformation retraction p_H of H onto a. However, examining the above construction of p, we see that p is constructed from the characteristic maps φ_σ and the radial deformation retractions r^n and these maps do not depend upon the open set V, whence $p_H = p \restriction H \times I$ and so $p(H \times I) \subseteq H$.

LEMMA 3.5. *For every $a \in \bigoplus_B T(A_\lambda)$ and every open set V containing a there is an open set U, $a \in U \subseteq V$, and a strong deformation retraction p of U onto a. Hence, $\bigoplus_B T(A_\lambda)$ is locally arcwise connected and locally simply connected.*

PROOF. Since $\bigoplus_B T(A_\lambda)$ is a topological group, we may assume without loss of generality that $a = 0$ and V is a neighborhood of 0. Hence V contains an open set of the form $\prod_B V_\lambda \cap \bigoplus_B T(A_\lambda)$ where for each λ, $0 \in V_\lambda$ and V_λ is open in $T(A_\lambda)$. Each $T(A_\lambda)$ is a CW complex. Let $U_\lambda \subseteq V_\lambda$, $0 \in U_\lambda$, be the open set of Lemma 3.4

and p_λ the strong deformation retraction of U_λ onto 0. Let $U = \prod_B U_\lambda \cap \bigoplus_B T(A_\lambda) \subseteq \prod_B V_\lambda \cap \bigoplus_B T(A_\lambda)$. Define a map $p\colon U \times I \to U$ by

$$p(\langle x_\lambda \rangle, t) = \langle p_\lambda(x_\lambda, t) \rangle$$

for every $\langle x_\lambda \rangle \in U$ and $t \in I$. Since $x_\lambda = 0$ a.e., $p_\lambda(x_\lambda, t) = 0$ a.e. and $p(\langle x_\lambda \rangle, t) \in U$. We claim that p is a strong deformation retraction of U onto 0.

We need only show that p is continuous. Take any $y \in U$ and an open set $G = \prod G_\lambda \cap \bigoplus_B T(A_\lambda)$ with $y \in G$. Without loss of generality, we may assume $G_\lambda \subseteq U_\lambda \subseteq V_\lambda$. Suppose $p(x, t) = y$, where $x = \langle x_\lambda \rangle$, $x_\lambda = 0$ a.e. For each λ such that $x_\lambda = 0$, appealing to Lemma 3.4, we select an open set $H_\lambda \subseteq G_\lambda$ with $0 \in H_\lambda$ such that $p_\lambda(H_\lambda \times I) \subseteq H_\lambda \subseteq G_\lambda$. Assume only $x_{\lambda_1}, \ldots, x_{\lambda_m}$ are unequal to 0. Since each p_{λ_i}, $i = 1, \ldots, m$, is continuous, there is an open set H_{λ_i} of $T(A_\lambda)$ with $x_{\lambda_i} \in H_{\lambda_i}$ and an open set I_{λ_i} of I with $t \in I_{\lambda_i}$ such that $p(H_{\lambda_i}, I_{\lambda_i}) \subseteq G_{\lambda_i}$. Let $H = \prod H_\lambda \cap \bigoplus_B T(A_\lambda)$ and $I' = \bigcap_{i=1}^m I_{\lambda_i}$. Then $(x, t) \in H \times I'$ and $p(H \times I') \subseteq G$, whence p is continuous.

Since $\bigoplus_B T(A_\lambda)$ is arcwise connected and locally simply connected, we can construct the covering space $T(A)$ of $\bigoplus_B T(A_\lambda)$ with fundamental group isomorphic to π_0 and hence to A_G. Since $\bigoplus_B T(A_\lambda)$ is a topological group, so is $T(A)$ [**4**, p. 217]. We complete the proof of Theorem 3.1 by showing that $T(A)$ is a topological R-module.

We define scalar multiplication in $T(A)$ in the obvious way. Recall that the elements of $T(A)$ are equivalence classes $\{w\}_{\pi_0}$ of paths w with $w(0) = 0$. Two paths w and v are equivalent if there exists a loop u, $[u] \in \pi_0$, with $w \approx_H u * v$.

