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FORMAL SPACES WITH FINITE-DIMENSIONAL RATIONAL HOMOTOPY

BY

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ABSTRACT. Let S be a simply connected space. There is a certain principal fibration $K_1 \to E \xrightarrow{\pi} K_0$ in which K_1 and K_0 are products of rational Eilenberg-Mac Lane spaces and a continuous map $\phi: S \to E$ such that in particular $\phi_0 = \pi \circ \phi$ maps the primitive rational homology of S isomorphically to that of K_0. A main result of this paper is the

THEOREM. *If* $\dim \pi_*(S) \otimes \mathbf{Q} < \infty$ *then* ϕ *is a rational homotopy equivalence if and only if all the primitive homology in* $H_*(S; \mathbf{Q})$ *and* $H_*(K_0, S; \mathbf{Q})$ *can* (*up to integral multiples*) *be represented by spheres and disk-sphere pairs.*

COROLLARY. *If* S *is formal,* ϕ *is a rational homotopy equivalence.*

1. Introduction. Let S be a simply connected space with $\dim H_p(S) < \infty$ for all p. ($H_*(S)$ denotes singular cohomology with rational coefficients.) Let $P_*(S) \subset H_*(S)$ denote the primitive subspace of the coalgebra $H_*(S)$, and fix a homogeneous basis $\alpha_i \in P_{m_i}(S)$.

Set $K_0 = \prod_i K(\mathbf{Q}; m_i)$ and let $\beta_j \in H^{m_j}(K_0)$ be the image of the fundamental class of $K(\mathbf{Q}; m_j)$ in K_0. Choose a continuous map $\phi_0: S \to K_0$ so that $\langle (\phi_0)_* \alpha_i, \beta_j \rangle = \delta_{ij}$; then $(\phi_0)_*: P_*(S) \xrightarrow{\cong} P_*(K_0)$. The relative homology $H_*(K_0, S)$ is a comodule over $H_*(K_0)$; let $P_*(K_0, S)$ denote the primitive subspace with homogeneous basis $\gamma_i \in P_{n_i}(K_0, S)$.

Because the $\phi_0^* \beta_j$ are dual to $P_*(S)$ they generate the algebra $H^*(S)$, and so ϕ_0^* is surjective. We may thus interpret $H^*(K_0, S)$ as an ideal in $H^*(K_0)$. Set $K_1 = \prod_i K(\mathbf{Q}; n_i - 1)$ and let

$$K_1 \to E \to K_0$$

be a principal fibration such that if $\omega_i \in H^{n_i}(K_0)$ is the transgressed fundamental class of $K(\mathbf{Q}; n_i - 1)$ then $\omega_i \in H^{n_i}(K_0, S)$ and $\langle \gamma_i, \omega_j \rangle = \delta_{ij}$. Standard obstruction theory shows that ϕ_0 lifts to a continuous map $\phi_1: S \to E$.

Call classes $\alpha \in H_p(S)$, $\beta \in H_p(K, S)$ *spherical* if some integral multiple of α (respectively, β) can be represented by S^p (respectively, by (D^p, S^{p-1})). Spherical

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homology is always primitive, but the reverse inclusion usually fails. Indeed, a main theorem of this paper reads

THEOREM I. *Suppose S is a simply connected space such that* $\dim \pi_*(S) \otimes \mathbf{Q} < \infty$. *Then* (*with the notation above*) *the following two conditions are equivalent*:

(1) *the primitive classes in $H_*(S)$ and $H_*(K_0, S)$ are all spherical*,

(2) *the continuous map ϕ_1: $S \to E$ is a rational homotopy equivalence*.

Moreover, *when they hold*, *the integers n_i are all even*; *the classes ω_i form a prime sequence in the free commutative graded algebra $H^*(K_0)$*; *and $H^*(S) \cong H^*(K_0)/I$, where I is the ideal generated by the ω_i.*

REMARK. A *prime* (*or regular*) *sequence* in an algebra H is a sequence $\omega_1,\ldots$ such that in the factor algebra obtained by setting $\omega_1 = \cdots = \omega_{i-1} = 0$, the image of ω_i is not a zero divisor ($i = 1, 2, \ldots$).

Theorem I can be restated in an apparently very different form. Recall that the Eilenberg-Moore spectral sequence for S **[9]** is a 2nd quadrant spectral sequence, converging to $H^*(\Omega S)$, which is a stronger invariant than the algebra $H^*(S)$. (Indeed d_1: $E_1^{-2,*} \to E_1^{-1,*}$ is simply the map $H^+(S) \otimes H^+(S) \to H^+(S)$.) The higher differentials are a further (but still incomplete—cf. [**7**, §8.13]) invariant of the rational homotopy type of S.

For certain spaces however (called *formal spaces*—the precise definition is given below) the rational homotopy type is a formal consequence of the cohomology algebra. Thus two formal spaces with isomorphic cohomology algebras have the same rational homotopy type. If a simply connected commutative graded algebra H over $\mathbf{Q}$ has the property that $H(S) \cong H \Rightarrow S$ is formal, then H is called *intrinsically formal.*

An algebra of the form

$$\wedge X = \text{exterior algebra}\,(X^{\text{odd}}) \otimes \text{symmetric algebra}\,(X^{\text{even}})$$

is intrinsically formal. (Such algebras are exactly the cohomology algebras for a product of $K(\mathbf{Q}; n)$'s, n possibly varying.) More generally if H is the quotient of $\wedge X$ by an ideal generated by a prime sequence then H is intrinsically formal (cf. Remark 3.1). We call algebras of this form *hyperformal.* Since a wedge of odd spheres is intrinsically formal [**7**, Theorem 1.5] but usually not hyperformal, none of the implications

$$H(S) \text{ hyperformal} \Rightarrow H(S) \text{ intrinsically formal} \Rightarrow S \text{ formal}$$

can be reversed.

On the other hand the Eilenberg-Moore spectral sequence of a formal space collapses at E_2. Spaces whose Eilenberg-Moore sequence collapses at E_2 will therefore be called *weakly formal.*

If, for some n,

$$E_2 = E_3 = \cdots = E_n$$

then the space is called *weakly n-formal.* A space which is n-formal in the sense of [**7**] is easily seen to be weakly n-formal.

We can approximate weak formality in another way. Call a space *spherically n-formal* if

$$E_2^{-i,*} = E_\infty^{-i,*}, \qquad i \geqslant n+1.$$

Evidently

$$\begin{aligned} S \text{ formal} &\Rightarrow S \text{ weakly formal} \Rightarrow \cdots \Rightarrow S \text{ spherically } l\text{-formal} \\ &\Rightarrow \cdots \Rightarrow S \text{ spherically 0-formal}. \end{aligned}$$

In [7, §8.13] it is shown that spherically 0-formal $\not\Rightarrow$ weakly formal $\not\Rightarrow$ formal. We shall give examples showing also that spherically 0-formal $\not\Rightarrow$ spherically 1-formal $\not\Rightarrow$ weakly formal and conjecture that in fact spherically l-formal $\not\Rightarrow$ spherically $(l+1)$-formal. All this is in contrast with our restatement of Theorem I which reads

THEOREM II. *Assume S is simply connected and* $\dim \pi_*(S) \otimes \mathbf{Q} < \infty$. *Then S is spherically* 1*-formal* $\Leftrightarrow$ *$H(S)$ is hyperformal.*

REMARK. When S is a homogeneous space then spherically 1-formal can be replaced by spherically 0-formal in the theorem [**3**, Chapter 11, Theorem IV], but this is not true more generally even if both $H(S)$ and $\pi_*(S) \otimes \mathbf{Q}$ have finite dimension, as is shown in §3. The corollary S formal $\Leftrightarrow$ $H(S)$ hyperformal, under the hypotheses $\dim H(S) < \infty$, $\dim \pi_*(S) \otimes \mathbf{Q} < \infty$, S simply connected, has a short and elegant proof [**2**]. This result has also been established by A. Pazitnev.

