

ON THE CONSTRUCTION AND DISTRIBUTION OF A LOCAL MARTINGALE WITH A GIVEN ABSOLUTE VALUE

BY

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ABSTRACT. A local martingale is constructed on an appropriate Loeb space whose absolute value equals a given nonnegative local submartingale. Nonstandard analysis is used to reduce the problem to the discrete time setting where the original construction of D. Gilat is fairly simple. This approach has the advantage of allowing explicit computations. In particular, the distribution of the local martingale is described in terms of the Doob-Meyer decomposition of the original local submartingale.

1. Introduction and statement of results. In [6] D. Gilat proved that every nonnegative submartingale x is equal in law to the absolute value of a martingale m . His construction, however, did not shed any light on the nature of this martingale. Several other authors, including Protter and Sharpe [13], Maisonneuve [11], and Barlow and Yor [4] have given more transparent constructions of m on an enlargement of the space on which x is defined (by an enlargement of (Ω, \mathcal{F}, P) we mean a space of the form $(\Omega \times \Omega', \mathcal{F} \times \mathcal{F}', P \times P')$). These constructions have made additional assumptions on x —notably $x(t) \wedge x(t^-) > 0$ for all $t \geq 0$, which is the case in [11 and 13]. Recently M. T. Barlow has shown in [3] that if x is a nonnegative submartingale on $(\Omega, \mathcal{F}, P, \mathcal{F}_t)$ and if there is a random variable on (Ω, \mathcal{F}, P) which has a continuous distribution function and is independent of $\bigvee_{t \geq 0} \mathcal{F}_t$, then there is a filtration $\{\mathcal{N}_t | t \geq 0\}$ and a martingale m on $(\Omega, \mathcal{F}, P, \mathcal{N}_t)$ such that $|m| = x$, $\mathcal{F}_t \subset \mathcal{N}_t$, and every \mathcal{F}_t -martingale is an \mathcal{N}_t -martingale. It is our aim to use nonstandard analysis to give an intuitive construction of a (local) martingale, m , whose absolute value equals a given nonnegative (local) submartingale, x . Our construction was carried out independently of Barlow's work and complements his results since the finite-dimensional distribution of m are described in terms of the Doob-Meyer decomposition of x . The construction uses nonstandard analysis to reduce the continuous time problem to a discrete time problem, where the construction of Gilat is fairly simple.

To state a precise version of the main result we will need the following definition that appears in Aldous [1]. The notation $(\Omega, \mathcal{F}, P, \mathcal{F}_t)$ indicates that $\{\mathcal{F}_t | t \geq 0\}$ is a right-continuous filtration on the complete probability space (Ω, \mathcal{F}, P) and that \mathcal{F}_0 includes all the P -null sets.

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DEFINITION 1.1. Suppose x^i is an adapted process on $(\Omega^i, \mathcal{F}^i, P^i, \mathcal{F}_t^i)$ ($i = 1, 2$) with sample paths in

$$D(M) = \{y: [0, \infty) \rightarrow M \mid y(\cdot) \text{ is right continuous with left limits}\},$$

where M is a Polish space. If $\phi: D(M) \rightarrow \mathbf{R}^d$ is bounded and Borel measurable (give $D(M)$ the Skorokhod J_1 topology as in Billingsley [5] and Stone [14]), let $L_\phi^i(t)$ denote a right-continuous version of $E^i(\phi(x^i) \mid \mathcal{F}_t^i)$ ($i = 1, 2$). Then (x^1, \mathcal{F}_t^1) and (x^2, \mathcal{F}_t^2) are synonomous (we write $(x^1, \mathcal{F}_t^1) \equiv (x^2, \mathcal{F}_t^2)$) if $\mathcal{L}(L_\phi^1) = \mathcal{L}(L_\phi^2)$ for all ϕ and d , where $\mathcal{L}(L_\phi^i)$ is the law of L_ϕ^i on $D(\mathbf{R}^d)$. \square

The above definition is easily shown to be equivalent to that in Aldous [1]. Note that if $(x^1, \mathcal{F}_t^1) \equiv (x^2, \mathcal{F}_t^2)$, then x^1 and x^2 have the same distribution on $D(M)$.

Our main result, which holds with or without the expressions in parentheses, is the following:

THEOREM 1.2. *If y is a nonnegative (local) submartingale on $(X, \mathcal{G}, Q, \mathcal{G}_t)$, there is a (local) martingale m on some $(\Omega, \mathcal{F}, P, \mathcal{F}_t)$ such that*

$$(1.1) \quad (|m|, a, \mathcal{F}_t) \equiv ((y, b), \mathcal{G}_t),$$

where a and b are the increasing processes in the Doob-Meyer decompositions of $|m|$ and y , respectively. Moreover, m may be chosen so that

$$(1.2) \quad E(m(0) \mid |m|, a) = 0 \quad a.s.,$$

and if $e_0 \in \{-1, 0, 1\}$, $\{e_1, \dots, e_n\} \subset \{-1, 1\}$, $B_0, \dots, B_n \in \mathcal{B}((0, \infty))$, and $0 = t_0 < t_1 < \dots < t_n$, then

$$(1.3) \quad \begin{aligned} &P((m(t_0), \dots, m(t_n)) \in e_0 B_0 \times \dots \times e_n B_n) \\ &= \int I((y(t_0), y(t_1), \dots, y(t_n)) \in |e_0| B_0 \times B_1 \times \dots \times B_n) \\ &\quad \times (e_0^2 + 1)^{-1} 2^{-n} \prod_{i=1}^n \left(1 + e_{i-1} e_i \exp \left\{ - \int_{t_{i-1}}^{t_i} y(s)^{-1} db^c(s) \right\} \right. \\ &\quad \left. \times \prod_{t_{i-1} < s \leq t_i} (1 - (y(s^-) + \Delta b(s))^{-1} \Delta b(s)) \right) dQ, \end{aligned}$$

where $b^c(t) = \int_0^t I(\Delta b(s) = 0) db(s)$. \square

Since the filtration $\{\mathcal{G}_t\}$ is an integral part of the submartingale y , it seems natural to require that the filtration for m , $\{\mathcal{F}_t\}$, be equivalent to $\{\mathcal{G}_t\}$ insofar as they relate to $(|m|, a)$ and (y, b) , respectively. This formulation of the existence problem has the advantage of leading to corresponding uniqueness results which are presented in [12].

In §2 we construct a local martingale on a Loeb space whose absolute value equals a given nonnegative local submartingale. The finite-dimensional distributions of m are computed in §3. The proof of Theorem 1.2 then amounts to showing that the class of probability spaces considered in §2 is large enough. In §4 we establish an important property of m , which in [12] is shown to uniquely determine several

additional properties of any local martingale $(m(t), \mathcal{F}_t)$ that satisfies (1.1). §5 contains the proof of a technical lifting theorem needed in §2.

The presentation assumes a knowledge of nonstandard methods in probability theory. A good introduction to the subject may be found in Loeb [10]. Keisler [8] and Hoover and Perkins [7] contain further background material.

2. Construction of the local martingale. Our setting is that used in Hoover and Perkins [7] (hereafter abbreviated [H.P.]). We work in an ω_1 -saturated enlargement of a superstructure $V(S)$, where $\mathbf{R} \subset S$. Consider an internal probability space $(\Omega^1, \mathcal{Q}^1, \bar{P}^1)$ on which $\{\mathcal{Q}_t^1 \mid t \in T\}$ is a nondecreasing internal sequence of $*$ -sub- σ -fields indexed by $T = \{k\Delta t \mid k \in *N_0\}$ (Δt is a fixed positive infinitesimal and $*N_0 = *N \cup \{0\}$). Elements of T are denoted by \underline{t} , \underline{u} , \underline{v} , etc. The Loeb space generated by the above space is $(\Omega^1, \mathcal{F}^1 = L(\mathcal{Q}^1), P^1 = L(\bar{P}^1))$ (i.e., P^1 is the unique extension of ${}^\circ\bar{P}^1$ to $\sigma(\mathcal{Q}^1)$, and \mathcal{F}^1 is the P^1 -completion of $\sigma(\mathcal{Q}^1)$), and a filtration on $(\Omega^1, \mathcal{F}^1)$ is defined by

$$\mathcal{F}_t^1 = \left(\bigcap_{\circ\underline{t} > t} \sigma(\mathcal{Q}_{\underline{t}}^1) \right) \vee \mathcal{N}^1,$$

where \mathcal{N}^1 is the class of all P^1 -null sets. It is clear that $\{\mathcal{F}_t^1 \mid t \geq 0\}$ is right continuous. The 4-tuple $(\Omega^1, \mathcal{F}^1, P^1, \mathcal{F}_t^1)$ constructed in this manner is called the adapted Loeb space generated by $(\Omega^1, \mathcal{Q}_1^1, \bar{P}^1, \mathcal{Q}_t^1)$. The reader is referred to Loeb [9] and Anderson [2] for background material on Loeb spaces. We will need the following elementary results from [H.P.].

LEMMA 2.1 [H. P., LEMMA 3.3 AND THEOREM 3.2]. (a) *If Y is an internal S -integrable random variable on $(\Omega^1, \mathcal{Q}^1, \bar{P}^1)$ and \mathcal{Q} is an internal $*$ -sub- σ -field of \mathcal{Q}^1 , then*

$${}^\circ\bar{E}^1(Y \mid \mathcal{Q}) = E^1({}^\circ Y \mid L(\mathcal{Q})) \quad a.s.$$

(b) *If $y: \Omega^1 \rightarrow \mathbf{R}^d$ is \mathcal{F}_t^1 -measurable for some $t \geq 0$, then there is a $\underline{t} \approx t$ and an internal $\mathcal{Q}_{\underline{t}}^1$ -measurable random variable $Y: \Omega^1 \rightarrow *R^d$ such that ${}^\circ Y = y$ a.s. \square*

Unless otherwise indicated, all stochastic processes have sample paths in $D = D(\mathbf{R})$. Lower case letters denote standard processes on the appropriate Loeb space, while upper case letters are used to denote internal processes on some internal probability space.

