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VARIETIES OF COMBINATORIAL GEOMETRIES

BY

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ABSTRACT. A *hereditary class* of (finite combinatorial) geometries is a collection of geometries which is closed under taking minors and direct sums. A *sequence of universal models* for a hereditary class $\mathfrak{T}$ of geometries is a sequence (T_n) of geometries in $\mathfrak{T}$ with rank $T_n = n$, and satisfying the universal property: if G is a geometry in $\mathfrak{T}$ of rank n, then G is a subgeometry of T_n. A *variety* of geometries is a hereditary class with a sequence of universal models.

We prove that, apart from two degenerate cases, the only varieties of combinatorial geometries are

(1) the variety of free geometries,

(2) the variety of geometries coordinatizable over a fixed finite field, and

(3) the variety of voltage-graphic geometries with voltages in a fixed finite group.

1. Introduction. The notion of free objects in a variety is one of the most important and pervasive ideas in universal algebra. In this paper, we investigate how that notion can be interpreted in the context of the theory of combinatorial geometries (or matroids). No detailed knowledge of universal algebra is required for reading this paper. However, we do assume familiarity with the basic concepts of the theory of combinatorial geometries [**2**, **4** and **13**].

To fix our terminology, let G be a finite geometric lattice. Its maximum and minimum are denoted by $\hat{1}$ and $\hat{0}$. Let S be the set of points (or atoms) in G. The lattice structure of G induces the structure of a combinatorial geometry, also denoted by G, on S. The cardinality $|G|$ of the geometry G is the cardinality of the set S of points. All geometries appearing in this paper are assumed to be of finite cardinality.

Let T be a subset of S. The *deletion* of T from G is the geometry on the point set $S \setminus T$ obtained by restricting G to the subset $S \setminus T$. The *contraction* G/T of G by T is the geometry induced by the geometric lattice $[\bar{T}, \hat{1}]$ on the set S' of all flats in G covering $\bar{T}$ (the closure of T). Thus, in our convention, the contraction of a geometry is always a geometry—indeed, it is isomorphic to the simplification of the matroid G/T as it is usually defined. A geometry which can be obtained from G by deletions and contractions is called a *minor* of G. Minors are the subobjects in the category of combinatorial geometries and strong maps.

Finally, we note that we shall often use "is" for "is isomorphic to" when precision threatens to turn into pedantry.

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2. Varieties of combinatorial geometries. A collection $\mathfrak{T}$ of combinatorial geometries is a *hereditary class* if it is closed under taking minors and direct sums. More precisely, a hereditary class $\mathfrak{T}$ satisfies the following axioms.

H1. If G is in $\mathfrak{T}$ and H is isomorphic to a minor of G, then H is also in $\mathfrak{T}$.

H2. If G and H are in $\mathfrak{T}$, then so is their direct sum $G \oplus H$.

Hereditary classes of geometries have been discussed earlier, most notably in [**9**], but little is known about them.

Formally, the definition of a hereditary class is a direct analog of the definition of a variety in universal algebra (see, for example, [**3**]). However, in universal algebra, a variety $\mathfrak{V}$ of algebras is a very structured collection of objects. For example, a theorem of Birkhoff says that it can be defined by equations. A consequence of this is the existence for each n of a free algebra on n generators—that is, there exists an algebra F_n in $\mathfrak{V}$ defined by the *universal* property: if A is an algebra in $\mathfrak{V}$ generated by n elements, then A is a subobject of F_n.

What is the analogue of free algebras for hereditary classes? A reasonable definition is the following. A *sequence of universal models* for a hereditary class $\mathfrak{T}$ of geometries is a sequence (T_n) of geometries such that

U1. T_n is in $\mathfrak{T}$ and is of rank n.

U2. If G is a geometry of rank n in $\mathfrak{T}$, then G is a subgeometry of T_n.

In contrast to the situation in universal algebra, sequences of universal models need not exist. Take for example, the hereditary class of all geometries, or the class of all totally unimodular geometries.

DEFINITION. A *variety* $\mathfrak{T}$ of combinatorial geometries is a hereditary class of geometries with a sequence of universal models.

There are two important examples of varieties. Let $\mathfrak{L}(q)$ be the collection of all geometries coordinatizable over the finite field $GF(q)$. The projective geometries $P_n(q)$—here, n is the rank of $P_n(q)$ as a geometric lattice—are universal models for $\mathfrak{L}(q)$. A less well-known example is the collection $\mathfrak{X}(A)$ of voltage-graphic geometries with voltages in a finite group A; the universal models here are the Dowling geometries $Q_n(A)$ (see [**7**]). These voltage-graphic geometries were discovered and studied by Zaslavsky [**14**, **15**]. We shall need only the basic properties of Dowling geometries and they will be recounted at the appropriate moment. We should mention that when A is the trivial group, $Q_n(A)$ is the graphic geometry of the complete graph (on $n + 1$ vertices) and the associated variety $\mathfrak{X}(A)$ is the variety of graphic geometries.

One is at this point hard put to think of additional examples. The reason for this is that, apart from degenerate cases, there are no others. This statement, which will be made precise in §9, is our main result. We proceed first to its proof.

