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BALANCED HOWELL ROTATIONS OF THE TWIN PRIME POWER TYPE

BY

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ABSTRACT. We prove by construction that a balanced Howell rotation for n players always exists if $n = p^r q^s$ where p and $q \neq 3$ are primes and $q^s = p^r + 2$. This generalizes a much weaker previous result. The construction uses properties of a Galois domain which is a direct sum of two Galois fields.

1. Introduction. A balanced Howell rotation for $n = 2k$ players, denoted by BHR(n), consists of a set of n players (denoted by $\infty, 0, 1, \ldots, n-2$) and a set of $n-1$ boards (denoted by $0, 1, \ldots, n-2$). For each board i the n players are divided into k ordered pairs (a_{ij}, b_{ij}), $j = 1, \ldots, k$, where a_{ij} and b_{ij} are said to *oppose* each other on board i, and a_{ij} and each of $a_{ij'}$, $j' \neq j$, are said to *compete* with each other on board i. The partitions on the $n-1$ boards together must also satisfy the following two conditions.

(i) Each player opposes every other player exactly once.

(ii) Each player competes with every other player exactly $k-1$ times.

A BHR(n) can also be represented by an $(n-1) \times n$ array $A = (a_{ij})$ where the rows are boards and the columns are players. Define $a_{ij} = k$ if (j, k) is an opposing pair for board i and define $a_{ij} = -k$ if (k, j) is such a pair. Let A^* be obtained from A by adding a row ∞ such that $a_{\infty j} = j$. Then the signs in A^* constitute a Hadamard matrix, and the numbers in A^* constitute a latin square $L = (l_{ij})$ with the property $l_{ij} = k \Rightarrow l_{ik} = j$ (called a *tournament latin square*). Of course, superimposing a Hadamard matrix on a tournament latin square does not automatically generate a BHR(n) unless for each row $i \neq \infty$, the signs of $a_{ij} = k$ and $a_{ik} = j$ are different for all j.

Direct constructions for BHR(n)'s have been given mostly when n is related to a prime power, for example,

1. $n = P + 1$ where $P = 4k + 3$ is a prime power, $k \geqslant 1$ [**1, 5**].
2. $n = 2(P + 1)$ where $P = 2^m k + 1$ is a prime power, $m \geqslant 1$, $k \geqslant 1$ and k is odd [**2, 4, 6**].

In [**3**], an attempt was made to construct BHR(n)'s when n is related to a product of two prime powers differing by 2 (called *twin prime powers*). More specifically, it was proved (where GF*(P) is the multiplicative group of GF(P)) that

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THEOREM 1 [3]. *A* BHR(n) *exists if*

(i) $n - 1 = PQ$ *where* P *and* Q *are twin prime powers, and*

(ii) *there exist generators* x *of* GF*(P) *and* y *of* GF*(Q) *with* $x^a \equiv 2 \pmod P$, $P - 2 \geqslant a \geqslant 0$, $y^b \equiv 2 \pmod Q$, $Q - 2 \geqslant b \geqslant 0$, *such that one of the following three cases holds*: $b = a + 1$, $(P-1)/2 \geqslant b = a \geqslant 0$, *and* $P - 2 \geqslant b - 2 \geqslant (P+1)/2$.

In this paper we look again into the twin prime power case and prove a much stronger result.

THEOREM 2. *A* BHR(n) *exists if* $n - 1 = PQ = p^r q^s$ *where* P *and* Q *are twin prime powers*, $P < Q$ *and* $q \neq 3$.

2. Some preliminary results. Let x and y generate GF*(p^r) and GF*(q^s), respectively. Let G be the Galois domain (see [7]) $G = \mathrm{GF}(p^r) \oplus \mathrm{GF}(q^s)$ (direct sum), and let $U = \{(u,0)\colon u \in \mathrm{GF}(p^r)\}$, $V = \{(0,v)\colon v \in \mathrm{GF}(q^s)\}$. Define $d = (P-1)(Q-1)/2$. The two cyclotomic classes in G are

$$C_0 = \{(x^i, y^i), i = 0, 1, \ldots, d-1\} = \{(x^i, y^j), i = j \pmod 2\},$$
$$C_1 = \{(-x^i, -y^i), i = 0, 1, \ldots, d-1\} = \{(x^i, y^j), i \neq j \pmod 2\}.$$

It is well known [7] that $C_0 + U$ forms a difference set. Therefore $C_1 + V - \{0\}$ is also a difference set.

