Abstract. It is proved that Fourier series with asymptotically even coefficients and satisfying \( \lim_{\lambda \to \infty} \limsup_{n \to -\infty} \sum_{j=1}^{\lambda n} j^{p-1} |\hat{f}(j)|^p = 0 \), for some \( 1 < p \leq 2 \), converge in \( L^1 \)-norm if and only if \( \| \hat{f}(n)E_n + \hat{f}(-n)E_{-n} \| = o(1) \), where \( E_n(t) = \sum_{k=0}^{n} e^{ikt} \). Recent results of Stanojević [1], Bojanic and Stanojevic [2], and Goldberg and Stanojevic [3] are special cases of some corollaries to the main theorem.

1. Introduction. The space \( L^1(T) \) of complex functions integrable on \( T = \mathbb{R}/2\pi \mathbb{Z} \) does not admit convergence in norm. Consequently, convergence in norm of the partial sums \( S_n(f) = S_n(f, t) = \sum_{|j| \leq n} \hat{f}(j)e^{ijt} \) to \( f \in L^1(T) \) cannot be characterized in terms of Fourier coefficients without additional assumptions about the sequence \( \{ \hat{f}(n) \} \).

In the case of even coefficients \( (\hat{f}(n) = \hat{f}(-n) \) for all integers \( n \) satisfying certain regularity and/or speed conditions, it is well known that

\[
\| S_n(f) - f \| = o(1), \quad n \to \infty,
\]

is equivalent with

\[
\hat{f}(n)\log n = o(1), \quad n \to \infty.
\]

(A survey of classical and recent results of this kind can be found in [1, 2 and 3].)

Most recent results concerning the equivalence between (1.1) and (1.2) are due to Stanojević [1], Bojanic and Stanojevic [2] and Goldberg and Stanojevic [3].

In [1] it is proved that if \( \{ \hat{f}(n) \} \) is even and satisfies

\[
\frac{1}{n} \sum_{k=1}^{n} k|\Delta \hat{f}(k)| = o(1), \quad n \to \infty,
\]

and

\[
n|\Delta \hat{f}(n)| = O(1), \quad n \to \infty,
\]

then (1.1) is equivalent with (1.2).

Goldberg and Stanojević [3] proved that if

\[
\{(\hat{f}(n) - \hat{f}(-n))\log n\} \text{ is a null-sequence of bounded variation,}
\]

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and if for some $1 < p \leq 2$

$$\frac{1}{n} \sum_{j=n}^{2n} j^p |\Delta \hat{f}(j)|^p = o(1), \quad n \to \infty,$$

then (1.1) if and only if (1.2). An earlier result of Bojanic and Stanojević [2] is a corollary to the Goldberg-Stanojević theorem.

In this paper I shall extend and generalize the Goldberg-Stanojević theorem in two ways. Instead of (1.5), a weaker condition will be assumed, i.e.,

$$\frac{1}{n} \sum_{j=1}^{n} |\hat{f}(j) - \hat{f}(-j)| \lg j = o(1), \quad n \to \infty,$$

(AE)

$$\lim_{n \to \infty} \limsup_{j \to n} \sum_{j=n}^{\lambda \cdot n} |\Delta(\hat{f}(j) - \hat{f}(-j))| \lg j = 0,$$

and (1.6) will be relaxed as follows:

(HK) $$\lim_{\lambda \to 1} \limsup_{n \to \infty} \sum_{j=n}^{\lambda \cdot n} j^{p-1} |\Delta \hat{f}(j)|^p = 0,$$

for some $1 < p \leq 2$.

A sequence of complex numbers satisfying (AE) is called asymptotically even. Clearly, every even sequence satisfies (AE).

The condition (HK) is a Tauberian condition of Hardy-Karamata [4] kind. Plainly (1.6) implies (HK).