3. Modular flats. Henceforth, (T_n) is a sequence of universal models for a variety $\mathfrak{T}$ of geometries.

LEMMA 1. *Let G be a geometry in $\mathfrak{T}$ and let t_i be a subgeometry of G isomorphic to T_i. Then t_i is a modular flat of G.*

PROOF. First observe that t_i is of rank i and its closure is a subgeometry of T_i by universality. Hence, t_i is a closed set. Now, let u be a modular complement of t_i. The function from the interval $[u, \hat{1}]$ to $[\hat{0}, t_i]$ defined by $x \mapsto x \wedge t_i$ is a surjection. Since u is a modular complement, $[u, \hat{1}]$ contains a copy of T_i as a subgeometry (viz. the image of $[0, t_i]$ under the map $x \mapsto x \vee u$); as T_i is universal, $[u, \hat{1}]$ is isomorphic to T_i. Thus, the surjection is in fact a bijection.

In particular, if v strictly contains u, $v \wedge t_i \neq \hat{0}$. Thus, none of the flats lying above u can be a complement of t_i: that is, no two complements of t_i are comparable. By a result of Stanley [**11**, Theorem 1], t_i is modular. □

An immediate consequence of the lemma is that the universal model T_n is supersolvable. Recall that a geometric lattice of rank n is *supersolvable* if there exists a (saturated) chain $\hat{0} = x_0 < x_1 < \cdots < x_{n-1} < x_n = \hat{1}$ with rank $x_i = i$ such that every flat x_i is modular [**12**].

PROPOSITION 2. *The universal model T_n is supersolvable with a chain $\hat{0} = t_0 < t_1 < \cdots < t_n = \hat{1}$ of modular flats such that the subgeometry t_i is isomorphic to T_i.*

PROOF. As $\mathfrak{T}$ is closed under direct sum, $T_{n-1} \oplus T_1$ occurs as a subgeometry of T_n; in particular, there is a subset t_{n-1} in T_n isomorphic to T_{n-1}. This set t_{n-1} is a modular copoint. We can now repeat the argument on the subgeometry t_{n-1}. □

Another consequence is a technical tool we shall use time and again.

LEMMA 3 (THE PROJECTION ARGUMENT). *Let t_{n-1} be a copoint of T_n isomorphic to T_{n-1} and let $u = \{x_1, \ldots, x_m\}$ be a set of points of T_n. Let z be a point of T_n which is not in the union $t_{n-1} \cup \bar{u}$. Then the map $x_i \to \overline{x_i z} \wedge t_{n-1}$ determines an isomorphism of u with a subgeometry of t_{n-1}.*

PROOF. By Lemma 1, $\overline{x_i z} \wedge t_{n-1}$ is a point. □

We abbreviate the above situation by the locution: The configuration u is projected onto the copoint t_{n-1} via contraction by z.

4. Two disconnected examples. The *order* of the sequence (T_n) of universal models is the positive integer $|T_2| - 1$. We first consider the case when q equals one; that is, when T_2 is a two-point line.

PROPOSITION 4. *Let $q = 1$. Then T_n is the Boolean algebra* (*or free geometry*) *B_n on n elements.*

PROOF (GREENE [**8**]). We proceed by induction on the index n. By hypothesis, all the copoints in T_n are isomorphic to the Boolean algebra B_{n-1} and hence all n-element sets are independent. Thus, T_n is a truncated Boolean algebra. But if it were nontrivially truncated, there would exist a rank two upper minor containing a three-point line. □

The Boolean algebras B_n are the universal models of the *variety of free geometries*. They are also the only geometries in this variety.

We can now assume the *the order q is greater than one.*

LEMMA 5. *Suppose that T_3 is connected. Then T_n is connected for all n.*

PROOF. It suffices to show that for $n \geq 4$, if T_{n-2} and T_{n-1} are connected, then so is T_n. Suppose that T_n is disconnected. As T_{n-1} is connected and T_n contains $T_{n-1} \oplus T_1$ as a subgeometry, we conclude that $T_n \cong T_{n-1} \oplus T_1$. But T_n also contains

$T_{n-2} \oplus T_2$. Let t_{n-2} be the subset in T_n isomorphic to T_{n-2}; then $T_n/t_{n-2} \cong T_2$. Now, t_{n-2}, being connected, must be contained in the summand isomorphic to T_{n-1}. Hence, $T_n/t_{n-2} \cong (T_{n-1}/t_{n-2}) \oplus T_1 \cong T_1 \oplus T_1$. This implies that T_2 is a two-point line, contrary to our standing assumption that $q \geqslant 2$. $\square$

Is there a sequence (M_n) of universal models with M_3 disconnected? If so, we must have $M_3 \cong M_2 \oplus M_1$ where M_2 is a line with $q+1$ points. A moments thought yields: $M_4 \cong M_2 \oplus M_2$. We can now show by induction that

$$M_{2n+1} \cong M_2 \oplus \cdots \oplus M_2 \oplus M_1 \quad \text{and} \quad M_{2n} \cong M_2 \oplus \cdots \oplus M_2$$

(where there are n copies of M_2 in each sum). By taking all subgeometries of the geometries M_n, we obtain a variety of geometries. The reader should have no difficulty in seeing why we call it the *variety of matchstick geometries of order* q.

We can now assume that *the universal models* T_n *are all connected.*

5. Having enough points. We have arrived at the first branch (of two) in our proof. First, a definition. We say that a geometry *splits* if it is the union of two of its proper flats (where we regard flats as point sets).

From here until §8, we assume that *no* T_n *splits*. This condition ensures that there are "enough points" to perform certain contractions.

