A REMAINDER FORMULA AND LIMITS OF CARDINAL SPLINE INTERPOLANTS

T. N. T. GOODMAN AND S. L. LEE

Abstract. A Peano-type remainder formula

\[ f(x) - S_n(f; x) = \int_{-\infty}^\infty K_n(x, t) f^{(n+1)}(t) \, dt \]

for a class of symmetric cardinal interpolation problems C.I.P. (E, F, x) is obtained, from which we deduce the estimate \( \| f - S_n(f; \cdot) \|_\infty \leq K \| f^{(n+1)} \|_\infty \). It is found that the best constant \( K \) is obtained when \( x \) comprises the zeros of the Euler-Chebyshev spline function. The remainder formula is also used to study the convergence of spline interpolants for a class of entire functions of exponential type and a class of almost periodic functions.

1. Introduction. As in [5], for \( x = (x_0, x_1, \ldots, x_m) \), \( 0 = x_0 < x_1 < \cdots < x_m = 1 \) and incidence matrices \( E = \{ E_{ij} \}_{i,j=0}^m \) and \( F = \{ F_{ij} \}_{i,j=0}^m \) with \( E_{0j} = E_{mj} \) and \( F_{0j} = F_{mj} \), \( j = 0, \ldots, n \), let \( \mathcal{S}(F, x) := \{ f: \mathbb{R} \to \mathbb{C}; \forall \nu \in \mathbb{Z}, f((\nu + x_i, \nu + x_{i+1}) \in \mathcal{P}_n, i = 0, \ldots, m - 1, \) and \( f^{(n-j)}(\nu + x_{\nu}) = f^{(n-j)}(\nu + x_{\nu}^+) \) \( \forall (i, j) \) with \( F_{ij} = 0 \}, \) and refer to the following 'cardinal' interpolation problem as the C.I.P. (E, F, x):

For sequences of numbers \( \{ y^{(i,j)}(\nu) \} = \{ y^{(i,j)} \}; 0 \leq i < m \) and \( E_{ij} = 1 \), find \( S \in \mathcal{S}(F, x) \) satisfying \( S^{(i,j)}(\nu + x_i) = y^{(i,j)} \).

Sufficient conditions for C.I.P. (E, F, x) to be poised, i.e. existence of a unique \( S \in \mathcal{S}(F, x) \), \( S(x) = O(|x|^\gamma) \) as \( x \to \pm \infty \) satisfying \( S^{(i,j)}(\nu + x_i) = y^{(i,j)} \) when \( y^{(i,j)} = O(|\nu|^\gamma) \), are given in [5].

Suppose that the C.I.P. (E, F, x) is poised; then given a sufficiently smooth function \( f \) of power growth \( \exists \) a unique \( S_n(f; \cdot) \in \mathcal{C}(F, x) \) of power growth which interpolates \( f \) in the sense that

\[ S_n^{(i,j)}(f; \nu + x_i) = f^{(i,j)}(\nu + x_i), \quad \nu \in \mathbb{Z}, E_{ij} = 1. \]

The following problem then arises.

Problem. Find necessary and sufficient conditions so that \( S_n(f; \cdot) \to f \) uniformly as \( n \to \infty \).

This question was first raised by Schoenberg [9] who also found a sufficient condition for the convergence of \( S_n(f; \cdot) \) for the case where \( m = 1, n \) is odd, \( E = F \) and

\[ E_{0j} = E_{1j} = \begin{cases} 0, & j = 1, 2, \ldots, n, \\ 1, & j = 0. \end{cases} \]

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In this case \( \mathcal{C}(F, x) \) comprises odd degree cardinal splines with integer knots. Schoenberg \[9\] proves the following

**Theorem A.** Let

\[
(1.3) \quad f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{itx} \, d\alpha(u)
\]

where \( \alpha(u) \) is a function of bounded variation in \([ -\pi, \pi ]\) and let \( A = \alpha(-\pi + 0) - \alpha(-\pi), B = \alpha(\pi) - \alpha(\pi - 0). \) Then

\[
\lim_{k \to \infty} S_{2k-1}(f; x) = f(x) + \frac{i(A - B)}{2\pi} \sin \pi x
\]

uniformly on \( \mathbb{R} \).

A partial converse of Theorem A was given by Richards and Schoenberg \[8\]. Subsequently, Marsden and Riemenschneider \[6\] generalised Theorem A to the case of cardinal Hermite interpolation which corresponds to \( m = 1, E = F \) and, for some \( 1 \leq r \leq \frac{1}{2}(n + 1) \),

\[
E_{0j} = E_{1j} = \begin{cases} 
1, & j = 0, 1, \ldots, r - 1, \\
0, & j = r, \ldots, n.
\end{cases}
\]

In this case, a sufficient condition for convergence of \( S_{f}(f; \cdot) \) analogous to that of Richards and Schoenberg was given by Goodman \[4\].

Recently, in an attempt to obtain more information on the convergence problem, I. J. Schoenberg \[10\] obtained a Peano type remainder formula

\[
(1.4) \quad f(x) - S_{2k-1}(f; x) = \int_{-\infty}^{\infty} K_{2k-1}(x; t) f^{(2k)}(t) \, dt \quad (x \in \mathbb{R})
\]

for the case \( m = 1, n = 2k - 1, E = F \) and \( E \) satisfies \((1.2)\), where \( K_{2k-1}(x, t) := (x - t)^{2k-1} - S_{2k-1}(x, t)^{2k-1} \). From \((1.4)\), Schoenberg \[10\] deduced Theorem A and also obtained a convergence result for a class of almost periodic functions.

In this paper we shall consider the symmetric C.I.P. \((E, F, x)\)

\[
x_i = 1 - x_{m-i}, \quad i = 0, \ldots, m, \\
F_{ij} = 1 \quad \text{iff } i = 0 \text{ or } m \text{ and } j = 0, \ldots, r - 1,
\]

for some \( 1 \leq r \leq n + 1 \), and either

- (a) \( n + r \) is even, \( m = r \) and \( E_{ij} = 1 \) iff \( j = 0 \) and \( i = 0, \ldots, m \), or
- (b) \( n + r \) is odd, \( m = r + 1 \) and \( E_{ij} = 1 \) iff \( j = 0 \) and \( i = 1, \ldots, m - 1 \).