As a consequence of the main theorem it will follow that the condition (1.3) is superfluous, and that (1.4) can be weakened if a certain speed of $\|\sigma_n(f) - f\|$ is assumed, where $\sigma_n(f)$ is the Fefér sum of $S_n(f)$.

2. Main theorem. Fourier series considered throughout this section are the series with asymptotically even coefficients. That is

$$\frac{1}{n} \sum_{j=1}^{n} |\hat{f}(j) - \hat{f}(-j)| \lg j = o(1), \quad n \to \infty,$$

(2.1.1)

$$\lim_{\lambda \to 1} \limsup_{n \to \infty} \sum_{j=n}^{\lambda \cdot n} |\Delta(\hat{f}(j) - \hat{f}(-j))| \lg j = 0.$$

Main Theorem. Let $S[f] \sim \sum_{|n| \to \infty} \hat{f}(n)e^{int}$ be the Fourier series of $f \in L^1(T)$ with asymptotically even coefficients.

If for some $1 < p \leq 2$

(HK) $$\lim_{\lambda \to 1} \limsup_{n \to \infty} \sum_{j=n}^{\lambda \cdot n} j^{p-1} |\Delta \hat{f}(j)|^p = 0,$$

then $\|S_n(f) - f\| = o(1), n \to \infty$, if and only if

$$\|\hat{f}(n)E_n + \hat{f}(-n)E_{-n}\| = o(1), \quad n \to \infty,$$

where $E_n(t) = \sum_{k=0}^{n} e^{ikt}$. 

Proof. It suffices to show that

$$
(2.3) \limsup_{n \to \infty} \left| \left| S_n(f) - f \right| - \left| \hat{f}(n)E_n + \hat{f}(-n)E_{-n} \right| \right| = 0.
$$

Let $\lambda > 1$ and $n > 1$. Then the following identity can be established:

$$
S_n(f, t) - f(t) - (\hat{f}(n)E_n(t) + \hat{f}(-n)E_{-n}(t))
$$

$$
= \left[ \frac{\lambda n}{[\lambda n]} - n \right] \left[ \sigma_{[\lambda n]}(f, t) - f(t) \right] - \frac{n + 1}{[\lambda n]} \left[ \sigma_n(f, t) - f(t) \right]
$$

$$
\quad - \frac{1}{[\lambda n]} - n \sum_{j=1}^{[\lambda n]} \frac{1}{\lambda n} \left[ \Delta \hat{f}(j) \right] E_j(t)
$$

$$
\quad - \frac{1}{[\lambda n]} - n \sum_{j=1}^{[\lambda n]} \frac{1}{\lambda n} \left[ \Delta \hat{f}(-j) \right] E_{-j}(t).
$$

The Dirichlet kernel can be written as

$$
D_j(t) = E_j(t) + E_{-j}(t) - 1.
$$

Thus the third and the fourth terms on the right-hand side of (2.4) can be grouped in the following way:

$$
I_{1n} = \frac{1}{[\lambda n]} - n \sum_{j=1}^{[\lambda n]} \hat{f}(j)D_j(t)
$$

$$
\quad - \frac{1}{[\lambda n]} - n \sum_{j=1}^{[\lambda n]} (\hat{f}(j) - \hat{f}(-j))E_{-j}(t) + \frac{1}{[\lambda n]} - n \sum_{j=1}^{[\lambda n]} \hat{f}(j);
$$

and the fifth and the sixth terms as

$$
I_{2n} = \frac{1}{[\lambda n]} - n \sum_{j=1}^{[\lambda n]} \left[ \frac{\lambda n}{[\lambda n]} + \frac{1}{\lambda n} - 1 \right] \frac{\Delta \hat{f}(j)}{\Delta \hat{f}(j)} D_j(t)
$$

$$
\quad - \frac{1}{[\lambda n]} - n \sum_{j=1}^{[\lambda n]} \left[ \frac{\lambda n}{[\lambda n]} + \frac{1}{\lambda n} - 1 \right] \frac{\Delta \hat{f}(j) - \hat{f}(-j)}{\Delta \hat{f}(j) - \hat{f}(-j)} E_{-j}(t)
$$