Let $x^1(t)$ be a nonnegative local submartingale on $(\Omega^1, \mathcal{F}^1, P^1, \mathcal{F}_t^1)$. That is, there is a sequence of stopping times $\{U_j\}$, increasing to ∞ a.s., such that $x^1(U_j \wedge t)$ is an $\{\mathcal{F}_t^1\}$ -submartingale for all j . Since x^1 is locally of class D , it has a Doob-Meyer decomposition $x^1 = n^1 + a^1$ (n^1 is a local martingale and a^1 is an increasing predictable process that equals zero at $t = 0$). In practice we will construct x^1 and $(\Omega^1, \mathcal{Q}^1, \bar{P}^1, \mathcal{Q}_t^1)$ from the (local) submartingale, y , of Theorem 1.2 as follows. Let

$$(\Omega^1, \mathcal{Q}^1, \bar{P}^1, \mathcal{Q}_t^1) = *(X, \mathcal{G}, Q, \mathcal{G}_t).$$

Then x^1 will be the standard part of $*y(\cdot, \omega)$ in the Skorokhod J_1 topology. Using Lemma 2.1 and some elementary properties of synonymity, one can show that

$((x^1, a^1), \mathfrak{F}^1) \equiv ((y, b), \mathfrak{G})$. (The details of these arguments are presented at the end of §3 with the proof of Theorem 1.2.) Therefore in order to prove Theorem 1.2, we may replace $((y, b), \mathfrak{G})$ with $((x^1, a^1), \mathfrak{F}^1)$. We will construct a local martingale whose absolute value is x^1 in three steps:

1. Lift n^1 and a^1 to internal processes on $T \times \Omega$. (This reduces the problem to the simpler discrete time setting.)
2. Construct an internal martingale M that solves the discrete time problem.
3. “Push M down” to the required local martingale.

As the first step is the most technical, the proof of the main “lifting theorem” is deferred until §5. Some terminology from [H.P.] is required in order to state the result.

Terminology 2.2. An internal nondecreasing sequence of $*$ -sub- σ -fields of \mathcal{Q}^1 , $\{\mathfrak{B}_t \mid t \in T\}$ is an internal filtration if $\mathfrak{F}_t^1 = (\bigcap_{s > t} \sigma(\mathfrak{B}_s)) \vee \mathcal{N}^1$. A \mathfrak{B}_t -stopping time is an internal mapping V from Ω^1 to $T \cup \{\infty\}$ such that $\{V = t\} \in \mathfrak{B}_t$ for all t in T . If ${}^\circ V = U$ a.s., then V is a \mathfrak{B}_t -lifting of U . In what follows, we consider a fixed internal filtration $\{\mathfrak{B}_t\}$.

An internal stochastic process $X: T \times \Omega^1 \rightarrow *R^d$ (i.e. X is internal and $X(t, \cdot)$ is \mathcal{Q}^1 -measurable for all t in T) is of class SD if for a.a. ω , ${}^\circ X(t) \in R^d$ for all t in $ns(T) = \{t \in T \mid {}^\circ t < \infty\}$, and

$$(2.1) \quad st(X)(t) = \lim_{{}^\circ t \downarrow t} {}^\circ X(t)$$

defines a function, $st(X)$, in $D(R^d)$. We define $st(X) = 0$ for ω such that the right side of (2.1) is not well defined for some $t \geq 0$. Note that if X is SD and \mathfrak{B}_t -adapted (i.e., $X(t)$ is \mathfrak{B}_t -measurable for all t in T) then $st(X)$ is \mathfrak{F}_t^1 -adapted. An SD process X is of class SDJ if for a.a. ω , $X(t) \approx X(0)$ for all $t \approx 0$, and for each t in $[0, \infty)$ there is a $\underline{t}(\omega) \approx t$ such that $X(\underline{u}) \approx st(X)(t)$ if $\underline{u} \geq \underline{t}$ and $\underline{u} \approx t$, and $X(\underline{u}) \approx st(X)(t^-)$ if $\underline{u} < \underline{t}$ and $\underline{u} \approx t$. We say X is an SD (respectively, SDJ) lifting of x if X is SD (respectively, SDJ) and $st(X)(t) = x(t)$ for all $t \geq 0$ a.s. Note that these definitions include the case when $X(t, \omega) = X(t)$ is deterministic.

A \mathfrak{B}_t -adapted SD process $X: T \times \Omega \rightarrow *R^d$ is S -locally integrable if there is a nondecreasing sequence of \mathfrak{B}_t -stopping times $\{V_n\}$ such that $\|X(V_n \wedge t)\|$ is S -integrable (see Anderson [2, Definition 3]) for all (t, n) in $T \times \mathbf{N}$, $\lim_{n \rightarrow \infty} {}^\circ V_n = \infty$ a.s., ${}^\circ V_n < \infty$ a.s. and ${}^\circ X(V_n) = st(X)({}^\circ V_n)$ a.s. The sequence $\{V_n\}$ is said to reduce X .

A \mathfrak{B}_t - $*$ -increasing process is a \mathfrak{B}_t -adapted process $A: T \times \Omega^1 \rightarrow *R$ of class SDJ such that for all ω , $A(\cdot, \omega)$ is nondecreasing and $A(0, \omega) = 0$. If A is an SDJ lifting of a , we say A is a \mathfrak{B}_t - $*$ -increasing lifting of a .

A \mathfrak{B}_t -martingale is a \mathfrak{B}_t -adapted process $M: T \times \Omega^1 \rightarrow *R$ such that $\{(M(t), \mathfrak{B}_t) \mid t \in T\}$ is an internal martingale and $\sup_{(t, \omega)} |M(t, \omega)| \in *R$. If, in addition, M is S -locally integrable then M is an S -local martingale (with respect to $\{\mathfrak{B}_t\}$). We say that M is a \mathfrak{B}_t -local martingale lifting of m if M is an S -local martingale and an SDJ lifting of m .

Finally, if $X: T \times \Omega \rightarrow *R$, let $\Delta(X)(t) = X(t + \Delta t) - X(t)$. \square

The correspondence between the above nonstandard definitions and their standard counterparts is studied in [H.P.] and will be used in §5. For now we only need the following simple result.

THEOREM 2.3 [H.P., THEOREM 5.2]. *Let $\{\mathfrak{B}_t\}$ be an internal filtration.*

(a) *If M is a \mathfrak{B}_t -martingale and $\{V_n\}$ is a sequence of \mathfrak{B}_t -stopping times such that $\lim_{n \rightarrow \infty} {}^\circ V_n = \infty$ a.s. and $|M(V_n)|$ is S -integrable, then M is SD .*

(b) *If M is an S -local martingale, then $st(M)$ is an \mathfrak{F}_t^1 -local martingale. \square*

We are ready to state the main lifting theorem.

THEOREM 2.4. *There is an internal filtration $\{\mathfrak{B}_t\}$, a \mathfrak{B}_t -local martingale lifting of n^1 , N^1 , a \mathfrak{B}_t^* -increasing lifting of a^1 , A^1 , and a sequence of \mathfrak{B}_t^* -stopping times $\{W_n^1\}$ that satisfy the following conditions:*

(2.2) *Both A^1 and N^1 are reduced by $\{W_n^1\}$, and $A^1(t + \Delta t)$ is \mathfrak{B}_t -adapted.*

(2.3) *For a.a. ω if $t_1 \approx t_2 \in ns(T)$, ${}^\circ \Delta N^1(t_1) \neq 0$ and ${}^\circ \Delta A^1(t_2) \neq 0$ then $t_2 \leq t_1$.*

(2.4) *There is a positive infinitesimal δ and an infinite natural number γ such that $\delta \leq X^1(t, \omega) = N^1(t, \omega) + A^1(t, \omega) \leq \gamma$ for all (t, ω) in $T \times \Omega^1$.*

PROOF. See §5. \square

Note that X^1 need not be SDJ since N^1 and A^1 may have jumps at infinitely close, but distinct, times. Whenever this occurs, (2.3) implies that the jump in A^1 takes place first. As there were no additional conditions on the original internal filtration $\{\mathcal{Q}_t^1\}$ we may, and shall, assume that $\mathfrak{B}_t = \mathcal{Q}_t^1$.

We now enlarge our space as follows. Let $(\Omega^2, \mathcal{Q}^2, \bar{P}^2) = (*[0, 1]^T, * \mathfrak{B}([0, 1]^T, *m^T)$, where m is Lebesgue measure on $[0, 1]$, and $(*[0, 1]^T, * \mathfrak{B}([0, 1]^T)$ denotes the internal product space. Let \mathcal{Q}_t^2 be the internal σ -algebra generated by the coordinate mappings on Ω^2 up to and including t . Then define

$$(\Omega, \mathcal{Q}, \bar{P}, \mathcal{Q}_t) = (\Omega^1 \times \Omega^2, \mathcal{Q}^1 \times \mathcal{Q}^2, \bar{P}^1 \times \bar{P}^2, \mathcal{Q}_t^1 \times \mathcal{Q}_t^2)$$

and let $\pi_1: \Omega \rightarrow \Omega_1$ be the projection map. The adapted Loeb space generated by $(\Omega, \mathcal{Q}, \bar{P}, \mathcal{Q}_t)$ is denoted by $(\Omega, \mathfrak{F}, P, \mathfrak{F}_t)$. Integration with respect to P^1 and P is denoted by E^1 and E , respectively.

LEMMA 2.5. (a) *π_1 is measure-preserving and $\mathfrak{F}_s | \mathfrak{F}_s^1$ -measurable for all $s \geq 0$.*

(b) *If $\phi: \Omega^1 \rightarrow [0, \infty)$ is \mathfrak{F}_s^1 -measurable then for all $s \geq 0$, $E(\phi \circ \pi_1 | \mathfrak{F}_s) = E^1(\phi | \mathfrak{F}_s^1) \circ \pi_1$ a.s.*

PROOF. (a) Since $\mathfrak{F}_1 \times \mathfrak{F}_2 \subset \mathfrak{F}$ (see Anderson [2, Theorem 22]), π_1 is measurable and therefore is clearly measure-preserving. If $A \in \mathfrak{F}_s^1$, then by Lemma 2.1 there is an $\underline{s} \approx s$ and a B in $\mathcal{Q}_{\underline{s}}^1$ such that $P^1(A \Delta B) = 0$. Therefore $P(\pi_1^{-1}(A) \Delta \pi_1^{-1}(B)) = 0$ and hence $\pi_1^{-1}(A) \in \mathfrak{F}_{\underline{s}}$ since $\pi_1^{-1}(B) \in \mathcal{Q}_{\underline{s}} \subset \mathfrak{F}_{\underline{s}}$.

(b) By truncating we may assume ϕ is bounded. Let Φ be a bounded \mathcal{Q}^1 -measurable lifting of ϕ (see Loeb [9]). By Lemma 2.1 there exists $\underline{s} \approx s$ such that

$$E(\phi \circ \pi_1 | \mathfrak{F}_s) = E(\phi \circ \pi_1 | \sigma(\mathcal{Q}_{\underline{s}})) = {}^\circ \bar{E}(\Phi \circ \pi_1 | \mathcal{Q}_{\underline{s}}) \quad \text{a.s.,}$$

and

$$E^1(\phi | \mathcal{F}_s^1) \circ \pi_1 = E^1(\phi | \sigma(\mathcal{Q}_s^1)) \circ \pi_1 = {}^\circ E^1(\Phi | \mathcal{Q}_s^1) \circ \pi_1 \quad \text{a.s.}$$

The result follows since the right sides of the above equalities are clearly the same. \square

Let $x = x^1 \circ \pi_1$, $n = n^1 \circ \pi_1$ and $a = a^1 \circ \pi_1$. The above lemma implies that n is an \mathcal{F}_t -local martingale and that $((x^1, a^1), \mathcal{F}_t^1)$ and $((x, a), \mathcal{F}_t)$ are synonymous. In particular, by Aldous [1, Theorem 19.6] $a(t)$ is a predictable increasing process with respect to $\{\mathcal{F}_t\}$. It follows that $x(t)$ is a nonnegative local submartingale with respect to $\{\mathcal{F}_t\}$ and $x = n + a$ is its Doob-Meyer decomposition. We also define $A = A^1 \circ \pi_1$, $N = N^1 \circ \pi_1$ and $X = X^1 \circ \pi_1$. Clearly A , N and X satisfy the same properties as A^1 , N^1 and X^1 , only with \mathcal{Q}_t in place of \mathcal{Q}_t^1 and $W_n = W_n^1 \circ \pi_1$ in place of W_n^1 . Note also that $N(t)$ is an internal martingale with respect to the larger \ast - σ -fields $\{\mathcal{Q}_t^1 \times \mathcal{Q}^2 | t \in T\}$.