A simple translation of [**7**, Theorem 8.12] shows that S is spherically 0-formal if and only if every primitive homology class in $H_*(S)$ is spherical. Slightly more subtly, we shall establish the

3.6. PROPOSITION. *S is spherically* 1*-formal if and only if every primitive homology class in $H^*(S)$ and in $H^*(K_0, S)$ is spherical.* (K_0 *is as in Theorem* I.)

This at least suggests why Theorem II is closely related to Theorem I.

The definitions above and Theorem II extend to the categories of path-connected topological spaces S and c-connected commutative graded differential algebras (c-connected c.g.d.a.'s) (A, d_A) over a field Γ of characteristic zero, with the following modifications.

(i) $\pi_*(S)$ must be replaced by $\pi_\psi^*(S)$ or $\pi_\psi^*(A, d_A)$.

(ii) Homotopy type must be suitably defined (over Γ).

(iii) For nonnilpotent spaces S the Eilenberg-Moore sequence although convergent may not converge to $H(\Omega S)$.

The extension runs as follows:

Let (A, d_A) be a c-connected c.g.d.a over Γ. (Thus $A = \sum_{p \geqslant 0} A^p$; if $a \in A^p$, $b \in A^q$ then $ab = (-1)^{pq} ba$, d_A is a derivation of degree 1 and square zero, and $H^0(A, d_A) = \Gamma$.) A (*minimal*) *model* for A is a c.g.d.a. homomorphism ϕ: $(\wedge X, d) \to (A, d_A)$ for which $(\wedge X, d)$ is a (minimal) KS complex and ϕ^*: $H(\wedge X) \to H(A)$ is an isomorphism. If $(\wedge X, d)$ is minimal then $X^0 = 0$. (A *KS* (Koszul-Sullivan) *complex* is a differential algebra of the form $(\wedge X, d)$ in which $X = \sum_{p \geqslant 0} X^p$ and admits a well-ordered homogeneous basis x_α such that dx_α is a polynomial in the x_β with $\beta < \alpha$. It is *minimal* if these polynomials have no linear term.)

A basic result of Sullivan [**10**, §5; **5**] asserts the existence and uniqueness (up to c.g.d.a. isomorphism) of minimal models. If $(\wedge X, d)$ is a model for (A, d_A) then d induces a quotient differential $Q(d)$ in the space $Q(\wedge X) = \wedge^+ X / \wedge^+ X \cdot \wedge^+ X$ of indecomposables. $Q(d) = 0$ if and only if $(\wedge X, d)$ is minimal. The cohomology space $H(Q(\wedge X), Q(d))$ (which is isomorphic to X if $(\wedge X, d)$ is minimal) is independent of the model and is denoted by $\pi_\psi^*(A, d_A)$.

Now suppose S is a path-connected topological space. Then Sullivan defines a c.g.d.a. $(A(S), d)$ over Γ [**10**, §7 or **5**, Chapter 15], natural in S and whose cohomology is naturally isomorphic with $H(S; \Gamma)$. The minimal model of $(A(S), d)$ is called the *minimal model* of S, and we write $\pi_\psi^*(A(S), d) = \pi_\psi^*(S)$.

A second basic result of Sullivan [**10**, Theorem 8.1; **1**] asserts that if S_1, S_2 are simply connected with finite rational Betti numbers, and if $\Gamma = \mathbf{Q}$ then

(i) $\pi_\psi^*(S_i) \cong \operatorname{Hom}_Z(\pi_*(S_i); \mathbf{Q})$, naturally in S_i, and

(ii) S_1 and S_2 have the same rational homotopy type $\Leftrightarrow$ they have isomorphic minimal models.

The definition of rational homotopy type is thus extended by

DEFINITION. Two path-connected spaces (or c-connected c.g.d.a.'s over Γ) have the same Γ-homotopy type if their minimal models (over Γ) are isomorphic as c.g.d.a.'s.

The definition of formality is

DEFINITION. A path-connected space S (respectively, a c-connected c.g.d.a. (A, d_A)) is *formal* if $(A(S), d)$ and $(H(S), 0)$ (respectively, (A, d_A) and $(H(A), 0)$) have minimal models isomorphic as c.g.d.a.'s.

The Eilenberg-Moore spectral sequence for (A, d_A) is obtained by filtering the bar construction on (A, d_A); details can be found in [7, §7]. The Eilenberg-Moore sequence for a path-connected space S is the sequence for $(A(S), d)$. With these conventions the definitions of weakly formal and spherically l-formal given earlier apply verbatim and we have

THEOREM III. *Let (A, d_A) be a c-connected c.g.d.a. with* $\dim \pi_\psi^*(A, d_A) < \infty$. *Then (A, d_A) is spherically 1-formal $\Leftrightarrow$ $H(A)$ is hyperformal.*

Clearly this theorem implies the identical result for path-connected spaces (replace A by $A(S)$) and hence contains the topological Theorem II.

The proofs of the theorems rely on the filtered models of [7]. After some preliminaries in §2, these are described in §3 where also are the examples and the proof of Proposition 3.1. The actual proofs of the theorems are in §4; these, however, depend on the results of §5.

The second major ingredient in these proofs is a careful analysis of finitely generated models whose cohomology algebra is also finitely generated, and this is deferred to §5.

As a byproduct of this analysis we obtain one final result. Let (A, d_A) have finitely generated cohomology, and suppose $\dim \pi_\psi^*(A, d_A) < \infty$. Set

$$\chi_\pi(A, d_A) = \dim \pi_\psi^{\mathrm{even}}(A, d_A) - \dim \pi_\psi^{\mathrm{odd}}(A, d_A)$$

and

$$f_{H(A)}(t) = \sum_{p \geq 0} \dim H^p(A) t^p.$$

Similarly, if $\wedge X$ is the minimal model we set $f_{\wedge X}(t) = \Sigma_{p \geq 0} \dim(\wedge X)^p t^p$.

Because $\dim X < \infty$, $f_{\wedge X}(t)$ is convergent for $|t| < 1$. Because $H(\wedge X) = H(A)$, $\dim(\wedge X)^p \geq \dim H^p(A)$ and so $f_{H(A)}(t)$ is convergent for $|t| < 1$. Set (following Hsiang)

$$\rho_0(H(A)) = \inf\left\{\alpha \;\middle|\; \lim_{t \to 1-} (1-t)^\alpha f_{H(A)}(t) = 0\right\}.$$

As is shown, for instance in [**4**, Proposition 2], $\rho_0(H(A))$ is the Krull dimension of the commutative algebra $H^{\text{even}}(A)$.