For each equivalence class $\{w\}_{\pi_0}$ and for each $r \in R$, we define $r \cdot \{w\}_{\pi_0} = \{rw\}_{\pi_0}$, where rw is the path given by $(rw)(t) = r(w(t))$ for all $t \in I$. We verify that multiplication is well defined. Suppose $w \simeq_H u * v$, $\{u\}_H \in \pi_0$. Then

$$rw \simeq_H r(u*v) = ru * rv.$$

Since $\{u\} \in \pi_0$, u is homotopic to a loop $\langle w_{a_\lambda} \rangle$ where $\langle a_\lambda \rangle \in A_G$ and each w_{a_λ} is the loop representing a_λ as in Theorem 2.1 (i). Hence

$$\begin{aligned} ru * rv &\simeq_H r\langle w_{a_\lambda} \rangle * rv \\ &= \langle rw_{a_\lambda} \rangle * rv \\ &= \langle w_{ra_\lambda} \rangle * rv. \end{aligned}$$

Since $[\langle w_{ra_\lambda} \rangle] \in \pi_0$, $rv \in \{rw\}_{\pi_0}$.

The proof that scalar multiplication is continuous is routine and left to the reader.

We can define scalar multiplication on $\pi_0 = \{[w_a]\colon a \in A\} \subseteq \pi_1(\bigoplus T(A_\lambda), 0)$ by $r[w_a] = [w_{ra}]$. Let $\tilde{w}_a$ be w_a lifted to $T(A)$. Since $\pi_1(T(A)) = \{[\tilde{w}_a]\colon a \in A\}$, we can likewise define scalar multiplication on $\pi_1(T(A))$ in such a way that

COROLLARY 3.1. *$\pi_1(T(A))$ and A are isomorphic as R-modules.*

Taking $R = Z$ in Theorem 3.1, we have

COROLLARY 3.2. *Every abelian group is isomorphic to the fundamental group of some topological group.*

4. Applications to Universal Algebra. Theorem 3.1 has applications to algebraic structures other than R-modules and, in fact, was inspired by Walter Taylor's paper *Varieties obeying homotopy laws* [7]. In this section we define some terms from Universal Algebra which we will use to describe these applications.

An *algebra* $\mathfrak{A}$ of type $(n_t)_{t\in T}$ consists of a nonempty set A together with a family $(f_t)_{t\in T}$ of operations such that $f_t\colon A^{n_t} \to A$ for each $t \in T$. By a *variety* V, we mean a class of algebras for which there exists a set Σ of equations such that $\mathfrak{A} \in V$ if and only if e holds true in $\mathfrak{A}$ for every $e \in \Sigma$. For example, one speaks of the variety of groups, the variety of R-modules, and the variety of lattices. By a *topological algebra in a variety* V, we mean a structure $(A, \mathfrak{T}, f_t)_{t\in T}$, where $(A, \mathfrak{T})$ is a Hausdorff space, $(A, f_t)_{t\in T} \in V$, and each $f_t\colon A^{n_t} \to A$ is continuous with the product topology on A^{n_t}.

A *groupoid* is a small category in which every morphism is an isomorphism. A *groupoid in a variety* V is a structure $(G, D, \cdot, {}^{-1}, f_t)_{t\in T}$ such that $(G, f_t)_{t\in T} \in V$, $(G, D, \cdot, {}^{-1})$ is a groupoid, where $D \subseteq G \times G$ is the domain of the binary operation $\cdot$, and $D \to G$ and $G \xrightarrow{-1} G$ are V-homomorphisms (i.e., they commute with the operations f_t, $t \in T$). A *group in a variety* V is a one object groupoid in V.

Let V be a variety and let $\Pi_V = \{G\colon G$ is a group isomorphic to the fundamental group of some arc-component of some topological algebra in $V\}$. Let $\mathcal{G}_V = \{G\colon G$ is the automorphism group of some object of some groupoid in $V\}$.

Taylor proves in [7] that $\Pi_V \subseteq \mathcal{G}_V$ and the group laws obeyed by all groups in Π_V coincide with those laws obeyed by all groups in $\mathcal{G}_V$. Whether or not $\Pi_V = \mathcal{G}_V$ remains an open question. However, as a corollary to Theorem 3.1, we prove (Theorem 4.1) that under certain conditions $\mathcal{A}_V \subseteq \Pi_V$ where $\mathcal{A}_V \subseteq \mathcal{G}_V$ consists of all abelian groups in V.