$$
\quad + \frac{1}{[\lambda n]} - n \sum_{j=1}^{[\lambda n]} \left[ \frac{\lambda n}{[\lambda n]} + \frac{1}{\lambda n} - 1 \right] \Delta \hat{f}(j).
$$
Taking the norm of both sides of (2.5) we obtain

\[ \|I_{1n}\| \leq \frac{1}{[\lambda n] - n} \left\| \sum_{j=n}^{[\lambda n]} \hat{f}(j) D_j(t) \right\| \]

\[ + \frac{1}{[\lambda n] - n} \sum_{j=n}^{[\lambda n]} |\hat{f}(j) - \hat{f}(-j)| \log j \]

(2.7)

\[ + \frac{1}{[\lambda n] - n} \sum_{j=n}^{[\lambda n]} |\hat{f}(j)| \]

\[ = J_n + \frac{[\lambda n]}{[\lambda n] - n} \left( \frac{1}{[\lambda n]} \sum_{j=1}^{[\lambda n]} |\hat{f}(j) - \hat{f}(-j)| \log j \right) \]

\[ + \frac{[\lambda n]}{[\lambda n] - n} \left( \frac{1}{[\lambda n]} \sum_{j=1}^{[\lambda n]} |\hat{f}(j)| \right). \]

Applying first the Hölder inequality and then the Hausdorff-Young equality to \( J_n \), we have

\[ J_n \leq A_p \frac{[\lambda n]}{[\lambda n] - n} \left( \sum_{j=n}^{[\lambda n]} |\hat{f}(j)|^p \right)^{1/p} = A_p \frac{[\lambda n]}{[\lambda n] - n} \left( \frac{1}{[\lambda n]} \sum_{j=1}^{[\lambda n]} |\hat{f}(j)|^p \right)^{1/p} \]

where \( A_p \) is an absolute constant depending on \( p \), and \( 1/p + 1/q = 1 \). The last term in (2.7) is majorized by

\[ \frac{[\lambda n]}{[\lambda n] - n} \left( \frac{1}{[\lambda n]} \sum_{j=1}^{[\lambda n]} |\hat{f}(j)|^p \right)^{1/p} \]

Hence

\[ \|I_{1n}\| \leq C_1 \frac{[\lambda n]}{[\lambda n] - n} \left( \sum_{j=1}^{[\lambda n]} |\hat{f}(j)|^p \right)^{1/p} \]

\[ + C_2 \frac{[\lambda n]}{[\lambda n] - n} \left( \sum_{j=1}^{[\lambda n]} |\hat{f}(j) - \hat{f}(-j)| \log j \right), \]

where \( C_1 \) and \( C_2 \) are absolute constants.

In a similar manner we obtain

\[ \|I_{2n}\| \leq C_3 \sum_{j=n}^{[\lambda n]} |\Delta(\hat{f}(j) - \hat{f}(-j))| \log j + C_4 \left( \sum_{j=n}^{[\lambda n]} j^{p-1} |\Delta \hat{f}(j)|^p \right)^{1/p} \]

where \( C_3 \) and \( C_4 \) are absolute constants.
Combining estimates for both $\|I_{1n}\|$ and $\|I_{2n}\|$ we get

$$
\|S_n(f) - f\| - \|\hat{f}(n)E_n + \hat{f}(-n)E_{-n}\| 
\leq \frac{[\lambda n]}{[\lambda n] - n} \|\sigma_{[\lambda n]}(f) - f\| 
+ \frac{n + 1}{[\lambda n] - n} \|\sigma_n(f) - f\| + C_1 \frac{[\lambda n]}{[\lambda n] - n} \left( \frac{1}{[\lambda n]} \sum_{j=1}^{[\lambda n]} |\hat{f}(j)|^p \right)^{1/p}
$$