Let the n players be denoted by the elements in $G \cup \{\infty\}$. Suppose we can partition the n players into $n/2$ pairs a_i vs. b_i, $i = 1, 2, \ldots, n/2$, which meet the following two requirements.

(R1) $\pm(a_i - a_j)$ over all i, except the pair involving ∞, runs through the set of nonzero elements of G.

(R2) $\pm(a_i - a_j)$, $\pm(b_i - b_j)$ over all a_i, a_j, b_i, b_j, except ∞, covers each nonzero element of G an equal number of times.

Then a cyclic development of this set of $n/2$ pairs (which defines a board) yields a BHR(n), with requirement (R1) guaranteeing condition (i) and requirement (R2) guaranteeing condition (ii), since the cyclic development preserves differences.

By letting $\{a_1, a_2, \ldots, a_{n/2}\} = C_0 + U + \{\infty\}$, $\{b_1, b_2, \ldots, b_{n/2}\} = C_1 + V - \{0\}$, requirement (R2) is automatically satisfied. It suffices to produce a pairing between $\{a_i\}$ and $\{b_j\}$ which satisfies requirement (R1). We first prove some lemmas.

LEMMA 1. *Suppose that* j, k, l, m *satisfy the conditions*

$$x^{2k} + x^j = x^m, \qquad 0 \leqslant m - j \leqslant P - 2, \qquad -2y^{j+2l} = 1.$$

Furthermore, suppose that (i) *when* $0 \leqslant m - j \leqslant (P-1)/2$, *then* $2j + 2l - m - (P+1)/2$ *is either* 0 *or* 1, (ii) *when* $(P-1)/2 \leqslant m - j \leqslant P - 2$, *then* $2j + 2l - m - (P+1)/2$ *is either* 1 *or* 2. *Then there exists a pairing satisfying requirements* (R1) *and* (R2).

PROOF. We demonstrate pairings between elements in $C_0 + U + \{\infty\}$ and elements in $C_1 + V - \{0\}$ satisfying requirement (R1) for both case (i) and case (ii).

Case (i). The pairing is:

(1) (x^{i+2k}, y^i) vs. $(-x^{i+j}, -y^{i+j+2l})$, $(P-1)/2 \leqslant i \leqslant d-1$,

(2) (x^{i+2k}, y^i) vs. $(0, y^i)$, $0 \leqslant i \leqslant (P-3)/2$,

(3) $(-x^{i+j}, 0)$ vs. $(-x^{i+j}, -y^{i+j+2l})$, $0 \leqslant i \nleqslant (P-3)/2$,

(4) $(-x^{i+j}, 0)$ vs. $(0, y^{i+2j+2l-m-(P+1)/2})$,
$(P-1)/2 \leqslant i \leqslant m+(P-3)/2-j$,

(5) $(-x^{i+j}, 0)$ vs. $(0, y^{i+2j+2l-m-(P+1)/2+1})$,
$m+(P-1)/2-j \leqslant i \leqslant P-2$,

(6) $(0,0)$ vs. $(0, y^{j+2l-1})$,

(7) ∞ vs. $(0, y^P)$, if $2j+2l-m-(P+1)/2=0$,
∞ vs. $(0, y^{(P-1)/2})$, if $2j+2l-m-(P+1)/2=1$.

The symmetric differences are:

(1) $\pm(x^{i+m}, -y^{i+j+2l})$, $(P-1)/2 \leqslant i \leqslant d-1$,

(2) $\pm(x^{i+2k}, 0)$, $0 \leqslant i \leqslant (P-3)/2$,

(3) $\pm(0, y^{i+j+2l})$, $0 \leqslant i \leqslant (P-3)/2$,

(4) $\pm(x^{i+j}, y^{i+2j+2l-m-(P+1)/2})$, $(P-1)/2 \leqslant i \leqslant m+(P-3)/2-j$,
$= \pm(x^{i+m}, y^{i+j+2l-(P+1)/2})$, $(P-1)/2-m+j \leqslant i \leqslant (P-3)/2$,
$= \pm(x^{i+m}, -y^{i+j+2l})$, $(P-1)/2-m+j \leqslant i \leqslant (P-3)/2$,

(5) $\pm(x^{i+j}, y^{i+2j+2l-m-(P+1)/2+1})$, $m+(P-1)/2-j \leqslant i \leqslant P-2$,
$= \pm(x^{i+m-(P-1)/2}, y^{i+j+2l})$, $0 \leqslant i \leqslant (P-3)/2-m+j$,
$= \pm(x^{i+m}, -y^{i+j+2l})$, $0 \leqslant i \leqslant (P-3)/2-m+j$,

(6) $\pm(0, y^{j+2l-1}) = (0, -y^{(P-1)/2+j+2l}) = \pm(0, y^{(P-1)/2+j+2l})$.

