TREES, GLEASON SPACES, AND COABSOLUTES OF $\beta N \sim N$

BY

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Abstract. For a regular Hausdorff space $X$, let $\mathcal{A}(X)$ denote its absolute, and call two spaces $X$ and $Y$ coabsolute ($\mathcal{A}$-absolute) when $\mathcal{A}(X)$ and $\mathcal{A}(Y)$ ($\beta \mathcal{A}(X)$ and $\beta \mathcal{A}(Y)$) are homeomorphic. We prove $X$ is $\mathcal{A}$-absolute with a linearly ordered space iff the POSET of proper regular-open sets of $X$ has a cofinal tree; a Moore space is $\mathcal{A}$-absolute with a linearly ordered space iff it has a dense metrizable subspace; a dyadic space is $\mathcal{A}$-absolute with a linearly ordered space iff it is separable and metrizable; if $X$ is a locally compact noncompact metric space, then $\beta X \sim X$ is coabsolute with a compact linearly ordered space having a dense set of $P$-points; CH implies but is not implied by "if $X$ is a locally compact noncompact space of $\pi$-weight at most $2^\omega$ and with a compatible complete uniformity, then $\beta X \sim X$ and $\beta N \sim N$ are coabsolute."

A tree $T$ is a POSET (partially ordered set) in which $\preceq$, the set of predecessors of $t$, is well ordered for each $t \in T$. The trees most familiar to topologists are the Cantor tree, the Souslin trees, and the Aronszajn trees [Ku], [Ru]. In §1 we study conditions under which a given POSET contains a cofinal tree.

Recall [Po], [P.S.] that if $X$ is a space, then the absolute $\mathcal{A}(X)$ of $X$ is the unique (up to a homeomorphism) extremally disconnected space that can be mapped irreducibly onto $X$ by a perfect map. Following [C.N.2] call $\beta \mathcal{A}(X)$ the Gleason space of $X$ and denote it by $\mathfrak{G}(X)$. Two spaces $X$ and $Y$ are coabsolute ($\mathcal{A}$-absolute) whenever $\mathcal{A}(X)$ and $\mathcal{A}(Y)$ (respectively, $\mathfrak{G}(X)$ and $\mathfrak{G}(Y)$) are homeomorphic. Designate $\mathfrak{R}(X)$ for the Boolean algebra of regular-open sets of $X$—then it is known that $\mathfrak{G}(X) \cong \mathfrak{G}(Y)$ iff $\mathfrak{R}(X) \cong \mathfrak{R}(Y)$.

In §2, we begin an application of §1 to topology with several theorems. We prove:

(2.1) $X$ is $\mathfrak{G}$-absolute with a linearly ordered space if, and only if, $(\mathfrak{R}(X) \sim \{X\}, \supseteq)$ contains a cofinal tree.

(2.3) (2.8)) A first countable (Moore) space is $\mathfrak{G}$-absolute with a linearly ordered space iff it has a dense linearly ordered (metrizable) subspace.

(2.10) A dyadic space is $\mathfrak{G}$-absolute with a linearly ordered space iff it is separable and metrizable.

We also give (2.6 and 2.7) sufficient conditions (dependent on certain cardinal functions) for a space $X$ to have a dense linearly ordered subspace.

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1All spaces are infinite Hausdorff and regular.
In §3 we consider coabsolutes of Stone-Čech remainders. For a noncompact completely regular regular space $X$, let $X^* = \beta X \sim X$. We prove

(3.5) A locally compact noncompact metric space $X$ has $X^*$ coabsolute with a linearly ordered space having a dense set of $P$-points.

(3.9) If $X$ is a locally compact noncompact metric space of density at most $2^\omega$, then $X^*$ is coabsolute with one of $N^*$, $R^*$, or $N^* + R^*$.

(3.13) The following is implied by CH, and consistent with and independent of $\neg$CH: If $X$ is locally compact noncompact, has $\piw(X) \leq 2^\omega$, and if $X$ admits a complete uniformity, then $X^*$ is coabsolute with $N^*$.

A result of importance in §§2 and 3 is the Stone duality theorem [C.N.2]. The Stone space of a Boolean algebra $B$ is denoted by $\text{St}(B)$. "$\equiv$" between POSETS or Boolean algebras means "is isomorphic to".

We assume ZFC. CH is the continuum hypothesis, SH is Souslin’s hypothesis, and MA is Martin’s axiom. All cardinals and ordinals are von Neumann ordinals, so $\beta < \alpha$ means $\beta \in \alpha$. $\omega_0$ is denoted by $\omega$ and if $\kappa$ is a cardinal, $2^\kappa$ is the cardinal number of the set of subsets of $\kappa$. $| X |$ means the cardinality of $X$. If $A$ and $B$ are sets $A^B$ denotes the set of functions from $A$ to $B$. The standard binary tree [Ku] of height $\lambda$, an ordinal, is $\{ f \in ^\omega a : \alpha \in \lambda \}$ ordered by $f \leq g$ when $f = g \upharpoonright \text{dom}(f)$ and is denoted by $\text{TREE}(\lambda)$.

$N$ is the space of natural numbers, $\text{Irr}$ is the space of irrationals, and $R$ is the space of reals. "int" and "cl" are the interior and closure operators. "$\cong$" between spaces means "is homeomorphic" and $A \sim B$ means the complement of $B$ in $A$. "$\Sigma"$ and "$+"" denote free union.

1. Trees in POSETS. Suppose $P$ is a POSET, $p \in P$, and $Q \subseteq P$. $Q$ is cofinal if $r \in P \Rightarrow \exists q \in Q \text{ with } r \leq q$. $Q$ is a filter if $p > q \in Q \Rightarrow p \in Q, p \in P$ is compatible\(^2\) with $Q$ if, for each $q \in Q$, $\exists r \text{ with } p \leq r \text{ and } q \leq r$. $P$ is separative if, for each pair $p, q \in P, p \nleq q \Rightarrow \exists r \geq q \text{ with } r \text{ and } p \text{ incompatible}$. If $F$ is a cofinal filter in a separative POSET $P$, then each maximal incompatible family of $F$ is a maximal incompatible family of $P$. On the other hand, if $I$ is a maximal incompatible family of $P$, then $\{ p \in P : \exists i \in I, i \leq p \}$ is a cofinal filter of $P$.

Suppose $T$ is a tree and $\alpha$ is an ordinal, then

$$lv(T, \alpha) = \{ t \in T : \downarrow t \text{ has order type } \alpha \}$$

is the $\alpha$th level of $T$ also denoted by $lv(\alpha)$ when there is no confusion. $T(\alpha) = \bigcup \{ lv(\beta) : \beta \in \alpha \}$ is the $\alpha$th subtree. The height of $T$ is

$$h(T) = \inf \{ \alpha : lv(\alpha) = \emptyset \}.$$  

A branch $b$ of $T$ is a maximal linearly ordered subset of $T$ and $\text{ord}(b)$ denotes the order type of $b$.

If every linearly ordered subset of a POSET $P$ is bounded above, then Zorn’s lemma provides $P$ with a cofinal tree of height 1. However, within any POSET $P$ we may build a tree $T$, recursively, by the subtrees $T(\alpha)$, having special properties:

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\(^2\)This is the first of several traditional [Bu], [Je] definitions (also separative, $\kappa$-distributive, $\kappa$-closed) for which we have reversed the usual order relation to maintain the orientation upwards for trees.
1.1. Lemma. If $P$ is a POSET, then there is a tree $T \subseteq P$ satisfying:

1. $\downarrow(0)$ is a maximal incompatible family of $P$.
2. If, for $\alpha \in h(T)$, $b$ is a branch of $T(\alpha)$ bounded above in $P$, then $\downarrow(\alpha)$ contains a maximal incompatible family of $\cap \{t \to [t \in b] \}$, where each $t \to [t = \{p \in P: t < p\}$.
3. $\cap \{t \to [t \in b] \} \subseteq T$ for each branch $b$ of $T$.

Whenever a tree $T$ satisfies 1.1(1)–(3) in a POSET $P$, $T$ will be called an unbounded tree of $P$. Since a cofinal tree in a POSET will be unbounded it is useful to define the ordinal invariants (under isomorphism)

$$\#(P) = \inf \{h(T): T \text{ is an unbounded tree of } P\}$$

and $w\#(P) = \inf \{\#(\{p, \to [\}): p \in P\}$.

1.2. Theorem. For a tree $T$ in a POSET $P$ the following hold:

1. $T$ is unbounded if $p \in P$, $(p \notin t \forall t \in T) \Rightarrow p$ is compatible with at least two incompatible elements of $T$.
2. If $P$ is separative and $T$ is unbounded, then $(p \notin t \forall t \in T) \Rightarrow p$ is compatible with two elements of some level of $T$.
3. If $P$ is separative, then $\#(\{p, \to [\}) \leq \#(P) \forall p \in P$.
4. [Ny] If every compatible pair of elements of $P$ is linearly ordered and if $T$ is unbounded, then $T$ is cofinal in $P$.

Proof. (1) Certainly $T$ satisfies 1.1(1), for otherwise some element of $P$ would be compatible with no elements of $T$. If $\lambda \leq h(T)$ and if $p$ is a successor of each member of a branch $b$ of $T(\lambda)$, then $b = \{t \in T(\lambda): p$ and $t$ are compatible}. So 1.1(3) is immediate while 1.1(2) uses, as well, the observations for 1.1(1).