Fix $p \in [0, 1]$, once and for all.

Notation 2.6. If $\omega \in \Omega$, we write $\omega = (\omega^1, \omega^2) = (\omega^1, (\omega_s^2)_{s \in T})$. Define $\phi: T \times \Omega \rightarrow \{-1, 1\}$ by

$$\phi(0, \omega) = \begin{cases} 1 & \text{if } \omega_0^2 \leq p, \\ -1 & \text{if } \omega_0^2 > p, \end{cases}$$

and

$$\phi(t, \omega) = \begin{cases} -1 & \text{if } \omega_t^2 \leq \Delta A(t - \Delta t, \omega)(2(X(t - \Delta t, \omega) + \Delta A(t - \Delta t, \omega)))^{-1}, \\ 1 & \text{otherwise.} \end{cases}$$

Let $Z(t) = \prod_{s=0}^t \phi(s)$, and define $M(t) = Z(t)X(t)$. \square

Hence $|M(t)| = X(t)$ and M changes sign whenever $\phi = -1$. These added jumps are weighted to counteract the increasing part of X , and hence make M an \mathcal{Q}_t -martingale.

LEMMA 2.7. *M is an \mathcal{Q}_t -martingale.*

PROOF. It is clear from its definition that M is an \mathcal{Q}_t -adapted process and $|M(t)| = X(t) \leq \gamma$ (by (2.4)). Note that

$$\begin{aligned} \bar{E}(M(t + \Delta t) | \mathcal{Q}_t) &= Z(t) \bar{E}((X(t) + \Delta N(t) + \Delta A(t))\phi(t + \Delta t) | \mathcal{Q}_t) \\ &= Z(t)((X(t) + \Delta A(t))\bar{E}(\phi(t + \Delta t) | \mathcal{Q}_t) \\ &\quad + \bar{E}(\Delta N(t)\phi(t + \Delta t) | \mathcal{Q}_t)), \end{aligned}$$

because $A(t + \Delta t)$ is \mathcal{Q}_t -measurable. Since N is an $\mathcal{Q}_t^1 \times \mathcal{Q}^2$ -martingale and $\phi(t + \Delta t)$ is $\mathcal{Q}_t^1 \times \mathcal{Q}^2$ -measurable, the last term becomes

$$\bar{E}(\phi(t + \Delta t)\bar{E}(\Delta N(t) | \mathcal{Q}_t^1 \times \mathcal{Q}^2) | \mathcal{Q}_t) = 0.$$

An easy computation now implies

$$\begin{aligned} \bar{E}(M(t + \Delta t) | \mathcal{Q}_t) &= Z(t)(X(t) + \Delta A(t))(1 - \Delta A(t)(X(t) + \Delta A(t))^{-1}) \\ &= Z(t)X(t) = M(t). \quad \square \end{aligned}$$

In order to apply Theorem 2.3 to obtain the required local martingale, we need the following lemma:

LEMMA 2.8. *If $U = U^1 \circ \pi_1$ for some \mathfrak{F}^1 -measurable random variable U^1 , then*

$$P(\exists \underline{t} \in \text{ns}(T) \ni \underline{t} \approx U, \quad \circ X(\underline{t}) > 0, \phi(\underline{t} + \Delta t) = -1 \text{ and } \circ \Delta A(\underline{t}) = 0) = 0.$$

PROOF. It clearly suffices to show that $P(B_{n,\varepsilon}) = 0$ for each n in \mathbf{N} and positive infinitesimal ε , where

$$B_{n,\varepsilon} = \{ \omega \mid \exists \underline{t} \leq n \text{ such that } |\underline{t} - V| < \varepsilon, X(\underline{t}) \geq n^{-1}, \\ \phi(\underline{t} + \Delta t) = -1 \text{ and } \Delta A(\underline{t}) \leq \varepsilon \}$$

and $V = V^1 \circ \pi_1$, for some \mathcal{Q}^1 -measurable lifting of U^1, V^1 . By Lemma 2.1 we have

$$\begin{aligned} P(B_{n,\varepsilon} \mid \sigma(\mathcal{Q}^1 \times \{ \phi, \Omega_2 \})) &= \circ \bar{P}(B_{n,\varepsilon} \mid \mathcal{Q}^1 \times \{ \phi, \Omega_2 \}) \quad \text{a.s.} \\ &\leq \circ \sum_{\underline{t} \leq n} I_{\{ |\underline{t} - V| < \varepsilon, \Delta A(\underline{t}) \leq \varepsilon, X(\underline{t}) \geq n^{-1} \}} \\ &\quad \times \bar{P}(\phi(\underline{t} + \Delta t) = -1 \mid \mathcal{Q}^1 \times \{ \phi, \Omega_2 \}) \\ &\leq \circ \sum_{\underline{t} \leq n} I_{\{ |\underline{t} - V| < \varepsilon, \Delta A(\underline{t}) \leq \varepsilon \}} \Delta A(\underline{t}) n/2 \\ &= 0 \quad \text{a.s.} \end{aligned}$$

(the last line follows because A is of class SDJ and $\varepsilon \approx 0$). The proof is completed by taking expectations in the above. \square

THEOREM 2.9. *The process M is SD and $m = \text{st}(M)$ is an $\{\mathfrak{F}_t\}$ -local martingale such that $|m|$ and x are indistinguishable. Moreover, $((|m|, a), \mathfrak{F}) \equiv ((x^1, a^1), \mathfrak{F}^1)$, where a and a^1 are the increasing processes in the Doob-Meyer decompositions of $|m|$ and x^1 , respectively. If x^1 is a submartingale, m is a martingale.*

PROOF. Since $|M(W_n)| = X(W_n)$ is S -integrable, M is SD by Theorem 2.3(a). We claim that M is an S -local martingale. If $\circ M(W_n) \neq m(\circ W_n)$ ($m \equiv \text{st}(M)$), then, since $\circ |M(W_n)| = |m(\circ W_n)|$ and $\circ A(W_n) = a(\circ W_n)$ a.s., with probability one there is a positive infinitesimal \underline{t} such that $\phi(W_n + \underline{t}) = -1$, $\circ X(W_n + \underline{t} - \Delta t) = x(\circ W_n) > 0$, and $\circ \Delta A(W_n + \underline{t} - \Delta t) = 0$. Therefore

$$(2.5) \quad \{ \circ M(W_n) \neq m(\circ W_n) \} \subset \{ \exists \underline{t} \in \text{ns}(T) \ni \underline{t} \approx W_n, \circ X(\underline{t}) > 0, \\ \phi(\underline{t} + \Delta t) = -1 \text{ and } \circ \Delta A(\underline{t}) = 0 \} \cup \Lambda$$

for some null set Λ . The previous lemma implies that $\circ M(W_n) = m(\circ W_n)$ a.s. and hence proves the claim. By Theorem 2.3(b), m is an $\{\mathfrak{F}_t\}$ -local martingale, and

$$|m| = |\text{st}(M)| = \text{st}(|M|) = x \quad \text{a.s.}$$

Lemma 2.5 implies that $((|m|, a), \mathfrak{F}) \equiv ((x^1, a^1), \mathfrak{F}^1)$. If x^1 is a submartingale, then so is $|m| = x$. From this and the fact that m is a local martingale, it follows easily that m must be a martingale. \square

3. The distribution of m . In this section we describe the finite-dimensional distributions of m in terms of the law of (x, a) . In order to facilitate the computations we will modify the definition of m by adding a positive infinitesimal to the lifting N of n obtained in Theorem 2.4 (see Remark 3.8). A series of technical lemmas precedes the proof of the main result (Theorem 3.11).

Notation 3.1. Let \mathcal{C} denote the internal σ -algebra of internal subsets of T . If $F: T \rightarrow \mathbf{*R}$ is an internal function, let ν_F denote the internal signed measure on (T, \mathcal{C}) defined by $\nu_F(\{\underline{t}\}) = F(\underline{t} + \Delta t) - F(\underline{t})$. If $F: T \times \Omega \rightarrow \mathbf{*R}$ is an internal stochastic process, then ν_F denotes the internal random signed measure defined by the above. \square

The following two elementary results [H.P., Lemma 2.7 and Theorem 4.7(b)] are frequently used.

LEMMA 3.2. *If $F: T \rightarrow \mathbf{*R}$ is an internal, nondecreasing function of class SD such that $F(\underline{t}) \approx F(0)$ for all $\underline{t} \approx 0$, and $f = \text{st}(F)$, then for each B in $\mathfrak{B}([0, \infty))$, $L(\nu_F)(\{\underline{t} \in T \mid \circ \underline{t} \in B\}) = \int_B df$. \square*

LEMMA 3.3. *If $X: T \times \Omega \rightarrow \mathbf{*R}^d$ is an SD internal stochastic process then for each $t \geq 0$ there is a $\underline{t}^1 \approx t$ such that for almost all ω if $\underline{t} \approx t$ and $\underline{t} \geq \underline{t}^1$ then $\circ X(\underline{t}) = \text{st}(X)(t)$. \square*

To obtain a formula for the distribution of m , it is clear we need to examine $\text{sgn}(m(t))$. The following result allows us to study $\text{sgn}(m(t))$ by means of the internal process $Z(t)$.