Case (ii). The pairing is:

(1) (x^{i+2k}, y^i) vs. $(-x^{i+j}, y^{i+j+2l})$, $(P-1)/2 \leqslant i \leqslant d$,

(2) (x^{i+2k}, y^i) vs. $(0, y^i)$, $0 \leqslant i \leqslant (P-3)/2$,

(3) $(-x^{i+j}, 0)$ vs. $(-x^{i+j}, y^{i+j+2l})$, $0 \leqslant i \leqslant (P-3)/2$,

(4) $(-x^{i+j}, 0)$ vs. $(0, y^{i+2j+2l-m-(P+1)/2-1})$, $(P-1)/2 \leqslant i \leqslant m-j-1$,

(5) $(-x^{i+j}, 0)$ vs. $(0, y^{i+2j+2l-m-(P+1)/2})$, $m-j \leqslant i \leqslant P-2$,

(6) $(0,0)$ vs. $(0, y^{j+2l-(P+3)/2})$,

(7) ∞ vs. $(0, y^P)$, if $2j+2l-m-(P+1)/2=1$,
∞ vs. $(0, y^{(P-1)/2})$ if $2j+2l-m-(P+1)/2=2$.

The symmetric differences are:

(1) $$\pm\left(x^{i+m}, y^{i+j+2l}\right), \qquad (P-1)/2 \leqslant i \leqslant d-1,$$

(2) $$\pm\left(x^{i+2k}, 0\right), \qquad 0 \leqslant i \leqslant (P-3)/2,$$

(3) $$\pm\left(0, y^{i+j+2l}\right), \qquad 0 \leqslant i \leqslant (P-3)/2,$$

(4) $$\begin{aligned} &\pm\left(x^{i+j}, y^{i+2j+2l-m-(P+1)/2-1}\right), \qquad (P-1)/2 \leqslant i \leqslant m-j-1,\\ &\quad = \pm\left(x^{i+j}, y^{i+2j+2l-m+(P+1)/2-1}\right), \qquad (P-1)/2 \leqslant i \leqslant m-j-1,\\ &\quad = \pm\left(x^{i+m-(P-1)/2}, y^{i+j+2l}\right), \qquad P-1-m+j \leqslant i \leqslant (P-3)/2,\\ &\quad = \pm\left(x^{i+m}, -y^{i+j+2l}\right), \qquad P-1-m+j \leqslant i \leqslant (P-3)/2, \end{aligned}$$

(5) $$\begin{aligned} &\pm\left(x^{i+j}, y^{i+2j+2l-m-(P+1)/2}\right), \qquad m-j \leqslant i \leqslant P-2,\\ &\quad = \pm\left(x^{i+m}, y^{i+j+2l-(P+1)/2}\right), \qquad 0 \leqslant i \leqslant P-2-m+j,\\ &\quad = \pm\left(x^{i+m}, -y^{i+j+2l}\right), \qquad 0 \leqslant i \leqslant P-2-m+j, \end{aligned}$$

(6) $$\pm\left(0, y^{j+2l-(P+3)/2}\right) = \pm\left(0, y^{(P-1)/2+j+2l}\right).$$

In both cases, it is straightforward to verify that the pairings and the symmetric differences are indeed what we want. Note that if $m-j=(P-1)/2$, then subcases (i)(5) and (ii)(4) do not occur.

LEMMA 2. *Suppose that k, m, z satisfy the following conditions*:

$$x^{2k}+1=x^m, \qquad 0 \leqslant m \leqslant P-2, \qquad 2=y^z.$$

Furthermore, suppose that (i) *when* $0 \leqslant m \leqslant (P-1)/2$, *then* $z-m$ *is either* 0 *or* 1, (ii) *when* $(P-1)/2 \leqslant m \leqslant P-2$, *then* $z-m$ *is either* 1 *or* 2. *Then there exists a pairing satisfying requirements* (R1) *and* (R2).