(2) We note that if $I$ is a maximal incompatible family of a separative POSET $P$ and if $p \in P$ is compatible with precisely one $i \in I$, then $i \leq p$. Otherwise, we may find $q \in P$ so that $p < q$ with $q$ and $i$ and hence $I \cup \{q\}$) incompatible.

So if $p \in P$ is compatible with at most one element of each level, then 1.1(1) and (2) force $p$ to be an upper bound of a branch of $T$. So $p \in T$, by 1.1(3).

(3) Suppose $U$ is an unbounded tree of $P$, $\#(P) = h(U)$, and $p \in P$. If $p \leq u$ for some $u \in U$, then $U \cap [p, \to [$ is an unbounded tree of $[p, \to [$. So assume $p \notin u \forall u \in U$. From (2) $\exists$ a first $\alpha_0 \in h(U)$ such that $p$ is compatible with two elements of $\downarrow(U, \alpha_0)$. We may find a maximal incompatible family $I$ of $\{q \in P: \exists u \in \downarrow(U, \alpha_0), u < q, p < q\}$. Starting with

$$T(\alpha_0 + 1) = U(\alpha_0) \cup I \cup \{u \in \downarrow(U, \alpha_0): p$ and $u$ are incompatible\}
$$

we can build, using (2) to obtain each level, an unbounded tree $T$ of $P$ such that if $\alpha \in h(T)$ and $t \in \downarrow(T, \alpha)$, then there is a $u \in \downarrow(U, \alpha) \exists t \leq u$. So $h(U) = h(T)$. Since $I \subseteq [p, \to [$, 1.1 shows that $T \cap [p, \to [ is an unbounded tree of $[p, \to [$. Therefore,

$$h([p, \to ]) \leq h(T) = \#(P).$$

(4) This generalization of “every linearly ordered set has a well-ordered cofinal subset” is proved similarly to (2).
For a POSET $P$ and a cardinal $\kappa$, $P$ is called $\kappa$-distributive if each intersection of at most $\kappa$ cofinal filters in $P$ is cofinal; $P$ is $\kappa$-closed whenever each increasing sequence of length at most $\kappa$ is bounded above. In forcing arguments the above properties have seen considerable activity [Bu].

1.3. Lemma [Bu, 3.11]. Let $P$ be a POSET and $\kappa$ a cardinal. 
(1) If $P$ is $\kappa$-closed, it is $\kappa$-distributive.
(2) If $\lambda$ is the first cardinal such that $P$ is not $\lambda$-distributive (or $\lambda$-closed), then either $\lambda = \omega$ or $\lambda$ is a successor cardinal.

1.4. Lemma. Suppose $P$ is a POSET with no maximal elements, $T$ is an unbounded tree of $P$, and $\kappa$ is a cardinal.
(1) If $P$ is $\kappa$-distributive and has no maximal elements, then $\kappa^+ \leq h(T)$ and $\operatorname{lv}(T, \alpha)$ is a maximal incompatible family of $P \forall \alpha \in \kappa^+$.
(2) If $P$ is $\kappa$-closed, $\kappa < \operatorname{cf}(\operatorname{ord}(b)) \forall \text{ branches } b$ of $T$.

Proof. As (2) is obvious, we suppose $P$ is $\kappa$-distributive. Let $\alpha$ be the first ordinal such that $\operatorname{lv}(\alpha)$ is not a maximal incompatible family of $P$. If $\alpha \in \kappa^+$

$$C = \bigcap \{ \bigcup \{ \langle t, \rightarrow \rangle : t \in \operatorname{lv}(\beta) \} : \beta \in \alpha \}$$

is a cofinal filter of $P$. Given $c \in C$ we may choose $t_\beta \in \operatorname{lv}(\beta) \forall \beta \in \alpha$ such that $t_\beta \leq c$. As $T$ is a tree, $c$ is an upper bound of the branch $b = \{ t_\beta : \beta \in \alpha \}$ of $T(\alpha)$. From 1.1(2), $c$ is compatible with some element of $\operatorname{lv}(\alpha)$ bounding $b$ from above. Thus, $\operatorname{lv}(\alpha)$ is a maximal incompatible family of $C$, and hence, of $P$. As this is absurd, we must have $\alpha \geq \kappa^+$. □

1.5. Theorem. If $P$ is a separative POSET without maximal elements, then $w\#(P) = \sup \{ \kappa^+ : P \text{ is } \kappa\text{-distributive} \}$.

Proof. We denote the right-hand side of the above equation by $\lambda$. Then 1.4(1) shows $\lambda \leq \#(P)$. We consider two cases.

Case 1. $w\#(P) = \#(P)$. Suppose $\{ F_\alpha : \alpha \in \lambda \}$ is a family of cofinal filters in $P$. Let $p \in P$ be arbitrary and $J = \{ q \in P : p$ and $q$ are incompatible}$). We build, recursively, a tree $T$ in $p$, $\rightarrow$.

Let $T(0) = \emptyset$. Suppose we are given an ordinal $\alpha$,$\alpha \leq \lambda$, such that for each $\beta < \lambda$, we have found $T(\beta)$ subject to the restriction

(*) If $\gamma < \beta$, then $\operatorname{lv}(T, \gamma)$ is a maximal incompatible family of $F_\gamma \cap \{ p, \rightarrow \}$. 

If $\alpha$ is a limit ordinal, we must set $T(\alpha) = \bigcup \{ T(\beta) : \beta < \alpha \}$. If $\alpha$ is a successor ordinal, then $\mid \alpha \mid < \lambda$ since $\lambda$ is a cardinal. Set

$$F = F_\alpha \cap \{ \bigcup \{ \langle t, \rightarrow \rangle : t \in \operatorname{lv}(T, \beta) \} : \beta + 1 < \alpha \}$$

So $J \cup F$ is a cofinal filter of $P$. Now suppose $r \in J \cup F$ and $r > p$.

Clearly $r \in F$ and $\exists r_\beta$ for each $\beta < \alpha$ such that $r_\beta < r$ and $r \in \operatorname{lv}(T, \beta)$. Since $T(\alpha - 1)$ is a tree, $\{ r_\beta : \beta + 1 < \alpha \}$ is a branch of $T(\alpha - 1)$. Since $F$ is now a cofinal filter of $p$, $\rightarrow$, there is a maximal incompatible family $I$ of $\{ p, \rightarrow \}$ contained in $F$. Set $T(\alpha) = F \cup T(\alpha - 1)$. As our hypothesis (*) is now met, the construction of $T$ is complete when we let $T = T(\lambda)$. 
Now let us suppose that \( \lambda < \#(P) \). Then \( T \) is not an unbounded tree of \( |p, \rightarrow | \). From (*) each \( lv(T, \alpha) \) is a maximal incompatible family of \( |p, \rightarrow | \), so 1.1 implies that \( T \) has a branch \( b \) bounded above. If \( \text{ord}(b) < \lambda \), then there is a \( t \in lv(T, \text{ord}(b)) \) compatible with every element of \( b \). Since \( T \) is a tree, \( t \) is an upper bound for \( b \). As the latter is impossible, \( \text{ord}(b) = \lambda \). Now if \( s \) is an upper bound for \( b \), then (*) implies \( p < s \) and \( s \in \cap \{ E_{\alpha}: \alpha < \lambda \} \). As \( p \) was chosen arbitrarily, \( \cap \{ E_{\alpha}: \alpha < \lambda \} \) is a cofinal filter of \( P \). So \( \lambda \leq \lambda^+ \). This is a contradiction. Therefore, \( \lambda = \#(P) \).

Case 2. \( R(\#(P)) < \#(P) \). Choose a maximal incompatible family \( I \subseteq P \) with

\[
\text{w}(p, \rightarrow ) = \#(p, \rightarrow ) \forall p \in I.
\]

Since \( \bigcup \{ p, \rightarrow : p \in I \} \) is a cofinal filter of \( P \), \( \#(P) = \sup \{ \#(p, \rightarrow ) : p \in I \} \). On the other hand, we may add \( \bigcup \{ p, \rightarrow : p \in I \} \) to any family of cofinal filters. So, from case 1, \( \#(p, \rightarrow ) < \lambda \forall p \in I \). Therefore, \( \lambda = \#(P) \).

We observe that case 2 of 1.5 also shows

1.6. Corollary. \( \#(P) \) is a cardinal whenever \( P \) is a separative POSET.

1.7. Theorem. Suppose \( P \) is a separative POSET for which \( \#(P) = \text{w}(\#(P)) \). If \( P \) has a cofinal family the union of \( \#(P) \) incompatible families, then \( P \) has a cofinal tree of height \( \#(P) \).