That this problem is poised follows from Corollary 4.3 of \[5\] and was earlier shown by Micchelli \[7\]. In this case the class of cardinal spline functions \( \mathcal{C}(F, x) \) is usually denoted by \( \mathcal{C}_{n,r} \) and clearly

\[
\mathcal{C}_{n,r} \equiv \mathcal{C}(F, x) = \{ S \in C^{n-r}(\mathbb{R}) \mid S\big|_{(\nu, \nu + 1)} \in \pi_{\nu}, \forall \nu \in \mathbb{Z} \}.
\]

For \( r, n \) as above, we define \( \mathcal{C}_{n,r} \equiv \{ f \in C^{n+1-}(\mathbb{R}) \mid f\big|_{(\nu, \nu + 1)} \in C^{n}[(\nu, \nu + 1)] \} \) and \( f^{(n)} \) bounded and absolutely continuous on \( (\nu, \nu + 1), \forall \nu \in \mathbb{Z} \).
For \( f \in \mathcal{S}_{n,r} \) of power growth, we let \( S_{n,r}(f; \cdot) \) denote the unique function of power growth that interpolates \( f \) as in (1.1). Following the approach of Schoenberg [10] we derive a formula for the remainder \( f - S_{n,r}(f; \cdot) \) and deduce the following result.

**Theorem 1.** For fixed \( n, r \) and \( x \), \( \exists K \) such that for any \( f \in \mathcal{S}_{n,r} \) with \( f^{(n+1-r)} \) of power growth and \( \| f^{(n+1)} \|_{\infty} < \infty \),

\[
\| f - S_{n,r}(f; \cdot) \|_{\infty} \leq K \| f^{(n+1)} \|_{\infty},
\]

and equality is attained for some \( f \in \mathcal{S}_{n,r} \). For fixed \( n, r, K \) is a minimum when \( x \) comprises the zeros of the Euler-Chebyshev spline \( \mathcal{S}_{n+1,r} \) (see [1] and [4]) and in this case equality is attained for \( f = \mathcal{S}_{n+1,r} \).

For \( r = n + 1 \), this result reduces to a classical result on optimal constants in the remainder for Lagrange interpolation by polynomials (see [3, p. 64]).

Now let \( B_{\sigma} = \{ f; f \) is the restriction to \( \mathbb{R} \) of an entire function of exponential type \( \leq \sigma \) and \( \| f \|_{\infty} < \infty \} \). By deriving bounds on the best constants \( K \) in (1.6), we prove the following results, all of which are generalisations of results of Schoenberg [10].

**Theorem 2.** For fixed \( x \) and \( r \geq 1 \), \( \exists K_r \) such that for all \( n \geq r - 1 \) and \( f \in B_{\sigma} \),

\[
\| f - S_{n,r}(f; \cdot) \|_{\infty} \leq K_r (\sigma r^n)^{n+1} \| f \|_{\infty}.
\]

**Theorem 3.** If \( f(x) = \int_{-\pi r}^{\pi r} e^{iux} d\alpha(u) \), where \( \alpha(u) \) is a function of bounded variation in \([-\pi r, \pi r] \), then

\[
\lim_{n \to \infty} S_{n,r}(f; x) = f(x) + C 2^{r-1} \prod_{i=1}^r \sin \pi(x - x_i)
\]

uniformly, where

\[
C = \begin{cases} 
(-1)^{r} \{ \alpha(r\pi) - \alpha(-r\pi) + \alpha(-r\pi - 0) - \alpha(-r\pi + 0) \} & \text{if } n + r \text{ is even,} \\
(-1)^{r-1} \{ \alpha(r\pi) - \alpha(-r\pi - 0) - \alpha(-r\pi + 0) \} & \text{if } n + r \text{ is odd.}
\end{cases}
\]

**Theorem 4.** If \( f \in B_{r\sigma} \) is almost periodic in the sense of Bohr, then

\[
\lim_{n \to \infty} S_{n,r}(f; x) = f(x) + C 2^{r} \prod_{i=1}^r \sin \pi(x - x_i)
\]

uniformly, where

\[
C = \begin{cases} 
(-1)^{r} \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} f(x) \sin r\pi x \, dx & \text{if } n + r \text{ is even,} \\
(-1)^{r-1} \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} f(x) \cos r\pi x \, dx & \text{if } n + r \text{ is odd.}
\end{cases}
\]

In §2 we apply the results of our preceding paper [5] to the study of the sign structure of the kernel, from which the corresponding remainder formula is derived. The proof of Theorem 1 is given in §3. Theorem 1 is then used in §4 to derive convergence results.
2. The remainder formula. Take $1 \leq r \leq n + 1$ and consider the poised, symmetric C.I.P. $(E, F, x)$ defined in §1. In this case the relation (1.5) and the conditions (a) and (b) imply that $0 < x_1 < \ldots < x_r \leq 1$ with equality iff $n + r$ is even. We shall also write $\tilde{S}_{n, r}$ for $\tilde{c}(F, x)$ and $\tilde{S}_{n, r}$ for $\tilde{c}(E, x)$ which corresponds to the dual C.I.P. $(F, E, x)$.

Specialising the results of [5] to this special case it is easy to see that the corresponding null space

$$\mathcal{C}^0 = \mathcal{C}^0(F, x) := \{ S \in \mathcal{C}(F, x); S(\nu + x_i) = 0 \forall \nu \in \mathbb{Z}, E_{i0} = 1 \}$$

has dimension $d$, where

$$d = \begin{cases} n - \frac{r}{2} & \text{if } n + r \text{ is even}, \\ n - r + 1 & \text{if } n + r \text{ is odd}, \end{cases}$$

and is spanned by $d$ eigensplines $S_j, j = 1, \ldots, d$, satisfying the functional relation

$$S_j(x + 1) = \lambda_j S_j(x) \quad \forall x \in \mathbb{R}. \tag{2.1}$$

The eigenvalues $\lambda_j, j = 1, \ldots, d$, of the C.I.P. $(E, F, x)$ are real, distinct, of sign $(-1)^\nu$ and are precisely the eigenvalues of the matrix $C = (C_{\mu\nu})_{d \times d}$ where $S^\mu(1) = C^\mu_{\nu} S^{(\nu)}(0), \mu, \nu = 0, \ldots, d$.

Now for $i = 1, \ldots, r$, we let $L_i$ denote the unique element of $\mathcal{C}(F, x)$ of power growth (actually of exponential decay) satisfying

$$E_j(\nu + x_i) = \delta_{ij}, \quad \forall \nu \in \mathbb{Z}, j = 1, \ldots, r. \tag{2.2}$$

Then, for $f \in \mathcal{C}_{n, r}$ of power growth, the unique function $S_{n, r}(f; x) \in \mathcal{C}(F, x)$ of power growth that interpolates $f$ for the C.I.P. $(E, F, x)$ is given by

$$S_{n, r}(f; x) = \sum_{i=1}^{r} \sum_{\nu=-\infty}^{\infty} f(\nu + x_i) L_i(x - \nu). \tag{2.3}$$

We let $\tilde{S}_{n, r}(f; x) \in \tilde{S}_{n, r}$ denote the unique spline function of power growth that interpolates $f$ for the dual C.I.P. $(F, E, x)$.