In [7], Taylor mentions the previously known fact that for each variety V there is a unital ring R_V such that the abelian groups in V are precisely the underlying abelian groups of R_V-modules. For any variety V, R_V is constructed by taking noncommuting free generators $\alpha_1^f, \ldots, \alpha_n^f$ for each n-ary operation f and setting

$$R_V = Z[\alpha_i^f\colon \text{all } f, \text{ all } i]/\Sigma,$$

where Z is the ring of integers and Σ is an ideal constructed as follows. For each n-ary operation f, write

$$f(x_1, \ldots, x_n) = \alpha_1^f x_1 + \cdots + \alpha_n^f x_n.$$

Let w_1 and w_2 be the coefficients of an arbitrary variable x_i in terms τ_1 and τ_2, where $\tau_1 = \tau_2$ is an equation of V. Then Σ is generated by all such $w_1 - w_2$. (For more details, see [7, p. 521].) For example, let V be the variety defined by the equations

$$(*) \qquad f(x, x) = x, \qquad f(f(x, y), x) = y.$$

Let G be an abelian group in V. Since $f\colon G^2 \to G$ is a group homomorphism, we have endomorphisms $\alpha_i\colon G \to G$, $i = 1, 2$, such that

$$f(x, y) = \alpha_1(x) + \alpha_2(y).$$

The equations $(*)$ imply

$$(**) \qquad \alpha_1 + \alpha_2 = 1, \qquad \alpha_1^2 + \alpha_2 = 0, \qquad \alpha_1\alpha_2 = 1.$$

Let Σ be the ideal generated by $\alpha_1 + \alpha_2 - 1$, $\alpha_1^2 + \alpha_2$, and $\alpha_1\alpha_2 - 1$. Then $R_V = Z[\alpha_1, \alpha_2]/\Sigma$. Clearly, every abelian group in V is an R_V-module and given any R_V-module G, we can use α_1 and α_2 to define a homomorphism $f\colon G^2 \to G$ satisfying $(*)$. Incidentally, since $\alpha_1 = (1 + \sqrt{3}\,i)/2$ and $\alpha_2 = (1 - \sqrt{3}\,i)/2$ satisfy $(**)$, $R_V = Z[(1 + \sqrt{3}\,i)/2]$.

If T is a topological R_V-module for any variety V, we can use $\alpha_1^{f_t}, \ldots, \alpha_{n_t}^{f_t}$ to define the n_t-ary operation f_t on the underlying topological space of T and $(T, f_t)_{t \in T}$ is a topological algebra in V. Hence, it follows from Theorem 3.1 that

THEOREM 4.1. *If V is a variety such that R_V is a countable, Noetherian ring, then every abelian group in V is isomorphic to the fundamental group of some arcwise connected topological algebra in V.*

If V is a variety of finite type, then R_V is countable and, by Hilbert's basis theorem, Noetherian, whence

THEOREM 4.2. *If V is a variety of finite type, then every abelian group in V is isomorphic to the fundamental group of some arcwise connected topological algebra in V.*

In [7] Taylor proves that if every group in V obeys a group law λ and $(A, \mathfrak{T}, f_t)_{t \in T}$ is a topological algebra in V with one element subalgebra $\{a\}$, then $\pi_1(A, a)$ obeys λ. From Theorem 4.1, we can prove the partial converse.

COROLLARY 4.1. *Let V be a variety such that R_V is a countable, Noetherian ring. Suppose that for every topological algebra $(A, \mathfrak{T}, f_t)_{t \in T}$ in V with one element subalgebra $\{a\}$, $\pi_1(A, a)$ obeys λ. Then every abelian group in V obeys λ.*

PROOF. Let G be an abelian group in V. Then, by Theorem 4.1, $G \simeq \pi_1(A, a)$ where $(A, \mathfrak{T}, f_t)_{t \in T}$ is a topological algebra in V and $a \in A$. $(A, \mathfrak{T}, f_t)_{t \in T} \in V$ is constructed from an R_V-module whose underlying set is A and a is the zero of A as an R-module, whence $f_t(a, \ldots, a) = a$ for every $t \in T$ and $\{a\}$ is a one element subalgebra of A. By assumption $\pi_1(A, a)$ obeys λ, whence G obeys λ.

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