Proof. WLOG we assume \( \bigcup \{ I_{\alpha}: \alpha \in \#(P) \} \) is cofinal in \( P \) where each is a maximal incompatible family of \( P \) containing no maximal elements of \( P \). We construct an unbounded tree \( T \), modifying the successor ordinal steps in 1.1, as follows:

If \( \alpha < \#(P) \), \( |\alpha| < \#(P) \) from 1.5. By 1.4

\[
C = \left( \bigcup \{ [i, \rightarrow : i \in I_{\alpha}] \} \right)
\]

\[
\cap \left( \bigcap \{ \bigcup \{ [t, \rightarrow : t \in lv(T(\alpha - 1), \beta)] : \beta < \alpha - 1 \} \right)
\]

is a cofinal filter of \( P \). So there is a maximal incompatible family \( I \) of \( P \not\subseteq I \subseteq C \). Set \( T(\alpha) = T(\alpha - 1) \cup I \).

It is clear that \( T \) is cofinal and \( h(T) = \#(P) \).

1.8. Corollary. (1) (Weiss) If \( P \) is a separative \( \kappa \)-closed POSET \( \forall \kappa \leq \lambda \) and \( P \) has a cofinal family which is the union of \( \lambda \) incompatible families, then \( P \) has a cofinal tree. (2) (Davies) If \( P \) is a separative \( \omega \)-distributive POSET and \( |P| \leq \omega_1 \), then \( P \) has a cofinal tree.

Proof. Observe that both results follow from 1.7 even if “separative” is removed from their hypothesis since we only used “\( P \) is \( \kappa \)-distributive \( \forall \kappa \leq \#(P) \)” in the proof of 1.7.

1.9. Theorem. If a separative POSET has a cofinal tree, then it has a cofinal tree of height \( \#(P) \).

Proof. In proving 1.5, we observed that there is a maximal incompatible family \( I \) of \( P \) such that, \( \forall i \in I, \text{w}(i, \rightarrow ) = \#(i, \rightarrow ) \) and

\[
\#(P) = \sup \{ \#(i, \rightarrow ) : i \in I \}.
\]
So WLOG we may assume \( P \) has no maximal elements and \( \#(P) = \omega \#(P) \).
Suppose \( T \) is a cofinal tree in \( P \), \( U \) is an unbounded tree in \( P \), and \( \text{h}(U) = \#(P) \).
From 1.4 each \( \text{lv}(U, \alpha) \) is a maximal incompatible family of \( P \). So we may choose a maximal incompatible family of \( P \),
\[
I_\alpha \subseteq \bigcup \{ [t, \to : t \in \text{lv}(U, \alpha)] \} \cap T.
\]
If \( \bigcup \{ I_\alpha : \alpha \in \#(P) \} \) is cofinal, the theorem follows from 1.7. So suppose \( t \in T \)
such that \( t \notin i \forall i \in I_\alpha \forall \alpha \in \#(P) \). Since \( P \) is separative, we have, from 1.2(2), \( \exists \alpha \in \text{h}(U) \) \( u_0, u_1 \in \text{lv}(U, \alpha) \) \( \exists t \) is compatible with each \( u_n \). Again using separativity \( \exists \), for each \( n \in \{0, 1\} \), a \( t_n \in T \) such that \( t < t_n \) and \( u_n < t_n \). \( \exists \) for each \( n, i_n \in I_\alpha \) such that \( u_n < i_n \) and \( t_n \) is compatible with \( i_n \). Since \( T \) is a tree, either \( i_n \geq t_n \geq t \) (a contradiction) or \( i_n < t_n \). In the latter case, \( t \) and \( i_n \) are compatible so
\[
0 < t \quad \text{and} \quad i_0 < t \quad \text{(a contradiction).} \quad \square
\]
From 1.7, \( 2^\kappa = \kappa^+ \) implies each \( \kappa \)-distributive POSET of cardinality at most \( 2^\kappa \) has a cofinal tree. Our next theorem represents an attempt at removing the set-theoretic hypothesis from this result.

1.10. Lemma. Suppose \( \kappa \) is an infinite cardinal, \( P \) is a separative \( \kappa \)-closed POSET, and \( p \in P \) has no maximal successor; then \( p \) has \( 2^\kappa \) incompatible successors.

Proof. We can construct a tree in \([p, \to] \) isomorphic to \( \text{TREE}(\kappa + 1) \) by applying “separative” at successor ordinals to get two incompatible elements, and “\( \kappa \)-closed” at limit ordinals to get a single successor of a branch. The final level of \( \text{TREE}(\kappa + 1) \) contains \( 2^\kappa \) incompatible elements. \( \square \)

1.11. Lemma. Suppose \( \kappa \) is an infinite cardinal and \( P = \{ p(\xi) : \xi \in 2^\kappa \} \) is a \( \kappa \)-closed separative POSET. If \( I = \{ i(\alpha) : \alpha \in 2^\kappa \} \) is an incompatible family in \( P \), then there is a family \( J \subseteq P \) subject to

(1) \( J = \{ j(\alpha, \xi) : (\alpha, \xi) \in 2^\kappa \times 2^\kappa \} \), where \( i(\alpha) < j(\alpha, \xi) \) for each \( \xi \).

(2) If \( p(\xi) \) is compatible with an element of \( J \) but \( p(\xi) \notin \bigcup J \), then
\[
| \{ i \in I : p(\xi) \text{ and } i \text{ are compatible} \} | \leq | \xi | .
\]

Proof. \( J \) is constructed recursively via a diagonalization argument—we examine the \( \beta \)th step:

Suppose \( \delta \) is the first element of \( 2^\kappa \) such that, \( \forall \alpha \in 2^\kappa, p(\delta) \not\in i(\alpha) \) and \( \forall \gamma \in \beta \forall \xi \in 2^\kappa p(\delta) \not\in j(\gamma, \xi) \), but \( p(\delta) \) and \( i(\beta) \) are compatible. For some \( q \in P \) with \( p(\delta) < q \) and \( i(\beta) < q \) we choose a family \( \{ j(\beta, \xi) : \xi \in 2^\kappa \} \) of maximal incompatible successors of \( i(\beta) \) to which \( q \) belongs. \( \square \)

1.12. Theorem. Let \( \kappa \) be an infinite cardinal. If \( P \) is a separative \( \kappa \)-closed POSET with \( | P | \leq 2^\kappa \), then \( P \) has a cofinal tree.

Proof. From 1.10, \( | P | \leq 2^\kappa \Rightarrow P \) has a cofinal tree of height 1. So WLOG assume \( P \) has no maximal elements and we have a listing of \( P \), \( \{ p(\xi) : \xi \in 2^\kappa \} \).

We can construct, as in 1.1, using 1.10 and 1.11, an unbounded tree \( T \) of \( P \) subject to the additional conditions:

(4) \( \text{lv}(0) \) contains a successor of \( p(0) \).
(5) If \( \alpha \in h(T) \) and \( b \) is a branch of \( T(\alpha) \) bounded above in \( P \), then \( |lv(\alpha) \cap (\{T, t \vdash t : t \in b\})| = 2^\kappa \).

(6) If \( p(\xi) \not\leq t \forall t \in T(\alpha) \) for \( \alpha \in h(T) \), then \( \alpha \leq \xi \) and \( |\{t \in T(\alpha) : p(\xi) \text{ and } t \text{ are incompatible}\}| \leq |\xi| \).

Suppose \( p = p(\xi) \in P \) and \( p \not\leq t \forall t \in T \); then there is, by 1.2(2), a first \( \alpha_0 \) such that \( p \) is compatible with two elements of \( lv(\alpha_0) \). Using 1.4(2) and following the proof of 1.10, we may build a tree \( S \subseteq T \) consisting of elements compatible with \( p \), each of whose levels is contained in a level of \( T \), and which is isomorphic to TREE(\( \kappa + 1 \)). Since \( \exists \lambda \in h(T) \) such that the last level of \( S \) is contained in \( lv(\lambda) \), \( p \) causes (6) to fail for \( \alpha = \lambda \). □

1.13. Lemma [Je, 29B]. If \( P \) is a separative POSET, then there exists a unique (up to an isomorphism) complete Boolean algebra \( \mathbb{B}(P) \) for which \( P \) is cofinally embedded in \( (\mathbb{B}(P) - \{0\}, \geq) \).

If \( B \) is a Boolean algebra such that \( (B - \{1\}, \leq) \) or, equivalently, \( (B - \{0\}, \geq) \), possesses (resp. a cofinal set \( P \) satisfying) the properties defined in this section for a POSET, we say for simplicity that \( B \) (resp. cofinally) possesses said property; therefore:

(i) Every Boolean algebra is separative.

(ii) No atomless \( \sigma \)-complete Boolean algebra is \( \omega \)-closed.³

(iii) \( \mathbb{B}(\text{TREE}(\omega_1)) \) is cofinally \( \omega \)-closed.

1.14. Corollary. Suppose \( \kappa \) is an infinite cardinal. If \( \kappa^+ = 2^\kappa \), then \( \mathbb{B}(\text{TREE}(2^\kappa)) \) is the only complete atomless Boolean algebra which is cofinally \( \kappa \)-closed and has a cofinal set of cardinal \( 2^\kappa \). If \( 2^{\kappa^+} = 2^\kappa \), then there are at least two complete atomless Boolean algebras cofinally \( \kappa \)-closed and having cofinal subsets of cardinal \( 2^\kappa \).