For $x, t \in \mathbb{R}$ we define

$$g_t(x) \equiv g_t(t) := (1/n!)(x - t)^n$$

and

$$K(x, t) \equiv K_t(x) \equiv \tilde{K}_t(x) := g_t(x) - S_{n, r}(g_t; x). \tag{2.4}$$

Clearly, from (2.1), we have

$$K(x + 1, t + 1) = K(x, t), \quad \forall x, t \in \mathbb{R}, \tag{2.5}$$

and

$$K_t(\nu + x_i) = 0, \quad \forall \nu \in \mathbb{R} \text{ and } i = 1, \ldots, r. \tag{2.6}$$

Now we see from (2.2) and (2.3) that, for $\rho = 0, \ldots, n$,

$$\tilde{K}_x^{(\rho)}(t) = (-1)^\rho g_t^{(\rho)}(x) - S_{n, r}^{(\rho)}(-1)^\rho g_t^{(\rho); x}. \tag{2.7}$$

But for $\nu \in \mathbb{Z}$ and $\rho = 0, \ldots, r - 1, g_t^{(\rho)} \in \tilde{c}(F, x)$ and so

$$\tilde{K}_x^{(\rho)}(\nu) = 0, \quad \forall x \in \mathbb{R}. \tag{2.8}$$
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Now we see from (2.2) that, for fixed \( x \), \( S_{n,r}(g_x; t) \) as a function of \( t \) lies in \( C(E, x) \) and has power growth. By (2.6) it interpolates \( \hat{g}_x \) for the C.I.P. \((F, E, x)\). Hence \( S_{n,r}(g_x; x) = \hat{S}_{n,r}(\hat{g}_x; t), \forall x, t \in \mathbb{R} \), and so

\[
(2.7) \quad K(x, t) = \hat{g}_x(t) - \hat{S}_{n,r}(\hat{g}_x; t).
\]

**Lemma 2.1.** For \( 1 < r < n \), there is a constant \( \beta > 0 \) such that the following hold for \( p = 0, \ldots, n \).

(a) For \( t \in \mathbb{R} \), \( K^{(p)}(x) = O(e^{-\beta|x|}) \) as \( x \to \pm \infty \) and the only zeros of \( K_i(t \notin \mathbb{Z}) \) are simple zeros at \( x + x_i, v \in \mathbb{Z}, i = 1, \ldots, r \).

(b) For \( x \in \mathbb{R} \), \( \tilde{K}^{(p)}(t) = O(e^{-\beta|t|}) \) as \( t \to \pm \infty \) and the only zeros of \( \tilde{K}_x(x \notin \mathbb{Z} + x_i) \) are isolated zeros of multiplicity \( r \) at the integers.

**Proof.** We shall prove only (a) as (b) follows similarly by duality, and only for even \( n + r \) as the result for odd \( n + r \) follows similarly. By (2.4) and (2.6) we may suppose \( 0 < t < 1 \).

Now, for \( n + r \) even the C.I.P. \((E, F, x)\) has \( n - r \) distinct eigenvalues \( \lambda_1, \ldots, \lambda_{n-r} \) of sign \((-1)^r \) with \( |\lambda_1| > |\lambda_2| > \cdots > |\lambda_{n-r}| > 0 \) and \( \lambda_{n-r-i+1} = \lambda_i^{-1}, i = 1, \ldots, n - r \).

We let \( S_1, \ldots, S_{n-r} \) denote the corresponding eigenvalues, which span \( C^{0} \). The eigenvalues \( \lambda_1, \ldots, \lambda_{n-r} \) are precisely the eigenvalues of the matrix \( C \) where \((-1)^r C \) is an oscillation matrix with corresponding eigenvectors \((S'_1(0), \ldots, S'_{n-r}(0)), i = 1, \ldots, n - r \). So by a theorem of Gantmacher and Krein we have

\[
S(S'_1(0), \ldots, S'_{n-r}(0)) = i - 1, \quad i = 1, \ldots, n - r
\]

(see also Micchelli [7]).

Now \( K_i \in C^{n-r}(\mathbb{R}) \) and for \( \nu = 1, 2, \ldots, K_i|_{\nu \in \nu + 1} \in \pi_n \). Thus, by (2.5), \( K_i|_{1, \infty} \) can be extended to an element of \( C^{0} \). Since \( K_i \) is of power growth, we therefore have

\[
K_i(x) = \sum_{i=p} c_i S_i(x), \quad \forall x \geq 1,
\]

where \( c_p \neq 0 \) and \( p > \frac{1}{2}(n - r) \). Similarly \( K_i(x) = \sum_{i=q} c_i S_i(x), \forall x \leq 0 \), where \( c_q \neq 0 \) and \( q < \frac{1}{2}(n - r) \).

The first part of (a) follows.

Now for \( p = 0, \ldots, n - r \) and \( \nu = 1, 2, \ldots, \)

\[
K_i^{(p)}(\nu) = \sum_{i=p} c_i \lambda_i S_i^{(p)}(0) \quad \text{and} \quad K_i^{(p)}(-\nu) = \sum_{i=q} c_i \lambda_i^{-r} S_i^{(p)}(0).
\]

So \( K_i^{(p)}(\nu) = c_p \lambda_p S_p^{(p)}(0) + O(\lambda_p^\nu) \) as \( \nu \to \infty \) and \( K_i^{(p)}(-\nu) = c_q \lambda_q^{-r} S_q^{(p)}(0) + O(\nu^{-r}) \) as \( \nu \to \infty \). Thus for large enough \( N \),

\[
(2.8) \quad S(K_i(N), \ldots, K_i^{(n-r)}(N)) = S(S_p^0(0), \ldots, S_p^{(n-r)}(0)) = p - 1
\]

and

\[
(2.9) \quad S(K_i(-N), \ldots, K_i^{(n-r)}(-N)) = S(S_q^0(0), \ldots, S_q^{(n-r)}(0)) = q - 1.
\]

We now show that \( K_i \) is oscillating in \((-N, N)\). For suppose \( K_i = 0 \) on some interval \((a, b) \in (-N, N)\). Since \( K_i \in C^{n-r}(\mathbb{R}) \) and is a piecewise polynomial with
knots at $Z \cup \{t\}$, $a, b \in Z \cup \{t\}$. Furthermore, either $a < t$ or $b > t$. Suppose $a < t$. Then $K_i^{(p)}(a) = 0, \rho = 0, \ldots, n - r$, and $K_i(a - 1 + x_i) = 0, i = 0, \ldots, r - 1$, implies that $K_i = 0$ on $(a - 1, a)$. Hence $K_i^{(p)}(a - 1) = 0, \rho = 0, \ldots, n - r$. By induction we have $K_i^{(p)}(-N) = 0, \rho = 0, \ldots, n - r$, which contradicts (2.9). Similarly $b > t$ leads to $K_i^{(p)}(N) = 0, \rho = 0, \ldots, n - r$, which contradicts (2.8).