LEMMA 6 (UPPER HOMOGENEITY). *Let p be any point in T_{n+1}. Then there exists a copoint in T_{n+1} isomorphic to T_n and not containing p. In particular, the contraction T_{n+1}/p is isomorphic to T_n.*

PROOF. The result being trivial for $n = 1$, we assume that n is at least 2. Let p be a point in T_{n+1} and consider an embedding i_{2n} of T_{n+1} into T_{2n}. As T_{2n} contains the direct sum $T_n \oplus T_n$ and the image of p can be in at most one of the summands, there exists a flat t_n in T_{2n} isomorphic to T_n not containing the point $i_{2n}(p)$. Now as T_{2n} does not split, there exists a point z such that z is not in the union $(t_n \vee i_{2n}(p)) \cup i_{2n}(T_{n+1})$. By our choice of z, the image of $i_{2n}(T_{n+1})$ in the contraction T_{2n}/z is isomorphic to T_{n+1} and the image t_n' of t_n is a flat in T_{2n}/z isomorphic to T_n and not containing the image of $i_{2n}(p)$. By universality, there is an embedding of T_{2n}/z into T_{2n-1}. Let i_{2n-1} be the composition of the maps

$$T_{n+1} \to T_{2n} \to T_{2n}/z \to T_{2n-1}.$$

In T_{2n-1}, there is a flat (the image of t_n') which is isomorphic to T_n and does not contain $i_{2n-1}(p)$. As $T_{2n-1}, T_{2n-2}, \ldots, T_{n+2}$ do not split, we can repeat this argument to obtain an isomorphism i_{n+1} of T_{n+1} into T_{n+1} such that there is a flat t_n'' of T_{n+1} isomorphic to T_n and not containing $i_{n+1}(p)$. The inverse image $i_{n+1}^{-1}(t_n'')$ is a copoint of T_{n+1} not containing p. $\square$

Now, consider the set of all copoints in T_n isomorphic to T_{n-1}. By the lemma, their intersection is $\hat{0}$. Thus, there exists a set of n such copoints for which the intersection is $\hat{0}$. Let $t^1, \ldots, t^n$ be such a set of copoints. As these copoints are modular, the meet sublattice they generate in T_n is a Boolean algebra. In particular, the intersections

$$p_i = t^1 \wedge \cdots \wedge t^{i-1} \wedge t^{i+1} \wedge \cdots \wedge t^n$$

are points in T_n. The subgeometry of T_n obtained by restricting T_n to the set of points $\bigcup t^i$ is denoted by $\langle p_1,\ldots,p_n\rangle$; it is called the *frame of* T_n *generated by the joints* $p_1,\ldots,p_n$.

What is the span of $p_1,\ldots,p_k$? This span, being the intersection $t_{k+1} \wedge \cdots \wedge t_n$ of modular copoints, is a modular flat of rank k. It is in fact isomorphic to T_k. (For, by modularity, $[\hat{0}, \overline{p_1 \cdots p_k}] \cong [u, \hat{1}]$, where u is any complement of $\overline{p_1 \cdots p_k}$; but T_n is upper homogeneous.)

We now arrive at the second branch in our proof. Choose a frame in T_3. Two possibilities may occur: the frame is all of T_3; or there may be more points in T_3. The first possibility leads to Dowling geometries while the second leads to projective geometries.

6. Projective geometries. We assume in this section that T_3 *contains more points than a frame*. Ultimately, we shall show that this implies the order q of (T_n) is a prime power and T_n is the projective geometry $P_n(q)$.

We begin with an easy fact.

LEMMA 7. *Let* (T_n) *be of order* q. *Then*

$$|T_n| \leqslant 1 + q + q^2 + \cdots + q^{n-1},$$

with equality if and only if T_n *is a projective geometry of order* q.

PROOF. For $n \geqslant 1$, let $a_n = |T_n| - |T_{n-1}|$. We prove by induction that $a_n \leqslant q^{n-1}$; this will clearly imply the inequality. For $n = 1$ or 2, the assertion holds trivially. Now consider $n \geqslant 3$. Let $t_{n-2} < t_{n-1}$ be a chain in T_n consisting of a coline and a copoint isomorphic to T_{n-2} and T_{n-1} (respectively). There are q other copoints covering t_{n-2} and by induction, these copoints u satisfy: $|u| - |t_{n-2}| \leqslant q^{n-2}$. Hence, $|T_n| - |T_{n-1}| \leqslant q \cdot q^{n-2}$.

Suppose now that equality is attained. As T_n is connected, it suffices to show that T_n is a modular geometry [**2**, p. 93]; as is easy to show, this is equivalent to showing that every line is modular, which we prove by showing that every line contains at least $q + 1$ points. We proceed by induction. In T_1 and T_2, every line has $q + 1$ points. Now, let t_{n-2} be a coline isomorphic to T_{n-2} in T_n. By the equality, every copoint lying above t_{n-2} is isomorphic to T_{n-1}. Using the induction hypothesis, every line contained in one of these copoints has $q + 1$ points. Finally, consider a line l not contained in any of these copoints. This line l must intersect every copoint covering t_{n-2} and hence must have $q + 1$ points. □