Proof. The Pressing Down Lemma [Ku] shows that, for each \( \kappa \), \( \mathbb{B}(\text{TREE}(\kappa^+)) \neq \mathbb{B}(\text{TREE}(\kappa^{++})) \). On the other hand (v) in the construction of \( T \) in 1.12 shows that if \( \kappa^{++} = 2^\kappa \) and \( P \) is a \( \kappa \)-closed separative POSET without maximal elements of cardinality \( 2^\kappa \), then \( P \) has a cofinal tree isomorphic to \( \bigcup \{lv(\text{TREE}(\kappa^+) \setminus \lambda) : \lambda \text{ is a limit ordinal in } \kappa^+ \} \). □

1.15. On products. Suppose \( \kappa \) is a cardinal and \( P(\alpha) \) is a POSET for each \( \alpha \in \kappa \); there are two traditional definitions for partial orders on the Cartesian product \( \prod = \prod\{P(\alpha) : \alpha \in \kappa\} \):

(1) The lexicographic product, \( \text{lex} \prod \), is ordered by “\( f < g \) whenever \( \exists \alpha \in \kappa \) with \( f(\alpha) < g(\alpha) \) and \( f(\beta) = g(\beta) \forall \beta < \alpha \).” It is easy to see that \( \text{lex} \prod \) has a cofinal tree whenever \( P(\alpha) \) has a cofinal tree \( \forall \alpha \in \kappa \).

(2) The usual product on \( \prod \), denoted by \( \times \{P(\alpha) : \alpha \in \kappa \} \), is ordered by “\( f < g \) whenever \( f(\alpha) \leq g(\alpha) \forall \alpha \in \kappa \).” An easy application of the Pressing Down Lemma shows \( \text{TREE}(\omega_1 \times \text{TREE}(\omega_1)) \) has no cofinal tree. However, 1.7 shows that \( P \times Q \) has a cofinal tree whenever each of \( P \) and \( Q \) have a cofinal tree and \( w\#(P) = w\#(Q) \).

1.16. Remarks. (1) Is it consistent that “every \( \omega \)-distributive POSET of cardinality \( \omega_1 \) is \( \omega \)-closed?” Not in a model of ZFC + SH; however, Franklin Tall has

³In [Wo2], [Wo3] cofinally \( \omega \)-closed Boolean algebras are called Cantor-separable.
communicated to the author Peter Davies' affirmative answer under the assumption of the consistency of certain large cardinal axioms.

(2) For the POSET $P$ of nonempty clopen subsets of $\beta N \sim N$ (under $\supseteq$) 1.5, 1.9, and 1.12 were proved, independently, in [B.P.S.].

(3) 1.8(2) is due to Peter Davies. 1.8(1) is an observation William Weiss made from one of our early results.

(4) Is it consistent with $\text{ZFC} + \text{CH}$ that "there is precisely one complete atomless cofinally $\omega$-closed Boolean algebra with a cofinal set of cardinality $2^\omega$?" See 3.13.

(5) For many POSETS $P$, $\#(P)$ is well defined by considering cofinal subsets of $P$. With proof similar to 1.2(3) and (4), this is true when $P$ is either separative or when every compatible pair of elements of $P$ are linearly ordered or when $P$ is directed.

2. $\mathcal{G}$-absolutes of linearly ordered spaces. Recall [Ju2] if $(X, \tau)$ is a space, then a cofinal subset of $(\tau - \{\emptyset\}, \supseteq)$ is known as a $\pi$-base (pseudobase in [C.N.2]) and the $\pi$-weight, $\pi w(X)$, is the least cardinal possessed by a $\pi$-base for $X$. The weight, $w(X)$, is the least cardinal possessed by a base for $\tau$.

2.1. Theorem. For a space $X$, the following are equivalent:

(1) $X$ has a $\pi$-base with a cofinal tree.

(2) Every $\pi$-base of $X$ has a cofinal tree.

(3) $X$ is $\mathcal{G}$-absolute with a linearly ordered space.

Proof. If $Y$ is the set of isolated points of $X$, then $Y$ is a subset of every $\pi$-base for $X$, $Y$ is the set of atoms of $\mathcal{R}(X)$, and

$$\mathcal{R}(X) = \mathcal{R} ( Y \cup \text{int} (X - Y)) .$$

Since the free union of linearly ordered spaces is linearly ordered, we need only prove the theorem for $X = Y$. WLOG we assume $X$ has no isolated points.

(1) $\Rightarrow$ (2). Let $P$ be a $\pi$-base for $X$. Since a cofinal subset of a $\pi$-base is a $\pi$-base, we suppose that $(T, \supseteq)$ is a tree of nonempty open subsets of $X$ such that $T$ has the minimum possible height for a tree $\pi$-base for $X$. Since $X$ has no isolated points, $h(T)$ and ord($b$) are limit ordinals whenever $b$ is a branch of $T$. We now construct, recursively, two trees $S_1 \subseteq T$ and $S_2 \subseteq P$.

For $i \in \{1,2\}$ let $S_i(0) = \emptyset$. Suppose we are given an ordinal $\alpha < h(T)$ such that $S_i(\beta)$ has been found, for each $\beta < \alpha$ and each $i \in \{1,2\}$, subject to the restriction $\gamma < \beta$.

(a) $\text{lv}(S_i, \gamma)$ is a pairwise-disjoint family of nonempty open sets.

(b) If $b$ is a branch of $S_i(\gamma)$ and if $\text{int}(\cap b) \neq \emptyset$, then

$$\text{int}(\cap b) \subseteq \text{cl}(\{ p \in \text{lv}(S_2, \gamma) : p \subseteq \cap b \}).$$

(c) $\text{lv}(S_1, \gamma) \subseteq T \sim T(\gamma)$.

(d) If $p \in \text{lv}(S_2, \gamma)$, then $p \subseteq \text{cl}(t \in \text{lv}(S_1, \gamma) : t \subseteq p)$.

If $\alpha$ is a limit ordinal, we set $S_i(\alpha) = \bigcup \{ S_i(\beta) : \beta < \alpha \} \forall i$. If $\alpha$ is a successor ordinal, the choice of $\text{lv}(S_i, \alpha - 1)$ is straightforward (given a $\pi$-base $Q$ of a space $Y$.
and a nonempty open set $G$ of $Y$, $G$ has a dense set which is the union of a family of pairwise-disjoint members of $Q$). Let $S_i(\alpha) = S_i(\alpha - 1) \cup \text{lv}(S_i, \alpha - 1)$. Our recursion hypothesis is clearly met. So our construction of $S_i \forall i \in \{1, 2\}$ is complete when we set each $S_i = S_i(h(T))$.

Now (b) and (d) imply $S_2$ is a $\pi$-base for $X$ iff $S_1$ is cofinal in $T$. (b) and (d) also imply that if $\alpha < h(T)$ and if $b$ is a branch of $S_1(\alpha)$, then

$$\text{int}(\cap b) \subseteq \text{cl}(\{t \in \text{lv}(S_1, \alpha): t \subseteq \cap b\}).$$

From (c) and 1.1, $S_1$ is an unbounded tree of $T$. 1.2(4) shows $S_1$ is cofinal in $T$.

$(2) \Rightarrow (3)$. Since an infinite Hausdorff space contains an infinite family of non-empty pairwise-disjoint open sets, we suppose $T$ is a cofinal tree in $\mathbb{R}(X)$ satisfying

(iv) if $\alpha \in h(T)$ and $b$ is a branch of $T(\alpha)$ bounded above in $\mathbb{R}(X)$, then $\text{lv}(T, \alpha)$ contains infinitely many elements each of whose closure is a subset of $\cap b$.

Following the standard [Ku] collapsing of Souslin and Aronszajn trees, order each $\text{lv}(T, \alpha)$ so that $\text{lv}(\alpha)$ and in the case of (iv), the successors of $b$ in $\text{lv}(\alpha)$, form a discrete linearly ordered set without endpoints.

Set $L = L(T) = \{b: b$ is a branch of $T\}$ and for $b_0, b_1 \in L$ define $b_0 < b_1$ if for some $\alpha \in h(T)$

$$(b_0 \cap \text{lv}(T, \alpha)) < (b_1 \cap \text{lv}(T, \alpha)) \quad \text{while} \quad b_0 \cap T(\alpha) = b_1 \cap T(\alpha).$$

Thus, $L$ is linearly ordered. Set $\psi(t) = \{b \in L: t \in b\}$ for each $t \in T$.

Since $\psi(t)$ has no endpoints, $\psi(t)$ is clopen; further, if in $L$, $b_0 < b_1 < b_2$, then by (iv), $\exists t \in b_1 \sim (b_0 \cup b_1) \supseteq \psi(t) \subseteq ]b_0, b_1[$. So $\psi$ embeds $T$ cofinally within $\mathbb{R}(L)$. Now apply 1.13 and the Stone duality.