(2.8)

$$
+ C_2 \frac{[\lambda n]}{[\lambda n] - n} \left( \frac{1}{[\lambda n]} \sum_{j=1}^{[\lambda n]} |\hat{f}(j) - \hat{f}(-j)||\lg j \right)

+ C_3 \sum_{j=1}^{[\lambda n]} |\Delta(\hat{f}(j) - \hat{f}(-j))||\lg j + C_4 \left( \sum_{j=1}^{[\lambda n]} j^{p-1}|\Delta \hat{f}(j)|^p \right)^{1/p}.
$$

Since for $\lambda > 1$ we have $\lambda n/(\lambda n - n) \sim \lambda/\lambda - 1$, $n \to \infty$, it follows that

$$
\limsup_{n \to \infty} \frac{[\lambda n]}{[\lambda n] - n} C_n = 0,
$$

for any null-sequence $\{C_n\}$.

After taking the limit superior of both sides of (2.8) we get

$$
\limsup_{n \to \infty} \left| \|S_n(f) - f\| - \|\hat{f}(n)E_n + \hat{f}(-n)E_{-n}\| \right|
$$

(2.9)

$$
\leq C_3 \limsup_{n \to \infty} \sum_{j=n}^{[\lambda n]} |\Delta(\hat{f}(j) - \hat{f}(-j))||\lg j

+ C_4 \limsup_{n \to \infty} \left( \sum_{j=n}^{[\lambda n]} j^{p-1}|\Delta \hat{f}(j)|^p \right)^{1/p}.
$$

For $\|\sigma_n(f) - f\| = o(1)$, $n \to \infty$, $\hat{f}(n) = o(1)$, $n \to \infty$, and

$$
\frac{1}{[\lambda n]} \sum_{j=1}^{[\lambda n]} |\hat{f}(j) - \hat{f}(-j)||\lg j = o(1), \quad n \to \infty,
$$

because of (2.1.1).

Taking the limit as $\lambda \to 1$ of both sides of (2.9) we obtain

$$
\lim_{\lambda \to 1} \limsup_{n \to \infty} \left| \|S_n(f) - f\| - \|\hat{f}(n)E_n + \hat{f}(-n)E_{-n}\| \right|
$$

$$
\leq C_3 \lim_{\lambda \to 1} \limsup_{n \to \infty} \sum_{j=n}^{[\lambda n]} |\Delta(\hat{f}(j) - \hat{f}(-j))||\lg j

+ C_4 \lim_{\lambda \to 1} \limsup_{n \to \infty} \left( \sum_{j=n}^{[\lambda n]} j^{p-1}|\Delta \hat{f}(j)|^p \right)^{1/p}.
$$
Because of (2.1.2) and (HK) we finally have (2.3), i.e.

$$\limsup_{n \to \infty} \left| \| S_n(f) - f \| - \| \hat{f}(n) E_n + \hat{f}(-n) E_{-n} \| \right| = 0.$$ 

This completes the proof of the main theorem.

3. Corollaries and additional results. By strengthening either (2.1.2) or (HK), or both, one can obtain a number of corollaries that, as a special case, contain the results of Stanojević [1], Bojanic and Stanojević [2], and Goldberg and Stanojević [3].

The class of complex null-sequences \( \{c_n\} \) satisfying

$$\frac{1}{n} \sum_{k=1}^{n} k|\Delta c_k| = o(1), \quad n \to \infty,$$

includes as a proper subclass null-sequences of bounded variation.