PROOF. *Case* (i). The pairing is:

(1) $\left(x^{i+2k}, y^i\right)$ vs. $\left(-x^i, -y^i\right)$, $\quad (P-1)/2 \leqslant i \leqslant d-1$,

(2) $\left(x^{i+2k}, y^i\right)$ vs. $\left(0, y^i\right)$, $\quad 0 \leqslant i \leqslant (P-3)/2$,

(3) $\left(-x^i, 0\right)$ vs. $\left(-x^i, -y^i\right)$, $\quad 0 \leqslant i \leqslant (P-3)/2$,

(4) $\left(-x^i, 0\right)$ vs. $\left(0, y^{i+z-m}\right)$, $\quad (P-1)/2 \leqslant i \leqslant (P-3)/2+m$,

(5) $\left(-x^i, 0\right)$ vs. $\left(0, y^{i+z-m+1}\right)$, $\quad (P-1)/2+m \leqslant i \leqslant P-2$,

(6) $(0,0)$ vs. $\left(0, y^P\right)$, $\quad$ if $z-m=0$,
$(0,0)$ vs. $\left(0, y^{(P-1)/2}\right)$, $\quad$ if $z-m=1$,

(7) ∞ vs. $\left(0, y^{z+(P-1)/2}\right)$.

The symmetric differences are:

(1) $$\pm\left(x^{i+m}, y^{i+z}\right), \qquad (P-1)/2 \leqslant i \leqslant d-1,$$

(2) $$\pm\left(x^{i+2k}, 0\right), \qquad 0 \leqslant i \leqslant (P-3)/2,$$

(3) $$\pm\left(0, y^i\right), \qquad 0 \leqslant i \leqslant (P-3)/2,$$

(4) $$\begin{aligned} &\pm(x^i, y^{i+z-m}), \quad (P-1)/2 \leqslant i \leqslant (P-3)/2+m,\\ &\quad = \pm(x^{i+m}, y^{i+z}), \quad (P-1)/2-m \leqslant i \leqslant (P-3)/2, \end{aligned}$$

(5) $$\begin{aligned} &\pm(x^i, y^{i+z-m+1}), \quad (P-1)/2+m \leqslant i \leqslant P-2,\\ &\quad = \pm(x^{i+m}, y^{i+z+1}), \quad (P-1)/2 \leqslant i \leqslant P-2-m,\\ &\quad = \pm(x^{i+m}, y^{i+z}), \quad 0 \leqslant i \leqslant (P-3)/2-m, \end{aligned}$$

(6) $$\begin{aligned} &\pm(0, y^P) \pm (0, y^P) = \pm(0, y^{(P-1)/2}), \quad \text{if } z-m=0,\\ &\pm(0, y^{(P-1)/2}), \quad \text{if } z-m=1. \end{aligned}$$

Case (ii). The pairing is:

(1) (x^{i+2k}, y^i) vs. $(-x^i, -y^i)$, $\quad (P-1)/2 \leqslant i \leqslant d-1$,

(2) (x^{i+2k}, y^i) vs. $(0, y^i)$, $\quad 0 \leqslant i \leqslant (P-3)/2$,

(3) $(-x^i, 0)$ vs. $(-x^i, -y^i)$, $\quad 0 \leqslant i \leqslant (P-3)/2$,

(4) $(-x^i, 0)$ vs. $(0, y^{i+z-m-1})$, $\quad (P-1)/2 \leqslant i \leqslant m-1$,

(5) $(-x^i, 0)$ vs. $(0, y^{i+z-m})$, $\quad m \leqslant i \leqslant P-2$,

(6) $(0,0)$ vs. $(0, y^P)$, $\quad$ if $z-m=1$,
$(0,0)$ vs. $(0, y^{(P-1)/2})$, $\quad$ if $z-m=2$,

(7) ∞ vs. $(0, y^{z-1})$.