$(3) \Rightarrow (1)$. Suppose $L$ is a linearly ordered space without isolated points; then $P = \{G \in \mathbb{R}(L): G$ is an open interval of $L\}$ is cofinal in $\mathbb{R}(L)$. Suppose $T$ is any unbounded tree of $P$ and $]x, y[ \in P$ such that $t \subseteq ]x, y[$ for $\forall t \in T$. According to 1.2(2) $\exists \beta \in h(T), ]x_i, y_i[ \in \text{lv}(T, \beta)$ for $i \in \{0, 1\}$, such that $]x, y[ \cap ]x_i, y_i[ \neq \emptyset$, where

$$(*) \quad x_0 < x < y_0 < y, \quad x < x_1 < y < y_1,$$

and there are no points between $y_0$ and $x_1$. Since $\mathbb{R}(L)$ is closed under intersection, $]x, y_0[ \in P$. Again applying 1.2(2) $\exists \alpha \in h(T), \alpha > \beta, \exists x_j, y_j[ \in \text{lv}(T, \alpha)$ for $j \in \{2, 3\}$, such that $]x_0, y_0[ \cap ]x_j, y_j[ \neq \emptyset$, $x_2 < x < y_2 < y_0$, $x < x_3 < y_0 < y_3$, and there are no points between $x_2$ and $x_3$. In particular, $y_0 \in ]x_3, y_3[ \sim ]x_0, y_0[$. Since $T$ is a tree, $\beta < \alpha$, and $L$ has no isolated points, $]x_3, y_3[ \cap ]x_0, y_0[ = \emptyset$ which contradicts (*). So $T$ is cofinal in $P$. Now apply the Stone duality. □

2.2. Corollary. For a space $X$, the following are equivalent:

(1) $X$ has a $\sigma$-disjoint $\pi$-base.

(2) $\mathbb{R}(X)$ has a cofinal tree and $\#(\mathbb{R}(X)) \leq \omega$.

(3) $\exists$ a metric space $M$ $\Theta$-absolute with $X$.

Proof. $(1) \Rightarrow (2)$. This is a corollary of 1.7 since $\mathbb{R}(X)$ is $n$-closed for finite cardinals $n$.

$(2) \Rightarrow (3)$. In 2.1 $(2) \Rightarrow (3)$ the tree $T$ may be assumed to have height $\omega$ (from 1.9). Thus, the space $L$ is metrizable via $\rho(b_0, b_1) = 2^{-n}$ when $b_0 \cap \text{lv}(n) \neq b_1 \cap \text{lv}(n)$ and $b_0 \cap T(n) = b_1 \cap T(n)$. 
(3) ⇒ (1). Every metric space has a σ-disjoint base. □

2.3. Theorem. Suppose X is a space whose every point has a well-ordered local base; then X is SO-absolute with a linearly ordered space iff X has a dense linearly ordered subspace.

Proof. We need only show "⇒". WLOG assume X has no isolated points. For each \( x \in X \) let \( \mathcal{R}(x) \subseteq \mathcal{R}(X) \) be a well-ordered, by "⊃", local base at \( x \). From 2.1, \( \mathcal{R}(X) \) has a cofinal tree \( U \). We construct a tree \( T \) and a function \( \psi: T \to X \) as follows:

Let \( T(1) = U(1) \) and \( \psi(t) \in t \) be arbitrarily chosen for each \( t \in T(1) \). Suppose we are given an ordinal \( \lambda \) such that \( T(\alpha) \) and \( \psi(t) \in t \) have been constructed \( \forall \alpha \in \lambda \forall t \in T(\alpha) \) subject to the restrictions:

(i) \( T(\alpha) \) satisfies 1.1(1) and (2).
(ii) \( T(\alpha) \) satisfies (iv) of 2.1(2) ⇒ (3).
(iii) If \( \beta < \alpha \) and \( b \) is a branch of \( T(\beta) \), then

\[
\left| \{ t \in \text{lv}(T(\alpha), \beta) : t \subseteq \cap b \text{ and } t \notin U \} \right| \leq 1,
\]

where equality holds only if \( \psi \) is constant on a tail of \( b \).

(iv) If \( \beta < \alpha \) and \( t \in T(\beta) \), then \( T(\alpha) \) has a branch \( b \) with \( t \in b \) and \( \psi(b) = \{ \psi(t) \} \).

(v) If \( s, t \in T(\alpha), t \subsetneq s \), and if \( \psi(t) = \psi(s) \), then \( t \in \mathcal{R}(\psi(s)) \).

If \( \lambda \) is a limit ordinal, set \( T(\lambda) = \bigcup \{ T(\alpha) : \alpha \in \lambda \} \). If \( \lambda \) is a successor ordinal, then we will assign to each branch \( b \) of \( T(\lambda - 1) \) a family \( I(b) \) and we will set

\[
T(\lambda) = T(\lambda - 1) \cup \bigcup \{ I(b) : \text{\( b \) is a branch of \( T(\lambda - 1) \)} \}.
\]

For \( t \in T(\lambda) \sim T(\lambda - 1) \) and \( t \subseteq \cap b \), we assign \( \psi(t) = \psi(s) \) if \( \psi \) is constantly \( \psi(s) \) on a tail of \( b \) and if \( t \in \mathcal{R}(\psi(s)) \); otherwise, \( \psi(t) \in t \) may be arbitrarily chosen.

Let \( I(b) = \emptyset \) whenever \( \text{int}(\cap b) = \emptyset \). If \( \text{int}(\cap b) \neq \emptyset \) and \( \psi \) is not constant on a tail of \( b \), we may choose, since \( X \) has no isolated points, a pairwise disjoint subfamily \( I(b) \) of \( U \) with \( \bigcup \{ \text{cl}(u) : u \in I(b) \} \) a dense subset of \( \text{int}(\cap b) \). If \( \text{int}(\cap b) \neq \emptyset \) and \( \psi \) is constantly \( \psi(s) \) on a tail of \( b \) with \( s \in b \), then (v) implies \( \text{int}(\cap b) \) is a nbhd of \( \psi(s) \). Choose \( t \in \mathcal{R}(\psi(s)) \) with \( \text{cl}(t) \subsetneq \cap b \). Then \( I(b) \) will be the union of \( \{ t \} \) and an infinite pairwise disjoint subfamily of \( U \) such that \( \bigcup \{ \text{cl}(u) : u \in I(b) \sim \{ t \} \} \) is a dense subset of \( \text{int}(\cap b) \sim \text{cl}(t) \).

As the induction hypothesis is clearly satisfied, we can continue it until we have an unbounded tree \( T \). Since \( \psi(t) \in t \forall t \in T, \psi(T) \) is dense if \( T \) is cofinal in \( \mathcal{R}(X) \). So we suppose \( u \in U \) such that \( t \not\subseteq u \forall t \in T \). By 1.2(2) there is a first ordinal \( \beta \in h(T) \) such that \( u \cap t_0, u \cap t_1 \in \mathcal{R}(X) \sim \{ \emptyset \} \) for two distinct elements \( t_0, t_1 \in \text{lv}(T, \beta) \). So \( u \subseteq \cap b \) for a branch \( b \) of \( T(\beta) \), and \( t_0 \cup t_1 \subseteq \cap b \). Since \( U \) is a tree, neither \( t_0 \) nor \( t_1 \) is in \( U \). This contradicts (iii) for \( \alpha = \beta + 1 \). So \( T \) is cofinal in \( \mathcal{R}(X) \).

From (iv) and (v) there is for each \( t \in T \) precisely one branch \( b(t) \) of \( T \) such that \( t \in b(t) \) and \( b(t) \) is a local base at \( \psi(t) \). Thus when \( \{ b(t) : t \in T \} \) inherits the order given in 2.1(2) ⇒ (3), the map \( b(t) \to \psi(t) \) gives \( \psi(T) \) a linear order generating the subspace topology inherited from \( X \). □
2.4. COROLLARY [Wh]. A first countable space has a σ-disjoint π-base iff it has a dense metrizable subspace.

PROOF. In 2.3 each \( \mathcal{R}(x) \) can be assumed to have order type \( \omega \), and, according to 2.2, \( h(U) = \omega \). So \( h(T) = \omega \). Following the proof of 2.2(2) \( \Rightarrow \) (3) we see that \( \psi(T) \) is metrizable. \( \square \)

As a Souslin line has no dense metrizable subspace, it is consistent with ZFC that “σ-disjoint π-base” cannot be replaced by “\( \mathfrak{g} \)-absolute with a linearly ordered space” in 2.4. However, it is replaceable for the class of first countable spaces \( X \) which have \( \#(\mathcal{R}(X)) \neq \omega \). Therefore, we consider some cardinal functions which affect \( \#(\mathcal{R}(X)) \).

Henceforth, we shall use \( \#[w#] \) to denote \( \#(\mathcal{R}(X))[w#(\mathcal{R}(X))] \) when there is no confusion. A topological translation of 1.5 and 1.9 yields

2.5. LEMMA. Let \( X \) be a space and \( \{I_\alpha: \alpha \in \kappa\} \) be a collection of families of pairwise-disjoint nonempty open sets such that \( \text{cl}(\bigcup I_\alpha) = X \forall \alpha \in \kappa \). If \( \kappa \leq w# \), then there is an unbounded tree \( T \) of \( \mathcal{R}(X) \) satisfying:

1. \( h(T) = \# \).
2. \( s, t \in T, s \nsubseteq t \Rightarrow \text{cl}(s) \subseteq t \).
3. \( \text{cl}(\bigcup I_\alpha) = X \forall \alpha \in w# \).
4. \( T \) satisfies (iv) of 2.1(2) \( \Rightarrow \) (3).
5. \( i \in I_\alpha \Rightarrow \exists t \in I_\alpha \exists t \subseteq i \).