In [**4**, Proposition 2] it is shown that $\chi_\pi(A, d_A) - \rho_0(H(A)) \leq 0$. On the other hand in [**4**] is defined a fourth quadrant spectral sequence $E_i^{p,q}$, converging to $H(A) = H(\wedge X)$, called the odd spectral sequence. (The definition is recalled in 2.2.) An immediate consequence of Proposition 5.6 and Lemma 5.8 in this paper is

THEOREM IV. *Let A be a c-connected c.g.d.a. such that $\pi_\psi^*(A, d_A)$ is finite-dimensional and $H(A)$ is a finitely generated algebra. Let k be the largest integer such that* (*in the odd spectral sequence*) $E_\infty^{*,-k} \neq 0$. *Then*

$$k = \rho_0(H(A)) - \chi_\pi(A, d_A).$$

2. Preliminaries. In this section we recall material which will be needed in the sequel. There are three distinct parts: $\wedge$-extensions, Koszul complexes, and dimension theory for commutative rings.

2.1. $\wedge$*-extensions*. A $\wedge$*-extension* is a sequence of KS complexes

$$(\wedge X, d) \xrightarrow{i} (\wedge X \otimes \wedge Y, d) \xrightarrow{\rho} (\wedge Y, \bar{d})$$

in which i and ρ are the obvious inclusion and projection. Note that the differential in $\wedge X \otimes \wedge Y$ need *not* be of the form $d \otimes 1 \pm 1 \otimes \bar{d}$. If ϕ: $(\wedge X, d) \to (A, d_A)$ is any c.g.d.a. homomorphism between c-connected c.g.d.a.'s, and if $(\wedge X, d)$ is a KS complex then there is a commutative diagram

$$\begin{tikzcd} & (A, d_A) & \\ (\wedge X, d) \arrow[ur, "\phi"] \arrow[r, "i"'] & (\wedge X \otimes \wedge Y, d) \arrow[u, "\psi"'] \arrow[r, "\rho"'] & (\wedge Y, \bar{d}) \end{tikzcd}$$

in which the bottom row is a $\wedge$-extension, ψ^* is an isomorphism, and $(\wedge Y, \bar{d})$ is minimal—cf. [**5**, Theorem 6.1].

2.2. *Koszul complexes*. Suppose A is an algebra, $a_1, \ldots, a_m$ are in the centre of A, and let X be a space with basis $x_1, \ldots, x_m$. A differential space $(A \otimes \wedge X, d)$ is defined by $d(A) = 0$, $dx_i = a_i$ and indeed this is the classical Koszul complex [**8**]. Since d is homogeneous of degree -1 with respect to the grading $A \otimes \wedge X = \Sigma_k A \otimes \wedge^k X$, a grading is induced in the cohomology, and we write this $H(A \otimes \wedge X) = \Sigma_k H_k(A \otimes \wedge X)$. Of course if $A = \Sigma A^p$ is a c.g.a. and we set $\deg x_i = \deg a_i - 1$

then $A \otimes \wedge X$ becomes a c.g.d.a. with bigraded cohomology $H = \Sigma H_k^p$. Note in any case that $H_0 = A/I$, I the ideal generated by the a_i.

Suppose now that $(\wedge Z, d)$ is a connected KS complex. Write $Z^e = Z^{\text{even}}$, $Z^o = Z^{\text{odd}}$ and define an associated Koszul complex $(\wedge Z, d) = (\wedge Z^e \otimes \wedge Z^o, d_\sigma)$ by $d_\sigma(Z^e) = 0$, $d_\sigma(Z^0) \subset \wedge Z^e$ and

$$dz - d_\sigma z \in \wedge Z^e \otimes \wedge^+ Z^o, \qquad z \in Z^o.$$

In this case the lower gradation is given by the grading $\Sigma_k \wedge Z^e \otimes \wedge^k Z^o$, and we write $H(\wedge Z, d_\sigma) = \Sigma H_k^p(\wedge Z, d_\sigma)$.

Filter $\wedge Z$ by setting $F^p(\wedge Z)^r = \Sigma_{k \geqslant p-r}(\wedge Z^e \otimes \wedge^k Z^o)^r$. The resulting spectral sequence (introduced in [**4**] and called the odd spectral sequence) converges to $H(\wedge Z, d)$ and its E_0, E_1 and E_2 terms are given by

$$(E_0, d_0) = (\wedge Z, d_\sigma) \quad \text{and} \quad E_1^{p,q} = E_2^{p,q} = H_{-q}^{p+q}(\wedge Z, d_\sigma).$$

Thus it is a fourth quadrant spectral sequence.

2.3. *Dimension theory*. Let R be a noetherian integral domain over our ground field Γ. Any ideal $I \subset R$ is the finite irredundant intersection of primary ideals Q_j whose prime ideals P_j are called the associated prime ideals of I. Following [**11**] we write $\underline{\dim}\, P_j =$ transc. degree of the quotient field of R/P_j and $\underline{\dim}\, I = \inf_j \underline{\dim}\, P_j$.

The chief result we need is a straightforward consequence of [**11**, Theorem 2.1, p. 195; Theorem 26, p. 203]. The result asserts that if R is a polynomial algebra over Γ on n variables then $x_1, \ldots, x_s$ is a prime sequence if and only if the ideal I generated by $x_1, \ldots, x_s$ satisfies $\underline{\dim}\, I = n - s$. In this case I is unmixed, i.e. every associated prime ideal P of I satisfies $\underline{\dim}\, P = n - s$. As a consequence we have that any permutation of a prime sequence in R is a prime sequence.

For any ideal $I \subset R$ the *dimension* of I in R depends only on R/I and is called the Krull dimension of R/I.

Let $a_1, \ldots, a_s$ be a sequence of elements of even degree in a graded commutative algebra A. Let Y have as basis $y_1, \ldots, y_s$ with $\deg y_i = \deg a_i - 1$ and consider the Koszul complex $(A \otimes \wedge Y, d)$ with $dy_i = a_i$. Using the argument of [**4**, Lemma 2] it is easy to see that

2.4. LEMMA. *The sequence $a_1, \ldots, a_s$ is prime if and only if $H_+(A \otimes \wedge Y) = 0$.*

3. Filtered models. Let H be a connected c.g.a. The minimal model $(\wedge X, d)$ of $(H, 0)$ carries an additional structure [7, §3]: X has a second, lower grading $X = \Sigma_{k \geqslant 0} X_k$ such that, for the induced grading in $\wedge X$, d is homogeneous of degree -1 and $H_+(\wedge X, d) = 0$, $H_0(\wedge X, d) = H$. In particular X_0 is a minimal set of generators for H. The model $(\wedge X, d)$ is called the *bigraded model*.