**Corollary 3.1.** Let \( S[f] = \sum_{|n| < \infty} \hat{f}(n)e^{int} \) be the Fourier series of \( f \in L^1(T) \), and let \( \{(\hat{f}(n) - \hat{f}(-n))\lg n\} \) satisfy (3.1). If (HK) holds then

$$\| S_n(f) - f \| = o(1), \quad n \to \infty,$$

if and only if

$$\hat{f}(n)\lg n = o(1), \quad n \to \infty.$$

**Proof.** The condition (2.1.1) is satisfied. It remains to show that (2.1.2) holds. Since for \( \lambda > 1 \)

$$\sum_{j=n}^{[\lambda n]} |\Delta(\hat{f}(j) - \hat{f}(-j))| \lg j \leq \frac{1}{n} \sum_{j=n}^{[\lambda n]} j|\Delta(\hat{f}(j) - \hat{f}(-j))| \lg j$$

$$\leq \frac{\lambda}{[\lambda n]} \sum_{j=1}^{[\lambda n]} j|\Delta(\hat{f}(j) - \hat{f}(-j))| \lg j$$

$$\leq \frac{\lambda}{[\lambda n]} \sum_{j=1}^{[\lambda n]} j|\Delta((\hat{f}(j) - \hat{f}(-j))\lg j) + \frac{\lambda}{[\lambda n]} \sum_{j=1}^{[\lambda n]} |\hat{f}(j) - \hat{f}(-j)| \lg(1 + 1/j)^{-j},$$

it follows that if \( \{(\hat{f}(n) - \hat{f}(-n))\lg n\} \) satisfies (3.1) then (2.1.2) holds.

A special case of Corollary 3.1 is the Goldberg-Stanojević theorem. Indeed, let \( 1 < \lambda \leq 2 \). Then

$$\sum_{j=n}^{[\lambda n]} j^{\lambda - 1}|\Delta \hat{f}(j)|^p \leq \frac{2}{n} \sum_{j=n}^{2n} j^p|\Delta \hat{f}(j)|^p.$$

**Corollary 3.2.** Let \( S[f] = \sum_{|n| < \infty} \hat{f}(n)e^{int} \) be the Fourier series of \( f \in L^1(T) \), and let (2.1.1) hold. If (HK) holds and if

$$n|\Delta(\hat{f}(n) - \hat{f}(-n))| \lg n = O(1), \quad n \to \infty,$$

then (1.1) is equivalent with (1.2).
Proof. Due to (3.2) we have \( \sum_{j=n}^{[\lambda n]} |\Delta(\hat{f}(j) - \hat{f}(-j))| \lg j \leq C \lg \lambda \), where \( C \) is an absolute constant. Hence \( \{\hat{f}(n)\} \) is an asymptotically even sequence.

Since (3.2) is a summability condition in the sense of Hardy [6], from (2.1.1) it follows that

\[
(f(n) - \hat{f}(-n)) \lg n = o(1), \quad n \to \infty.
\]

But (3.3) implies that (2.2) is equivalent with (1.2), for \( \|D_n\| = (4/\pi^2)\lg n + O(1), n \to \infty \).

Corollary 3.3. Let \( S[f] \sim \sum_{|n|<\infty} \hat{f}(n)e^{int} \) be the Fourier series of \( f \in L^1(T) \) with even coefficients. If (HK) holds then (1.1) is equivalent with (1.2).

A special case of this corollary is the main theorem of Bojanic and Stanojevic [2].

Proof. Every even sequence is asymptotically even.

Corollary 3.4. Let \( S[f] \sim \sum_{|n|<\infty} \hat{f}(n)e^{int} \) be the Fourier series of \( f \in L^1(T) \) with even coefficients. If \( n\Delta \hat{f}(n) = O(1), n \to \infty \), then

\[
\|S_n(f) - f\| = o(1), \quad n \to \infty
\]

if and only if

\[
\hat{f}(n) \lg n = o(1), \quad n \to \infty.
\]

Proof. The condition \( n\Delta \hat{f}(n) = O(1), n \to \infty \) implies (HK), for

\[
\sum_{j=n}^{[\lambda n]} j^{p-1} |\Delta \hat{f}(j)|^p \leq C \lg \lambda,
\]

where \( C \) is an absolute constant.