Further, if \( X \) is \( \mathfrak{g} \)-absolute with a linearly ordered space, then \( T \) may be assumed to be cofinal in \( \mathcal{R}(X) \).

Recall [C.N.2] that for a cardinal \( \kappa \), a space \( X \) is \( \kappa \)-Baire if the intersection of at most \( \kappa \) many open dense subsets of \( X \) is dense [so Baire = \( \omega \)-Baire]; and \( x \in X \) is a \( P_\kappa \)-point if it has a local base \( \lambda \)-closed \( \forall \lambda < \kappa \) [so \( P \)-point = \( P_\omega \)-point]. \( X \) is an \( \kappa \)-point \( \kappa \)-space [Le] if the intersection of less than \( \kappa \) many nonempty open subsets of \( X \) has nonempty interior (equivalently, if \( \mathcal{R}(X) \) is cofinally \( \lambda \)-closed \( \forall \lambda < \kappa \)). A base (or \( \pi \)-base) for a space will be called \( \kappa \)-disjoint when it is the union of a collection of \( \kappa \) many families of pairwise disjoint sets (WLOG each family may be assumed to have union dense in \( X \)).

The following should be compared to [C.N.1, 3.1] and [C.N.2, 6.15].

2.6. THEOREM. For a space \( X \) with a \( \kappa \)-disjoint \( \pi \)-base \( B \),

1. \( \kappa \geq \# \) (so \( \pi w(X) \geq \# \)),
2. if \( \kappa = w# \), \( X \) is \( \kappa \)-Baire, and if \( B \) is a base, then \( X \) has a dense subset of \( P_\kappa \)-points which is linearly orderable.

PROOF. (1) From 1.6 there is a family \( J \) of pairwise-disjoint nonempty regular-open subsets of \( X \) such that \( \bigcup J \) is dense in \( X \), \( w#(\mathcal{R}(G)) = \#(\mathcal{R}(G)) \forall G \in J \), and \( \# = \sup(\#(\mathcal{R}(G))): G \in J \). From 2.5(5) \( \kappa \geq \#(\mathcal{R}(G)) \) for each \( G \).

(2) Let \( B = \bigcup \{I_\alpha: \alpha \in \kappa\} \), where each \( I_\alpha \) is pairwise-disjoint and has dense union. Let \( T \) be the tree guaranteed in 2.5 and set \( D = \{ \cap b: b \text{ is a branch of } T, \text{ord}(b) = \kappa, \text{ and } \cap b \neq \emptyset \} \). Since \( X \) is \( \kappa \)-Baire, 2.5(3) implies \( \bigcup D \) is dense. Since \( B \) is a base and \( T \) is a tree, 2.5(5) implies \( b \) is a well-ordered local base at \( \cap b \) and
| ∩ b | = 1 whenever ∩ b ∈ D. To see that ∪ D is linearly orderable, follow the last paragraph of 2.3. □

2.7. Theorem. Suppose X is a space whose diagonal is the intersection of \( \kappa \) many open subsets of \( X \times X \); then \( \kappa \geq \# \). Further, if \( \kappa = w\# \), \( X \) is \( \kappa \)-Baire, and if \( X \) is the intersection of at most \( \kappa \) many open subsets of \( \beta X \), then \( X \) has a dense subset of \( \mathcal{P}_\kappa \)-points which is linearly orderable.

**Proof.** Let \( \{ \tilde{c}(\alpha): \alpha \in \kappa \} \) be a collection of open sets of \( X \times X \) whose intersection is \( \Delta = \{ (x, x): x \in X \} \). Given \( x \in X \) and \( \alpha \in \kappa \), there is a nbhd \( G \) of \( x \) such that \( G \times G \subseteq \tilde{c}(\alpha) \). Hence, for each \( \alpha \in \kappa \) there is a pairwise-disjoint collection \( I_\alpha \subseteq X(X) \) such that \( \cl(\bigcup I_\alpha) = X \) and \( G \times G \subseteq \tilde{c}(\alpha) \forall G \in I_\alpha \). Let \( T \) be the tree of 2.5 constructed from \( \{ I_\alpha: \alpha \in \kappa \} \). If \( \lambda < w\# \), then 2.5(5) implies \( \Delta \neq \cap \{ \tilde{c}(\alpha): \alpha \in \lambda \} \). Following 2.5(1) shows \( \# \leq \kappa \).

For the further, let \( \{ H(\alpha): \alpha \in \lambda \} \) be a family of open subsets of \( \beta X \) whose intersection is \( X \), and suppose \( \lambda \leq \kappa \). For \( \alpha \in \kappa \setminus \lambda \) set \( H(\alpha) = \beta X \). For each \( \alpha \in \kappa \), let

\[
J_\alpha = \{ H(\alpha) \cap \text{int}_{\beta X}(\cl_{\beta X}(G)): G \in I_\alpha \}.
\]

Let \( S \subseteq X(\beta X) \) be the tree of 2.5 constructed for \( \beta X \) from \( \{ J_\alpha: \alpha \in \kappa \} \). Using 2.5(2) we see \( \cap b \) is a nonempty compact subset of \( \beta X \) whenever \( b \) is a branch of \( S(\eta) \) for \( 0 < \eta \leq \kappa \). So \( b \) is always a local base for \( \cap b \) when \( \eta \) is a limit ordinal. Since \( \cap \{ H(\alpha): \alpha \in \lambda \} = X \), 2.5(5) implies \( \cap b \subseteq X \) whenever \( b \) is a branch of \( S(\eta) \) for \( \lambda = \eta \leq \kappa \). Since \( \Delta = \cap \{ \tilde{c}(\alpha): \alpha \in \kappa \} \), 2.5(5) implies \( | \cap b | = 1 \) whenever \( b \) is a branch of \( S \) and \( \text{ord}(b) = \kappa \). As \( X \) is \( \kappa \)-Baire, 2.5(3) implies \( \{ x \in \cap b: b \text{ is a branch of } S, \text{ord}(b) = \kappa \} \) is dense in \( X \). Now follow the last paragraph of 2.3. □

2.8. Corollary. A Moore space is \( \mathfrak{G} \)-absolute with a linearly ordered space iff it has a dense metrizable subspace.

**Proof.** A Moore space is 1st countable so 2.3 applies. As it also has a \( G_\delta \)-diagonal 2.7, and hence 2.2, applies. □

2.9. Corollary. For a Čech-complete space \( X \), the following hold:

(1) If \( \#(\mathfrak{G}(X)) > \omega \), then \( X \) has a dense open locally compact subspace.

(2) If \( \#(\mathfrak{G}(X)) > \omega \) and \( X \) is perfectly normal, then \( X \) is not \( \omega_1 \)-Baire.

(3) If \( X \) has a \( G_\delta \)-diagonal, then \( X \) has a dense metrizable linearly orderable \( G_\delta \)-set.

**Proof.** (1) Consider \( \text{lv}(T, \omega) \) in the “further” of 2.7. (2) The Pressing Down Lemma implies \( \text{lv}(T, \omega_1) = \emptyset \) in the “further”. (3) This follows immediately from the “further” and 2.2. □

Recall [Jul] the cardinal \( \kappa \) is a *caliber* for a space \( X \) if each collection of \( \kappa \) many nonempty open subsets of \( X \) contains a centered subfamily of cardinality \( \kappa \). [Jul, A2.2] shows that if \( X = *^2 \) is given the Tychonov product topology, then \( \omega_1 \) is a caliber for \( X \) whenever \( \kappa \geq \omega_1 \).

Suppose \( X \) is a space for which \( \kappa \) is a caliber; if \( Y \) is the image of \( X \) under a continuous surjection and \( T \) is an unbounded tree of \( \mathfrak{G}(Y) \) satisfying 2.5(2) for \( Y \), then \( | T | < \kappa \) since the inverse image of \( T \) satisfies 2.5(2).
The next theorem is essentially the theme of [Gv]. The preceding paragraph yields for us a shorter proof.

2.10. Theorem. A dyadic space is $\Theta$-absolute with a linearly ordered space iff it is separable and metrizable.

Proof. (\(\Rightarrow\)) \(D\) is a dyadic space, if, by definition, \(D\) is a dense subset of a continuous image \(Y\) of \(2^\omega\), where, by [Ju1, 4.9] \(\kappa = \pi w(Y) = w(Y)\). From 2.1 and the above \(\kappa = \omega\) and so \(Y\) is a compact metric. \(\square\)

2.11. Corollary [Po]. If a dyadic space is coabsolute with a metric space, it is metrizable.

2.12. Remarks. (1) A space is non-Archimedean (see [Ny] for a survey) provided it has a base in which every pair of members are either related by \(\subseteq\) or disjoint. By virtue of 1.2(4), these are spaces having inverted trees as bases. Observe that the space \(L\) of 2.1(2) \(\Rightarrow\) (3) and \(\psi(T)\) of 2.3 are non-Archimedean, while the dense subspaces in 2.6 and 2.7 are actually \(\kappa\)-metrizable. A technical modification of 2.1(2) \(\Rightarrow\) (3) shows that “linearly ordered” in 2.1(3) can be replaced by “subspace of linearly ordered”.