Let $\langle p_1 p_2 p_3\rangle$ be a frame in T_3. By assumption, there is a point—an *exterior point*—in T_3 not in the frame. Consider now a frame $\langle p_1 p_2 p_3 p_4\rangle$ in T_4. Let z be an exterior point of the plane $\overline{\langle p_2 p_3 p_4\rangle}$ and let p be the intersection of the lines $\overline{p_4 z}$ and $\overline{p_2 p_3}$. Contracting by the point z, we project the line $\overline{p_1 p_4}$ onto the line $\overline{p_1 p}$. We conclude that $\overline{p_1 p}$ has $q + 1$ points, of which $q - 1$ are exterior points. Similarly, each contraction (of T_4) by one of these $q - 1$ points projects $\overline{p_3 p_4}$ onto a line containing p_4 and $q - 1$ exterior points in $\overline{p_1 p_2 p_4}$. This gives at least $(q - 1)^2$ exterior points in $\overline{p_1 p_2 p_4}$ (hence in T_3), which by the lemma makes T_3 a projective plane of order q.

Our final step is to show that T_n is a projective geometry. Again, we proceed by induction, having already proved the assertion for $n \leq 3$. Suppose that the assertion is true for $m \leq n-1$. Let $\langle p_1 \cdots p_{n+1} \rangle$ be a frame in T_{n+1} and let $t = \overline{p_1 \cdots p_{n-2}}$. As observed earlier, t is isomorphic to T_{n-2} and is thus a projective geometry of rank $n-2$. Let u be any flat covering t and contained in the modular flat $\overline{p_1 \cdots p_n}$. The flat $\overline{p_1 \cdots p_n}$ is isomorphic to T_n. We shall show that it is in fact a projective geometry. Consider the intersection $u \wedge \overline{p_{n-1} p_n}$. As $\overline{p_{n-1} p_n}$ is a modular line, this intersection is a point a. Obviously, u is the closure of $p_1, \ldots, p_{n-2}, a$. Now, in T_{n+1}, consider the line $\overline{ap_{n+1}}$. This line is contained in the projective plane $\overline{p_{n-1} p_n p_{n+1}}$ and hence there exists a point z in $\overline{ap_{n+1}}$ distinct from a and p_{n+1}. Contracting by the point z, we project the projective geometry (of rank $n-1$) $\overline{p_1 \cdots p_{n-2} p_{n+1}}$ onto u (u being contained in $\overline{p_1 \cdots p_n}$). We conclude that u is a projective geometry of rank $n-1$; indeed, every flat contained in $\overline{p_1 \cdots p_n}$ covering t is a projective geometry. Counting up all the points, we obtain the equality

$$|T_n| = |\overline{p_1 \cdots p_n}| = 1 + q + \cdots + q^{n-1}.$$

By the lemma, T_n is a projective geometry of rank n. Finally, we note that, as is well known, projective geometries of order q and rank higher than three exist only if q is a prime power.

7. Dowling geometries. We now tackle the possibility that T_3 *consists only of a frame*. Our first result is that this property propagates upwards.

LEMMA 8. *If T_3 consists only of a frame, then T_n also consists only of a frame.*

PROOF. Let n be the first index for which the lemma is false. Choose a frame $\langle p_1, \ldots, p_n \rangle$ in T_n and let z be a point in T_n not in the frame. Consider the intersection $\overline{p_1 \cdots p_{n-1}} \wedge \overline{p_n z}$. By modularity, this is a point contained in the flat $\overline{p_1 \cdots p_{n-1}}$ (which is isomorphic to T_{n-1}), but it is not contained in the frame $\langle p_1, \ldots, p_{n-1} \rangle$. This is a contradiction. □

Our next task is to determine the structure of T_3. Let $\langle p_1 p_2 p_3 \rangle$ be a frame for T_3. There are exactly three modular lines: $l_{12} = \overline{p_1 p_2}$, $l_{23} = \overline{p_2 p_3}$ and $l_{13} = \overline{p_1 p_3}$. Let a be an *interior* point (i.e. a point distinct from the joints) of l_{12}, and let b be any interior point of l_{23}. The line $\overline{ab}$ meets the modular line l_{13} in a unique point c, and consists precisely of a, b and c. Such lines $\overline{ab}$ are called *transversals*. We interpret the transversal $\{a, b, c\}$ as the equation $ab = c$ and form a multiplication table with the interior points of l_{12} labelling the rows, the interior points of l_{23} labelling the columns, and the interior points of l_{13} as the entries. This multiplication table is a Latin square. Our next step is to show that it can be labelled so as to form the multiplication table of a group.

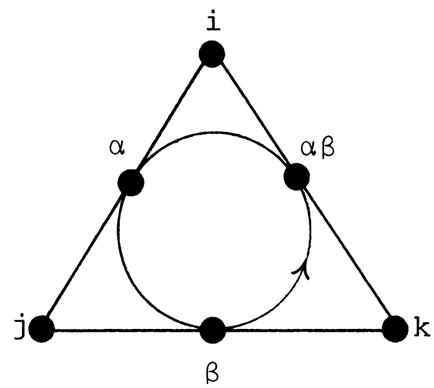
To do so, we move up to T_4; there are four copies of T_3 (called *facets*) in T_4, viz. $t_{ijk} = \overline{p_i p_j p_k}$ where i, j and k are elements from the set $\{1, 2, 3, 4\}$. We shall always choose i, j and k so that $i < j < k$. Again, we denote the modular line $\overline{p_i p_j}$ by l_{ij}.