LEMMA 3.4. *For each $t \geq 0$ there is a $\underline{t}^1 \approx t$ such that with probability one, $m(t) = Z(\underline{t})x(t)$ for all $\underline{t} \geq \underline{t}^1$ that satisfy $\underline{t} \approx t$. If $t = 0$, we may take $\underline{t}^1 = 0$.*

PROOF. Since (M, X) is an SD lifting of (m, x) , by Lemma 3.3 there is a $\underline{t}^1 \approx t$ such that $\circ(M(\underline{t}), X(\underline{t})) = (m(t), x(t))$ for all $\underline{t} \geq \underline{t}^1$ satisfying $\underline{t} \approx t$ and a.a. ω . Therefore for a.a. ω if $\underline{t} \geq \underline{t}^1$ and $\underline{t} \approx t$, then

$$m(t) = \circ M(\underline{t}) = Z(\underline{t}) \circ X(\underline{t}) = Z(\underline{t})x(t).$$

If $t = 0$ and $x(0) = 0$, then $m(0) = Z(0)x(0) = 0$ a.s. If $t = 0$ and $x(0) > 0$, then $\circ X(\underline{t}) > 0$ for all $\underline{t} \approx 0$ a.s. and therefore, by Lemma 2.7 (with $U = 0$), $\phi(\underline{t} + \Delta t) = 1$ for all infinitesimal \underline{t} in T a.s. It follows that for a.a. ω and for all $\underline{t} \approx 0$,

$$m(0) = Z(\underline{t})x(0) = Z(0)x(0). \quad \square$$

LEMMA 3.5.

$$L(\nu_A)(\{\underline{t} \in \text{ns}(T) \mid \circ X(\underline{t}) \neq x(\circ \underline{t}^-) \text{ or } \circ \Delta A(\underline{t}) \neq \Delta a(\circ \underline{t})\}) = 0 \quad \text{a.s.}$$

PROOF. Let $\{V_i \mid i \in \mathbf{N}\}$ be a sequence of \mathfrak{A}_t -stopping times such that

$$\{\underline{t} \in \text{ns}(T) \mid \circ |X(\underline{t}) - X(\underline{t} - \Delta t)| > 0\} = \{V_i \mid \circ V_i \neq \infty\} \quad \text{a.s.}$$

(such a sequence clearly exists since X is the sum of two *SDJ* processes). Therefore,

$$\{\underline{t} \in \text{ns}(T) \mid \circ X(\underline{t}) \neq x(\circ \underline{t}^-)\} \subset \bigcup_{i \in \mathbf{N}} \bigcap_{m \in \mathbf{N}} ([V_i, V_i + m^{-1}) \cap T) \quad \text{a.s.}$$

Moreover by (2.3) we have $\circ A(V_i) = a(\circ V_i)$ a.s. and, hence,

$$\begin{aligned} L(\nu_A) \left(\bigcup_{i \in \mathbf{N}} \bigcap_{m \in \mathbf{N}} ([V_i, V_i + m^{-1}) \cap T) \right) \\ \leq \sum_{i=1}^{\infty} \lim_{m \rightarrow \infty} L(\nu_A) ([V_i, V_i + m^{-1}) \cap T) = 0 \quad \text{a.s.} \end{aligned}$$

Thus it follows that $L(\nu_A)(\{\underline{t} \in \text{ns}(T) \mid \circ X(\underline{t}) \neq x(\circ \underline{t}^-)\}) = 0$ a.s. A similar argument shows that $L(\nu_A)(\{\underline{t} \in \text{ns}(T) \mid \circ \Delta A(\underline{t}) \neq \Delta a(\circ \underline{t}^-)\}) = 0$ a.s., and the proof is complete. \square

Let

$$a^d(t) = \int_0^t I(\Delta a(s) > 0) da(s) \quad \text{and} \quad a^c(t) = \int_0^t I(\Delta a(s) = 0) da(s)$$

be the pure jump and continuous parts of a , respectively. A corresponding decomposition of A is given by the following:

LEMMA 3.6. For some H in ${}^*\mathbf{N} - \mathbf{N}$, $A^c(t) = \sum_{\underline{s} < t} \Delta A(\underline{s}) I(\Delta A(\underline{s}) < H^{-1})$ and $A^d = A - A^c$ are *SDJ* liftings of a^c and a^d , respectively. \square

The simple proof is left for the reader. Fix H and define A^c and A^d as in the above. Let $\{r_i \mid i \in \mathbf{N}\}$ be an enumeration of the nonnegative rationals and choose $\underline{r}_i \approx r_i$ ($\underline{r}_i \in T$) such that $m(\underline{r}_i) = Z(\underline{r}_i)x(\underline{r}_i)$ and $\circ A(\underline{t}_i) = a(\underline{r}_i)$ a.s. The existence of \underline{r}_i follows from Lemmas 3.3 and 3.4. Since $A(0) = a(0) = 0$ we may let $\underline{r}_i = 0$ if $r_i = 0$.

LEMMA 3.7. There is a K in ${}^*\mathbf{N} - \mathbf{N}$ such that $KH^{-1} \approx 0$, and for all (i, j) with $r_i < r_j$,

$$\begin{aligned} (3.1) \quad & \circ \int I(r_i \leq \underline{s} < \underline{r}_j) (K^{-1} + X(\underline{s}) + \Delta A(\underline{s}))^{-1} d\nu_{A^c} \\ & = \int I(r_i < s \leq r_j) x(s)^{-1} da^c(s) \quad \text{a.s.} \end{aligned}$$

and

$$\begin{aligned} (3.2) \quad & \circ \int I(r_i \leq \underline{s} < \underline{r}_j) (K^{-1} + X(\underline{s}) + \Delta A(\underline{s}))^{-1} d\nu_{A^d} \\ & = \int I(r_i < s \leq r_j) (x(s^-) + \Delta a(s))^{-1} da^d(s) \quad \text{a.s.} \end{aligned}$$

PROOF. If $r_i < r_j$ we claim that (3.1) and (3.2) hold for K in \mathbf{N} . Indeed if $K \in \mathbf{N}$,

$$\begin{aligned} & \circ \int I(r_i \leq \underline{s} < r_j)(K^{-1} + X(\underline{s}) + \Delta A(\underline{s}))^{-1} d\nu_{A^d} \\ &= \int I(r_i \leq \underline{s} < r_j)(K^{-1} + x(\circ \underline{s}^-) + \Delta a(\circ \underline{s}))^{-1} dL(\nu_{A^d}) \quad \text{a.s.} \\ & \hspace{20em} \text{(Lemma 3.5)} \\ &= \int I(r_i < \circ \underline{s} \leq r_j)(K^{-1} + x(\circ \underline{s}^-) + \Delta a(\circ \underline{s}))^{-1} dL(\nu_{A^d}) \quad \text{a.s.} \\ & \hspace{15em} \text{(since } \circ A(r_k) = a(r_k) \text{ a.s.)} \\ &= \int I(r_i < s \leq r_j)(K^{-1} + x(s^-) + \Delta a(s))^{-1} da^d(s) \quad \text{a.s.} \\ & \hspace{15em} \text{(Lemmas 3.2 and 3.6).} \end{aligned}$$

The same result holds with A^c and a^c in place of A^d and a^d , and hence the claim is proved. Let

$$\begin{aligned} X_{i,j,K}^c &= \arctan \left(\int I(r_i \leq \underline{s} < r_j)(K^{-1} + X(\underline{s}) + \Delta A(\underline{s}))^{-1} d\nu_{A^c} \right), \\ X_{i,j,K}^d &= \arctan \left(\int I(r_i \leq \underline{s} < r_j)(K^{-1} + X(\underline{s}) + \Delta A(\underline{s}))^{-1} d\nu_{A^d} \right), \end{aligned}$$

and let $X_{i,j}^c$ and $X_{i,j}^d$ be uniformly bounded liftings of

$$\begin{aligned} & \arctan \left(\int I(r_i < s \leq r_j)x(s)^{-1} da^c(s) \right) \quad \text{and} \\ & \arctan \left(\int I(r_i < s \leq r_j)(x(s^-) + \Delta a(s))^{-1} da^d(s) \right), \end{aligned}$$

respectively. Since (3.1) and (3.2) hold for K in \mathbf{N} , the monotone convergence theorem implies that

$$\lim_{K \rightarrow \infty} \circ \bar{E} (|X_{i,j,K}^c - X_{i,j}^c| + |X_{i,j,K}^d - X_{i,j}^d|) = 0,$$

whenever $r_i < r_j$. Therefore for each n in \mathbf{N} there is a K_n in ${}^*\mathbf{N} - \mathbf{N}$ such that whenever $K \in {}^*\mathbf{N} - \mathbf{N}$ satisfies $K \leq K_n$,

$$\max_{\substack{i,j \leq n \\ r_i < r_j}} \bar{E} (|X_{i,j,K}^c - X_{i,j}^c| + |X_{i,j,K}^d - X_{i,j}^d|) \leq 2^{-n}.$$

Extend $\{K_n \mid n \in \mathbf{N}\}$ internally to ${}^*\mathbf{N}$ such that $K_n \geq n$. Let $K = (\min_{n < \gamma} K_n) \wedge H^{1/2}$ for some γ in ${}^*\mathbf{N} - \mathbf{N}$. Then K is the required number. \square

REMARK 3.8. If N^1 and X^1 are replaced by $N^1 + K^{-1}$ and $X^1 + K^{-1}$, respectively (A^1 remains unchanged), then the conditions of Theorem 2.4 remain satisfied. Thus we may, and shall, assume that

$$(3.3) \quad X(t)^{-1}H^{-1} \approx 0 \quad \text{for all } (t, \omega),$$

$$\begin{aligned}
 (3.4) \quad & \circ \int I(r_i \leq \underline{s} < r_j)(X(\underline{s}) + \Delta A(\underline{s}))^{-1} d\nu_{A^c} \\
 & = \int I(r_i < s \leq r_j)x(s)^{-1} da^c(s) \quad \text{for all } r_i < r_j \text{ a.s.,}
 \end{aligned}$$

and

$$\begin{aligned}
 (3.5) \quad & \circ \int I(r_i \leq \underline{s} < r_j)(X(\underline{s}) + \Delta A(\underline{s}))^{-1} d\nu_{A^d} \\
 & = \int I(r_i < s \leq r_j)(x(s^-) + \Delta a(s))^{-1} da^d(s) \quad \text{for all } r_i < r_j \text{ a.s.} \quad \square
 \end{aligned}$$

LEMMA 3.9. Let $F: T \rightarrow *R$ be an SDJ nondecreasing internal function, $f = \text{st}(F)$ and let $t_1 < t_2 \text{ ns}(T)$ such that for $i = 1, 2$, $\circ F(t_i) = f(\circ t_i)$. If $G: T \rightarrow *[0, \infty)$ and $g: [0, \infty) \rightarrow [0, \infty)$ satisfy $\circ G(\underline{s}) = g(\circ \underline{s})$, $L(\nu_F)$ -a.s. and

$$\circ \int I(t_1 \leq \underline{s} < t_2)G(\underline{s}) d\nu_F = \int I(\circ t_1 < s \leq \circ t_2)g(s) df(s) < \infty,$$

then $H(t) = \int I(t_1 \leq \underline{s} < t \wedge t_2)G(\underline{s}) d\nu_F$ is an SDJ lifting of

$$h(t) = \int I(\circ t_1 < s \leq t \wedge \circ t_2)g(s) df(s).$$

PROOF. If $t_1 \leq t \leq t_2$ and $\circ F(t) = f(\circ t)$, then

$$\begin{aligned}
 \circ \int I(t_1 \leq \underline{s} < t)G(\underline{s}) d\nu_F & \geq \int I(t_1 \leq \underline{s} < t)g(\circ \underline{s})L(\nu_F) \\
 & = \int I(\circ t_1 < \circ \underline{s} \leq \circ t)g(\circ \underline{s})L(\nu_F) \\
 & = \int I(\circ t_1 < s \leq \circ t)g(s) df(s) \quad (\text{Lemma 3.2}).
 \end{aligned}$$

