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FREE PRODUCTS OF C^*-ALGEBRAS

BY

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ABSTRACT. Small ("spatial") free products of C^*-algebras are constructed. Under certain conditions they have properties similar to those proved by Paschke and Salinas for the algebras $C_r^*(G_1 * G_2)$ where G_1, G_2 are discrete groups. The free-product analogs of noncommutative Bernoulli shifts are discussed.

0. Introduction. Let K be a field. Consider the category of unital algebras over K. It is well known that this category admits coproducts: free products of algebras [2]. Heuristically, the free product of algebras is the algebra generated by them, with no relations except for the identification of unit elements.

If $K = \mathbf{C}$, the complex numbers, and we consider unital $*$-algebras, we can easily define a $*$-operation on the free products.

Let A, B be unital C^*-algebras, and $A * B$ their free product, which is a unital $*$-algebra. The question arises: in what ways may one define a pre-C^* norm on $A * B$ that extends the norms on A and B? Guided by analogy with tensor products, we expect to have a choice among many pre-C^* norms, giving rise to many "C^* free products" of A and B.

One natural norm is

$$\|c\| = \sup\{\|\pi(c)\| : \pi \ * \text{-representation of } A * B\}.$$

The $*$-representations of $A * B$ are in 1-1 correspondence with pairs of $*$-representations of A and B, which act on the same Hilbert space. Let $A \check{*} B$ be the completion of $A * B$ in this norm. It is easy to see that this construction defines a coproduct in the category of C^*-algebras, and that $A \check{*} B$ is the "biggest free product" of A and B, analogous to the biggest tensor product $A \check{\otimes} B$. If G_1, G_2 are discrete groups we obtain

$$C^*(G_1) \check{*} C^*(G_2) \simeq C^*(G_1 * G_2)$$

where $G_1 * G_2$ is the free product group, and this is analogous to the relation

$$C^*(G_1) \check{\otimes} C^*(G_2) \simeq C^*(G_1 \times G_2).$$

This paper is motivated by the question: Is there a "smallest C^*-product", $A \star B$, in analogy to the smallest tensor product, $A \check{*} B$, satisfying a relation

$$C_r^*(G_1 * G_2) \simeq C_r^*(G_1) \star C_r^*(G_2)$$

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analogous to

$$C_r^*(G_1 \times G_2) \simeq C_r^*(G_1) \underset{*}{\otimes} C_r^*(G_2)$$

for G_1, G_2 discrete groups?

There seems to be no natural way to define a free product of representations of two C^*-algebras on Hilbert spaces. Therefore, the question as posed must be answered in the negative. However, if one considers representations together with designated cyclic vectors, one can define a "free product representation" with a designated cyclic vector. Equivalently: given states ϕ, ψ on C^*-algebras A, B, there is a naturally defined state $\phi * \psi$ on $A \check{*} B$ in analogy to the tensor product state $\phi \otimes \psi$ on $A \check{\otimes} B$. This state is defined in §§1, 2. If ϕ_i is the standard trace on $C_r^*(G_i)$ for $i = 1, 2$ then $\phi_1 * \phi_2$ is the standard trace on $C_r^*(G_1 * G_2)$.

The representations corresponding to the states $\phi_1 * \phi_2$, for different choices of faithful states ϕ_1, ϕ_2, are *not* weakly equivalent, so we have not one "small free product" but a whole family of them.

The results of Paschke and Salinas [**8**] on the algebras $C_r^*(G_1 * G_2)$ can sometimes be generalized to the small C^*-free products defined here. This is done in §3. Thus, the construction of small C^*-free products gives new examples of simple, uniquely traced C^*-algebras.

§4 discusses some automorphisms of free products that are analogous to Bernoulli shifts on tensor products.

All C^*-algebras in this paper are assumed to be unital. Results for nonunital algebras may be obtained by adjoining units. This is, of course, the case for tensor products also. For a general reference on tensor products of C^*-algebras see [**9**].

1. Free products of linear functionals on algebras.

1.0. In this section let A, B be unital algebras over a field K. We identify the units of K, A, B, $A * B$ so $K \subseteq A \cap B \subseteq A \cup B \subseteq A * B$. Let ϕ, ψ be linear functionals on A, B respectively, such that $\phi(1) = \psi(1) = 1$, and let $A_0 = \ker \phi$, $B_0 = \ker \psi$.

1.1. PROPOSITION. *There is a unique linear functional*, $\phi * \psi$ *on* $A * B$, *such that* $(\phi * \psi)(1) = 1$ *and*

$$(\phi * \psi)(c_1 \cdots c_m) = 0,$$

where $m \geq 1$, $c_i \in A_0 \cup B_0$ *and for all* i, $c_i \in A_0$ *iff* $c_{i+1} \in B_0$.

PROOF. Products $c_1 \cdots c_m$, where $c_i \in A$ iff $c_{i+1} \in B$ for all i, span $A * B$. I will define $\phi * \psi$ for such products, then show that the definition is consistent with linearity and with the identification of units.