We now label the interior points in the six modular lines l_{ij}. Choose interior points a_{12}, a_{13} and a_{14} in the lines l_{12}, l_{13} and l_{14} respectively. Label these points ε; ε will turn out to be the identity element of the group. Further, label the interior points in l_{12} (by Greek letters, say). We will make use of the following two procedures for extending the labelling.

(A) Suppose that the points b on l_{ij} and a_{jk} on l_{jk} have been labelled α and ε respectively, and that b' is the point of l_{ik} for which $\{b, a_{jk}, b'\}$ is a transversal. Then label b' by α.

(B) Suppose that the points a_{ij} on l_{ij} and b on l_{ik} have been labelled ε and α respectively, and that b' is the point of l_{jk} for which $\{a_{ij}, b', b\}$ is a transversal. Then label b' by α.

The reader should keep in mind here that multiplication in the quasigroup corresponding to t_{ijk} is given by:



The following steps extend the labelling to all interior points of the line l_{ij}.

(1) Apply (B), first with $(i, j, k) = (1,2,3)$ and $b = a_{13}$, then with $(i, j, k) = (1,2,4)$ and $b = a_{14}$. This gives us points labelled ε on l_{23} and l_{24}.

(2) Apply (A) with $(i, j, k) = (1,2,3)$ and b ranging over all interior points of l_{12} to obtain labels for the points of l_{13}. Repeat, replacing 3 by 4, to obtain labels for l_{14}.

(3) Apply (B) with $(i, j, k) = (1,2,3)$ and b ranging over the interior points of l_{13} to obtain labels for l_{23}. Repeat, replacing 3 by 4, to obtain labels for l_{24}.

(4) Apply (B) with $(i, j, k) = (1,3,4)$ and b ranging over the interior points of l_{14} to obtain labels for l_{34}.

This completes the labelling. Notice that step (4) above was one of four different ways of obtaining a labelling of l_{34}, the other three being application of (A) with $(i, j, k) = (1,3,4)$, and application of (A) or (B) with $(i, j, k) = (2,3,4)$. We will show momentarily that all of these procedures lead to the same labelling. Here, the basic observation is

LEMMA 9. *Let $\{a, b, c\}$, $\{c, d, e\}$ and $\{a, f, e\}$ be transversals on three distinct facets of T_4. Then, $\{b, d, f\}$ is a transversal on the fourth facet.*

PROOF. Without loss of generality, we may assume that $\{a, b, c\}$, $\{c, d, e\}$ and $\{a, f, e\}$ are transversals on the facets t_{123}, t_{134} and t_{124} respectively. In the contraction of T_4 by b, a is projected onto c, $\overline{ae}$ onto $\overline{ce}$ and l_{24} onto l_{34}. Therefore, $f = \overline{ae} \wedge l_{24}$ is projected onto $\overline{ce} \wedge l_{34} = d$. □

In what follows we use α_{ij} to denote the point of l_{ij} which is labelled α.

We may now verify that each of the three alternate procedures mentioned above produces our present labelling of l_{34}. We first observe that since $\{\varepsilon_{12}, \varepsilon_{23}, \varepsilon_{13}\}$, $\{\varepsilon_{12}, \varepsilon_{24}, \varepsilon_{14}\}$ and $\{\varepsilon_{13}, \varepsilon_{34}, \varepsilon_{14}\}$ are transversals, Lemma 9 says that $\{\varepsilon_{23}, \varepsilon_{34}, \varepsilon_{24}\}$ is also. But then, as $\{\alpha_{12}, \varepsilon_{23}, \alpha_{13}\}$ and $\{\alpha_{12}, \varepsilon_{24}, \alpha_{14}\}$ are also transversals, $\{\alpha_{13}, \varepsilon_{34}, \alpha_{14}\}$ must be one as well. That is, we would have obtained the same labelling of l_{34} by applying (A) with $(i, j, k) = (1, 3, 4)$. That the remaining two procedures give the same result is shown similarly: $\{\alpha_{23}, \varepsilon_{34}, \alpha_{24}\}$ is a transversal because $\{\varepsilon_{12}, \alpha_{23}, \alpha_{13}\}$, $\{\alpha_{13}, \varepsilon_{34}, \alpha_{14}\}$ and $\{\varepsilon_{12}, \alpha_{24}, \alpha_{14}\}$ are; and $\{\varepsilon_{23}, \alpha_{34}, \alpha_{24}\}$ is a transversal because $\{\varepsilon_{12}, \varepsilon_{23}, \varepsilon_{13}\}$, $\{\varepsilon_{12}, \alpha_{24}, \alpha_{14}\}$ and $\{\varepsilon_{13}, \alpha_{34}, \alpha_{14}\}$ are.

Now each of the four facets t_{ijk} defines a multiplication as described above. We need to know that these multiplications are the same. But contraction by a_{34} (which equals ε_{34}) fixes α_{12} and projects β_{23} onto β_{24}, $(\alpha\beta)_{13}$ onto $(\alpha\beta)_{14}$, so that t_{123} and t_{124} define the same multiplication. Similarly we can equate the multiplications of t_{124} and t_{134} by contracting a_{23} and of t_{134} and t_{234} by contracting a_{12}. Thus all multiplications are the same.