A similar argument shows that

$$\circ \int I(t \leq \underline{s} < t_2)G(\underline{s}) d\nu_F \geq \int I(\circ t < s \leq \circ t_2)g(s) df(s).$$

Since the sum of the above inequalities leads to an equality, it follows that $\circ H(t) = h(\circ t)$. Similarly, if $\circ F(t) = f(\circ t^-)$ and $t_1 < t \leq t_2$, then $\circ H(t) = h(\circ t^-)$. Note also that if $t \geq t_2$ or $t \leq t_1$ then $\circ H(t) = h(\circ t)$. The result follows because F is SDJ and $f = \text{st}(F)$. \square

LEMMA 3.10. Suppose f is a nondecreasing pure jump function in $D(R)$, (i.e., $f(t) = \sum_{s \leq t} \Delta f(s)$) and $F: T \rightarrow *R$ is a nondecreasing SDJ lifting of f . Let $h: [0, \infty) \rightarrow R$ be Lipschitz continuous and satisfy $h(0) = 0$. If $g(t) = \sum_{s \leq t} h(\Delta f(s))$ and $G(t) = \sum_{s < t} h(\Delta F(\underline{s}))$, then $\circ F(t) = f(\circ t)$ implies $\circ G(t) = g(\circ t)$.

PROOF. Fix $t \geq 0$ and $\underline{t} \approx t$ such that ${}^\circ F(\underline{t}) = f(t)$. Let $\{t_i \mid i \in I\} = \{s \leq t \mid \Delta f(s) > 0\}$ and for each t_i choose $\underline{t}_i \approx t_i$ such that ${}^\circ \Delta F(\underline{t}_i) = \Delta f(t_i)$. Since ${}^\circ F(\underline{t}) = f({}^\circ \underline{t})$, it follows that $\{t_i \mid i \in I\} \subset [0, \underline{t}]$. If $|h(x)| \leq c|x|$, then

$$\begin{aligned} \limsup_{n \rightarrow \infty} \sum_{s \leq \underline{t}} {}^\circ I(s \notin \{t_1, \dots, t_n\}) |h(\Delta F(s))| \\ \leq c \limsup_{n \rightarrow \infty} \sum_{s \leq \underline{t}} {}^\circ I(s \notin \{t_1, \dots, t_n\}) \Delta F(s) \\ = c \limsup_{n \rightarrow \infty} \left(f(t) - \sum_{i=1}^n \Delta f(t_i) \right) = 0. \end{aligned}$$

Therefore

$${}^\circ G(\underline{t}) = \lim_{n \rightarrow \infty} \sum_{i=1}^n {}^\circ h(\Delta F(t_i)) = \lim_{n \rightarrow \infty} \sum_{i=1}^n h(\Delta f(t_i)) = g(t). \quad \square$$

We are finally ready for the main result of this section.

THEOREM 3.11. *Suppose $\{B_0, \dots, B_n\} \subset \mathfrak{B}((0, \infty))$, $\{e_1, \dots, e_n\} \subset \{-1, 1\}$, $e_0 \in \{-1, 0, 1\}$, and $0 = t_0 < t_1 < \dots < t_n$. Then $E(m(0) \mid a, x) = (2p - 1)x(0)$ a.s. and*

$$\begin{aligned} P((m(t_0), \dots, m(t_n)) \in e_0 B_0 \times \dots \times e_n B_n) \\ = (pI(e_0 = 1) + (1 - p)I(e_0 = -1) + I(e_0 = 0)) \\ \times E \left(I((x(t_0), \dots, x(t_n)) \in |e_0| B_0 \times B_1 \times \dots \times B_n) \right. \\ (3.6) \quad \times 2^{-n} \prod_{i=1}^n \left(1 + \hat{e}_{i-1} \hat{e}_i \exp \left\{ - \int_{t_{i-1}}^{t_i} x(s)^{-1} da^c(s) \right\} \right. \\ \left. \left. \times \prod_{t_{i-1} < s \leq t_i} (1 - (x(s^-) + \Delta a(s))^{-1} \Delta a(s)) \right) \right), \end{aligned}$$

where

$$\hat{e}_i = \begin{cases} e_i & \text{if } i > 0 \text{ or } e_i \neq 0, \\ 2p - 1 & \text{if } i = 0 \text{ and } e_0 = 0. \end{cases}$$

PROOF. By a routine limiting argument it suffices to prove the result when t_1, \dots, t_n are rationals. By the definition of $\{t_i\}$, and (3.4) and (3.5), there are points $0 = t_0 < \dots < t_n$ such that ${}^\circ \underline{t}_i = t_i$, $m(t_i) = Z(\underline{t}_i)x(t_i)$ a.s., ${}^\circ A(\underline{t}_i) = a(t_i)$ a.s.,

$$\begin{aligned} (3.7) \quad {}^\circ \int I(t_i \leq s < t_{i+1})(X(s) + \Delta A(s))^{-1} dv_{A^c} \\ = \int I(t_i < s \leq t_{i+1})x(s)^{-1} da^c(s) \quad \text{a.s.,} \end{aligned}$$

and

$$(3.8) \quad \begin{aligned} & \circ \int I(t_i \leq \underline{s} < t_{i+1})(X(\underline{s}) + \Delta A(\underline{s}))^{-1} dv_{A^d} \\ & = \int I(t_i < s \leq t_{i+1})(x(s^-) + \Delta a(s))^{-1} da^d(s) \quad \text{a.s.} \end{aligned}$$

Then

$$(3.9) \quad \begin{aligned} & P((m(t_0), \dots, m(t_n)) \in e_0 B_0 \times \dots \times e_n B_n) \\ & = E(I((Z(0)x(0), x(t_1), \dots, x(t_n)) \in e_0 B_0 \times B_1 \times \dots \times B_n) \\ & \quad \times I((Z(t_1), \dots, Z(t_{n-1})) = (e_1, \dots, e_{n-1})) \\ & \quad \times P(Z(t_n) = e_n | L(\mathcal{Q}^1 \times \mathcal{Q}_{t_{n-1}}^2))) \end{aligned}$$

We claim that

$$(3.10) \quad \begin{aligned} & P(Z(t_n) = e_n | L(\mathcal{Q}^1 \times \mathcal{Q}_{t_{n-1}}^2)) \\ & = \frac{1}{2} \left(1 + e_n Z(t_{n-1}) \exp \left\{ - \int_{t_{n-1}}^{t_n} x(s)^{-1} da^c(s) \right\} \right. \\ & \quad \left. \times \prod_{t_{n-1} < s \leq t_n} (1 - (x(s^-) + \Delta a(s))^{-1} \Delta a(s)) \right). \end{aligned}$$

The definition of Z implies that

$$(3.11) \quad \begin{aligned} & \bar{E}(Z(t_n) | \mathcal{Q}^1 \times \mathcal{Q}_{t_{n-1}}^2) = Z(t_{n-1}) \bar{E} \left(\prod_{t_{n-1} < \underline{s} \leq t_n} \phi(\underline{s}) | \mathcal{Q}^1 \times \mathcal{Q}_{t_{n-1}}^2 \right) \\ & = Z(t_{n-1}) \prod_{t_{n-1} \leq \underline{s} < t_n} (1 - (X(\underline{s}) + \Delta A(\underline{s}))^{-1} \Delta A(\underline{s})), \end{aligned}$$

where we have used the fact that $\{\phi(\underline{s}) | t_{n-1} < \underline{s} \leq t_n\}$ are conditionally independent given $\mathcal{Q}^1 \times \mathcal{Q}_{t_{n-1}}^2$.

To prove (3.10), assume first that

$$\int (t_{n-1} < s \leq t_n)(x(s^-) + \Delta a(s))^{-1} da(s) = \infty.$$

Then (3.7) and (3.8) imply that for a.a. ω ,

$$\sum_{t_{n-1} \leq \underline{s} < t_n} (X(\underline{s}) + \Delta A(\underline{s}))^{-1} \Delta A(\underline{s}) = \infty$$

and, therefore by (3.11), $\circ \bar{E}(Z(t_n) | \mathcal{Q}^1 \times \mathcal{Q}_{t_{n-1}}^2) = 0$ a.s. This implies that

$$P(Z(t_n) = e_n | L(\mathcal{Q}^1 \times \mathcal{Q}_{t_{n-1}}^2)) = \circ \bar{P}(Z(t_n) = e_n | \mathcal{Q}^1 \times \mathcal{Q}_{t_{n-1}}^2) \quad \text{a.s.} \tag{Lemma 2.1}$$

$$= 1/2 \quad \text{a.s.}$$

Clearly the right side of (3.10) is also $\frac{1}{2}$ by assumption, and the claim is proved in this case.

Assume now that

$$\int I(t_{n-1} < s \leq t_n)(x(s^-) + \Delta a(s))^{-1} da(s) < \infty.$$

If ${}^\circ(X(\underline{s}) + \Delta A(\underline{s}))^{-1}\Delta A(\underline{s}) = 1$ for some \underline{s} in $[t_{n-1}, t_n)$, then by (3.11), ${}^\circ\bar{E}(Z(t_n) | \mathcal{Q}^1 \times \mathcal{Q}_{t_{n-1}}^2) = 0$ a.s. and therefore $P(Z(t_n) = e_n | L(\mathcal{Q}^1 \times \mathcal{Q}_{t_{n-1}}^2)) = \frac{1}{2}$ a.s., as before. On the other hand, (3.7), (3.8) and Lemma 3.9 imply that for a.a. ω , ${}^\circ\underline{s} > t_{n-1}$ and $(x({}^\circ\underline{s}^-) + \Delta a({}^\circ\underline{s}))^{-1}\Delta a({}^\circ\underline{s}) = 1$. This implies that the right side of (3.10) is also $\frac{1}{2}$. Hence, we may assume that

$${}^\circ\max_{t_{n-1} \leq s < t_n} (X(\underline{s}) + \Delta A(\underline{s}))^{-1}\Delta A(\underline{s}) = y(\omega) < 1.$$

From (3.11) we obtain

$$\begin{aligned} (3.12) \quad & \log\left(\bar{E}\left(Z(t_n)/Z(t_{n-1}) \mid \mathcal{Q}^1 \times \mathcal{Q}_{t_{n-1}}^2\right)\right) \\ &= \sum_{t_{n-1} \leq \underline{s} < t_n} \log\left(1 - (X(\underline{s}) + \Delta A(\underline{s}))^{-1}\Delta A^c(\underline{s})\right) \\ &+ \sum_{t_{n-1} \leq \underline{s} < t_n} \log\left(1 - (X(\underline{s}) + \Delta A(\underline{s}))^{-1}\Delta A^d(\underline{s})\right) \\ &= \sum_1 + \sum_2. \end{aligned}$$