The symmetric differences are:

(1) $\pm(x^{i+m}, y^{i+z})$, $\quad (P-1)/2 \leqslant i \leqslant d-1$,

(2) $\pm(x^{i+2k}, 0)$, $\quad 0 \leqslant i \leqslant (P-3)/2$,

(3) $\pm(0, y^i)$, $\quad 0 \leqslant i \leqslant (P-3)/2$,

(4) $$\begin{aligned} &\pm(x^i, y^{i+z-m-1}), \quad (P-1)/2 \leqslant i \leqslant m-1,\\ &\quad = \pm(x^i, y^{i+z-m+P}), \quad (P-1)/2 \leqslant i \leqslant m-1,\\ &\quad = \pm(x^{i+m-(P-1)/2}, y^{i+z-(P+1)/2}), \quad P-1-m \leqslant i \leqslant (P-3)/2,\\ &\quad = \pm(x^{i+m}, y^{i+z}), \quad P-1-m \leqslant i \leqslant (P-3)/2, \end{aligned}$$

(5) $$\begin{aligned} &\pm(x^i, y^{i+z-m}), \quad m \leqslant i \leqslant P-2,\\ &\quad = \pm(x^{i+m}, y^{i+z}), \quad 0 \leqslant i \leqslant P-2-m, \end{aligned}$$

(6) $$\begin{aligned} &\pm(0, y^P) = (0, y^{(P-1)/2}), \quad \text{if } z-m=1,\\ &\pm(0, y^{(P-1)/2}), \quad \text{if } z-m=2. \end{aligned}$$

Note that when $m = (P-1)/2$, then subcases (i)(5) and (ii)(4) do not occur.

3. Proof of Theorem 2. Let x be a generator of $GF^*(P)$. For $u \in GF^*(P)$, define $\log_x u = i$ if $u = x^i$, $0 \leqslant i \leqslant P-2$. Similarly, we can define $\log_y v$ for $v \in GF^*(Q)$. Let $\log_y 2 = z$. Then $z \neq (P+1)/2$ since $2 = y^z = y^{(P+1)/2} = -1$ implies $q = 3$, a contradiction to our assumption. We consider four other possible cases.

Case (i). $1 \leqslant z \leqslant (P-1)/2$, $\log_x(x^z - 1) \equiv 1 \pmod 2$.

Set $j = 0$ or 1 where $j \equiv (P+1)/2 - z \pmod 2$,

$$2l = 3(P+1)/2 - z - j, \qquad 2k = 2j + 2l - 3 + \log_x(x^z - 1),$$
$$m = 2j + 2l - (P+1)/2 - 2.$$

We now verify that the conditions in Lemma 1(ii) are satisfied.

First of all it is easily seen that both $2l$ and $2k$ are even. So k and l are well defined. Furthermore

$$\begin{aligned} x^{2k} + x^j &= x^{2j+2l-3+\log_x(x^z-1)} + x^j \\ &= x^{m+(P-1)/2}(x^z - 1) + x^j = -x^m(x^{3(P+1)/2-j-2l} - 1) + x^j \\ &= -x^m(x^{(P+1)/2-j-2l+2} - 1) + x^j = -x^j + x^m + x^j = x^m, \\ -2y^{j+2l} &= -2y^{3(P+1)/2-z} = -2(-1)\left(\tfrac{1}{2}\right) = 1. \end{aligned}$$

Finally,

$$2j + 2l - m - (P+1)/2 = 2,$$

and

$$m - j = j + 2l - (P+1)/2 - 2 = P + 1 - z - 2 = P - 1 - z$$

imply $(P-1)/2 \leqslant m - j \leqslant P - 2$. Thus Theorem 2 follows from Lemma 1(ii).

Case (ii). $1 \leqslant z \leqslant (P-1)/2$, $\log_x(x^z - 1) \equiv 0 \pmod 2$.

Set $m = z$, $2k = \log_x(x^z - 1)$. We now verify that the conditions in Lemma 2(i) are satisfied. Clearly, $2k$ is even. Furthermore

$$x^{2k} + 1 = x^z - 1 + 1 = x^m.$$

Finally, by our assumptions,

$$y^z = 2, \qquad 0 \leqslant m \leqslant (P-1)/2,$$

and $z - m = 0$.

Case (iii). $(P+3)/2 \leqslant z \leqslant P$, $\log_x(x^{z-2} - 1) \equiv 1 \pmod 2$.

Set $j = 0$ or 1 where $j \equiv (P+1)/2 - z \pmod 2$,

$$2l = 3(P+1)/2 - z - j, \qquad 2k = 2j + 2l - 1 + \log_x(x^{z-2} - 1),$$
$$m = 2j + 2l - (P+1)/2.$$

The verification that the conditions in Lemma 1(i) are satisfied is similar to case (i).