Now $K_i$ has exact degree $n$ in $(0, 1)$ and so we may apply Theorem 2.1 of [5] to

$$g := K_i|_{[-N, N]}$$

to give

$$Z(g) \leq (2N - 1)r + 1 + S^- \left( g(-N^+), \ldots, g^{(n)}(-N^-) \right) - S^+ \left( g(N^-), \ldots, g^{(n)}(N^+) \right) - \left( q + 1 - r \right) - p \quad \text{(since } g(-N^+) = g(N^-) = 0)$$

$$= 2Nr + q - p \leq 2Nr - 1.$$ But $K_i$ has $2Nr - 1$ zeros in $(-N, N)$ at points as in (2.5) and hence these are the only zeros of $K_i$ in $(-N, N)$. Since $N$ can be arbitrarily large, (a) follows. □

We must also consider the special cases $r = n$ or $n + 1$, for which it is easy to see the following. If $t \in (\nu, \nu + 1)$, then $K_i$ vanishes outside $[\nu, \nu + 1]$ and vanishes in $(\nu, \nu + 1)$ only at $\nu + x_i, i = 1, \ldots, r$. Similarly if $x \in (\nu, \nu + 1)$, then $\tilde{K}_i$ vanishes outside $[\nu, \nu + 1]$ and vanishes nowhere in $(\nu, \nu + 1)$.

**Theorem 2.1.** If $f \in C_{n,r}$ and $f^{(n+1-r)}$ is of power growth, then

$$f(x) - S_{n,r}(f; x) = \int_{-\infty}^{\infty} K(x, t)f^{(n+1)}(t)\, dt.$$ 

**Proof.** By (2.4) we may assume $0 < x \leq 1$. We may also assume $x \neq x_i, i = 1, \ldots, r$, since otherwise (2.10) is trivially satisfied.

Now integrating by parts and applying (2.6) gives, for any $\nu \in Z$,

$$\int_{\nu}^{\nu + 1} K(x, t)f^{(n+1)}(t)\, dt = (-1)^r \int_{\nu}^{\nu + 1} K^{(r)}(t)f^{(n+1-r)}(t)\, dt.$$ So

$$\int_{-\infty}^{\infty} K(x, t)f^{(n+1)}(t)\, dt = (-1)^r \int_{-\infty}^{\infty} K^{(r)}(t)f^{(n+1-r)}(t)\, dt$$

which converges since $f^{(n+1-r)}$ is of power growth and $K^{(r)}$ decays exponentially. Also $f^{(p)}$ is of power growth for $0 \leq p \leq n - 1 - r$ and so we may integrate by parts to give

$$(-1)^r \int_{-\infty}^{\infty} K^{(r)}(t)f^{(n+1-r)}(t)\, dt = (-1)^n \int_{-\infty}^{\infty} K^{(n)}(t)f'(t)\, dt.$$ Now $(-1)^n K^{(n)}(t) = (x - t)^0 - \sum_{i=1}^{r} \sum_{\nu = -\infty}^{\infty} (\nu + x_i - t)^0 L_i(x - \nu)$ and so

$$(-1)^n \int_{-\infty}^{\infty} K^{(n)}(t)f'(t)\, dt = f(x) - \sum_{i=1}^{r} \sum_{\nu = -\infty}^{\infty} f(\nu + x_i)L_i(x - \nu)$$

$$= f(x) - S_{n,r}(f; x). \quad \Box$$

**3. Proof of Theorem 1.** We see immediately from Theorem 2.1 that for $f \in C_{n,r}$ with $f^{(n+1-r)}$ of power growth and $\|f^{(n+1)}\|_{\infty} < \infty$,

$$\|f - S_{n,r}(f; \cdot)\|_{\infty} \leq \sup_{x \in \mathbb{R}} \left\{ \int_{-\infty}^{\infty} |K(x, t)|\, dt \right\} \|f^{(n+1)}\|_{\infty}.$$
Now there is an eigenspline $E_{n+1,r} \in S_{n+1,r}$ with eigenvalue $(-1)^r$ which vanishes at $\nu + x_i$, $\forall \nu \in \mathbb{Z}$ and $i = 1, \ldots, r$. We assume it is normalised so that $|E_{n+1,r}(x)| = 1$.

Putting $f = E_{n+1,r}$ in (2.10), we see from Lemma 2.1(b) that

$$|E_{n+1,r}(x)| = \int_{-\infty}^{\infty} |K(x, t)| dt, \quad \forall x \in \mathbb{R},$$

so that the constant $K$ in (1.6) is given by

$$K = \|E_{n+1,r}\|_\infty.$$

Following the definitions of Cavaretta [1] and Goodman [4], we let $\mathcal{E}_{n+1,r} \in S_{n+1,r}$ denote the Euler-Chebyshev spline, normalised so that $|\mathcal{E}_{n+1,r}(x)| = 1$. Now the zeros of $\mathcal{E}_{n+1,r}$ are points $\nu + \beta_i$, $\nu \in \mathbb{Z}$, where $\beta_i$, $i = 1, \ldots, r$, are symmetric about $x = \frac{1}{2}$, and $0 < \beta_1 < \cdots < \beta_r \leq 1$, with equality iff $n + r$ is even. Furthermore

$$\mathcal{E}_{n+1,r}(x + 1) = (-1)^r \mathcal{E}_{n+1,r}(x), \quad \forall x \in \mathbb{R},$$

and $f = \mathcal{E}_{n+1,r}$ minimises $\|f\|_\infty$ over all $f \in S_{n+1,r}$ with $\|f^{(n+1)}\|_\infty = 1$. The result follows. □

4. Convergence of $S_{n,r}(f, x)$. Henceforth we shall examine the behaviour of $S_{n,r}(f, x)$ as $n \to \infty$. Analogous problems were studied in [6, 9 and 10]. We first derive an estimate for $\|E_{n+1,r}\|_\infty$.