By (3.7), (3.8) and Lemma 3.9 we see that for a.a. ω satisfying the above assumptions, the hypotheses of Lemma 3.10 are satisfied, where

$$\begin{aligned} h(x) &= \log(1 - (x \vee (-y(\omega))) \wedge y(\omega)), \\ f(t) &= \int I(t_{n-1} < s \leq t_n \wedge t)(x(s^-) + \Delta a(s))^{-1} da^d(s), \\ F(\underline{t}) &= \int I(t_{n-1} \leq \underline{s} < t_n \wedge \underline{t})(X(\underline{s}) + \Delta A(\underline{s}))^{-1} dv_{A^d}. \end{aligned}$$

Therefore Lemma 3.10 implies that for a.a. ω satisfying the above conditions,

$$\begin{aligned} (3.13) \quad {}^\circ\sum_2 &= {}^\circ\sum_{t_{n-1} \leq \underline{s} < t_n} h\left((X(\underline{s}) + \Delta A(\underline{s}))^{-1}\Delta A^d(\underline{s})\right) \\ &= \sum_{t_{n-1} < s \leq t_n} \log\left(1 - (x(s^-) + \Delta a(s))^{-1}\Delta a(s)\right). \end{aligned}$$

For \sum_1 , let $\log(1 - z) = -z - z^2\theta(z)$ for $|z| < 1$ and note that

$$\begin{aligned} {}^\circ\sum_1 &= - {}^\circ\sum_{t_{n-1} \leq \underline{s} < t_n} (X(\underline{s}) + \Delta A(\underline{s}))^{-1}\Delta A^c(\underline{s}) \\ &- {}^\circ\sum_{t_{n-1} \leq \underline{s} < t_n} (X(\underline{s}) + \Delta A(\underline{s}))^{-2}(\Delta A^c(\underline{s}))^2 \\ &\quad \times \theta\left((X(\underline{s}) + \Delta A(\underline{s}))^{-1}\Delta A^c(\underline{s})\right). \end{aligned}$$

Since $\Delta A^c(s) \leq H^{-1}$, we may use (3.3) and (3.7) to bound the latter summation a.s. in absolute value by

$$\begin{aligned} & \circ\theta(0) \circ \left(\max_{t_{n-1} \leq s < t_n} (X(s) + \Delta A(s))^{-1} \Delta A^c(s) \right) \\ & \quad \times \int_{t_{n-1}}^{t_n} (x(s^-) + \Delta a(s))^{-1} da^c(s) = 0. \end{aligned}$$

Therefore by (3.7),

$$(3.14) \quad \circ\Sigma_1 = - \int_{t_{n-1}}^{t_n} x(s)^{-1} da^c(s) \quad \text{a.s.}$$

Substitute (3.13) and (3.14) into (3.12) to see that

$$\begin{aligned} E(Z(t_n) | L(\mathcal{Q}^1 \times \mathcal{Q}_{t_{n-1}}^2)) &= \circ\bar{E}(Z(t_n) | \mathcal{Q}^1 \times \mathcal{Q}_{t_{n-1}}^2) \quad \text{a.s.} \\ &= Z(t_{n-1}) \exp \left\{ - \int_{t_{n-1}}^{t_n} x(s)^{-1} da^c(s) \right\} \\ & \quad \times \prod_{t_{n-1} < s \leq t_n} (1 - (x(s^-) + \Delta a(s))^{-1} \Delta a(s)). \end{aligned}$$

Thus (3.10) follows immediately and the claim is proved. Using (3.10) inductively in (3.9) we obtain

$$\begin{aligned} P((m(t_0), \dots, m(t_n)) \in e_0 B_0 \times \dots \times e_n B_n) \\ &= E \left(I((Z(0)x(0), x(t_1), \dots, x(t_n)) \in e_0 B_0 \times B_1 \times \dots \times B_n) \right. \\ & \quad \times 2^{-n} \prod_{i=1}^n \left(1 + e'_{i-1} e'_i \exp \left\{ - \int_{t_{i-1}}^{t_i} x(s)^{-1} da^c(s) \right\} \right. \\ & \quad \left. \left. \times \prod_{t_{i-1} < s \leq t_i} (1 - (x(s^-) + \Delta a(s))^{-1} \Delta a(s)) \right) \right), \end{aligned}$$

where $e'_i = e_i$ if $i > 0$ and $e'_0 = Z(0)$. Since $Z(0)$ is independent of (a, x) and $P(Z(0) = 1) = p = 1 - P(Z(0) = -1)$, (3.6) follows immediately. Finally $E(m(0) | a, x) = (2p - 1)x(0)$ a.s. because $m(0) = Z(0)x(0)$ (Lemma 3.4). \square

REMARK 3.12. Note that if B_1, \dots, B_n are any Borel subsets of \mathbf{R} , and $0 \leq t_1 < \dots < t_n$, then $P((m(t_1), \dots, m(t_n)) \in B_1 \times \dots \times B_n)$ is a linear combination of the probabilities obtained in the above theorem. \square

It is now easy to prove Theorem 1.2. We give the proof in the local submartingale setting.

PROOF OF THEOREM 1.2. If $y(t) \geq 0$ is a local submartingale on $(X, \mathcal{G}, Q, \mathcal{G}_t)$ with Doob-Meyer decomposition $y = l + b$ (b is the increasing process), let $(\Omega^1, \mathcal{Q}^1, \bar{P}^1) = (*X, *\mathcal{G}, *\mathcal{Q})$, and for t in $T = \{k\Delta t | k \in *\mathbf{N}_0\}$, let $\mathcal{Q}_t^1 = *\mathcal{G}_t$. Let $(\Omega^1, \mathcal{F}^1, P^1, \mathcal{F}_t^1)$ be the adapted Loeb space generated by $(\Omega^1, \mathcal{Q}^1, \bar{P}^1, \mathcal{Q}_t^1)$. Let $x^1 = \text{st}(*y)$, $n^1 = \text{st}(*l)$ and $a^1 = \text{st}(*b)$. Then by [H.P., Remark 10.10], (x^1, a^1) is a well-defined \mathcal{F}_t^1 -adapted process and satisfies $((x^1, a^1), \mathcal{F}_t^1) \equiv ((y, b), \mathcal{G}_t)$. Moreover, the same result shows

that n^1 is an \mathfrak{F}_t^1 -local martingale and, by Aldous [1, Theorem 19.6], a^1 is an increasing \mathfrak{F}_t^1 -predictable process. Hence $x^1(t)$ is a nonnegative local submartingale on $(\Omega^1, \mathfrak{F}^1, P^1, \mathfrak{F}_t^1)$ with Doob-Meyer decomposition $x^1 = n^1 + a^1$. Construct x and m on $(\Omega, \mathfrak{F}, P, \mathfrak{F}_t)$ as in §2. By Theorem 2.9 m is a local martingale and satisfies

$$(3.15) \quad ((|m|, a), \mathfrak{F}) \equiv ((x^1, a^1), \mathfrak{F}^1) \equiv ((y, b), \mathcal{G}),$$

where a is the increasing process in the Doob-Meyer decomposition of $|m|$. Finally (1.2) and (1.3) follow from Theorem 3.11 by setting $p = \frac{1}{2}$ and noting that (3.15) implies that (y, b) and (x, a) have the same distribution. \square

4. An additional property of m . One could study several other properties of m , such as its jump structure across zero or its quadratic variation, by examining the corresponding internal properties of M . Instead of this we establish a key fact about m (Corollary 4.4) which is shown in [12] to uniquely determine the probabilistic behaviour of the above properties when m is any local martingale satisfying (1.1). In fact, in [12] we show that under an additional hypothesis on x (which is readily checked and is satisfied in most cases of interest) any m satisfying the conditions ((4.4) and (4.5)) of Corollary 4.4, as well as (1.1) and (1.2), must also satisfy (1.3) and hence is unique in law. Moreover in many cases of interest the conditions (4.4) and (4.5) are always satisfied (see §4 of [12]).

The notation is that introduced in §2.

LEMMA 4.1. *If $y(t, \omega) = y^1(t, \pi_1(\omega))$ for some process y^1 on $(\Omega^1, \mathfrak{F}^1, P^1)$ with sample paths in $D(\mathbf{R})$, then*

$$(4.1) \quad P(\exists \underline{t} \in \text{ns}(T) \text{ such that } \phi(\underline{t} + \Delta t) = -1, \\ \circ X(\underline{t}) > 0, \circ \Delta A(\underline{t}) = 0 \text{ and } \Delta y(\circ \underline{t}) \neq 0) = 0$$

and

$$(4.2) \quad P(\exists t \in [0, \infty) \text{ such that } m(t^-)m(t) < 0, \\ \Delta a(t) = 0 \text{ and } \Delta y(t) \neq 0) = 0.$$

PROOF. There is a sequence of \mathfrak{F}^1 -measurable random variables $\{U_n^1\}$ such that $\{t \mid \Delta y^1(t) \neq 0\} \subset \{U_n^1\}$ a.s. The probability in (4.1) is bounded above by

$$P\left(\bigcup_{n=1}^{\infty} \left\{ \exists \underline{t} \in \text{ns}(T) \text{ such that } \underline{t} \approx U_n^1 = U_n^1 \circ \pi, \right. \right. \\ \left. \left. \phi(\underline{t} + \Delta t) = -1, \circ X(\underline{t}) > 0, \circ \Delta A(\underline{t}) = 0 \right\} \right),$$

which is zero by Lemma 2.8. If $m(t^-)m(t) < 0$ then clearly $\phi(\underline{t} + \Delta t) = -1$ for some $\underline{t} \approx t$. Moreover (2.3) implies that $\circ X(\underline{t}) \geq x(t^-) \wedge x(t) > 0$ a.s. (the exceptional null set is independent of t) and hence (4.2) is an immediate consequence of (4.1). \square

Recall that if U is a stopping time with respect to a filtration $\{\mathfrak{F}_t \mid t \geq 0\}$, then $\hat{\mathfrak{F}}_U$ is the σ -field generated by $\{A \cap \{t < U\} \mid A \in \hat{\mathfrak{F}}_t, t \geq 0\} \cup \hat{\mathfrak{F}}_0$.

LEMMA 4.2. Let $(\Omega, \mathfrak{F}, P, \hat{\mathfrak{F}})$ be the adapted Loeb space generated by $(\Omega, \mathcal{A}, \bar{P}, \hat{\mathcal{A}}_t)$, where $\{\hat{\mathcal{A}}_t\}$ is some internal filtration.