Next, if (A, d_A) is any c-connected c.g.d.a. the bigraded model $(\wedge X, d)$ of $H(A)$ can be perturbed to a (not necessarily minimal) model $(\wedge X, D)$ for (A, d_A) so that $D - d$: $X_k \to \Sigma_{j<k-1}(\wedge X)_j$. This is called the *filtered model* for (A, d_A)—cf. [7, §4] —and is minimal if and only if (A, d_A) is weakly formal [7, Theorem 7.20].

3.1. REMARK. If H is hyperformal then $X_{\geqslant 2} = 0$ and so no perturbations are possible. Thus H is intrinsically formal.

Set $\overline{X}^{p,q} = X^{p+q+1}_{-p-1}$ and extend the bigrading to $\wedge\overline{X} = \Sigma_{p\leqslant -1}(\wedge\overline{X})^{p,*}$. Identity $Q(\wedge X) \cong X = \overline{X}$ and denote the differential $Q(D)$, transported to $\overline{X}$ and extended to $\wedge\overline{X}$, by $\overline{D}$. Filtering $\wedge\overline{X}$ by the left-hand degree produces a spectral sequence which (from E_2 on) is isomorphic with the spectral sequence of Eilenberg-Moore [7, Theorem 7.14]. It follows easily that (A, d_A) is spherically 1-formal if and only if $\overline{X}^{-1,*} \oplus \overline{X}^{-2,*} \to H(\overline{X}, \overline{D})$ is injective. This is equivalent to the condition

$$X_0 \oplus X_1 \to H(X, Q(D)) \quad \text{is injective.} \tag{3.2}$$

On the other hand let $(\wedge Z, d)$ be any model for (A, d_A), and let $U \subset \ker d$ map isomorphically to a minimal subspace of $H^+(\wedge Z)$ which generates the algebra $H(\wedge Z)$. The projection $\wedge^+ Z \to Z$ induces a map $U \to H(Z, Q(d))$ and in [7, Theorem 8.12] it is shown that either of the conditions

(3.3) $X_0 \to H(X, Q(D))$ is injective (($\wedge X$, D) the filtered model), or

(3.4) $U \to H(Z, Q(d))$ is injective

is equivalent to spherical 0-formality for (A, d_A).

We come now to the examples. Because ([**10**, §8]) a KS complex $(\wedge Z, d)$ with $Z^1 = Z^0 = 0$ and $\dim Z^p < \infty$, all p, can be realized as the model of a simply connected space we need only construct the KS complex.

3.5. Example. Our first example is a filtered model which is spherically 1-formal but not weakly formal. In fact, the cohomology algebra H is even intrinsically spherically 1-formal.

Let H be the algebra $\wedge(u_7, u_7', u_9, u_9', u_9'', u_{11})/I$ where I is generated by u_7u_7', $u_7u_{11} - u_9u_9'$, $u_7'u_{11} - u_9'u_9''$, and subscripts denote degrees. If $(\wedge X, d)$ denotes the bigraded model for $(H, 0)$ then X_0, X_1, X_2, X_3, X_4 have bases

$$X_0: u_7, u_7', u_9, u_9', u_9'', u_{11}, \qquad X_1: v_{13}, v_{17}, v_{17}',$$
$$X_2: w_{19}, w_{19}', w_{32}, \ldots, \qquad X_3: z_{25}, z_{25}', \ldots, \qquad X_4: y_{31}, y_{31}', \ldots$$

in which the missing elements all have degrees $\geqslant 32$. Moreover d is given by

$$\begin{aligned} dv_{13} &= u_7u_7', & dv_{17} &= u_7u_{11} - u_9u_9', & dv_{17}' &= u_7'u_{11} - u_9'u_9'', \\ dw_{19} &= v_{13}u_7, & dw_{19}' &= v_{13}u_7', & dw_{32} &= v_{17}u_7u_9', \\ dz_{25} &= w_{19}u_7, & dz_{25}' &= w_{19}'u_7', & & \end{aligned}$$

and

$$dy_{31} = z_{25}u_7, \qquad dy_{31}' = z_{25}'u_7'.$$

Now define a perturbation D by setting

$$D = d \quad \text{in } X_0, X_1, \qquad Dw_{19} = v_{13}u_7 - u_9'u_{11}, \qquad Dw_{19}' = dw_{19}',$$
$$Dz_{25} = w_{19}u_7 - v_{17}u_9', \qquad Dz_{25}' = dz_{25}', \qquad Dy_{31} = z_{25}u_7 - w_{32}.$$

Because $H^p = 0$, $p \geqslant 30$, this operator extends to a differential D in $\wedge X$ such that $(D - d)\colon X_k \to \Sigma_{j<k-1}(\wedge X)_j$. Note that $Q(D)y_{31} = w_{32}$.

The resulting filtered model is trivially even intrinsically spherically 1-formal, but not minimal. Hence it is not weakly formal.

3.5′. EXAMPLE. We construct a spherically 0-formal model which satisfies $\dim \pi_\psi^* < \infty$, $\dim H^* < \infty$ but is not spherically 1-formal. This shows that the hypothesis of spherically 1-formal in Theorem III is essential, and also that spherically 0-formal $\nRightarrow$ spherically 1-formal.

Define a minimal KS complex $(\wedge Z, d)$ as follows. Z has as basis $u_3, v_3, w_5, a_4, b_4, x_7, y_7$ (subscripts denote degrees) and $du = dv = da = db = 0$, $dw = uv$, $dx = uw - a^2$, $dy = vw - b^2$. The spherical cohomology is then $[u], [v], [a], [b]$. A straightforward computation shows that a basis of the cohomology is given by 1, $[u]$, $[v]$, $[a]$, $[b]$, $[a][u]$, $[a][v]$, $[b][u]$, $[b][v]$, $[a][b]$, $[a]^2$, $[b]^2$, $[a][b][u]$, $[a][b][v]$, $[b]^2[u]$, $[a]^3$, $[a]^2[b]$, $[a][b]^2$, $[b]^3$, $[a][b]^2[u]$, $[b]^3[u]$, $[a]^3[b]$, $[a][b]^3$ and $[a][b]^3[u]$. Thus $U = (u, v, a, b)$ generates $H(\wedge Z)$ so that $(\wedge Z, d)$ is spherically 0-formal by (3.4). It is clearly *not* hyperformal and so, by Theorem III, not spherically 1-formal.

We establish next the geometric characterization of spherical 1-formality.

3.6. PROPOSITION. *A simply connected space S with finite rational Betti numbers is spherically* 1-*formal if and only if the primitive homology classes in $H_*(S)$ and $H_*(K_0, S)$ are all spherical* (K_0 *as in the introduction*).

PROOF. Because a generating space for $H^*(S)$ is dual to the primitive subspace of $H_*(S)$, we may assume that the map $\phi_0\colon S \to K_0$ of the introduction is represented by the inclusion $\wedge X_0 \to (\wedge X, D)$ in the filtered model.