**Lemma 4.1.** For fixed $x$ and $r \geq 1$, $\exists K$, such that

$$\|E_{n+1,r}\|_\infty \leq K/(r\pi)^{n+1}, \quad \forall n \geq r - 1.$$

**Proof.** First, assume that $r$ is odd. For $x \in [0, 1]$, let

$$E_{n+1,r}(x) = a_0 E_{n+1}(x) + a_1 E_n(x) + \cdots + a_n E_1(x) + a_{n+1} E_0(x),$$

where $E_k(x)$, $k = 0, 1, \ldots, n + 1$, are the Euler polynomials. The conditions $E_{n+1,r}^{(\rho)}(1) = -E_{n+1,r}^{(\rho)}(0)$ for $\rho = 0, 1, \ldots, n - r + 1$ imply that $a_{n-k+1} = 0$ $\forall k = 0, 1, \ldots, n - r + 1$. Hence for $x \in [0, 1]$,

$$E_{n+1,r}(x) = a_0 E_{n+1}(x) + a_1 E_n(x) + \cdots + a_{r-1} E_{n-r+2}(x).$$

Now $E_{n+1,r}(x_i) = 0$, $i = 1, 2, \ldots, r$, gives a homogeneous system of equations

$$a_0 E_{n+1}(x_i) + a_1 E_n(x_i) + \cdots + a_{r-1} E_{n-r+2}(x_i) = 0, \quad i = 1, 2, \ldots, r,$$

whose determinant must be zero. Hence we can write

$$E_{n+1,r}(x) = \frac{\text{det}(E_{n-m+2}(\beta_l), m = 1)}{\text{det}(E_{n-m+2}(x_l), m = 2)} \quad \forall x \in [0, 1],$$

where

$$\beta_l = \begin{cases} x & \text{if } l = 1, \\ x_l & \text{if } l \neq 1. \end{cases}$$
Using the Fourier expansions of the Euler polynomials $E_k(x)$ we obtain
\[
|E_{n+1,r}(x)| = \frac{2}{\pi^{n+2}} \left| \frac{\det\left( \sum_{-\infty}^{\infty} e^{2k \pi i \beta_j}/(2k+1)^{n-m+3} \right)_{l,m=1}^r}{\det\left( \sum_{-\infty}^{\infty} e^{2k \pi i x_j}/(2k+1)^{n-m+3} \right)_{l,m=2}^r} \right|
\]
\[
= \left( \frac{2^r}{\pi^{n+2}} \right) \left| \sum_{k_1, k_2, \ldots, k_r} V(k_1, k_2, \ldots, k_r) \prod_{j=1}^r e^{2k_j \pi i \beta_j}/(2k_j+1)^{n+1} \right|
\]
\[
\sum_{k_2, k_3, \ldots, k_r} V(k_2, k_3, \ldots, k_r) \prod_{j=2}^r e^{2k_j \pi i x_j}/(2k_j+1)^{n+1} \right|
\]
where $V(a_1, a_2, \ldots, a_r)$ denotes the Vandermonde determinant $\det(a_m^i)_{l,m=1}^r = \prod_{1 \leq j < k \leq r} (a_k - a_j)$. A straightforward computation gives
\[
|E_{n+1,r}(x)|
\]
(4.2) \[
\left( \frac{2^r}{\pi^{n+2}} \right) \left| \sum_{k_1, k_2, \ldots, k_r} V(k_1, k_2, \ldots, k_r) \det(e^{2k_j \pi i \beta_j})_{l,m=1}^r \sum_{j=1}^r (2k_j+1)^{-(n+2)} \right|
\]
\[
\sum_{k_2, k_3, \ldots, k_r} V(k_2, k_3, \ldots, k_r) \det(e^{2k_j \pi i x_j})_{l,m=2}^r \sum_{j=2}^r (2k_j+1)^{-(n+1)} \right|
\]
The dominant term in the numerator of (4.2) is
\[
\prod_{j=1}^r (2j - r - 1)^{(n+2)} \left( V\left( -\frac{r-1}{2}, -\frac{r-3}{2}, \ldots, -\frac{r-1}{2} \right) \right) \left( e^{(2m-r-1)\pi i \beta_j} \right)_{l,m=1}^r
\]
(4.3)
\[
- V\left( -\frac{r+1}{2}, -\frac{r-1}{2}, \ldots, -\frac{r-3}{2} \right) \left( e^{(2m-r-3)\pi i \beta_j} \right)_{l,m=1}^r
\]
The dominant term in the denominator is
\[
\prod_{j=2}^r (2j - r - 2)^{(n+1)} V\left( -\frac{r-1}{2}, -\frac{r-3}{2}, \ldots, -\frac{r-3}{2} \right)
\]
(4.4)
\[
\times \det(e^{(2m-r-3)\pi i x})_{l,m=2}^r
\]
It follows from (4.2), (4.3) and (4.4) that $\forall x \in [0,1]$
\[
|E_{n+1,r}(x)| \leq \left( \frac{2^r}{\pi^{n+2}} \right) \left| \frac{\prod_{j=2}^r (2j - r - 2)^{n+1}}{\prod_{j=1}^r (2j - r)^{n+2}} \right| O(1),
\]
and (4.1) follows for odd $r$.

If $r$ is even, a similar argument shows that for $x \in [0,1]$ the eigensplines $E_{n+1,r}(x)$ may be expressed in terms of Bernoulli polynomials $B_k(x)$ as follows:
\[
E_{n+1,r}(x) = \frac{\det^*(B_{n-m+1}(\beta_j))_{l=1; m=2}^r}{\det^*(B_{n-m+3}(x_j))_{l=2; m=3}^r},
\]
where det* means that all the entries in the last row of the determinant are 1. Expanding each determinant along the last row and applying a similar method to each term gives the inequality (4.1) for \( r \) even.

Now recall the definition of \( B_\alpha \) in §1. We shall need Bernstein’s theorem that if \( f \in B_\alpha \) then, for each integer \( n, f^{(n)} \in B_\alpha \) and

\[
\| f^{(n)} \|_\infty \leq \sigma^n \| f \|_\infty.
\]

From (1.6), (3.1), (4.1) and (4.5), we can immediately deduce Theorem 2.