(a) Then $U: \Omega \rightarrow [0, \infty)$ is an $\hat{\mathfrak{F}}_t$ -stopping time if and only if it has an $\hat{\mathcal{A}}_t$ -lifting (see 2.2).

(b) If V is an $\hat{\mathcal{A}}_t$ -lifting of U and $U > 0$ a.s., then $\hat{\mathfrak{F}}_{U^-} \subset \sigma(\hat{\mathcal{A}}_V) \vee \mathfrak{N}$, where $\hat{\mathcal{A}}_V = \{A \in \mathcal{A} \mid A \cap \{V = t\} \in \hat{\mathcal{A}}_t \text{ for all } t \text{ in } T\}$ and \mathfrak{N} is the class of P -null sets.

PROOF. The proof of (a) is in [H.P.] (Theorem 4.7(a)).

For (b), fix $t \geq 0$ and A in $\hat{\mathfrak{F}}_t$. By Lemma 2.1 there is a $\underline{t} \approx t$ and $B \in \hat{\mathcal{A}}_{\underline{t}}$ such that $P(A\Delta B) = 0$. Since

$$A \cap \{t < U\} = \bigcup_{m=1}^{\infty} B \cap \{\underline{t} + m^{-1} < V\} \quad \text{a.s.,}$$

and $B \cap \{\underline{t} + m^{-1} < V\} \in \hat{\mathcal{A}}_V$, it follows that $A \cap \{t < U\} \in \sigma(\hat{\mathcal{A}}_V) \vee \mathfrak{N}$. If $t = 0$ we see that $\hat{\mathfrak{F}}_0 \subset \sigma(\hat{\mathcal{A}}_V) \vee \mathfrak{N}$ and hence $\hat{\mathfrak{F}}_{U^-} \subset \sigma(\hat{\mathcal{A}}_V) \vee \mathfrak{N}$. \square

THEOREM 4.3. If

$$(4.3) \quad P(\exists t \geq 0 \ni x(t^-) > 0, x(t) = 0 \text{ and } \Delta a(t) > 0) = 0,$$

then there is a sequence of disjoint predictable stopping times $\{T_i\}$ such that for each local martingale y^1 on $(\Omega^1, \mathfrak{F}^1, P^1, \mathfrak{F}_t^1)$ we have

$$(4.4) \quad \{t \mid \Delta y(t)I(m(t^-)m(t) < 0) \neq 0\} \subset \{T_i\} \quad \text{a.s.}$$

and

$$(4.5) \quad E(\Delta y(T_i)I(m(T_i^-)m(T_i) < 0, T_i < \infty) \mid \mathfrak{F}_{T_i^-}^1) = 0 \quad \text{a.s.}$$

for all i in \mathbf{N} , where $y(t, \omega) = y^1(t, \pi_1 \omega)$.

PROOF. Let $\{T_i^1\}$ be a sequence of disjoint $\{\mathfrak{F}_t^1\}$ -predictable stopping times such that $\{t \mid \Delta a^1(t) \neq 0\} = \{T_i^1 \mid T_i^1 < \infty, i \in \mathbf{N}\}$ a.s., and let $T_i = T_i^1 \circ \pi_i$. Clearly $\{T_i\}$ is a sequence of $\{\mathfrak{F}_t\}$ -predictable times that satisfies (4.4) (by (4.2)). For i fixed, let U and Y be \mathcal{A}^1 -measurable liftings of T_i^1 and $\Delta a^1(T_i^1)I(T_i^1 < \infty)$, respectively, and define

$$V_n^1 = \min\{t \geq U - n^{-1} \mid |\Delta A(t - \Delta t) - Y| \leq n^{-1}\}$$

($\min \emptyset = +\infty$). By saturation there is an \mathcal{A}^1 -measurable $V^1 = V_Y^1: \Omega^1 \rightarrow T \cup \{\infty\}$ (${}^\circ \gamma = \infty$) such that ${}^\circ V^1 = T_i^1$ a.s. and ${}^\circ \Delta A^1(V^1 - \Delta t) = \Delta a^1(T_i^1)$ a.s. on $\{T_i^1 < \infty\}$. Let $V_i = V^1 \circ \pi_1$ and $V_i' = (V_i - \Delta t)^+$.

We claim that

$$I(m(T_i^-)m(T_i) < 0, T_i < \infty) = I(x(T_i^-) > 0, \phi(V_i) = -1, T_i < \infty) \quad \text{a.s.}$$

If $m(T_i^-)m(T_i) < 0$ and $T_i < \infty$ then w.p. 1, $x(T_i^-) > 0$ and $\phi(\underline{t}) = -1$ for some $\underline{t} \approx T_i$. If $\underline{t} \neq V_i$ then w.p. 1, ${}^\circ \Delta A(\underline{t} - \Delta t) = 0$ (recall A is SDJ and ${}^\circ \Delta A(V_i) = \Delta a(T_i)$ a.s. on $\{T_i < \infty\}$), ${}^\circ X(\underline{t} - \Delta t) > 0$ (by (2.3) since $x(T_i^-) \wedge x(T_i) > 0$) and $\phi(\underline{t}) = -1$. By (4.2) this occurs with zero probability and hence $\phi(V_i) = -1$ (a.s.). Conversely if $x(T_i^-) > 0$, $\phi(V_i) = -1$ and $T_i < \infty$, then by (4.3) $x(T_i) > 0$ a.s. and

therefore (2.3) implies that ${}^\circ X(t) > 0$ for all $t \approx T_i$ a.s. Use (4.1) with $y = a$ to see that w.p. 1, $\phi(t) = 1$ for all $t \approx T_i$ such that $t \neq V_i$, and therefore for all t in $(V_i - \epsilon, V_i + \epsilon) - \{V_i\}$ for some real positive $\epsilon(\omega)$. Therefore $m(T_i^-)m(T_i) < 0$ a.s. (by the definition of m) and the claim is proven.

Let $\hat{\mathcal{Q}}_t = \mathcal{Q}^1 \times \mathcal{Q}_t^2$ and note that $\{V'_i = t\} \in \hat{\mathcal{Q}}_0$ and, by Lemma 4.2, $\sigma(\hat{\mathcal{Q}}_{V'_i}) \vee \mathfrak{N} \supset \mathfrak{F}_{T_i^-}$, where \mathfrak{N} is the class of all P -null sets. Therefore by the above result and Lemma 2.1,

$$\begin{aligned} & E(\Delta y(T_i)I(m(T_i^-)m(T_i) < 0, T_i < \infty) | \mathfrak{F}_{T_i^-}) \\ &= E(\Delta y(T_i)I(x(T_i^-) > 0, T_i < \infty) {}^\circ \bar{P}(\phi(V_i) = -1 | \hat{\mathcal{Q}}_{V'_i}) | \mathfrak{F}_{T_i^-}) \quad \text{a.s.} \end{aligned}$$

By considering separately the sets $\{V_i = s\}$, it is clear that on $\{V_i \in T - \{0\}\}$,

$$\bar{P}(\phi(V_i) = -1 | \hat{\mathcal{Q}}_{V'_i}) = \Delta A(V_i - \Delta t)(2(X(V_i - \Delta t) + \Delta A(V_i - \Delta t)))^{-1}.$$

Thus we obtain

$$\begin{aligned} & E(\Delta y(T_i)I(m(T_i^-)m(T_i) < 0, T_i < \infty) | \mathfrak{F}_{T_i^-}) \\ &= E(\Delta y(T_i)I(x(T_i^-) > 0, T_i < \infty)\Delta a(T_i) \\ &\quad \times (2(x(T_i^-) + \Delta a(T_i)))^{-1} | \mathfrak{F}_{T_i^-}) \quad \text{a.s.} \\ &\quad \quad \quad (\text{since } {}^\circ X(V_i - \Delta t) = x(T_i^-) \text{ a.s. by (2.3)}) \\ &= I(x(T_i^-) > 0, T_i < \infty)\Delta a(T_i)(2(x(T_i^-) + \Delta a(T_i)))^{-1}E(\Delta y(T_i) | \mathfrak{F}_{T_i^-}) \\ &= 0 \quad \text{a.s.} \end{aligned}$$

(the last because T_i is predictable). \square

The following corollary is now immediate from the previous result and Lemma 2.5. Recall that $n(t)$ is the local martingale part of the Doob-Meyer decomposition of x .

COROLLARY 4.4. *Assume (4.3) holds. If $\phi: D(\mathbf{R}^2) \rightarrow \mathbf{R}$ is bounded and measurable let $L_\phi(t)$ be a right continuous version of $E(\phi(x, a) | \mathfrak{F}_t)$. Then there is a sequence of predictable stopping times $\{T_i\}$ with disjoint graphs such that (4.4) and (4.5) hold whenever $y \in \{L_\phi | \phi \text{ bounded measurable}\} \cup \{n\}$. \square*

5. Proof of Theorem 2.4. Our setting is that described at the beginning of §2 except that the superscript ¹ is dropped for notational convenience. The following lifting theorem is proved in [H.P.] (Theorem 5.6, Remark 5.7(c), and Remark 4.5(a)).

THEOREM 5.1. *If $y(t)$ is a local martingale with respect to $\{\mathfrak{F}_t\}$ and y is reduced by $\{U_n\}$ (i.e. $y(U_n \wedge t)$ is a uniformly integrable martingale and U_n increase to ∞ a.s.), then there is an internal filtration $\{\mathfrak{B}_t\}$, a sequence of \mathfrak{B}_t -stopping times $\{W_n\}$ and a \mathfrak{B}_t -local martingale lifting of y , Y , such that $\{W_n\}$ reduces Y and ${}^\circ W_n = U_n$ a.s.*

(b) *If $b(t)$ is a right-continuous \mathfrak{F}_t -adapted process with nondecreasing sample paths, such that $b(0) = 0$ a.s., then b has a \mathfrak{B}_t -*-increasing lifting for every internal filtration $\{\mathfrak{B}_t\}$. \square*

We will also use the fact that if $F: T \rightarrow *R$ is an internal function then F is SDJ if and only if F is near-standard in the Skorokhod J_1 topology on $D(R)$, and the mapping st defined in §2 is the standard part map on this class of functions (see [H.P., Theorem 2.6]).

Recall that a is the increasing predictable process in the Doob-Meyer decomposition of the local submartingale x . The following lemma will be used to obtain a lifting of a satisfying (2.2) and (2.3).