Passing to cohomology we obtain the short exact sequence

$$0 \to H^*(K_0, S) \to \wedge X_0 \to H(\wedge X, D) = H(S) \to 0.$$

Since $D = d$ in X_1, and since $H(\wedge X, D) = H(\wedge X, d) = \wedge X_0/\wedge X_0 \cdot d(X_1)$, we can identify $H^*(K_0, S)$ with the ideal $\wedge X_0 \cdot d(X_1)$ in $\wedge X_0$ as a $\wedge X_0$-algebra. It follows at once that $d(X_1)$ is dual to the primitive subspace of $H_*(K_0, S)$. It is now easy to see that the inclusion

$$(\wedge X_0 \otimes \wedge X_1, d) \to (\wedge X, D) \tag{3.7}$$

represents $\phi_1 : S \to E$ of the introduction.

In the diagram

$$\begin{array}{ccccccc}
\pi_*(S)\otimes \mathbf{Q} & \overset{(\phi_0)_\sharp}{\rightarrow} & \pi_*(K_0)\otimes \mathbf{Q} & \rightarrow & \pi_*(K_0, S)\otimes \mathbf{Q} & \overset{\partial}{\rightarrow} & \pi_*(S)\otimes \mathbf{Q} \\
\downarrow h_1 & & \downarrow h_2 & & \downarrow h_3 & & \\
P_*(S) & \underset{(\phi_0)_*}{\overset{\cong}{\rightarrow}} & P_*(K_0) & \rightarrow & P_*(K_0, S) & &
\end{array}$$

(in which the h_i are the Hurewicz homomorphisms) note that $\operatorname{Im} h_2 \subset$ primitive subspace $\subset \operatorname{Im}(\phi_0)_*$. Hence h_3 factors over ∂ to yield a linear map $f\colon \ker(\phi_0)_\sharp \to P_*(K_0, S)$.

Since $(\phi_0)_\sharp$ is dual to the map $X_0 \to H(X, Q(D))$ it follows (by a straightforward check) that f is dual to the composite

$$g\colon X_1 \to H(X, Q(D)) \to H(X, Q(D))/\operatorname{Im} X_0.$$

To say that every primitive class is spherical in $H_*(K_0, S)$ is to say that f is surjective. This is equivalent to g injective. Similarly the primitive classes in $H_*(S)$ are all spherical if and only if $X_0 \to H(X, Q(D))$ is injective.

In view of (3.2) this completes the proof. Q.E.D.

4. The main theorems. Preliminary to the proof of Theorems I and III we use filtered models to put the minimal model of spherically 1-formal space in a desirable form.

Let $(\wedge X, D) \xrightarrow{m} (A, d_A)$ be the filtered model of a c-connected c.g.d.a. Apply 2.1 to the inclusion $(\wedge X_0 \otimes \wedge X_1, d) \to (\wedge X, D)$ to obtain a c.g.d.a. homomorphism $\psi: (\wedge X_0 \otimes \wedge X_1 \otimes \wedge T, d) \to (\wedge X, D)$ such that ψ^* is an isomorphism and $(\wedge T, \bar{d})$ is minimal.

4.1. Proposition. *If (A, d_A) is spherically 1-formal then*

$$m \circ \psi : (\wedge X_0 \otimes \wedge X_1 \otimes \wedge T, d) \to (A, d_A)$$

is the minimal model.

Proof. This is clearly a model. Since $(\wedge T, \bar{d})$ and $(\wedge X_0 \otimes \wedge X_1, d)$ are minimal we have (identifying $X_0 \oplus X_1 \oplus T = Q(\wedge X_0 \otimes \wedge X_1 \otimes \wedge T)$) that $T \xrightarrow{Q(d)} X_0 \oplus X_1 \xrightarrow{Q(d)} 0$. Since ψ^* is an isomorphism the linear part of ψ defines an isomorphism $Q(\psi)^* : H(X_0 \oplus X_1 \oplus T, Q(d)) \xrightarrow{\cong} H(X, Q(D))$—cf. [**5**, Theorem 7.2].

Now spherical 1-formality shows that $X_0 \oplus X_1 \to H(X_0 \oplus X_1 \oplus T, Q(d))$ is injective; hence $Q(d) = 0$ and the model is minimal. Q.E.D.

The main result is Theorem 4.2 below. Combined with its corollary and Proposition 3.6 (including the remark in the proof of Proposition 3.6 that $(\wedge X_0 \otimes \wedge X_1, d) \to (\wedge X, D)$ represents ϕ_1) it immediately implies both Theorems I and III.

4.2. Theorem. *Assume that (A, d_A) is spherically 1-formal, c-connected, and that $\dim \pi_\psi^*(A, d_A) < \infty$. Then the filtered model for (A, d_A) has the form $(\wedge X, d)$ with $X = X_0 \oplus X_1$, $X_1 = X_1^{\text{odd}}$, and $X_j = 0$, $j \geqslant 2$.*

4.3. Corollary. *If $x_1, \ldots, x_m$ is a basis for X_1 then $dx_1, \ldots, dx_m$ is a prime sequence in $\wedge X_0$.*

Proof. Apply Lemma 2.4, noting that necessarily $H_+(\wedge X_0 \otimes \wedge X_1, d) = 0$. Q.E.D.

4.4. Proof of Theorem 4.2. The minimal model for (A, d_A) has the form of Proposition 4.1, and hence $\dim X_0$, $\dim X_1$ and $\dim T$ are all finite. Since $H(A) = \wedge X_0 / \wedge X_0 \cdot d(X_1)$ it is a finitely generated algebra.

Now write $Y = X_0^{\text{even}}$, $W = X_0^{\text{odd}} \oplus X_1 \oplus T$, $Z = X_0 \oplus X_1 \oplus T$ so that $(\wedge X_0 \otimes \wedge X_1 \otimes \wedge T, d) = (\wedge Z, d) = (\wedge Y \otimes \wedge W, d)$ exhibits the minimal model of (A, d_A) as a $\wedge$-extension. If $y_1, \ldots, y_l$ is a basis for Y define $\bar{Y}$ to be a graded space with basis $\bar{y}_i$, and set $\deg \bar{y}_i = 2 \deg y_i - 1$. Extend d to $\wedge Z \otimes \wedge \bar{Y}$ by putting $d\bar{y}_i = y_i^2$.

The construction of $(\wedge X, D)$ is such that every cohomology class is represented by an element in $\wedge X_0$. This is therefore also true in $\wedge Z$. Hence the argument at the end of [**4**, Proposition 2] shows that $H(\wedge Z \otimes \wedge \bar{Y}, d)$ has finite dimension. But by [**6**, Corollary 5.13] this implies that

$$H(\wedge W) \otimes \wedge \bar{Y} = H(\wedge W \otimes \wedge \bar{Y}, \bar{d})$$

has finite dimension, where $\bar{d}$ denotes the differential induced by putting $Y = 0$. Thus the $\wedge$-extension $\wedge Y \otimes \wedge W$ satisfies the conditions in Proposition 5.1(ii) below.

We now apply the results in §5. In particular we can find $c_1, \ldots, c_r \in \wedge Y$ satisfying the conditions of Lemmas 5.2 and 5.7. As in Lemma 5.7 let U be a graded space with basis $u_1, \ldots, u_r$ and degree $u_i =$ degree $c_i - 1$. Extend $(\wedge Z, d)$ to $(\wedge Z \otimes \wedge U, D)$ by putting $Du_i = c_i$.