**Corollary 4.1.** If \( f \in B_\alpha \) and \( \sigma < r\pi \), then

\[
\lim_{n \to \infty} S_{n, r}(f; x) = f(x) \quad \text{uniformly on } \mathbb{R}.
\]

**Corollary 4.2.** If \( f \in B_\alpha \), then for all \( n \geq r - 1 \)

\[
\| f - S_{n, r}(f) \|_\infty \leq K_r \| f \|_\infty.
\]

We now follow a similar approach to that of Schoenberg [10] in proving Theorem 3. First we introduce the class \( B^{n*}_\alpha \) of functions which are uniform limits on \( \mathbb{R} \) of functions belonging to \( B_p \) for \( p < r\pi \), i.e. \( f \in B^{n*}_\alpha \) if and only if \( \exists f_j \in B_{p_j}, p_j < r\pi, j = 1, 2, 3, \ldots \), such that \( \| f - f_j \|_\infty \to 0 \) as \( j \to \infty \).

**Lemma 4.2.** If \( f \in B^{n*}_\alpha \) then

\[
\lim_{n \to \infty} S_{n, r}(f; x) = f(x) \quad \text{uniformly}.
\]

**Proof.** Suppose \( f_j \in B_{p_j}, p_j < r\pi, j = 1, 2, 3, \ldots \), and \( \| f - f_j \|_\infty \to 0 \) as \( j \to \infty \). Then \( f - S_{n, r}(f) = f - f_j - S_{n, r}(f_j; x) + f_j - S_{n, r}(f_j; x) \) and using Corollaries 4.1 and 4.2, the result follows as in [10].

Before we prove Theorems 3 and 4 we first study the behaviour of the spline functions \( S_{n, r}(\cos r\pi x) \) and \( S_{n, r}(\sin r\pi x) \) that interpolate \( \cos r\pi x \) and \( \sin r\pi x \) respectively at \( v + x_i, i = 1, 2, \ldots, r, v \in \mathbb{Z} \). In order to simplify writing, we define \( \alpha_1, \ldots, \alpha_r \) by

\[
\alpha_i = x_i, \quad i = 1, \ldots, r, \text{ if } n + r \text{ is odd},
\]

\[
\alpha_1 = 0 \quad \text{and} \quad \alpha_i = x_{i-1}, \quad i = 1, \ldots, r - 1, \text{ if } n + r \text{ is even}.
\]

We first introduce the exponential Euler splines

\[
S_{n, r}(x; u) = \sum_{s=1}^{r} e^{iu\alpha_s} \Omega_s(x, u) \quad \forall x \in \mathbb{R},
\]

where

\[
\Omega_s(x, u) = \frac{\det(S^\infty_{k=1} e^{u(2k \pi i \beta_l/\pi)} (u + 2k \pi)^{n-m+2})_{l, m=1}^r}{\det(S^\infty_{k=1} e^{u(2k \pi i \alpha_l/\pi)} (u + 2k \pi)^{n-m+2})_{l, m=1}^r},
\]

\((r - 2)\pi < u \leq r\pi, \text{ and}

\[
\beta_l = \begin{cases} x & \text{if } l = s, \\ \alpha_l & \text{if } l \neq s. \end{cases}
\]
Clearly $S_{n,r}(v + \alpha_i; u) = e^{i(u(v + \alpha_i))} \forall i = 1, 2, \ldots, r, v \in \mathbb{Z}$, and $S_{n,r}(x, r\pi) = S_{n,r}(\cos(r\pi x) + iS_{n,r}(\sin(r\pi x))$. Therefore we are interested in the limit of $S_{n,r}(x; r\pi)$ as $n \to \infty$. First we prove

**Lemma 4.3.** For $s = 1, 2, \ldots, r$,

(4.7) $\lim_{n \to \infty} \Omega_s(x, r\pi) = e^{\pi((s-1)\pi(\nu + \alpha_1))} \cos{\pi}\left(\alpha_s - x\right) \frac{V(e^{-2\pi i\beta_1}, e^{-2\pi i\beta_2}, \ldots, e^{-2\pi i\beta_s})}{V(e^{-2\pi i\alpha_1}, e^{-2\pi i\alpha_2}, \ldots, e^{-2\pi i\alpha_s})}$

uniformly on $\mathbb{R}$.

**Proof.** Assume $(r - 1)\pi < u < r\pi$. A straightforward computation shows that

$$
\Omega_s(x, u) = \sum_{k_1, k_2, \ldots, k_r} V(k_1, k_2, \ldots, k_r) \prod_{j=1}^{r} e^{2k_j \pi i \beta_j} / (u + 2k_j \pi)^{n+1}
$$

$$
= \sum_{k_1, k_2, \ldots, k_r} \det(e^{2k_j \pi i \alpha_j}) \prod_{j=1}^{r} (u + 2k_j \pi)^{-n-1}
$$

Taking the limit as $u \to r\pi$, after some simplification, we obtain

(4.8) $\Omega_s(x, u) = V(e^{-2\pi i \beta_1}, \ldots, e^{-2\pi i \beta_r})(1 + (-1)^{n+1} e^{-2\pi i \Sigma_j \beta_j}) + O((r - 2/r)^{n+1}) / V(e^{-2\pi i \alpha_1}, \ldots, e^{-2\pi i \alpha_r})(1 + (-1)^{n+1} e^{-2\pi i \Sigma_j \alpha_j}) + O((r - 2/r)^{n+1})$.

Now suppose $n$ even. If $r$ is even, $\alpha_1 = 0$ and

(4.9) $\sum_{i=1}^{r} \alpha_i = \frac{r - 2}{2} + \frac{1}{2}$.

If $r$ is odd, $\alpha_1 > 0$ and

(4.10) $\sum_{i=1}^{r} \alpha_i = \frac{r - 1}{2} + \frac{1}{2}$.