First let $\phi * \psi$ be defined as ϕ on A, and ψ on B. We already have $(\phi * \psi)(1) = 1$. Now for products $c_1 \cdots c_m$ where $m > 1$ let

$$\begin{aligned}(\phi * \psi)(c_1 \cdots c_m) = &\sum_{1 \leq i \leq m} (\phi * \psi)(c_i)(\phi * \psi)(c_1 \cdots \check{c}_i \cdots c_m) \\ &- \sum_{1 \leq i < j \leq m} (\phi * \psi)(c_i)(\phi * \psi)(c_j)(\phi * \psi)(c_1 \cdots \check{c}_i \cdots \check{c}_j \cdots c_m) \\ &+ \cdots + (-1)^{m+1}(\phi * \psi)(c_1) \cdots (\phi * \psi)(c_m),\end{aligned}$$

where the elements with $\check{}$ on top are to be omitted, and where the products containing them are simplified only by multiplying adjacent elements of the same algebra (not by identifying units). The products obtained on the right-hand side have length smaller than m, so this is a well-defined recursion. For motivation, note that the recursion is equivalent to

$$(\phi * \psi)[(c_1 - (\phi * \psi)(c_1)) \cdots (c_m - (\phi * \psi)(c_m))] = 0.$$

This also shows that if $\phi * \psi$ is well defined on $A * B$, then it satisfies the requirement of the proposition and is unique.

We still have to show that the definition is consistent with the identification of units, so that $\phi * \psi$ is actually defined on $A * B$ where such an identification is assumed. We have to prove that

$$(\phi * \psi)(c_1 \cdots c_{k-1} \cdot 1 \cdot c_{k+1} \cdots c_m) = (\phi * \psi)(c_1 \cdots (c_{k-1}c_{k+1}) \cdots c_m).$$

Use induction on m. By definition,

$$\begin{aligned}(\phi * \psi)&(c_1 \cdots c_{k-1} \cdot 1 \cdot c_{k+1} \cdots c_m)\\ &= \Big[\sum_{i \neq k} (\phi * \psi)(c_i)(\phi * \psi)(c_1 \cdots \check{c}_i \cdots c_m)\\ &\qquad + (\phi * \psi)(1)(\phi * \psi)(c_1 \cdots (c_{k-1}c_{k+1}) \cdots c_m)\Big]\\ &\quad - \Big[\sum_{k \neq i < j \neq k} (\phi * \psi)(c_i)(\phi * \psi)(c_j)(\phi * \psi)(c_1 \cdots \check{c}_i \cdots \check{c}_j \cdots c_m)\\ &\qquad + \sum_{i<k} (\phi * \psi)(c_i)(\phi * \psi)(1)(\phi * \psi)(c_1 \cdots \check{c}_i \cdots \check{1} \cdots c_m)\\ &\qquad + \sum_{k<j} (\phi * \psi)(1)(\phi * \psi)(c_j)(\phi * \psi)(c_1 \cdots \check{1} \cdots \check{c}_j \cdots c_m)\Big]\\ &\quad + \cdots .\end{aligned}$$

Now all the terms except the second term in the first brackets telescope to zero. One example will suffice:

$$\begin{aligned}&\sum_{i \neq k} (\phi * \psi)(c_i)(\phi * \psi)(c_1 \cdots \check{c}_i \cdots c_m)\\ &\quad - \sum_{i<k} (\phi * \psi)(c_i)(\phi * \psi)(1)(\phi * \psi)(c_1 \cdots \check{c}_i \cdots \check{1} \cdots c_m)\\ &\quad - \sum_{k<j} (\phi * \psi)(1)(\phi * \psi)(c_j)(\phi * \psi)(c_1 \cdots \check{1} \cdots \check{c}_j \cdots c_m) = 0,\end{aligned}$$

because by induction on m, the 1 in the kth place can be erased from all the terms in the first sum, and because $(\phi * \psi)(1) = 1$.

After all this telescoping we are left with the second term in the first brackets which equals

$$(\phi * \psi)(c_1 \cdots (c_{k-1}c_{k+1}) \cdots c_m). \quad \text{Q.E.D.}$$

1.2. PROPOSITION. *As linear spaces*

$$A * B = K \oplus A_0 \oplus B_0 \oplus (A_0 \otimes B_0) \oplus (B_0 \otimes A_0) \oplus (A_0 \otimes B_0 \otimes A_0) \oplus \cdots.$$

PROOF. Let $c_1 \cdots c_m$ be a product of elements of $A \cup B$ such that $c_i \in A$ iff $c_{i+1} \in B$. Write: $c_i = d_i + \alpha_i$ where d_i is in A_0 or B_0 and $\alpha_i \in \mathbf{C}$. Opening brackets in the product $(d_1 + \alpha_1) \cdots (d_m + \alpha_m)$, we get $d_1 \cdots d_m$, which is in the right-hand side of the equality in the proposition, and other terms that can be written as products of length less than m. So, $A * B$ is a sum of the linear spaces on the right-hand side.

We have to show that the sum is direct. Using only the identification $K \otimes V \simeq V$ for linear spaces V over K, we can write the sum as

$$\begin{aligned}(K \oplus A_0) \otimes (K + B_0 + B_0 \otimes A_0 + \cdots)\\ = (K \oplus A_0) \otimes (K \oplus (B_0 + B_0 \otimes A_0 + \cdots)).\end{aligned}$$

The first summation in the second factor is direct by the existence of $\phi * \psi$. Now continue:

$$\begin{aligned}B_0 + B_0 \otimes A_0 + B_0 \otimes A_0 \otimes B_0 + \cdots &= B_0 \otimes (K + A_0 + A_0 \otimes B_0 + \cdots)\\ &= B_0 \otimes (K \oplus (A_0 + A_0 \otimes B_0 + \cdots)),\end{aligned}$$

where, again, the existence of $\phi * \psi$ was used to make one summation sign direct. We already have

$$A * B = K \oplus A_0 \oplus B_0 \oplus (B_0 \otimes A_0) \oplus \cdots.$$

Continuing this process, we obtain the result stated in the proposition. Q.E.D.