A special case of Corollary 3.4 is the Stanojevic theorem.

In what follows it will be assumed that, for simplicity's sake, \( \{\hat{f}(n)\} \) are even sequences.

All classical conditions as well as (HK) imply that

\[
n^\alpha \Delta \hat{f}(n) = o(1), \quad n \to \infty, \quad \text{for some } 0 < \alpha < 1.
\]

It seems unlikely that (3.4) would imply that (1.1) \( \Leftrightarrow \) (1.2). But a slightly stronger form of (3.4) such as

\[
n^{1/p} \max_{n \leq j \leq [n/\lg n]} |\Delta \hat{f}(j)| = o(1), \quad n \to \infty,
\]

for some \( 1 < p \leq 2 \) and \( 1/p + 1/q = 1 \), and certain conditions on the speed with which \( \|\sigma_n(f) - f\| \) goes to zero as \( n \to \infty \) could imply that (1.1) \( \Leftrightarrow \) (1.2).

Proposition 3.1. Let \( S[f] \sim \sum_{|n|<\infty} \hat{f}(n)e^{int} \) be the Fourier series of \( f \in L^1(T) \) with even coefficients. If, for some \( 1 < p \leq 2 \) and \( 1/p + 1/q = 1 \), (3.5) holds and

\[
\lg n \|\sigma_n(f) - f\| = o(1), \quad n \to \infty,
\]

then (1.1) if and only if (1.2).
PROOF. Let \( m > n > 1 \). Then using the same technique as in the proof of the main theorem one can obtain the inequality

\[
\|S_n(f) - f\| - |\hat{f}(n)| \leq \frac{m+1}{m-n} \|\sigma_m(f) - f\| + \frac{n+1}{m-n} \|\sigma_n(f) - f\|
\]

\( (3.7) \)

\[
+ C_1 \left( \sum_{j=n}^m j^{p-1} |\Delta \hat{f}(j)|^p \right)^{1/p} \left( \frac{m}{n} \right)^{1/q}
\]

\[
+ C_2 \left( \frac{1}{m-n} \sum_{j=n}^m |\hat{f}(j)|^p \right)^{1/p} \left( \frac{m}{m-n} \right)^{1/q},
\]

where \( C_1 \) and \( C_2 \) are absolute constants.

Let \( m = n + \lfloor n/\log n \rfloor \). Then (3.7) becomes

\[
\|S_n(f) - f\| - |\hat{f}(n)| \leq B_1 \left[ \log \left( n + \left\lfloor n/\log n \right\rfloor \right) \right] \|\sigma_n + \lfloor n/\log n \rfloor (f) - f\|
\]

\[
+ B_2 [\log n] \|\sigma_n(f) - f\| + B_3 \left( \sum_{j=n}^{n + \lfloor n/\log n \rfloor} j^{p-1} |\Delta \hat{f}(j)|^p \right)^{1/p}
\]

\[
+ B_4 \left( \frac{1}{n} \sum_{j=n}^{n + \lfloor n/\log n \rfloor} |\hat{f}(j)|^p \right)^{1/p} \log n
\]

where \( B_1, \ldots, B_4 \) are absolute constants.

Due to (3.5) and (3.6), for sufficiently large \( n \) we have

\[
\|S_n(f) - f\| - |\hat{f}(n)| \log n = O \left( n [\log n]^{-1/p} \max_{n < j < n + \lfloor n/\log n \rfloor} |\Delta \hat{f}(j)| \right) + o(1).
\]

This completes the proof of Proposition 3.1.

If instead of \( \lfloor n/\log n \rfloor \) we take a sequence of integers \( \lfloor n/L(n) \rfloor \) where \( L(n) \) is a slowly varying function in the sense of Karamata [5], such that \( L(n + \lfloor n/L(n) \rfloor) \geq L(n) \) for all \( n \) greater than some \( n_0 \), we can obtain a generalization of Proposition 3.1.

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