Case (iv). $(P+3)/2 \leqslant z \leqslant P$, $\log_x(x^{z-2} - 1) \equiv 0 \pmod 2$.

Set $m = z - 2$, $2k = \log_x(x^{z-2} - 1)$.

The verification that the conditions in Lemma 2(ii) are satisfied is similar to case (ii). The proof is complete.

4. Examples.

EXAMPLE 1. $n = 16$, $P = 3$, $Q = 5$, $d = 4$.

$x = 2$ and $y = 2$ are generators of GF*(3) and GF*(5), respectively. Since $z = \log_y 2 = 1$ and $\log_x(x^3 - 1) \equiv 0 \pmod 2$, we set

$$m = z = 1, \qquad 2k = \log_x(x^z - 1) = 2,$$

and use the pairing of Lemma 2(i), i.e.,

$$
\begin{array}{ll}
 & (2,2) \text{ vs. } (1,3),\\
(1) & (1,4) \text{ vs. } (2,1),\\
 & (2,3) \text{ vs. } (1,2),\\
(2) & (1,1) \text{ vs. } (0,1),\\
(3) & (2,0) \text{ vs. } (2,4),\\
(4) & (1,0) \text{ vs. } (0,2),\\
(6) & (0,0) \text{ vs. } (0,3),\\
(7) & \infty \text{ vs. } (0,4).
\end{array}
$$

EXAMPLE 2. $n = 36, P = 5, Q = 7, d = 12$.

$x = 2$ and $y = 3$ are generators of GF*(5) and GF*(7), respectively. Since $z = \log_y 2 = 2$ and $\log_x(x^z - 1) \equiv 1 \pmod 2$, we set

$$j = 1 \equiv (P+1)/2 - z \pmod 2, \qquad 2l = 3(P+1)/2 - z - j = 6,$$

$$2k = 2j + 2l - 3 + \log_x(x^z - 1) = 8, \qquad m = 2j + 2l - (P+1)/2 - 2 = 3,$$

and use the pairing of Lemma 1(ii), i.e.,

$$
\begin{array}{llll}
(1) & (4,2) \text{ vs. } (2,1), & (3,6) \text{ vs. } (4,3), & (1,4) \text{ vs. } (3,2),\\
 & (2,5) \text{ vs. } (1,6), & (4,1) \text{ vs. } (2,4), & (3,3) \text{ vs. } (4,5),\\
 & (1,2) \text{ vs. } (3,1), & (2,6) \text{ vs. } (1,3), & \\
 & (4,4) \text{ vs. } (2,2), & (3,5) \text{ vs. } (4,6), & \\
(2) & (1,1) \text{ vs. } (0,1), & (2,3) \text{ vs. } (0,3), & \\
(3) & (3,0) \text{ vs. } (3,4), & (1,0) \text{ vs. } (1,5), & \\
(5) & (2,0) \text{ vs. } (0,4), & (4,0) \text{ vs. } (0,5), & \\
(6) & (0,0) \text{ vs. } (0,6), & & \\
(7) & \infty \quad \text{ vs. } (0,2). & &
\end{array}
$$

REFERENCES

1. E. R. Berlekamp and F. K. Hwang, *Constructions for balanced Howell rotations for bridge tournaments*, J. Combin. Theory Ser. A **12** (1972), 159–166.

2. D.-Z. Du and F. K. Hwang, *Symmetrical skew balanced starters and complete balanced Howell rotations*, Trans. Amer. Math. Soc. **271 (1982), 409–413.**

3. F. K. Hwang, *New constructions for balanced Howell rotations*, J. Combin. Theory Ser. A **21** (1976), 44–51.

4. F. K. Hwang, Q. D. Kang and J. E. Yu, *Complete balanced Howell rotations for* $16k + 12$ *teams* (to appear).

5. E. T. Parker and A. N. Mood, *Some balanced Howell rotations for duplicate bridge sessions*, Amer. Math. Monthly **62** (1955), 714–716.

6. P. J. Schellenberg, *Constructions for (balanced) Room squares*, Aequationes Math. **9** (1973), 75–90.

7. T. Storer, *Cyclotomy and difference sets*, Markham, Chicago, Ill., 1967.

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