We shall call a product $c_1 \cdots c_m$ as in Proposition 1.1 a *word*. Note that $\phi * \psi$ is zero on words and every element in the kernel of $\phi * \psi$ is a sum of words.

1.3. The linear functional $\phi * \psi$ can be seen as the projection onto K that is associated with the direct sum 1.2. We will consider some other interesting projections.

PROPOSITION. *Let P_A: $A * B \to K \oplus A_0 = A$ be the projection associated with the direct sum* 1.2. *P_A is a "conditional expectation", i.e.,*

$$P_A(ac) = aP_A(c), \qquad P_A(ca) = P_A(c)a$$

for all $a \in A$ and $c \in A * B$.

PROOF. It is enough to prove the identities with $a \in A_0$ and c an elementary tensor in 1.2. We will prove the first identity. The other is similar.

If c starts with an element of B_0, then $P_A(ac) = 0 = aP_A(c)$.

If c starts with $A_0 \otimes B_0$, let $c = a_1 b_1 d$ where $a_1 \in A_0$, $b_1 \in B_0$ and $d \in K$ or d starts with A_0. So,

$$ac = aa_1 b_1 d = (aa_1 - \phi(aa_1))b_1 d + \phi(aa_1)b_1 d.$$

Hence, $P_A(ac) = 0 = aP_A(c)$.

Finally, if $c \in A_0$, then

$$P_A(ac) = ac = aP_A(c). \quad \text{Q.E.D.}$$

1.4. PROPOSITION. *If ϕ, ψ are tracial, i.e., $\phi(a_1a_2) = \phi(a_2a_1)$, and similarly for ψ, then $\phi * \psi$ is tracial.*

PROOF. It is enough to show that $(\phi * \psi)(ac) = (\phi * \psi)(ca)$ for $a \in A_0$, c an elementary tensor in 1.2. We also need a similar identity with $b \in B_0$ instead of a, but the proof is identical.

Case 1. If c begins and ends with B_0,

$$(\phi * \psi)(ac) = 0 = (\phi * \psi)(ca).$$

Case 2. If c begins with B_0 and ends with A_0 (the opposite case is similar), let $c = b_1 d a_1$ where $a_1 \in A_0$, $b_1 \in B_0$, and d begins with A_0 and ends with B_0 or d is empty:

$$(\phi * \psi)(ac) = (\phi * \psi)(ab_1da_1) = 0,$$

$$\begin{aligned}(\phi * \psi)(ca) &= (\phi * \psi)(b_1dd_1a)\\ &= (\phi * \psi)(b_1d(a_1a - \phi(a_1a))) + (\phi * \psi)(\phi(a_1a)b_1d) = 0.\end{aligned}$$

Case 3. If c begins and ends with A_0, let $c = a_1da_2$, where $a_1, a_2 \in A_0$ and d begins and ends with B_0:

$$\begin{aligned}(\phi * \psi)(ac) &= (\phi * \psi)(aa_1da_2)\\ &= (\phi * \psi)((aa_1 - \phi(aa_1))da_2) + \phi(aa_1)(\phi * \psi)(da_2) = 0,\end{aligned}$$

and similarly $(\phi * \psi)(ca) = 0$.

Case 4. If $c \in A_0$,

$$(\phi * \psi)(ac) = \phi(ac) = \phi(ca) = (\phi * \psi)(ca). \quad \text{Q.E.D.}$$

1.5. The definitions and claims in this section extend to free products of any number, finite or infinite, of algebras.

The decomposition 1.2 becomes

$$* \{A^{(i)}: i \in I\} = \oplus \left\{A_0^{(i_1)} \otimes \cdots \otimes A_0^{(i_k)}: k \geqslant 0, i_j \neq i_{j+1} \text{ for all } j\right\},$$

where $A_0^{(i)} = \ker \phi^{(i)} \subseteq A^{(i)}$. For $k = 0$ the direct summand is K by convention.

An elementary tensor in $A_0^{(i_1)} \otimes \cdots \otimes A_0^{(i_n)}$ will be called a tensor of *type* $(i_1, \ldots, i_n)$. If $(i_1, \ldots, i_n) \neq \varnothing$, we will call such a tensor a *word*.

2. Small free products of C^*-algebras. Let A, B be C^*-algebras and ϕ, ψ states on them. It is not clear that $\phi * \psi$, defined in §2, is positive or that it extends to $A \mathbin{\bar{*}} B$. To show that, I will construct directly the representation $\pi_{\phi * \psi}$ of $A \mathbin{\bar{*}} B$, corresponding to the state $\phi * \psi$.