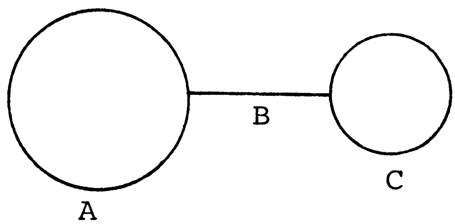
We are now ready to show that our multiplication table is the multiplication table of a group. The only axiom requiring proof is the associative law. Consider the transversals $\{\alpha_{12}, \beta_{23}, (\alpha\beta)_{13}\}$, $\{(\alpha\beta)_{13}, \gamma_{34}, (\alpha\beta)\gamma_{14}\}$, and $\{\beta_{23}, \gamma_{34}, (\beta\gamma)_{24}\}$. By the lemma, $\{\alpha_{12}, (\beta\gamma)_{24}, (\alpha\beta)\gamma_{14}\}$ is also a transversal. Hence, $(\alpha\beta)\gamma = \alpha(\beta\gamma)$. We denote the abstract group defined by our multiplication table by A. As Dowling has shown (his proof of Theorem 8 in [7] applies here), the group A is determined up to isomorphism by T_3.

What are the dependencies in T_n? The easiest way to describe them is to consider the following graphical representation. (The ideas and terminology here are based on Zaslavsky [**14**, **15**].) We construct the graph $\bar{K}_n$ on the vertex set $\{1, 2, \ldots, n\}$ by inserting between each pair of vertices i and j (with $i < j$) $|A|$ multiple edges labelled by the elements of A and directed from i to j. The graph $\bar{K}_n$ is thus a "flattened" representation of the geometry T_n. We identify the points of T_n with the vertices and edges of $\bar{K}_n$ by: $p_i \mapsto i$ and $\alpha_{ij} \mapsto$ the edge labelled α between the vertices i and j.

Let S be the union of $\{1, 2, \ldots, n\}$ and the edge set of $\bar{K}_n$. A subset $C \subseteq S$ is called a *cycle* if it consists of a single vertex or it is a cycle (which is not necessarily directed) in the graph-theoretic sense: that is, if the elements of C (which are all edges) can be rearranged so that e_1 is an edge between i_0 and i_1, e_2 is an edge between i_1 and $i_2, \ldots$ and e_m is an edge between i_{m-1} and i_0, and $i_0, \ldots, i_m$ are distinct vertices. Fix a cycle C of edges. Consider the *label map* λ: $C \to A$ defined on the edges of C by: for e_j labelled α, $\lambda(e_j) = \alpha$ if $i_{j-1} < i_j$ and α^{-1} if $i_{j-1} > i_j$. The cycle C is said to be *balanced* if $\lambda(e_1) \cdots \lambda(e_m) = \varepsilon$ (this condition is clearly independent of our choice of starting point); C is *unbalanced* otherwise. We also insist that a cycle consisting of a single vertex is unbalanced; cycles of this kind are called *unbalanced loops*. Besides cycles, we need three other kinds of subgraphs of $\bar{K}_n$. A *theta graph* is a set of edges $\{e_1, \ldots, e_k, f_1, \ldots, f_l, g_1, \ldots, g_m\}$ such that $\{f_1, \ldots, f_l\}$ is nonempty, and

$$\{e_1, \ldots, e_k, f_1, \ldots, f_l\}, \quad \{e_1, \ldots, e_k, g_1, \ldots, g_m\}, \quad \text{and} \quad \{f_1, \ldots, f_l, g_1, \ldots, g_m\}$$

are cycles. A *handcuff* is a subset of S of the form



where A and C are cycles (thus A or C may consist of a single vertex) and B is a simple path. In the special case that B is a single vertex, we also call the cycle a *figure-of-eight*. A *bicycle* (or *bicircular graph*) is a theta graph or a handcuff.

There are four kinds of circuits in the plane t_{ijk} (which equals the span of p_i, p_j and p_k), namely $\{\alpha_{ij}, \beta_{jk}, (\alpha\beta)_{ik}\}$, $\{r, \alpha_{rs}, s\}$, $\{r, \alpha_{rs}, \beta_{rs}\}$ and $\{\alpha_{rs}, \beta_{rs}, \gamma_{rs}\}$; here, in the latter three types, we use r and s to stand for any pair of elements from $\{i, j, k\}$. Circuits of the first type are balanced cycles; those of the remaining three types are bicycles containing no balanced cycles. We call these circuits the *atomic circuits* of T_n.

For brevity, we use the term *Z-set* for a set which is either a balanced cycle, or a bicycle none of whose cycles is balanced. Clearly, no Z-set properly contains another. We shall show that the Z-sets are all the circuits of T_n by showing, firstly, that any subset of S not containing a Z-set is independent, and secondly, that any Z-set is dependent.

Suppose that I is a dependent subset of S which contains no Z-set. We may assume that I is a circuit. Decompose I into its connected components (in the graph-theoretic sense). As I contains no Z-set, any connected component of I can contain at most one unbalanced cycle. Let m be the maximum size of an unbalanced cycle in I. Consider first the case $m \geqslant 3$. Let $\{e_1, \ldots, e_m\}$ (arranged so that e_{i-1} and e_i are incident) be an unbalanced cycle in I of size m. Take the circuit I and the atomic circuit $\{e_{m-1}, e_m, f\}$ (where $\{e_{m-1}, e_m, f\}$ is a transversal). By the circuit elimination property (see e.g. [**13**, p. 23]) we can eliminate the edge e_m from $I \cup \{f\}$, thus obtaining a circuit I' containing fewer unbalanced cycles of size m than I. Using this argument repeatedly, we can reduce to the case $m \leqslant 2$. If $m = 2$, let $\{e_1, e_2\}$ be an unbalanced cycle of size two in I. Consider the atomic circuit $\{e_1, e_2, i\}$, where i is one of the endpoints of e_1 (and e_2). By the circuit elimination property, we can obtain a circuit I' containing one less unbalanced cycle of size two than I. Iterating this argument, we may assume that I contains at most one unbalanced loop (which is the only possible unbalanced cycle of length one) in each connected component.