Lemma 5.7 asserts that $\dim H(\wedge Z \otimes \wedge U, D) < \infty$. If n is the top degree such that $H^n(\wedge Z \otimes \wedge U, D) \neq 0$, then Lemma 5.8 asserts that

$$\lambda^* : H^n(\wedge Z, d) \to H^n(\wedge Z \otimes \wedge U, D)$$

is surjective.

On the other hand write

$$\wedge Z \otimes \wedge U = \left(\wedge X_0 \otimes \wedge X_1^{\text{odd}} \otimes \wedge U\right) \otimes \left(\wedge X_1^{\text{even}} \otimes \wedge T\right).$$

This also exhibits $(\wedge Z \otimes \wedge U, D)$ as a $\wedge$-extension. By the proof of Proposition 5.6, $H(\wedge Z, d_\sigma)$ is finitely generated as a module over $\wedge(c_1, \ldots, c_r)$. Hence so is $H(\wedge Z, d)$, because the odd spectral sequence converges from $H(\wedge Z, d_\sigma)$ to $H(\wedge Z, d)$.

Let I, J, and $K \subset \wedge X_0$ be the ideals generated by $d(X_1)$, $d(X_1^{\text{odd}})$ and $d(X_1^{\text{odd}}) + (c_1, \ldots, c_r)$. Then $H(\wedge Z, d) = H(A) = \wedge X_0/I$ and so $\wedge X_0/I$ is a finitely generated $\wedge(c_1, \ldots, c_r)$ module. Since I is generated by J, together with finitely many elements of odd degree, $\wedge X_0/J$ is also a finitely generated $\wedge(c_1, \ldots, c_r)$ module. Hence $\dim \wedge X_0/K < \infty$ and it follows that

$$\dim H\left(\wedge X_0 \otimes \wedge X_1^{\text{odd}} \otimes \wedge U\right) < \infty.$$

On the other hand, since $(\wedge Z \otimes \wedge U, D)$ is minimal, [**6**, Corollary 5.13] shows that $\dim H(\wedge X_1^{\text{even}} \otimes \wedge T, \bar{D}) < \infty$. If we apply [**4**, Theorem 3] to each of $(\wedge X_0 \otimes \wedge X_1^{\text{odd}} \otimes \wedge U, D)$, $(\wedge X_1^{\text{even}} \otimes \wedge T, \bar{D})$ and $(\wedge Z \otimes \wedge U, D)$, we find that the top degrees n_1, n_2, n in which the cohomology is nonzero satisfy $n = n_1 + n_2$.

But we know from above that $H^n(\wedge Z) \to H^n(\wedge Z \otimes \wedge U)$ is nonzero. Since every cohomology class in $H(\wedge Z)$ can be represented by an element of $\wedge X_0$, it follows that $H^n(\wedge X_0 \otimes \wedge X_1^{\text{odd}} \otimes \wedge U) \to H^n(\wedge Z \otimes \wedge U)$ is nonzero. Hence $n_1 \geqslant n$ and so $n_2 = 0$.

We now have that $H^+(\wedge X_1^{\text{even}} \otimes \wedge T, \bar{D}) = 0$. Since $(\wedge X_1^{\text{even}} \otimes \wedge T, \bar{D})$ is minimal we conclude that $X_1^{\text{even}} = T = 0$, and hence that the minimal model for (A, d_A) has the form $(\wedge X_0 \otimes \wedge X_1, d)$ with $X_1^{\text{even}} = 0$. Q.E.D.

5. Finitely generated models with finitely generated cohomology. Let (A, d_A) be a c-connected c.g.d.a.

5.1. PROPOSITION. *The following two conditions on* (A, d_A) *are equivalent.*

(i) $\dim \pi_\psi^*(A, d_A) < \infty$ *and* $H(A)$ *is a finitely generated algebra.*

(ii) *There is a model for* (A, d_A) *of the form* $(\wedge Z, d) = \wedge Y \otimes \wedge W$ *in which* $(\wedge Y, 0) \to (\wedge Z, d) \xrightarrow{\rho} (\wedge W, \bar{d})$ *is a* $\wedge$*-extension*, $\dim H(\wedge W, \bar{d})$, $\dim Y$ *and* $\dim W$ *are finite and* Y *is evenly graded.*

PROOF. (ii) $\Rightarrow$ (i) Filter $\wedge Z$ using the degree of $\wedge Y$ to get a spectral sequence converging to $H(\wedge Z)$ with E_2-term $\wedge Y \otimes H(\wedge W)$. It follows that E_2 (hence also E_∞ and $H(\wedge Z)$) are finitely generated $\wedge Y$ modules. Thus $H(\wedge Z)$ is a finitely generated algebra.

(i) $\Rightarrow$ (ii) Let $(\wedge X, d)$ be the minimal model of (A, d_A) and let Y be an evenly graded space of finite dimension such that there is a homomorphism ψ: $(\wedge Y, 0) \to (\wedge X, d)$ which makes $H(\wedge X)$ into a finitely generated $\wedge Y$ module. Use 2.1 to produce a $\wedge$-extension $(\wedge Y, 0) \to (\wedge Z, d) \xrightarrow{\rho} (\wedge W, \bar{d})$ and an extension of ψ to a homomorphism ϕ: $(\wedge Z, d) \to (\wedge X, d)$ such that ϕ^* is an isomorphism. Do this so that $(\wedge W, \bar{d})$ is minimal.

Identify $Z = Y \oplus W$ and X with the respective indecomposable spaces for $\wedge Z$, $\wedge X$ and note [5, Theorem 7.2] that ϕ induces an isomorphism $Q(\phi)^*$: $H(Y \oplus W, Q(d)) \xrightarrow{\cong} X$. Because $Q(\bar{d}) = 0$, $Q(d)$: $W \to Y$ and it follows that $\dim W < \infty$. It remains to show that $\dim H(\wedge W, \bar{d}) < \infty$.

Define a graded space $\bar{Y}$ by $\bar{Y}^p = Y^{p+1}$ and extend $(\wedge Z, d)$ to $(\wedge Z \otimes \wedge \bar{Y}, d)$ by putting $d\bar{y} = y$ where $\bar{y}$ corresponds to y under the identification $\bar{Y} = Y$. Use the grading $\Sigma_k \wedge Z \otimes \wedge^k \bar{Y}$ to obtain a spectral sequence whose E_1 term is the Koszul complex $(H(\wedge Z) \otimes \wedge \bar{Y}, D)$ with $D\bar{y} = [y]$. Since $H(\wedge Z)$ is a finitely generated $\wedge Y$ module, $H_0(H(\wedge Z) \otimes \wedge \bar{Y})$ has finite dimension. But $H(H(\wedge Z) \otimes \wedge \bar{Y})$ is finitely generated as an $H_0(H(\wedge Z) \otimes \wedge \bar{Y})$ module, and it follows that $H(H(\wedge Z) \otimes \wedge \bar{Y})$ has finite dimension. This implies $\dim H(\wedge W, \bar{d}) = \dim H(\wedge Z \otimes \wedge \bar{Y}, d) < \infty$. Q.E.D.