2.1. Let $\{A^{(i)}: i \in I\}$ be C^*-algebras and ϕ_i states on them. Let $(\pi_i, \omega_i, H^{(i)})$ be the corresponding GNS objects. Let $H_0^{(i)} = \{\omega_i\}^{\perp} \subseteq H^{(i)}$. Consider the Hilbert space

$$H = \oplus \left\{H_0^{(i_1)} \otimes \cdots \otimes H_0^{(i_n)}: n \geqslant 0, i_j \neq i_{j+1} \text{ for all } j\right\}.$$

For $n = 0$, the direct summand is $\mathbf{C}$. $1 \in \mathbf{C}$, considered as a vector in H, will be denoted by Ω.

Given $i \in I$, there are natural identifications

$$H \simeq \left(\mathbf{C} \oplus H_0^{(i)}\right) \otimes \left(\bigoplus \left\{H_0^{(i_1)} \otimes \cdots \otimes H_0^{(i_n)}: i_1 \neq i\right\}\right)$$
$$\simeq H^{(i)} \otimes \left(\bigoplus \left\{H_0^{(i_1)} \otimes \cdots \otimes H_0^{(i_n)}: i_1 \neq i\right\}\right).$$

Let $U_i\colon H \to H^{(i)} \otimes (\bigoplus\{H_0^{(i_1)} \otimes \cdots \otimes H_0^{(i_n)}: i_1 \neq i\})$ be the isometry implementing this identification. For v an elementary tensor, $U_i v = v$ if v starts with $H_0^{(i)}$ and $U_i v = \omega_i \otimes v$ if v starts with $H_0^{(j)}$ for some $j \neq i$. $U_i \Omega = \omega_i$. Now let $A^{(i)}$ act on H by a multiple of π_i; formally: $a \in A^{(i)}$ acts by $U_i^*(\pi_i(a) \otimes I)U_i$. Denote by π the resulting representation of $*\{A^{(i)}: i \in I\}$ on H.

PROPOSITION. (i) Ω *is a* π*-cyclic vector.*

(ii) *For all* $c \in *\{A^{(i)}: i \in I\}$,

$$\langle \pi(c)\Omega, \Omega \rangle = (*\phi_i)(c).$$

(iii) *Let* $\bar{i} = (i_1, \ldots, i_k)$ *be a type* (*see* 1.5). *Let* $P_{\bar{i}}$ *be the projection of* $*\{A^{(i)}: i \in I\}$ *onto the span of tensors of type* $\bar{i}$ *that is associated with the decomposition* 1.2. *Let* $p_{\bar{i}}$ *be the projection of* H *onto* $H_0^{(i_1)} \otimes \cdots \otimes H_0^{(i_k)}$.

For all $c \in *\{A^{(i)}: i \in I\}$,

$$p_{\bar{i}}(\pi(c)\Omega) = \pi(P_{\bar{i}}(c))\Omega.$$

PROOF. Let $c \in A_0^{(i)}$ and let $v = v_{i_1} \otimes \cdots \otimes v_{i_k} \in H_0^{(i_1)} \otimes \cdots \otimes H_0^{(i_k)}$ such that $i_1 \neq i$. From the definition of π,

$$\pi(c)v = \pi_i(c)\omega_i \otimes v.$$

Repeating this m times we get: if $c = c_1 \cdots c_m$ is a tensor of type $(i_1, \ldots, i_m)$ then

$$\pi(c)\Omega = \pi_{i_1}(c_1)\omega_{i_1} \otimes \cdots \otimes \pi_{i_m}(c_m)\omega_{i_m}.$$

All the claims of the proposition follow immediately from this formula, and from the characterization 1.1 of $*\phi_i$. Q.E.D.

2.2. PROPOSITION. *Let* A, B *be* C^**-algebras and* ϕ, ψ *states on them.* $P_A\colon A * B \to A$, *defined in* 1.3, *extends to a conditional expectation* $P_A\colon \pi_{\phi*\psi}(A \check{*} B) \to \pi_{\phi*\psi}(A)$.

PROOF. Let $c \in A * B$. Since $\pi_{\phi*\psi|A}$ is a multiple of π_ϕ,

$$\begin{aligned}
\|\pi_{\phi*\psi}(P_A(c))\| &= \|\pi_\phi(P_A(c))\| \\
&= \sup\left\{\|\pi_\phi(P_A(c))\pi_\phi(a)\omega_1\|: \phi(a^*a) \leqslant 1\right\} \\
&= \sup\left\{\|\pi_\phi(P_A(c)a)\omega_1\|: \phi(a^*a) \leqslant 1\right\} \\
&= \sup\left\{\|\pi_{\phi*\psi}(P_A(c)a)\Omega\|: \phi(a^*a) \leqslant 1\right\} \\
&= \sup\left\{P_{\mathbf{C}\Omega \oplus H_0^{(1)}}\left(\pi_{\phi*\psi}(ca)\Omega\right)\|: \phi(a^*a) \leqslant 1\right\}
\end{aligned}$$

using 2.1(iii). Continuing,

$$\begin{aligned}
&\leqslant \sup\left\{\|\pi_{\phi*\psi}(ca)\Omega\|: \phi(a^*a) \leqslant 1\right\} \\
&\leqslant \sup\left\{\|\pi_{\phi*\psi}(c)\| \, \|\pi_{\phi*\psi}(a)\Omega\|: \phi(a^*a) \leqslant 1\right\} \\
&= \|\pi_{\phi*\psi}(c)\|. \quad \text{Q.E.D.}
\end{aligned}$$

2.3. If $\pi_{\phi*\psi}(A \check{*} B)$ is to be considered as a C^*-free product of A and B, then $A * B$ should be (faithfully) contained in it, that is, $\pi_{\phi*\psi}$ must be faithful on $A * B$. One case in which this happens is

PROPOSITION. *If the states ϕ, ψ on A, B are faithful, then Ω is a separating vector for $A * B$. In particular, $\pi_{\phi*\psi}$ is faithful on $A * B$.*

PROOF. By 2.1(iii) it is enough to check that $\pi_{\phi*\psi}(c)\Omega = 0$ implies $c = 0$, for c which is a sum of words of the same type. For ease of notation I will carry out the proof for some particular type, say $c = \sum_{i=1}^n a_i b_i a_i'$ where $a_i, a_i' \in A_0$ and $b_i \in B_0$.