LEMMA 5.2. *If $\{\mathfrak{B}_t\}$ is an internal filtration and $\{V_n\}$ is a sequence of \mathfrak{B}_t -stopping times, then $a(t)$ has a \mathfrak{B}_t -*-increasing lifting $A(t)$ such that $A(t + \Delta t)$ is \mathfrak{B}_t -adapted and ${}^\circ A(V_n) = a({}^\circ V_n)$ a.s. on $\{{}^\circ V_n < \infty\}$ for all n .*

PROOF. Extend $\{V_n | n \in N\}$ internally to $*N$ and let $Y(t) = \sum_{n=1}^\beta 2^{-n} I(t \geq V_n)$, where $\beta \in *N - N$. Let $a^d(t) = \sum_{s \leq t} \Delta a(s)$ and $a^c = a - a^d$. There is a disjoint sequence of positive predictable stopping times $\{T_n\}$ that covers the jumps of a , and for each n there is a sequence of stopping times $\{T_m^n | m \in N\}$ that announces T_n . If V_m^n is a \mathfrak{B}_t -lifting of T_m^n , a routine saturation argument shows the existence of a \mathfrak{B}_t -stopping time $V_{\gamma(n)}^n$ ($\gamma(n) \in *N - N$) such that ${}^\circ V_{\gamma(n)}^n = T_n$ a.s. and ${}^\circ Y(V_{\gamma(n)}^n) = st(Y)(T_n^-)$ a.s. Since $\Delta a(T_n)$ is $\mathfrak{F}_{T_n^-}$ -measurable it has a nonnegative $\mathfrak{B}_{V_{\gamma(n)}^n}$ -measurable lifting A^n by Lemma 4.2. Let

$$A_m(t) = \sum_{n=1}^m A^n I(t > V_{\gamma(n)}^n) \quad \text{and} \quad a_m(t) = \sum_{n=1}^m \Delta a(T_n) I(t \geq T_n).$$

If A^d is a \mathfrak{B}_t -*-increasing lifting of a^d (see Theorem 5.1) and ρ denotes a metric inducing the Skorokhod J_1 topology on $D(R)$, then, since $st(A_m) = a_m$ a.s., Theorem 2.6 in [H.P.] implies that

$$\lim_{m \rightarrow \infty} {}^\circ \rho(A_m, A^d) = \lim_{m \rightarrow \infty} \rho(a_m, a^d) = 0 \quad \text{a.s.}$$

Extend $\{(A_m, V_{\gamma(n)}^m, A^m) | m \in N\}$ internally to $*N$ and select α in $*N - N$ such that $A_\alpha(t + \Delta t)$ is \mathfrak{B}_t -adapted, $A_\alpha(t, \omega)$ is nondecreasing on T and is zero at $t = 0$ for all ω , $\Delta A_\alpha(V_{\gamma(n)}^m) = A^m$ for $m \leq \alpha$, and ${}^\circ \rho(A_\alpha, A^d) = 0$. The last condition and Theorem 2.6 in [H.P.] imply that A_α is an SDJ lifting of a^d . If A^c is a \mathfrak{B}_t -*-increasing lifting of a^c then the continuity of a^c implies that ${}^\circ A^c(t) = a^c({}^\circ t)$ for all t in $ns(T)$ a.s. Therefore $A(t) = A^c((t - \Delta t)^+) + A_\alpha(t)$ is a \mathfrak{B}_t -*-increasing lifting of a and $A(t + \Delta t)$ is clearly \mathfrak{B}_t -adapted. Suppose ${}^\circ A(V_m) \neq a({}^\circ V_m)$ and ${}^\circ V_m < \infty$. Then w.p. 1, $\Delta a({}^\circ V_m) > 0$ and therefore ${}^\circ V_m = T_n$ for some n . Since A is SDJ and ${}^\circ \Delta A(V_{\gamma(n)}^n) = {}^\circ A^n = \Delta a(T_n) > 0$ a.s., it follows that $V_m \leq V_{\gamma(n)}^n$ a.s. This implies that ${}^\circ Y(V_{\gamma(n)}^n) \geq {}^\circ Y(V_m) \geq st(Y)(T_n^-) + 2^{-m}$. Recalling that ${}^\circ Y(V_{\gamma(n)}^n) = st(Y)(T_n^-)$ a.s., we see that our assumption can only hold on a null set, that is ${}^\circ A(V_m) = a({}^\circ V_m)$ a.s. on $\{{}^\circ V_m < \infty\}$ for all m , as required. \square

LEMMA 5.3. *If $\{\mathfrak{B}_t\}$ is an internal filtration and A is a \mathfrak{B}_t -*-increasing lifting of a and $\{V_m | m \in N\}$ is a sequence of \mathfrak{B}_t -stopping times such that $\lim_{m \rightarrow \infty} {}^\circ V_m = \infty$ a.s., $E(a({}^\circ V_m)) < \infty$ and ${}^\circ A(V_m) = a({}^\circ V_m)$ a.s., then for some γ in $*N - N$, $(A \wedge \gamma)(V_m)$ is S -integrable for all m in N .*

PROOF. If Y_m is an S -integrable lifting of $a({}^\circ V_m)$, then for all n in \mathbf{N} ,

$$(5.1) \quad \bar{E} \left(\max_{m \leq n} |A(V_m) \wedge n - Y_m \wedge n| \right) < 2^{-n}.$$

Therefore (5.1) holds when n is replaced by some γ in ${}^*\mathbf{N} - \mathbf{N}$ and the result follows. \square

PROOF OF THEOREM 2.4. Let $\{U_n\}$ be a sequence of finite stopping times increasing to ∞ a.s. such that $E(a(U_n)) < \infty$ and $n(U_n \wedge t)$ is a uniformly integrable martingale. By Theorem 5.1 there is an internal filtration $\{\mathfrak{B}_t\}$, a sequence of \mathfrak{B}_t -stopping times $\{W_n\}$, and a \mathfrak{B}_t -local martingale lifting of n, N_1 , such that $\{W_n\}$ reduces N_1 and ${}^\circ W_n = U_n$ a.s. Let $\{W'_n\}$ be a sequence of \mathfrak{B}_t -stopping times such that $\{t \mid {}^\circ |\Delta N_1(t - \Delta t)| > 0\} \subset \{W'_n\}$ a.s. and let A_1 be the \mathfrak{B}_t -*increasing lifting of a obtained in Lemma 5.2 with $\{V_n\} = \{W_n\} \cup \{W'_n\}$. Since for a.a. ω , ${}^\circ W'_n < \infty$ implies ${}^\circ A_1(W'_n) = a({}^\circ W'_n)$, clearly A_1 and N_1 satisfy (2.3). Note that $E(a({}^\circ W_n)) = E(a(U_n)) < \infty$. Therefore Lemma 5.3 allows us to assume that $\{W_n\}$ reduces A_1 and that $\sup_{t, \omega} A_1(t, \omega) \in {}^*\mathbf{R}$, by replacing A_1 with $A_1 \wedge \gamma$ for some γ in ${}^*\mathbf{N} - \mathbf{N}$. If $X_1 = N_1 + A_1$, then X_1 is an SD lifting of x and since N_1 and A_1 satisfy (2.3), ${}^\circ X_1(t) \geq 0$ for all t in $\text{ns}(T)$ a.s. Choose a positive infinitesimal δ such that $X_1(t) + \delta \geq 0$ for all t in $\text{ns}(T)$ a.s. and let $X(t) = (X_1(t) + \delta)^+ + \delta$. Then

$$\begin{aligned} \Delta X(t) &= I(X_1(t) + \delta \geq 0)(\Delta X_1(t) + (X_1(t + \Delta t) + \delta)^-) \\ &\quad + I(X_1(t) + \delta < 0)(X_1(t + \Delta t) + \delta)^+ , \end{aligned}$$

and therefore,

$$(5.2) \quad \begin{aligned} X(t) &= (X(0) + \delta)^+ + \delta + (N_1(t) - N_1(0)) + N_2(t) \\ &\quad + \int I(\underline{s} < t, X_1(\underline{s}) + \delta \geq 0) \, d\nu_{A_1} + A_2(t), \end{aligned}$$

where

$$\begin{aligned} A_2(t) &= \sum_{\underline{s} < t} I(X_1(\underline{s}) + \delta \geq 0)(X_1(\underline{s} + \Delta t) + \delta)^- \\ &\quad + I(X_1(\underline{s}) + \delta < 0)(X_1(\underline{s} + \Delta t) + \delta)^+ \end{aligned}$$

and

$$N_2(t) = - \sum_{\underline{s} < t} I(X_1(\underline{s}) + \delta < 0) \Delta N_1(\underline{s}).$$

The choice of δ implies $A_2(t) = N_2(t) = 0$ for all t in $\text{ns}(T)$ a.s. By (5.2) $\{W_n\}$ reduces $N_2 + A_2$ since it reduces the other terms in (5.2). Since N_2 is an internal martingale, and since we may assume $W_n \leq \gamma$ for some γ in ${}^*\mathbf{N} - \mathbf{N}$ (replace W_n by $W_n \wedge \gamma$), it follows that

$$\begin{aligned} {}^\circ \bar{E}(|N_2(W_n)|) &= 2 {}^\circ \bar{E}(N_2^+(W_n)) \leq 2 {}^\circ \bar{E}(|N_2(W_n) + A_2(W_n)|) \\ &= 2 E({}^\circ |N_2(W_n) + A_2(W_n)|) \quad (\{W_n\} \text{ reduces } N_2 + A_2) \\ &= 0. \end{aligned}$$

Therefore $\{W_n\}$ reduces N_2 and hence also A_2 . If $A_2^p(t) = \sum_{\underline{s} < t} \bar{E}(\Delta A_2(\underline{s}) \mid \mathfrak{B}_s)$, then $\bar{E}(A_2^p(W_n)) = \bar{E}(A_2(W_n)) \approx 0$. Hence, $A_2^p(t) \approx 0$ for all t in $\text{ns}(T)$ a.s. and A_2^p is

reduced by $\{W_n\}$. Define

$$N(\underline{t}) = (X_1(0) + \delta)^+ + \delta + N_1(\underline{t}) - N_1(0) + N_2(\underline{t}) + A_2(\underline{t}) - A_2^p(\underline{t})$$

and

$$A(\underline{t}) = \int I(\underline{s} < \underline{t}, X_1(\underline{s}) + \delta \geq 0) dv_{A_1} + A_2^p(\underline{t}).$$

Then $\{W_n\}$ reduces N and A , $A(\underline{t} + \Delta t)$ is $\mathfrak{B}_{\underline{t}}$ -adapted, and for a.a. ω , $N(\underline{t}) \approx N_1(\underline{t})$ and $A(\underline{t}) \approx A_1(\underline{t})$ for all \underline{t} in $\text{ns}(T)$. It follows that $(X, A, N, \{W_n\})$ satisfies the required conditions. (Note that X satisfies (2.4) since $\sup_{(\underline{t}, \omega)} |X_1(\underline{t}, \omega)| \in \mathbf{*R}$.) \square

ADDED IN PROOF. Recently Hoover and Keisler [15] have shown that if x is a (local) submartingale defined on an adapted Loeb space $(\Omega, \mathfrak{F}, P, \mathfrak{F}_t)$, rich enough to support an \mathfrak{F}_t -Brownian motion then there is a (local) martingale, m , such that $|m| = x$ a.s. That is, there is no need to enlarge such a Loeb space. They use our results in their proof.

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