Now consider a $\wedge$-extension $\wedge Z = Y \otimes \wedge W$ satisfying the conditions of Proposition 5.1. Let ρ: $(\wedge Z, d) \to (\wedge W, \bar{d})$ be the projection and write $W^o = W^{\text{odd}}$, $W^e = W^{\text{even}}$.

5.2. LEMMA. *Let* N *be an integer divisible by the degrees of the homogeneous elements in* Y. *There is then a prime sequence in* $\wedge Y \otimes \wedge W^e$ *of the form*

$$a_1, \ldots, a_s, b_1, \ldots, b_t, c_1, \ldots, c_r, \tag{5.3}$$

and satisfying the following conditions.

(i) $a_i = d_\sigma v_i$, $b_j = d_\sigma v_{j+s}$ *for* (*not necessarily homogeneous*) *elements* $v_l \in W^o$; *and for any* $w \in W^o$, $a_1, \ldots, a_s, b_1, \ldots, b_t, d_\sigma w$ *is not prime.*

(ii) $\bar{d}_\sigma v_i$ $(1 \leq i \leq s)$ *is a prime sequence in* $\wedge W^e$.

(iii) $c_k \in (\wedge Y)^N$, $1 \leq k \leq r$.

(iv) $s = \dim W^e$ *and* $r + t = \dim Y$.

5.4. REMARK. The ideal J generated by the a_i, b_j, c_k satisfies $\underline{\dim}\, J = 0$ (cf. 2.3) and so $\dim \wedge Z^e/J < \infty$.

PROOF OF LEMMA 5.2. By [**4**, Proposition 1], $\dim H(\wedge W, \bar{d}_\sigma) < \infty$, and so [**4**, Lemma 8] yields $v_1, \ldots, v_s \in W^o$ with $s = \dim W^e$ so that $\bar{d}_\sigma v_i$ is a prime sequence in $\wedge W^e$. Then a basis of Y, followed by $d_\sigma v_1, \ldots, d_\sigma v_s$, is a prime sequence in $\wedge Z^e$. Since a permutation of a prime sequence is prime, $d_\sigma v_1, \ldots, d_\sigma v_s$ is a prime sequence in $\wedge Z^e$.

Extend this to a maximal prime sequence in $\wedge Z^e$ of the form $d_\sigma v_1, \ldots, d_\sigma v_s$, $d_\sigma v_{s+1}, \ldots, d_\sigma v_{s+t}$ with $v_l \in W^o$. Extend this in turn to a maximal prime sequence in $\wedge Z^e$ of the form $d_\sigma v_1, \ldots, d_\sigma v_{s+t}, c_1, \ldots, c_r$, with $c_k \in (\wedge Y)^N$.

The argument of [**4**, Lemma 8] shows that $(\wedge Y)^N$ is contained in one of the prime ideals P associated with the ideal J generated by this sequence. Since $(\wedge Y)^N$ contains a power of every homogeneous element of Y we conclude that $\wedge^+ Y \subset P$ and hence $\ker \rho \subset P$.

Moreover $\bar{d}_\sigma v_1, \ldots, \bar{d}_\sigma v_s \in \rho(P)$ and hence $\rho(P)$ is a prime ideal of $\underline{\dim}\, 0$ in $\wedge W^e$. It follows that $\underline{\dim}\, P = 0$ and so, by §2.3, $s + t + r = \dim Z^e$. Q.E.D.

5.5. LEMMA. *With the hypotheses and notation of Lemma* 5.2 *let k be the largest integer such that $H_k(\wedge Z, d_\sigma) \neq 0$. Then $k = \dim W^o - s - t$.*

PROOF. Let $I = \bigcap_j Q_j$ be the noetherian decomposition of the ideal I generated by $a_1, \ldots, b_t$ in $\wedge Z^e$ and let P_j be the prime ideal associated with the primary ideal Q_j. The maximality of $a_1, \ldots, b_t$ means that $d_\sigma w$ is in some P_j for each $w \in W^o$. Hence by the argument of [**4**, Lemma 8], $d_\sigma(W^o) \subset P_1$ say. Choose $q_j \in Q_j$ so that $q_j \notin P_1$ (possible because I is unmixed so that $P_i \not\subset P_1$ for any $i > 1$). Set $q = \prod_j q_j$.

Now $q \notin P_1$ and so $q \notin I$. Since some power of any $d_\sigma w$ is in Q_1, that power multiplied by q is in I. By multiplying q by suitable powers of the $d_\sigma w$ we find an element $\Phi \in \wedge Z^e$ such that $\Phi \notin I$ but such that $(d_\sigma w)\Phi \in I$, $w \in W^o$.

Choose now homogeneous elements $w_1, \ldots, w_k \in W^o$ which together with $v_1, \ldots, v_{s+t}$ give a basis. Thus $k = \dim W^o - s - t$. A projection

$$\pi : (\wedge Z, d_\sigma) \to (\wedge Z^e/I \otimes \wedge(w_1, \ldots, w_k), D)$$

is given by the obvious projection in $\wedge Z^e$ together with $\pi(v_l) = 0$, $\pi(w_i) = w_i$. Because the $d_\sigma v_l$ are a prime sequence π^* is an isomorphism of cohomology, homogeneous of lower degree zero.

In particular $H_p(\wedge Z, d_\sigma) = 0$, $p > k$. Moreover by construction $(Dw_i)\pi\Phi = 0$ $(1 \leq i \leq k)$ and $\pi\Phi \neq 0$. Thus $\pi\Phi \otimes w_1 \wedge \cdots \wedge w_k$ is a nonzero cocycle (and hence represents a nonzero class) in $H_k(\wedge Z^e/I \otimes \wedge(w_1, \ldots, w_k))$. Thus $H_k(\wedge Z, d_\sigma) \neq 0$. Q.E.D.

5.6. PROPOSITION. *With the notation and hypotheses above let k be the maximum integer such that $H_k(\wedge Z, d_\sigma) \neq 0$. Then*

$$\rho_0(H(\wedge Z, d)) = \rho_0(H(\wedge Z, d_\sigma)) = k + \chi_\pi(\wedge Z).$$

PROOF. Since the odd spectral sequence converges from $H(\wedge Z, d_\sigma)$ we have $\rho_0(H(\wedge Z, d)) \leqslant \rho_0(H(\wedge Z, d_\sigma))$. Next we claim that the inclusion $\wedge(c_1,\ldots,c_r) \to \wedge Z^e$ induces inclusions $\wedge(c_1,\ldots,c_r) \to H(\wedge Z, d_\sigma)$ and $\wedge(c_1,\ldots,c_r) \to H(\wedge Z, d)$.

Indeed, recall from the proof of Lemma 5.5 that one of the prime ideals P_1 for the ideal generated by $a_1,\ldots,b_t$ contains $d_\sigma(W^o)$. The argument of [**4**, Proposition 2] shows that the map $\wedge(c_1,\ldots,c_r) \to \wedge Z^e/P_1$ is an inclusion. Hence so is $\wedge(c_1,\ldots,c_r) \to \wedge Z^e/(d_\sigma W^o) = H_0(\wedge Z, d_\sigma)$.