(2) Is there (in ZFC) a compact first countable space not the compactification of a linearly ordered space? We observe that the Pressing Down Lemma shows that if \(X\) is a Souslin line, then \(X \times [0, 1]\) has no dense linearly ordered subspace.

(3) In [Ta] methods for recognizing Souslin trees in topologies are investigated, and equivalences and implications of SH are given. We observe (from 1.6 and 2.9) that SH is equivalent to the statement “if \(X\) is an \(\omega_1\)-Baire Čech-complete perfectly normal space, then \(\#(\mathcal{R}(X)) \leq \omega\)”.

(4) Does every compact almost P-space of weight \(2^\omega\) contain a dense linearly ordered subspace? From [C.N.I] the answer is yes if CH is assumed. From 1.12 and 2.3, a compact almost P-space of \(\pi\)-weight \(2^\omega\) contains a dense non-Archimedean linearly ordered subspace whenever it contains a dense set of points with well-ordered local bases. The last condition is necessary as every linearly ordered subspace of \(\mathcal{S}(\beta N \sim N)\) contains no isolated points.

(5) 2.7 (for \(\kappa = \omega\)) and 2.8 were originally proved independently of White’s result 2.4; however, 2.4 motivated 2.3. We observe that “Čech-complete” in 2.9 can be replaced by “\(p\)-space in the sense of Ar’hangelskii”, and a similar generality works in 2.7.

3. Coabsolutes of Stone-Čech remainders. If \(Z\) is a zero set of a completely regular space \(X\), then we let \(Z^* = \{x \in \beta X \sim X : Z \subseteq x\}\). So \(X^* = \beta X \sim X\). It is well known (see [En]) that \(X^*\) can be nearly anything for a suitable pseudocompact space, and if \(X^*\) is dyadic, then \(X\) is pseudocompact. Thus, 2.10 especially encourages us to restrict our attention to a class of spaces in which every pseudocompact closed subspace is compact—in this case we consider the class of spaces with a compatible complete uniformity (which we will call complete spaces).\(^4\) Further, if \(X\) is nowhere

\(^4\)Shirota’s theorem says that when we assume no measurable cardinals exist, a space is complete iff it is realcompact. So the reader may wish to replace “complete” with “realcompact” throughout this section.
locally compact, then \( X^* \) and \( X \) are dense in \( \beta X \). Therefore, we restrict our attention to complete locally compact noncompact spaces. With minor changes in proof the following is \([Wo2, 3.2]\):

### 3.1. Lemma
If \( X \) is complete, then \( X^* \) and \((\mathcal{E}(X))^*\) are coabsolute.

### 3.2. Lemma
If \( X \) is a locally compact extremally disconnected noncompact space, then \( \exists \kappa \geq 2^\omega \) and a family \( \{D(\alpha): \alpha \in \kappa\} \) satisfying:

1. Each \( D(\alpha) \) is the union of countably many pairwise-disjoint compact open subspaces of \( X \).
2. Either \( D(\alpha) \cong N \) or \( D(\alpha) \) has no isolated points.
3. \( \beta < \alpha \Rightarrow D(\beta)^* \cap D(\alpha)^* = \emptyset \).
4. \( X = \text{cl}(\cup \{D(\alpha)^*: \alpha \in \kappa\}) \) and each \( D(\alpha)^* \) is open in \( X^* \).

**Proof.** Let \( E(0) = \{ \{x\}: \{x\} \text{ is open in } X \} \) and choose, by local compactness, a maximal collection \( E(1) \) of pairwise disjoint nonempty open compact members of \( \mathfrak{R}(\text{int}(X \sim \cup E(0))) \). For each \( n \in \{0,1\} \) let \( \{e(\alpha, n): \alpha \in \kappa(n)\} \) be a listing of \( E(n) \) with a cardinal \( \kappa(n) \).

If \( \kappa(n) \) is finite, let \( I(n) = \emptyset \). If \( \kappa(n) \) is infinite, then choose \( I(n) \) to be a (maximal almost-disjoint) family of countably infinite subsets of \( \kappa(n) \) maximal w.r.t. “every intersection of distinct members is finite”. We may choose \( |I(n)| \geq 2^\omega \) since it is well known \([Ru]\) that \( \omega \) contains a maximal almost-disjoint family of cardinal \( 2^\omega \).

The desired family will be

\[ \{D(i, n): i \in I(n), n \in \{0,1\}\}, \quad \text{where } D(i, n) = \bigcup \{e(\alpha, n): \alpha \in \kappa(n)\}. \]

Its cardinality is at least \( 2^\omega \) since \( X \) is not compact. (1) and (2) are certainly satisfied. (3) follows since \( I(n) \) is almost-disjoint or empty and \( (\cup E(0)) \cap (\cup E(1)) = \emptyset \). If \( C \) is clopen in \( X \) and \( \emptyset \neq C^* \), then \( \exists n \in \{0,1\} \) and a countably infinite set \( j \subseteq \kappa(n) \) such that \( C \cap e(\alpha, n) \neq \emptyset \) for each \( \alpha \in j \) (otherwise, \( \cup (E(0) \cup E(1)) \) would not be dense in \( X \)). As \( I(n) \) is infinite and a maximal almost-disjoint family, \( i \cap j \) is infinite for some \( i \in I(n) \). (4) follows since

\[ \emptyset \neq \left( \bigcup \{C \cap e(\alpha, n): \alpha \in i \cap j\}\right)^* \subseteq C^*D(i, n)^*. \]

### 3.3. Lemma \([F.G., 3.1]\)
If \( X \) is realcompact and locally compact, then \( X^* \) is an almost \( P_{\omega_1} \)-space.

### 3.4. Theorem
If \( X \) is a complete locally compact noncompact space each of whose nonempty open sets contains a nonempty open set of \( \pi \)-weight at most \( 2^\omega \), then \( X^* \) is coabsolute with a linearly ordered space having a dense set of \( P \)-points.

**Proof.** From 3.1 we may assume \( X \) is extremally disconnected, locally compact, noncompact with the same \( \pi \)-weight conditions. Thus, in 3.2 we may assume \( \pi w(e(\alpha, n)) \leq 2^\omega \), and hence, from extremal disconnectedness \( 2^\omega \leq \pi w(D(\alpha, i)^*) \leq (2^\omega)^\omega = 2^\omega \). Since each \( D(\alpha, i) \) is \( \sigma \)-compact, each \( D(\alpha, i)^* \) is an almost \( P_{\omega_1} \)-space. From 1.3(2) and 1.12, each \( \mathcal{R}(D(\alpha, i)^*) \), and hence, \( \mathcal{R}(X^*) \) has a cofinal tree \( T \) whose branches fail to have countable cofinality. The desired space is the Dedekind completion (with endpoints) of the space \( L \) in 2.1(2) \( \Rightarrow (3) \).
As each locally compact metric space is the free union of $\sigma$-compact spaces which must have $\pi$-weight at most $\omega$, and as every locally compact metric space admits a complete uniformity, we have shown

3.5. **Corollary.** A locally compact noncompact metric space $X$ has $X^*$ coabsolute with a linearly ordered space having a dense set of $P$-points.

3.6. **Lemma.** If $X$ is a locally compact, extremally disconnected, noncompact space with $\pi w(X) = \omega$, then $X^*$ is homeomorphic to one of $N^*$, $(C(R))^*$, or $N^* + C(R)^*$.

**Proof.** Following the proof of 3.2, $\pi w(X) = \omega$ yields $X = E(0) \cup E(1)$, where $E(0)$ is the closure of all of the at most $\omega$ isolated points of $X$ and $E(1) = X \sim E(0)$. As $X$ is extremally disconnected $X^* = E(0)^* \cup E(1)^*$.

When $E(0)$ is not compact, each clopen set free ultrafilter on $E(0)$ traces to an ultrafilter on $\text{int}(E(0)) \cong N$; therefore, $E(0)^* \cong N^*$. When $E(1)$ is not compact, it is the disjoint union of many (since $\pi w(X) = \omega$) compact spaces of $\pi$-weight $\omega$ without isolated points. Applying the same argument to $R$, we have $E(1) \cong C(R)$. D

3.7. **Lemma.** If $K$ is a compact space and if, for each $n \in \{0,1\}$, $X(n)$ is the free union of a cardinal $\kappa(n)$ many copies of $K$, where $\omega \leq \kappa(n) \leq 2^\omega$, then $X(0)^*$ and $X(1)^*$ contain homeomorphic dense open subspaces.

**Proof.** Write $X(n) = \Sigma\{K(\alpha): \alpha \in \kappa(n)\}$, where $K(\alpha) \cong K$ for each $\alpha$. Since $\kappa(n) \leq 2^\omega$, $\kappa(n)$ contains precisely $2^\omega$ countably infinite subsets. Hence, following the argument in the proof of 3.2, we may choose a maximal almost-disjoint family $I(n)$ of countably infinite subsets of $\kappa(n)$ such that $|I(n)| = 2^\omega$. Then

$$\left\{\left\{x \in X(n)^*: \left(\sum\{K(\alpha): \alpha \in i\} \sim K(\gamma)\right) \in x \forall \gamma \in i\right\}: i \in I(n)\right\}$$

is a pairwise-disjoint family of clopen subsets of $X(n)^*$ whose union is dense in $X(n)^*$ and whose members are homeomorphic to $(\Sigma\{K(\alpha): \alpha \in \omega\})^*$. D

3.8. **Theorem.** If $X$ is a complete locally compact noncompact space of $\pi$-weight at most $2^\omega$, and if every nonempty open set of $X$ contains a nonempty open set of countable $\pi$-weight, then $X^*$ is coabsolute with one of $N^*$, $R^*$, or $N^* + R^*$.