Assume $\pi_{\phi*\psi}(c)\Omega = 0$. By the formula in the proof of 2.1 this means

$$\sum_{i=1}^n \pi_\phi(a_i)\omega_\phi \otimes \pi_\psi(b_i)\omega_\psi \otimes \pi_\phi(a_i')\omega_\phi = 0,$$

which can be written as

$$(\pi_\phi \otimes \pi_\psi \otimes \pi_\phi)\left(\sum_{i=1}^n a_i \otimes b_i \otimes a_i'\right)(\omega_\phi \otimes \omega_\psi \otimes \omega_\phi) = 0.$$

Now, $\omega_\phi \otimes \omega_\psi \otimes \omega_\phi$ is a separating vector for $\pi_\phi \otimes \pi_\psi \otimes \pi_\phi$ even on $A \underset{*}{\otimes} B \underset{*}{\otimes} A$ (see Appendix). So we get $\sum_{i=1}^n a_i \otimes b_i \otimes a_i' = 0$, so $c = 0$. Q.E.D.

It is not sufficient that π_ϕ, π_ψ be faithful in order for $\pi_{\phi*\psi}$ to be faithful on $A * B$. See 3.3 for an example. I do not know whether the proposition can be strengthened to give the result that $\phi * \psi$ is faithful on $\pi_{\phi*\psi}(A \check{*} B)$. Note, however, that if ϕ, ψ are traces, then $\phi * \psi$ is a trace, and is therefore faithful even on $\pi_{\phi*\psi}(A * B)''$.

2.4. Let G_1, G_2 be discrete groups. Let ϕ_1, ϕ_2 be the standard traces on $C_r^*(G_1)$, $C_r^*(G_2)$, such that $\phi_i(s) = 0$ for all $s \in G_i$ except $s = e$. By 1.1, $(\phi_1 * \phi_2)(t) = 0$ for all $t \in G_1 * G_2$ except $t = e$. This means that $\phi_1 * \phi_2$ is the standard trace of $G_1 * G_2$ and

$$C_r^*(G_1 * G_2) \simeq \pi_{\phi_1*\phi_2}(C_r^*(G_1) \check{*} C_r^*(G_2)).$$

This is analogous to the relation

$$C_r^*(G_1 \times G_2) \simeq C_r^*(G_1) \underset{*}{\otimes} C_r^*(G_2).$$

It is known that $C_r^*(G_1 * G_2)$ has a unique trace, unless $G_1 = G_2 = Z_2$ **[8]**. If ψ_1, ψ_2 are traces on $C_r^*(G_1)$, $C_r^*(G_2)$ that are not both equal to ϕ_1, ϕ_2 respectively, then $\psi_1 * \psi_2$ cannot be defined on $C_r^*(G_1 * G_2)$ because of the uniqueness of trace. It follows that $\pi_{\phi_1*\phi_2}$, $\pi_{\psi_1*\psi_2}$ are not weakly equivalent even if ψ_1, ψ_2 are faithful. This is in contrast to the situation in tensor products, and shows that we cannot define a unique "smallest C^*-free product".

3. Simplicity and uniqueness of trace.

3.0. It is known **[8]** that if G_1, G_2 are nontrivial discrete groups, not both equal to Z_2, then $C_r^*(G_1 * G_2)$ is simple, and has a unique trace. In this section, I will show how the same computations give a similar result for $\pi_{\phi*\psi}(A \check{*} B)$ under certain assumptions. I do not claim that these assumptions are relevant to the results, or that the results are definitive. My aim is merely to demonstrate that the computations in **[8]** do not depend on group structure, but on C^*-free product structure.

The following lemma, due to Choi [3], will be used.

LEMMA. *let $H = H_0 \oplus H_1$ be a Hilbert space. Let $u_1, \ldots, u_n$ be unitaries on H such that $u_i^* u_j H_1 \subseteq H_0$ for $i \neq j$. Let b be an operator on H such that $bH_0 \subseteq H_1$. Then*

$$\left\| \frac{1}{n} \sum_{i=1}^{n} u_i b u_i^* \right\| \leqslant \frac{2\|b\|}{n^{1/2}}.$$

3.1. PROPOSITION. *Let A, B be C^*-algebras and ϕ, ψ states on them. Let $a \in A_0$ and $b \in B_0$ be unitaries such that ϕ, ψ are invariant with respect to conjugation by a, b respectively. Let $c \in B_0$ be unitary such that $\psi(b^*c) = 0$.*

Then for all $x \in \pi_{\phi * \psi}(A \check{*} B)$.

$$(\phi * \psi)(x) \in \operatorname{cl\,conv}\{u^*xu\colon u \text{ unitary}\}$$

where cl conv *denotes norm-closed convex hull. It is enough to take u generated by a, b, c.*

PROOF. Let $W_0 \subseteq A * B$ be the span of words starting with A_0 or a multiple of b, and of constants. Let W_1 be the span of words starting with some $b' \in B_0$ such that $\psi(b^*b') = 0$. Let

$$H_i = \overline{\pi_{\phi*\psi}(W_i)\Omega} \subseteq H_{\phi*\psi}.$$

Then $H_{\phi*\psi} = H_0 \oplus H_1$.