Next, suppose that for some $\Phi \in \wedge(c_1,\ldots,c_r)$, $\Phi = d\Psi$, $\Psi \in \wedge Z$. Write $\Psi = \Psi_0 + \cdots + \Psi_m$, $\Psi_i \in \wedge Z^e \otimes \wedge^i W^o$. Then the component of $d\Psi$ in $\wedge Z^e$ is $d_\sigma \Psi_1$ so that $\Phi = d_\sigma \Psi_1$. This implies that $\Phi = 0$ by the above argument, so that the second map is also an inclusion. From this we deduce that $r \leqslant \rho_0(H(\wedge Z, d))$.

On the other hand, by the remark after Lemma 5.2, $\wedge Z^e$ is finitely generated as a module over $\wedge(a_1,\ldots,a_s, b_1,\ldots,b_t, c_1,\ldots,c_r)$. Hence $H_0(\wedge Z, d_\sigma)$ is finitely generated over $\wedge(c_1,\ldots,c_r)$. Thus $H(\wedge Z, d_\sigma)$ is finitely generated as a module over $\wedge(c_1,\ldots,c_r)$, and so $\rho_0(H(\wedge Z, d_\sigma)) \leqslant r$. The various inequalities we have derived give

$$\rho_0\big(H(\wedge Z, d)\big) = \rho_0\big(H(\wedge Z, d_\sigma)\big) = r.$$

Finally (using Lemma 5.5) $\chi_\pi(\wedge Z) + k = \dim Z^e - \dim W^o + \dim W^o - s - t$ $= (s + r + t) - s - t = r$. Q.E.D.

Let $\wedge Z$ be as in the previous lemmata. Choose the integer N of Lemma 5.2 so that $H_k^p(\wedge Z, d_\sigma) \neq 0$ for some $p < N$. Choose a graded space U with basis $u_1,\ldots,u_r$ and degree $u_i = N - 1$; thus $U = U^{\mathrm{odd}}$. Define a KS complex $(\wedge Z \otimes \wedge U, D)$ by putting $D = d$ in $\wedge Z$ and $Du_i = c_i$ (chosen as in Lemma 5.2). Then clearly $D_\sigma = d_\sigma$ in $\wedge Z$ and $D_\sigma u_i = c_i$.

5.7. LEMMA. *With the hypotheses and notation above*

(i) *$H(\wedge Z \otimes \wedge U, D)$ and $H(\wedge Z \otimes \wedge U, D_\sigma)$ have finite dimension.*

(ii) *$H_l(\wedge Z \otimes \wedge U, D_\sigma) = 0$, $l > k$, where $k = \dim W^o - s - t$.*

(iii) *The map $H_k(\wedge Z, d_\sigma) \to H_k(\wedge Z \otimes \wedge U, D_\sigma)$ is nonzero.*

PROOF. The remark after Lemma 5.2 shows that $\dim H_0(\wedge Z \otimes \wedge U, D_\sigma) < \infty$ and (i) follows. Let J be the ideal generated by $a_1,\ldots,c_r$. As in Lemma 5.5 we have a projection

$$\big(\wedge Z \otimes \wedge U, D_\sigma\big) \to \big(\wedge Z^e/J \otimes \wedge\,(w_1,\ldots,w_k)\big)$$

which induces a cohomology isomorphism, and (ii) follows.

Finally $\wedge Z \to \wedge Z \otimes \wedge U$ is an isomorphism in degrees $\leqslant N - 2$ and injective in degree $N - 1$. Thus $H^p(\wedge Z, d_\sigma) \to H^p(\wedge Z \otimes \wedge U, D_\sigma)$ is injective for $p \leqslant N - 1$. In particular $H_k^p(\wedge Z, d_\sigma) \to H_k^p(\wedge Z \otimes \wedge U)$ is nonzero for some p. Q.E.D.

Let $(\wedge Z \otimes \wedge U, D)$ be as in the previous lemma. Because $\dim Z \oplus U < \infty$ and $\dim H(\wedge Z \otimes \wedge U, D) < \infty$ we can apply the results of [**4**, §8]. These assert that $H(\wedge Z \otimes \wedge U, D)$ and $H(\wedge Z \otimes \wedge U, D_\sigma)$ are Poincaré duality algebras with fundamental classes of the same degree, say n. Moreover if k is as in Lemma 5.7 then $H^n(\wedge Z \otimes \wedge U, D_\sigma) = H_k^n(\wedge Z \otimes \wedge U, D_\sigma)$.

Consider now the inclusion λ: $(\wedge Z, d) \to (\wedge Z \otimes \wedge U, D)$. It induces a homomorphism of odd spectral sequences λ_i: $(E_i, d_i) \to (\check{E}_i, \check{d}_i)$ with

$$\lambda_0 = \lambda : (\wedge Z, d_\sigma) \to (\wedge Z \otimes \wedge U, D_\sigma)$$

and

$$\lambda_1 = \lambda_2 = \lambda_0^* : H(\wedge Z, d_\sigma) \to H(\wedge Z \otimes \wedge U, D_\sigma).$$

5.8. LEMMA. *With the hypotheses and notation above the maps*
(i) $(\lambda_0^*)^n : H_k^n(\wedge Z, d_\sigma) \to H_k^n(\wedge Z \otimes \wedge U, D_\sigma)$,
(ii) $\lambda_\infty^{n+k,-k} : E_\infty^{n+k,-k} \to \check{E}_\infty^{n+k,-k}$, *and*
(iii) $\lambda^* : H^n(\wedge Z, d) \to H^n(\wedge Z \otimes \wedge U, D)$
are surjective.

PROOF. (i) Because $\dim H_k^n(\wedge Z \otimes \wedge U, D_\sigma) = 1$ we need only show that $(\lambda_0^*)^n$ is nonzero. By Lemma 5.7 we can (for some p) find $\alpha \in H_k^p(\wedge Z, d_\sigma)$ with $\lambda_0^*(\alpha) \neq 0$. Since k is the top nonzero lower degree for $H(\wedge Z \otimes \wedge U, D_\sigma)$, Poincaré duality gives an element $\beta \in H_0^{n-p}(\wedge Z \otimes \wedge U, D_\sigma)$ such that $(\lambda_0^*\alpha) \cdot \beta \neq 0$.

But β is represented by some $\Phi \in (\wedge Z^e)^{n-p}$. Let γ be the class in $H_0^{n-p}(\wedge Z, d_\sigma)$ represented by Φ. Then clearly $\lambda_0^*(\gamma) = \beta$ and so $\lambda_0^*(\alpha \cdot \gamma) = (\lambda_0^*\alpha) \cdot \beta \neq 0$.

(ii) and (iii) Choose a class ε in $H_k^n(\wedge Z, d_\sigma)$ such that $\lambda_0^*\varepsilon \neq 0$. Because $H_i(\wedge Z, d_\sigma) = 0$, $i > k$, a simple spectral sequence argument shows that ε survives to $\bar{\varepsilon} \in E_\infty^{n+k,-k}$. Clearly $\lambda_0^*\varepsilon$ survives to $\lambda_\infty^{n+k,-k}(\bar{\varepsilon})$; hence by [**4**, Theorem 3] the latter class is nonzero. Q.E.D.

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