The result (4.7) then follows from (4.8), (4.9) and (4.10). For odd $n$,

$$
\sum_{i=1}^{r} \alpha_i = \begin{cases} (r - 1)/2 & \text{if } r \text{ is odd}, \\ r/2 & \text{if } r \text{ is even}, \end{cases}
$$

and (4.7) follows similarly. $\square$
Lemma 4.4. If $\alpha_1 = 0$, the following limits hold uniformly:

\begin{align}
\lim_{n \to \infty} S_{n,r}(\cos r\pi x) &= \cos r\pi x, \\
\lim_{n \to \infty} S_{n,r}(\sin r\pi x) &= \sin r\pi x + (-1)^{r-1} 2^{-r-1} \prod_{i=1}^{r} \sin (x - \alpha_i).
\end{align}

If $\alpha_1 > 0$, the following limits hold uniformly:

\begin{align}
\lim_{n \to \infty} S_{n,r}(\cos r\pi x) &= \cos r\pi x + (-1)^{r-1} 2^{-r-1} \prod_{i=1}^{r} \sin (x - \alpha_i), \\
\lim_{n \to \infty} S_{n,r}(\sin r\pi x) &= \sin r\pi x.
\end{align}

Proof. First we write

\begin{align*}
\frac{V(e^{-2\pi i\beta_1}, e^{-2\pi i\beta_2}, \ldots, e^{-2\pi i\beta_r})}{V(e^{-2\pi i\alpha_1}, e^{-2\pi i\alpha_2}, \ldots, e^{-2\pi i\alpha_r})} &= \prod_{1 \leq i < k \leq r} \frac{(e^{-2\pi i\beta_k} - e^{-2\pi i\beta_i})}{(e^{-2\pi i\alpha_k} - e^{-2\pi i\alpha_i})} \\
&= e^{(r-1)\pi i(\alpha_1 - \alpha_s)} \prod_{1 \leq i < k \leq r} \sin \pi (\beta_k - \beta_i) \\
&= \prod_{1 \leq i < k \leq r} \sin \pi (\alpha_k - \alpha_i) \\
&= e^{(r-1)\pi i(\alpha_1 - \alpha_s)} \prod_{k=1}^{r} \sin \pi (x - \alpha_k) \prod_{k=1}^{r} \sin \pi (\alpha_s - \alpha_k),
\end{align*}

where $\prod_{k=1}^{r}$ indicates that the factor involving $k = s$ is omitted. Hence it follows from (4.6) and (4.7) that

\begin{align*}
\lim_{n \to \infty} S_{n,r}(x, r\pi) &= \sum_{s=1}^{r} e^{r\pi i\alpha_s} \cos \pi (x - \alpha_s) \frac{\prod_{k=1}^{r} \sin \pi (x - \alpha_k)}{\prod_{k=1}^{r} \sin \pi (\alpha_s - \alpha_k)} \\
&\equiv \phi(x) + i\psi(x),
\end{align*}

where

\begin{align*}
\phi(x) &= \sum_{s=1}^{r} \cos r\pi \alpha_s \cos \pi (x - \alpha_s) \frac{\prod_{k=1}^{r} \sin \pi (x - \alpha_k)}{\prod_{k=1}^{r} \sin \pi (\alpha_s - \alpha_k)}, \\
\psi(x) &= \sum_{s=1}^{r} \sin r\pi \alpha_s \sin \pi (x - \alpha_s) \frac{\prod_{k=1}^{r} \sin \pi (x - \alpha_k)}{\prod_{k=1}^{r} \sin \pi (\alpha_s - \alpha_k)}.
\end{align*}
and
\[ \psi(x) = \sum_{s=1}^{r} \sin r\pi \alpha_s \cos \pi(x - \alpha_s) \frac{\prod_{k=1}^{r} \sin \pi(x - \alpha_k)}{\prod_{k=1}^{r} \sin \pi(\alpha_s - \alpha_k)}. \]

Clearly \( \lim_{n \to \infty} S_n, r(\cos r\pi x) = \phi(x), \lim_{n \to \infty} S_n, r(\sin r\pi x) = \psi(x). \)

Next, we want to simplify \( \phi(x) \) and \( \psi(x) \). First we consider \( r \) even. The Gauss trigonometric interpolation formula gives

\[ (4.15) \quad \phi(x) = \cos r\pi x + \lambda \prod_{k=1}^{r} \sin \pi(x - \alpha_k). \]

To determine \( \lambda \), we write

\[ (4.16) \quad \cos \pi(\alpha_s - x) \prod_{k=1}^{r} \sin \pi(x - \alpha_k) = \frac{1}{2(2i)^{r-1}} \left( e^{\pi i(x - \alpha_s)} + e^{-\pi i(x - \alpha_s)} \right) \]

\[ \times \left( \exp \left[ \pi i \left( (r - 1)x - \sum_{k=1}^{r} \alpha_k \right) \right] + \cdots + (-1)^{r-1} \exp \left[ -\pi i \left( (r - 1)x - \sum_{k=1}^{r} \alpha_k \right) \right] \right) \]

and

\[ (4.17) \quad \prod_{k=1}^{r} \sin \pi(x - \alpha_k) = \frac{1}{(2i)^{r}} \left( \exp \left[ \pi i \left( rx - \sum_{k=1}^{r} \alpha_k \right) \right] + \cdots \right. \]

\[ + (-1)^{r} \exp \left[ -\pi i \left( rx - \sum_{k=1}^{r} \alpha_k \right) \right] \right). \]

Equating the highest order terms in (4.15), it follows from (4.15)–(4.17) that

\[ \frac{1}{2(2i)^{r-1}} \sum_{s=1}^{r} A_s \left[ \cos \left( \pi \left( rx - \sum_{k=1}^{r} \alpha_k \right) \right) + i \sin \left( \pi \left( rx - \sum_{k=1}^{r} \alpha_k \right) \right) \right] \]

\[ + (-1)^{r-1} \left[ \cos \left( \pi \left( rx - \sum_{k=1}^{r} \alpha_k \right) \right) - i \sin \left( \pi \left( rx - \sum_{k=1}^{r} \alpha_k \right) \right) \right] \]

\[ = \cos r\pi x + \lambda \left[ \cos \left( \pi \left( rx - \sum_{k=1}^{r} \alpha_k \right) \right) + i \sin \left( \pi \left( rx - \sum_{k=1}^{r} \alpha_k \right) \right) \right] \]

\[ + (-1)^{r} \left[ \cos \left( \pi \left( rx - \sum_{k=1}^{r} \alpha_k \right) \right) - i \sin \left( \pi \left( rx - \sum_{k=1}^{r} \alpha_k \right) \right) \right], \]

where \( A_s = \cos \pi r \alpha_s / \prod_{k=1}^{r} \sin \pi(\alpha_s - \alpha_k). \)
Since \( r \) is even it follows that

\[
(4.18) \quad \frac{-2}{(2i)^r} \sum_{s=1}^{r} A_s \sin \pi \left( rx - \sum_{k=1}^{r} \alpha_k \right) = \cos r\pi x + \frac{2\lambda \cos \pi \left( rx - \sum_{k=1}^{r} \alpha_k \right)}{(2i)^r}.
\]

Now if \( \alpha_1 = 0 \), then \( \sum_{k=1}^{r} \alpha_k = r/2 - \frac{1}{2} \), so that (4.18) becomes

\[
(\frac{-1}{(2i)^r})^{(r+2)/2} \cos r\pi x \left( \sum_{s=1}^{r} A_s \right) = \cos \pi r x + \frac{(-1)^{(r+2)/2}2\lambda \sin r\pi x}{(2i)^r}.
\]

Hence \( \lambda = 0 \). This proves (4.11) for \( r \) even.