I claim: $(ba)^k H_1 \subseteq H_0$ for all $k \neq 0$. It is enough to show that $(ba)^k W_1 \subseteq W_0$. Indeed if $k > 0$, then $(ba)^k W_1$ is spanned by words starting with b. If $k < 0$ then

$$(ba)^k W_1 = (a^*b^*)^{-k} W_1,$$

which is spanned by words starting with a^*. Note that the last b^* in $(a^*b^*)^{-k}$ combines with any $b' \in B_0$ in the beginning of words in W_1 to give $b^*b' \in B_0$.

Now, let x be a word in $A * B$, and let j be an integer, bigger than the length of x. Since ϕ, ψ are a, b invariant, so is $\phi * \psi$, and we get

$$(\phi * \psi)\big((ab)^j x (ab)^{-j}\big) = 0.$$

So $(ab)^j x (ab)^{-j}$ is a sum of words. From the fact that j is bigger than the length of x, we get by a routine reduction that $(ab)^j x (ab)^{-j}$ is a linear combination of three kinds of words:

(1) $(ab)^n$, $n > 0$.

(2) $(b^*a^*)^n$, $n > 0$.

(3) Words starting with a and ending with a^*.

Using the facts that

$$\psi(bc^*) = \psi(c^*b) = \overline{\psi(b^*c)} = 0$$

and

$$\psi(cb^*) = \psi(b^*c) = 0$$

based on b-invariance of ψ, we get that

$$cac\big[(ab)^j x (ab)^{-j}\big] c^*a^*c^*$$

is a linear combination of words starting with c and ending with c^*. Hence

$$cac\left[(ab)^j x(ab)^{-j}\right]c^*a^*c^*W_0 \subseteq W_1,$$

Using again $\psi(c^*b) = 0$. So

$$cac\left[(ab)^j x(ab)^{-j}\right]c^*a^*c^*H_0 \subseteq H_1.$$

The same is true if x is a linear combination of words of length less than j.

Now using Lemma 3.0 we get

$$\left\| \frac{1}{N} \sum_{k=1}^{N} (ba)^k cac(ab)^j x\left((ba)^k cac(ab)^j\right)^* \right\| \leq \frac{2\|x\|}{N^{1/2}}.$$

By a routine approximation argument, x can be taken to be any element of $\pi_{\phi*\psi}(A \check{*} B)$ such that $(\phi * \psi)(x) = 0$. Q.E.D.

COROLLARY. (1) *If $\phi * \psi$ is faithful on $\pi_{\phi*\psi}(A \check{*} B)$ then this algebra is simple.*

(2) *Either $\phi * \psi$ is the only a, b, c-invariant state on $\pi_{\phi*\psi}(A \check{*} B)$ or there is no such state, depending on whether ψ is c-invariant.*

(3) *If ϕ, ψ are traces, then $\pi_{\phi*\psi}(A \check{*} B)$ is simple, with unique trace.*

Note that the corollary assumes the existence of a, b, c as in the proposition.

3.2. The conditions of 3.1 are not always satisfied, even if ϕ, ψ are traces. There may be no unitaries with zero trace. For example, if $A = \mathbf{C}^2$, and ϕ is the trace corresponding to the measure $\{p, 1-p\}$, $0 \leq p \leq 1$, there is a unitary with zero trace iff $p = \frac{1}{2}$. The conditions are satisfied in some interesting cases discussed in 3.4 including $M_n * M_m$ with respect to the traces.

3.3. The following example illustrates the relevance of the invariance assumptions for ϕ, ψ in 3.2. It also shows that even when π_ϕ, π_ψ are faithful, $\pi_{\phi*\psi}$ may not be faithful on $A * B$.

First let S be a discrete semigroup with unit e. Let $C_r^*(S)$ be the C^*-algebra on $l^2(S)$ generated by the operators of left multiplication by elements of S. The vector $e \in l^2(S)$ is cyclic, and the corresponding state δ_e satisfies $\delta_e(t) = 0$ for all $t \in S$ except $t = e$. In general, δ_e is not a trace and is not faithful. As in the group case, we obtain

$$C_r^*(S_1 * S_2) \simeq \pi_{\delta_e * \delta_e}\left(C_r^*(S_1) \check{*} C_r^*(S_2)\right).$$

Now, consider the special case $S = Z^+$, the nonnegative integers with addition. $C_r^*(Z^+)$ is the C^*-algebra of the one-sided shift. The representation $\pi_{\delta_e*\delta_e}$ is not faithful on $C_r^*(Z^+) * C_r^*(Z^+)$ because if V_1, V_2 are the generators of the two semigroups, then $V_1^*V_2 = 0$ in $C_r^*(Z^+ * Z^+)$. Note that V_2 takes every word to a word starting with a positive power of V_2, and on such words V_1^* is zero.