Now let i be an unbalanced loop in I occurring in a connected component of I which has size at least two. Let e be an edge of I incident on i and let j be the other endpoint of e. Consider the atomic circuit $\{i, e, j\}$; by circuit elimination, we can eliminate e from $I \cup \{j\}$ to obtain another circuit I' containing fewer edges than I. Similarly, if e is an edge (with endpoints i and j) in a component of I containing no unbalanced loop, we can eliminate e from the union of $\{i, e, j\}$ and I to obtain a circuit I containing fewer edges than I. All told, these elimination arguments allow us to assume that I contains no edges—that is to say, I is a union of unbalanced loops. But the set $\{1, 2, \ldots, n\}$ of unbalanced loops is an independent set. This contradiction proves that any set which contains no Z-set is independent.

The proof that all Z-sets are dependent consists in showing that every Z-set may be obtained from the atomic circuits by iterating circuit elimination. The details, which are easy but tedious, are left to the reader.

We have now shown that the circuits of T_n must be the Z-sets. That the Z-sets do indeed form the circuits of a geometry is proved in [**14** and **15**]. The universal model T_n thus obtained is the *Dowling geometry* $Q_n(A)$ *of rank n based on the finite group A* (see [7]). The variety specified by these universal models is called the variety of *voltage-graphic geometries with voltages in A*. (Our description of $Q_n(A)$ differs somewhat from that of [7]. For a proof of the equivalence when A is cyclic—which should indicate the proof in the general case—see [**5** or **6**].)

8. The split case: origami geometries. We return now to the first branch of the proof. Assume that some T_m splits and choose n to be the least integer for which T_{n+1} splits. Since T_2 is connected it does not split, and we may assume that n is at least two. We shall in fact show that n must be equal to two.

LEMMA 10. *Suppose* $T \in \mathfrak{T}$ *contains flats* t_m *and* t_r *isomorphic to* T_m *and* T_r *respectively and that the rank of their intersection* $t_m \wedge t_r$ *is k. Then if* $r \leqslant n$, $t_m \wedge t_r$ *is isomorphic to* T_k.

PROOF. We induct on r, the (smallest) case $r = k$ being trivial. Let $t_m \wedge t_r = t$.

Let t_{r-1} be a flat in t_r isomorphic to T_{r-1}, and let x be a point of t_r not in $t_{r-1} \cup t$. (Such a point exists since t_r does not split.) If we let t'_m and t'_{r-1} be the images of t_m and t_r under contraction by x, then t'_m and t'_{r-1} are flats of T/x isomorphic to T_m and T_{r-1} respectively. Moreover, $t'_m \wedge t'_{r-1}$ is isomorphic to t (since it is the image of t), and to T_k (by the induction hypothesis). Thus t is isomorphic to T_k. □

We must investigate in some detail the structure of the geometries T_m. A first step in this direction is

LEMMA 11. *For each* $m = n + 1, \ldots, 2n$, *there exist flats t and* t' *in* T_m, *isomorphic to* T_n, *such that* $t \vee t' = T_m$.

PROOF. The result is true for $m = 2n$ as T_{2n} contains the direct sum $T_n \oplus T_n$. Moreover, if it is true for $m + 1$ and there exists a point x in T_{m+1} but not in $t \cup t'$, then contraction by x gives the result for m.

So if the lemma is false, there must exist an m_0 ($> n$) such that the lemma holds for $m_0 + 1, \ldots, 2n$, and $T_{m_0+1} = t \cup t'$.

By connectivity $m_0 \leqslant 2n - 2$. By Lemma 1, we may regard T_{m_0+1} as a flat of T_{m_0+2}. Since the lemma is true for $m_0 + 2$, there is a flat t_n of T_{m_0+2} which is isomorphic to T_n and not contained in T_{m_0+1}. By Lemmas 1 and 10, $t_{n-1} = t_n \wedge T_{m_0+1}$ is isomorphic to T_{n-1}. Since t_{n-1} is contained in $t \cup t'$ and does not split, we may assume that t_{n-1} is contained in t. This proves the lemma for $m = n + 1$, since the flat $t \vee t_n$ is of rank $n + 1$, and by universality is a subgeometry of T_{n+1}.

Finally, the validity of the lemma for $m = n + 1$ rules out the existence of an m_0 as described above. For if u, v, w, x are isomorphic to T_n with $u \vee v = T_{n+1}$ and $w \vee x = T_{m_0+1}$ $(m_0 > n)$, then we may regard T_{n+1} as a flat of T_{m_0+1}, and it is easily seen (because $\operatorname{rank}(u \wedge v) > \operatorname{rank}(w \wedge x)$) that u and v cannot both be contained in $w \cup x$. $\square$

We see in particular that T_{n+1} is the union of two flats isomorphic to T_n. In fact we can show that for each $m \geqslant n$, T_{m+1} is the union of a copy of T_m and a copy of T_n, and can obtain a fairly precise description of how these additional copies of T_n must be attached.