If \( \alpha_1 \neq 0 \), then \( \sum_{k=1}^{r} \alpha_k = r/2 \), so that (4.18) becomes

\[
\frac{-2}{2^r} \sin r\pi x \left( \sum_{s=1}^{r} A_s \right) = \cos \pi r x + \frac{2\lambda \cos \pi r x}{2^r}.
\]

Hence \( \lambda = -2^{r-1} \). This proves (4.13) for \( r \) even. The proof of (4.12) and (4.14) for \( r \) even are the same.

Next we consider \( r \) odd. If \( \alpha_1 \neq 0 \), we let \( \alpha_0 = 0 \) and write

\[
(4.19) \quad \phi(x) = \frac{1}{\sin \pi x} \sum_{s=0}^{r} \cos r\pi \alpha_s \cos \pi(x - \alpha_s) \frac{\prod_{k=0}^{r} \sin(x - \alpha_k)}{\prod_{k=0}^{r} \sin(\alpha_s - \alpha_k)}.
\]

The Gauss interpolation formula again gives

\[
\sum_{s=0}^{r} \cos r\pi \alpha_s \sin \pi \alpha_s \cos \pi(x - \alpha_s) \frac{\prod_{k=0}^{r} \sin(x - \alpha_k)}{\prod_{k=0}^{r} \sin(\alpha_s - \alpha_k)} = \cos r\pi x \sin \pi x + \lambda \prod_{k=0}^{r} \sin(x - \alpha_k).
\]

A similar calculation gives \( \lambda = 2^{r-1} \), so that (4.19) gives \( \phi(x) = \cos r\pi x + 2^{r-1} \prod_{k=1}^{r} \sin(\pi(x - \alpha_k)) \). This proves (4.13) for \( r \) odd. The proof of (4.14) is similar.

If \( \alpha_1 = 0 \), we let \( \alpha_{k+1} = \frac{1}{2} \) and write

\[
(4.20) \quad \phi(x) = \frac{1}{\cos \pi x} \sum_{s=1}^{r+1} \cos r\pi \alpha_s \cos \pi(x - \alpha_s) \frac{\prod_{k=1}^{r+1} \sin(x - \alpha_k)}{\prod_{k=1}^{r+1} \sin(\alpha_s - \alpha_k)},
\]

and (4.11) and (4.12) for odd \( r \) are proved similarly. □

Proof of Theorem 3. Let

\[
\alpha_0(u) = \begin{cases} 
\alpha(-r\pi + 0) & \text{if } u = -r\pi, \\
\alpha(u) & \text{if } -r\pi < u < r\pi, \\
\alpha(r\pi - 0) & \text{if } u = r\pi.
\end{cases}
\]
Then $\omega_0(u)$ has no jumps at $\pm \pi$. Define

$$f_0(x) = \int_{-\pi}^{\pi} e^{iux} d\omega_0(u) \quad \forall x \in \mathbb{R}.$$  

Setting $A_1 = \omega(-\pi + 0) - \omega(-\pi)$, $A_2 = \omega(\pi) - \omega(\pi - 0)$, we can write $f(x) = f_0(x) + A_1 e^{-r\pi x} + A_2 e^{r\pi x}$, and setting $A = A_1 + A_2$, $B = i(A_2 - A_1)$ we obtain

(4.21) $f(x) = f_0(x) + A \cos r\pi x + B \sin r\pi x \quad \forall x \in \mathbb{R}.$

Now $f_0 \in B_{r\pi}^*$ since $f_0$ is the uniform limit of the sequence $\{f_j\}$, $f_j \in B_{p_j}$, defined by $f_j(x) = \int_{p_j}^{p_j+\pi} e^{iux} d\omega_0(u)$, with $0 < p_j < \pi$, $p_j \to \pi$ as $j \to \infty$. By Lemma 4.2 we conclude that $\lim_{n \to \infty} S_{n,r}(f_0; x) = f_0(x)$ uniformly on $\mathbb{R}$. The theorem now follows from (4.21) and Lemma 4.4.

Finally we consider the class $\mathcal{E}$ of almost periodic functions in the sense of Bohr. To every $f \in \mathcal{E}$ corresponds a Fourier series

$$f(x) \sim \sum_{\nu=1}^{\infty} A_{\nu} e^{i\lambda_{\nu} x},$$

where $\lambda_{\nu}$ are real numbers, called the Fourier exponents of $f$. Also for $\sigma \geq 0$,

$$\mathcal{E} \cap B_{\sigma} = \{ f : f \in \mathcal{E}, -\sigma \leq \lambda_{\nu} \leq \sigma \}.$$ 

**Proof of Theorem 4.** Suppose $f \in \mathcal{E} \cap B_{r\pi}$. Then its Fourier exponents $\lambda_{\nu}$, $\nu = 1, 2, 3, \ldots$, satisfy $-\pi \leq \lambda_{\nu} \leq \pi$.

Without loss of generality we may assume that $\lambda_1 = -\pi$, $\lambda_2 = \pi$ with the understanding that $A_1 = 0$ if the exponent $-\pi$ is absent, and similarly that $A_2 = 0$ if exponent $\pi$ is absent.

Let

$$A_1 e^{-r\pi x} + A_2 e^{r\pi x} = A \cos r\pi x + B \sin r\pi x,$$

where $A = A_2 + A_1$, $B = i(A_2 - A_1)$. It follows that the function

(4.22) $g(x) = f(x) - A \cos r\pi x - B \sin r\pi x$

has Fourier series $g(x) \sim \sum_{\nu=3}^{\infty} A_{\nu} e^{i\lambda_{\nu} x}$ where $-\pi < \lambda_{\nu} < \pi \forall \lambda_{\nu} = 3, 4, 5, \ldots$. A similar argument as in [10] shows that $g \in B_{r\pi}^*$. It follows from Lemma 4.2 that

(4.23) $S_{n,r}(g; x) \to g(x)$ uniformly on $\mathbb{R}$.

The theorem then follows from (4.22), (4.23) and Lemma 4.4. \[\square\]

**References**


DEPARTMENT OF MATHEMATICS, THE UNIVERSITY, DUNDEE DD1 4HN, SCOTLAND, UNITED KINGDOM

SCHOOL OF MATHEMATICAL SCIENCES, UNIVERSITY SCIENCE MALAYSIA, PENANG, MALAYSIA