Now let p be the one-dimensional projection on e in $l^2(Z^+ * Z^+)$. Then

$$p = 1 - V_1V_1^* - V_2V_2^* \in C_r^*(Z^+ * Z^+)$$

and $\delta_e(p) = 1$. However, all the operators in the norm-closed convex hull of the unitary orbit of p must be compact, so Proposition 3.1 fails. This is true even though there are many unitaries of zero δ_e-state, which are orthogonal to each other, in $C_r^*(Z^+)$. For example, let

$$u_i = 1 - 2p_{2^{-1/2}(e_i - e_0)} \in K(l^2(Z^+)) \subseteq C_r^*(Z^+)$$

where e_i is $i \in Z^+$ considered as a vector in $l^2(Z^+)$. See [**4**] for the fact that $C_r^*(Z^+)$ contains the compact operators. u_i is unitary, $u_i e_0 = e_i$, so $\delta_e(u_i) = 0$ and $\delta_e(u_i^* u_j) = 0$ if $i \neq j$. What goes wrong is that δ_e is not invariant by the unitaries.

Note, by the way, that [**5**, 3.1] $C_r^*(Z^+ * \cdots * Z^+)$ (k times) is an extension of K (the compact operators) by O_k (the Cuntz algebra), while $C_r^*(Z^+ * Z^+ * \cdots)$ is just O_∞, which is a simple algebra. So a free product with respect to a nonfaithful state may be simple.

3.4. PROPOSITION. *Let M be a W^*-algebra that has no nonzero Abelian ideal. Let ψ be a normal (finite) trace on M.*

Then M has two unitaries u, v such that $\psi(u) = \psi(v) = \psi(uv) = 0$.

PROOF. It is enough to prove the proposition for each direct summand of M.

For a type II finite algebra, let p_1, p_2, p_3 be disjoint projections of trace $\frac{1}{3}$ each. Take

$$u = v = p_1 + e^{2\pi i/3} p_2 + e^{-2\pi i/3} p_3,$$

so that

$$uv = p_1 + e^{-2\pi i/3} p_2 + e^{2\pi i/3} p_3.$$

Then u, v are unitaries and $\psi(u) = \psi(v) = \psi(uv) = 0$.

For an M_2 summand let

$$u = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad v = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad uv = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

For an M_n summand, $n > 2$, let $u = v = \operatorname{diag}(1, \zeta, \ldots, \zeta^{n-1})$ where $\zeta = \exp(2\pi i/n)$. Q.E.D.

We obtain: (M, ψ) satisfies the conditions for (B, ψ) in 3.1.

4. Shift automorphisms of free products.

4.0. Using the analogy between tensor and free products, we may define the free-product analogs of noncommutative Bernoulli shifts. They turn out to have interesting applications for noncommutative topological dynamics [**1**]. We will call an automorphism group uniquely ergodic if it has a unique invariant state.

4.1. Let $\{(A_j, \phi_j): j \in \Gamma\}$ be C^*-algebras and states on them. Let G be a group acting on Γ by permutations, and for each $s \in G$, $j \in \Gamma$ let an automorphism be given $A_j \to A_{sj}$, which takes ϕ_j to ϕ_{sj}. We get an action of G on

$$A = \pi_{*\phi_j}(\check{*}\{A_j : j \in \Gamma\})$$

by automorphisms, and this action preserves the state $\phi = * \phi_j$.

PROPOSITION. *Let G, A, ϕ, Γ be as above. Suppose each 2-element subset of Γ has infinitely many disjoint G-translates. Then (G, A) is uniquely ergodic. Moreover, if (G, B) is any other C^*-flow, then the invariant states of $(G, A \underset{*}{\otimes} B)$ are exactly $\{\phi \otimes \psi: \psi$ G-invariant on $B\}$.*

PROOF. I will prove the second claim, which in the case $B = \mathbf{C}$ gives unique ergodicity. Represent B faithfully and covariantly on some Hilbert space H, and consider $A \underset{*}{\otimes} B$ as acting on $H_\phi \otimes H$ covariantly with a representation u of G.

For γ, a 2-element subset of Γ, let H_γ be the span of elementary tensors in H_ϕ starting with $H_0^{(i)}$ for some $i \in \gamma$. Let $\{g_n\} \in G$ be an infinite sequence such that the sets $g_n\gamma$ are disjoint. For $n \neq m$,

$$u(g_n)(H_\gamma \otimes H) \perp u(g_m)(H_\gamma \otimes H).$$

Let $a \in A$ be a word starting and ending with $\{A_0^{(j)}: j \in \gamma\}$. For any $b \in B$,

$$(a \otimes b)\big((H_\gamma \otimes H)^\perp\big) \subseteq H_\gamma \otimes H.$$

By Lemma 3.0, for any N,

$$\left\| \frac{1}{N} \sum_{n=1}^{N} u(g_n)(a \otimes b)u(g_n)^* \right\| \leqslant \frac{2\|a \otimes b\|}{N^{1/2}}.$$

Let χ be any G-invariant state on $A \underset{*}{\otimes} B$. We obtain

$$|\chi(a \otimes b)| \leqslant \frac{2\|a \otimes b\|}{N^{1/2}},$$

and since N is arbitrary, $\chi(a \otimes b) = 0$. Since γ is arbitrary, this holds for every word a in A.