**Proof.** Following the proof of 3.4, we observe that each $D(\alpha, i)$ may also be assumed here to have countable $\pi$-weight, and from their construction in 3.2 there are precisely $2^\omega$ of the sets $D^*(\alpha, i)$ each of which is homeomorphic to one of the three spaces above by 3.6. Now apply 3.7 with $K = N^*$ and $K = R^*$. D

3.9. **Corollary.** Suppose $X$ is a locally compact noncompact metric space of density at most $2^\omega$; then $X^*$ is coabsolute with

1. $N^*$, if $X$ has a dense discrete subspace,
2. $R^*$, if the set of isolated points of $X$ has compact closure,

In [Wo1, Wo2] it is shown that if $CH$ is assumed $X^*$ is coabsolute with $N^*$ whenever $X$ is locally compact, noncompact, and either metric of density at most $2^\omega$
or with $|\mathcal{R}(X)| = 2^\omega$; however, this follows from 1.14 which shows $\text{CH}$ implies $\mathcal{R}(\mathcal{N}^*) = \mathcal{R}(Y)$ whenever $Y$ is an almost $P$-space with $\pi\omega(Y) = 2^\omega$ and no isolated points. We end this section with an example which shows $2^\omega$ is essential in 3.4 and which allows us to remove $\text{CH}$ from the hypothesis of Woods' results.

3.10. Example. Suppose $\kappa > \omega$ is a cardinal and $\mathfrak{D}(\kappa) = \Sigma\{\mathfrak{D}(\kappa, n): n \in \omega\}$, where each $\mathfrak{D}(\kappa, n) \cong 2^2$ given the Tychonov product topology. If $K$ denotes the linearly ordered space obtained from ordering $\omega \cdot 2$ lexicographically, then the following are equivalent:

1. $\kappa \leq 2^\omega$.
2. $\mathfrak{D}(\kappa)^*$ and $K$ are coabsolute.
3. $\mathfrak{D}(\kappa)^*$ is coabsolute with a linearly ordered space.

Proof. For each ordinal $\alpha < \omega_1$, we set

$$A(\alpha) = \{Z(f): \text{dom}(f) = \alpha\}$$

and $L(\alpha) = \{\text{int}(\bigcup\{Z(f, n): n \in \omega\}^*): \forall n \in \omega, Z(f, n) \in A(\alpha) \text{ and } Z(f, n) \subseteq D(\kappa, n)\}$. Then $T = \{L(\alpha): \alpha \in \omega_1\}$ is a tree in $\mathcal{R}(\mathfrak{D}(\kappa)^*)$ whose $\alpha$th level is $L(\alpha)$. We claim that $T$ is an unbounded tree.

First we observe that $\mathfrak{D}(\kappa)^*$ has a $\pi$-base $P(\kappa)$ of sets of the form $\text{int}(Z^*)$ such that for each $n \in \omega \exists h_n \in 2^2$ and a countable set $C_n \subseteq \kappa$ such that $g \in Z \cap \mathfrak{D}(\kappa, n)$ iff $g | C_n = h_n$. So if $\alpha < \omega_1$ is the first ordinal with $\omega_1 \cap \bigcup\{C_n: n \in \omega\} \subseteq \alpha$, then $\text{int}(Z^*)$ intersects two elements of $L(\alpha + 1)$. From 1.2(1), $T$ is unbounded.

$(1) \Rightarrow (2)$. From 1.9 and 1.12 $\mathcal{R}(\mathfrak{D}(\kappa)^*)$ has a cofinal tree of height $\omega_1$ since $\pi\omega(\mathfrak{D}(\kappa)^*) = 2^\omega$. From 3.3 it has a cofinal tree order isomorphic to the union of the limit ordinal levels of $\text{TREE}(\omega_1)$. If $L$ is constructed as in 2.1(2) $\Rightarrow (3)$, then $L$ is homeomorphic to a dense subspace of $K$.

$(2) \Rightarrow (3)$. Obvious.

$(3) \Rightarrow (1)$. As $\mathfrak{D}(\kappa)$ has caliber $\omega_1$, it follows that every pairwise-disjoint family of $P(\kappa)$ has cardinality at most $2^\omega$. On the other hand, $\pi\omega(\mathfrak{D}(\kappa)^*)$ is at least $2^\omega + \kappa$. So if $\mathfrak{D}(\kappa)^*$ is coabsolute with a linearly ordered space, then $\mathcal{R}(\mathfrak{D}(\kappa)^*)$ has a cofinal tree $S$ of height $\omega_1$. Since $|S| = 2^\omega \cdot \omega_1 = 2^\omega$ and $S$ is a $\pi$-base of $\mathfrak{D}(\kappa)^*, \kappa \leq 2^\omega$. $\Box$

3.11. Lemma. If $X$ is a locally compact complete noncompact space, then $\#(\mathcal{R}(X^*)) \leq \#(\mathcal{R}(\mathcal{N}^*))$.

Proof. From 3.1 and 3.2 $X = \Sigma\{X(n): n \in \omega\}$, where each $X(n)$ is compact and extremally disconnected, can be assumed. One readily observes that the map $X(n) \to n$ induces a function from $\mathcal{R}(\mathcal{N}^*)$ to $\mathcal{R}(X^*)$ which takes unbounded trees to unbounded trees isomorphic to their pre-images. $\Box$

3.12. Theorem. The following are equivalent statements in $\text{ZFC}$:

1. $\mathcal{N}^*$ and $D(\omega_1)^*$ are coabsolute.
2. $\#(\mathcal{R}(\mathcal{N}^*)) = \omega_1$.
3. If $X$ is a locally compact complete noncompact space and if $\pi\omega(X) \leq 2^\omega$, then $X^*$ and $\mathcal{N}^*$ are coabsolute.
Proof. Since \( \#(\mathcal{F}(\omega_1^*)) = \omega_1 \), (1) \( \Rightarrow \) (2). From 3.11 \( \#(\mathcal{R}(X^*)) = \omega_1 \) so (2) \( \Rightarrow \) (3) is similar to 3.10 (3) \( \Rightarrow \) (1). (3) \( \Rightarrow \) (1) is obvious. \( \Box \)

As we have observed before CH implies \( \#(\mathcal{R}(N^*)) = \omega_1 \). However, as remarked in [B.P.S.] there are numerous models of ZFC + \( \neg \)CH in which \( \#(\mathcal{R}(N^*)) = \omega_1 \), for example in any model \( \mathcal{M} \) for which \( \mathcal{M} + \omega = \omega_1 \) to have an \( \omega_1 \)-scale [He] and \( \mathcal{M} + \omega_1 < 2^\omega \). On the other hand, MA + \( \neg \)CH implies \( \mathcal{R}(N^*) \) is cofinally \( \kappa \)-closed \( \forall \kappa < 2^\omega \) [Ru], and hence, \( \#(\mathcal{R}(N^*)) = 2^\omega > \omega_1 \). Therefore, we have

3.13. Corollary. The following statement is implied by CH, and is consistent with and independent of \( \neg \)CH: If \( X \) is a locally compact complete noncompact space with \( \pi w(X) \leq 2^\omega \), then \( X^* \) and \( N^* \) are coabsolute.

3.14. Remarks. (1) 3.7 was communicated verbally to the author by S. Broverman for the case \( |K| = 1 \). The proof he gave is similar. R. G. Woods has informed us that 3.9(2) can also be proved using a recent result of E. K. van Douwen on remote points and a theorem in [Wo3].

(2) Are \( N^* \) and \( R^* \) coabsolute? As \( N^* \) is an almost \( P_\kappa \)-space iff \( R^* \) is an almost \( P_\kappa \)-space [vD1], we observe that a proof similar to that of 3.12 (2) \( \Rightarrow \) (3) shows \( N^* \) and \( R^* \) are coabsolute whenever \( N^* \) is an almost \( P_\kappa \)-space \( \forall \kappa < \#(\mathcal{R}(N^*)) \). The latter is true in a variety of situations (including MA). However, we do not know whether a negative answer is consistent.

(3) 3.10 was motivated by an example in [vD.vM.] called here \( \mathcal{D}(2^\omega) \).

(4) In the first draft of this paper we used a result in [vD2] to show 3.11. Originally we obtained 3.13 prior to 1.12 (and independently of [B.P.S.]); however, our proof was longer while a key lemma to our short proof in the first draft of this paper was false.

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Added in proof. The answer to the questions in 2.12(2) is "yes" (Todorčević); in 2.12(4) is "not \( \mathcal{D}(\omega_2) \)" (myself) and it is consistent that \( X^* \) has no dense linearly ordered subspace for every non-pseudo-compact space \( X \).

References


[vD1] E. K. van Douwen, Martin's axiom and pathological points in $\beta X \sim X$, unpublished.
[vD2] ______, Transfer of information about $\beta N \sim N$ via open remainder maps, preprint.

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