Now let $\psi = \chi_{|B}$, and let $a \in * \{A_j: j \in \Gamma\}$. For any $b \in B$,

$$\begin{aligned}\chi(a \otimes b) &= \chi((a - \phi(a)) \otimes b) + \phi(a)\chi(1 \otimes b) \\ &= 0 + \phi(a)\psi(b) = (\phi \otimes \psi)(a \otimes b),\end{aligned}$$

since $a - \phi(a)$ is a sum of words. Hence $\chi = \phi \otimes \psi$. Q.E.D.

The following lemma shows that the conditions on (G, Γ) are satisfied, for example, if G is infinite and acts freely on Γ.

LEMMA. *Let (G, Γ) be a permutation group such that G is infinite and every stability subgroup is finite. Then every finite subset of Γ has infinitely many G-translates.*

PROOF. Let $\gamma \subseteq \Gamma$ be finite. No finite subset of Γ intersects $s\gamma$ for all $s \in G$. Indeed, let $F \subseteq \Gamma$ be finite. If $f \in F$, then $sf \in \gamma$ happens for only finitely many $s \in G$, so $sF \cap \gamma \neq 0$ happens for only finitely many $s \in G$.

Now, suppose we have already chosen $s_1, \ldots, s_n \in G$ such that $s_i\gamma, \ldots, s_n\gamma$ are disjoint. The finite set $s_1\gamma \cup \cdots \cup s_n\gamma$ does not intersect $s\gamma$ for all $s \in G$, so there is $s_{n+1} \in G$ such that $s_i\gamma \cap s_{n+1}\gamma = \varnothing$ for $i = 1, \ldots, n$. Q.E.D.

4.2. Using the definitions of [**1** or **7**], we obtain the following: if G in 4.1 is amenable and ϕ is faithful, then (G, A) is a minimal C^*-flow, and its tensor products with any minimal C^*-flow is also minimal. This is in contrast to the commutative case, where a flow cannot be disjoint from itself [**6**]. See [**1**] for details.

4.3. By the remarks at the end of §3, and by 4.1, the Cuntz algebra O_∞ has a unique state which is invariant under all permutations of the generators S_i (or equivalently under a single infinite-order permutation).

Appendix: Tensor products of faithful states. Let A, B, C, D be C^*-algebras, and let $P: A \to C$, $Q: B \to D$ be completely positive maps. Then $P \otimes Q: A \otimes B \to C \otimes D$ extends to a completely positive map $P \otimes Q: A \underset{*}{\otimes} B \to C \underset{*}{\otimes} D$. Here $\underset{*}{\otimes}$ denotes the minimal (spatial) tensor product. [**9**, 4.23].

PROPOSITION. *Let A, B be C^*-algebras and $\{\phi_i: i \in I\}$ a faithful collection of states on A, i.e., $\{a \in A: \phi_i(a^*a) = 0$ for all $i \in I\} = 0$.*

The collection of completely positive maps, $\phi_i \otimes I_B: A \underset{*}{\otimes} B \to B$, is faithful.

PROOF. Let $T \in A \underset{*}{\otimes} B$ such that $(\phi_i \otimes I_B)(T^*T) = 0$ for all $i \in I$. Let ψ be any state of B. Then

$$(\phi_i \otimes I_{\mathbf{C}})(I_A \otimes \psi)(T^*T) = (I_{\mathbf{C}} \otimes \psi)(\phi_i \otimes I_B)(T^*T) = 0.$$

Identifying $A \underset{*}{\otimes} \mathbf{C}$ with A, and $\phi_i \otimes I_{\mathbf{C}}$ with ϕ_i, we get, since $\{\phi_i: i \in I\}$ is a faithful collection,

$$(I_A \otimes \psi)(T^*T) = 0.$$

Now let ϕ be any state of A. Then

$$(\phi \otimes \psi)(T^*T) = \phi(I_A \otimes \psi)(T^*T) = 0.$$

Since ϕ, ψ are arbitrary, $T = 0$. Q.E.D.

PROPOSITION. *Let A, B, C be C^*-algebras. Let $P: A \to C$ be a faithful, completely positive map. Then $P \otimes I_B: A \underset{*}{\otimes} B \to C \underset{*}{\otimes} B$ is faithful.*

PROOF. The collection $\{\phi P: \phi$ state of $C\}$ is a faithful collection of states of A. By the preceding proposition,

$$\{\phi P \otimes I_B: \phi \text{ state of } C\}$$

is a faithful collection of maps $A \underset{*}{\otimes} B \to B$.

Now let $T \in A \underset{*}{\otimes} B$ such that $(P \otimes I_B)(T^*T) = 0$. If ϕ is a state of C,

$$(\phi P \otimes I_B)(T^*T) = (\phi \otimes I_B)(P \otimes I_B)(T^*T) = 0.$$

So $T = 0$ and $P \otimes I_B$ is faithful.

COROLLARY. *Let A, B, C, D be C^*-algebras. Let $P: A \to C$, $Q: B \to D$ be faithful, completely positive maps. Then $P \otimes Q: A \underset{*}{\otimes} B \to C \underset{*}{\otimes} D$ is faithful.*

PROOF. $P \otimes Q = (P \otimes I_D)(I_A \otimes D)$. Q.E.D.

In particular, the tensor product of faithful states is faithful on the smallest tensor product algebra.

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