LEMMA 12. *Let $m \geqslant n$.*

(a) *There exist flats t_m and t_n in T_{m+1} with t_m isomorphic to T_m, t_n isomorphic to T_n, and $T_{m+1} = t_m \cup t_n$.*

(b) *Let $t_n \subseteq t_{n+1} \subseteq \cdots \subseteq t_m$ with t_i isomorphic to T_i, and for $n \leqslant r \leqslant m$ let $t_r = \bigcup_{i=1}^{r-n+1} t^i$ with t^i isomorphic to T_n. (This assumption is justified by* (a).) *If $k \leqslant n$, and t is a flat of t_m isomorphic to T_k, then $t \subseteq t^i$ for some $i \in \{1,\dots,m-n+1\}$. In particular, $t^1,\dots,t^{m-n+1}$ are the only copies of T_n in t_m.*

PROOF. (a) By Lemma 11 there is a collection of copies of T_n which span T_{m+1}. (For $m \geqslant 2n$ this follows from $T_{m+1} \supseteq T_{m-n+1} \oplus T_n$.) Thus, given a copy t_m of T_m in T_{m+1}, there exists a flat t_n of T_{m+1} isomorphic to T_n and not contained in t_m.

By Lemma 10, $t_m \wedge t_n$ is isomorphic to T_{n-1}.

Suppose there exists a point x in T_{m+1} not in $t_m \cup t_n$. If $t = t_m \wedge (t_n \vee x)$, then contraction by x projects t_n onto t, and shows that t is isomorphic to T_n. Now it is easy to show (since T_n does not split) that if $T_{n+1} = t' \cup t''$ with t', t'' isomorphic to T_n, then T_{n+1} contains no further copies of T_n. On the other hand, the flat $t_n \vee x$ is a subgeometry of T_{n+1} (by universality) and contains the copies t_n and t of T_n, plus the point x outside $t_n \cup t$. This is a contradiction and we have proved (a).

(b) Let r be the least index for which $t \subseteq t_r$. By (a), $t = (t \cap t^{r-n+1}) \cup (t \cap t_{r-1})$. Since $t \nsubseteq t_{r-1}$ and t does not split, we must have $t \subseteq t^{r-n+1}$. $\square$

Let $m \geqslant n$. By Lemma 12(b), there are exactly $m - n + 1$ copies of T_n in T_m. We take these to be the vertices of a graph G_m in which vertices t and t' are joined if and only if $t \wedge t'$ is isomorphic to T_{n-1}. We will speak of t both as a vertex of G_m and as a flat of T_m. We remark that G_m is connected by Lemma 12.

LEMMA 13. *G_m is a path. Moreover, if we take $t^0,\dots,t^{m-n}$ to be one of the* (*two*) *natural orderings of its vertices, then* $\operatorname{rank}(t^i \wedge t^j) = \max\{n - j + i, 0\}$ *for all* $0 \leqslant i \leqslant j \leqslant m$.

PROOF. Let VG_m denote the set of vertices of G_m. We first remark that for any $t, t', t'' \in VG_m$,

$$(*)\qquad \begin{aligned} n + \operatorname{rank}(t \wedge t') &\leqslant \operatorname{rank}[(t \wedge t'') \vee (t' \wedge t'')] + \operatorname{rank}(t \wedge t' \wedge t'') \\ &= \operatorname{rank}(t \wedge t'') + \operatorname{rank}(t' \wedge t''). \end{aligned}$$

(The equality of the last two expressions follows from Lemmas 10 and 1.)

The *length* of a path between two vertices is the number of edges in the path, and the *distance*, $d(t, t')$, between two vertices t and t' is the length of a shortest path joining them. An easy consequence of $(*)$ is:

$(**)$ If P is a path of length d joining t and t', then $\operatorname{rank}(t \wedge t') \geq n - d$. Moreover, if t'' is a vertex in P such that the segment of P from t to t'' has length c, and if $\operatorname{rank}(t \wedge t'') > n - c$, then $\operatorname{rank}(t \wedge t') > n - d$.

We first prove the lemma for $m = 2n$. As T_{2n} contains $T_n \oplus T_n$, there exist t and t' in VG_{2n} with $t \wedge t' = \hat{0}$. By $(**)$, $d(t, t') \geq n$. But G_{2n} has only $n + 1$ vertices, so it must be a path from t to t'. Moreover, if this path has vertex sequence $t = t^0, t^1, \ldots, t^n = t'$, then $(**)$ implies $\operatorname{rank}(t^i \wedge t^j) = n - j + i$ for $0 \leq i \leq j \leq 2n$. This proves the lemma for $m = 2n$.

For $m < 2n$, we regard T_m as a flat of T_{2n}. Then G_m is a connected subgraph of G_{2n} and the result follows.

For $m > 2n$, we proceed by induction. If $t^1, \ldots, t^{m-n}$ are the vertices of a connected subgraph of G_m, then their span (in T_m) is isomorphic to T_{m-1}. (Induction on distance shows that the rank of the span is at most $m - 1$; on the other hand, Lemma 12(b) shows that their union cannot be contained in a proper subgeometry of T_{m-1}.) Thus (by induction) every connected proper subgraph of G_m is a path, so that G_m is either a path or a cycle. But if G_m is a cycle, say $t^0, \ldots, t^{m-n}$, then our inductive hypothesis applied to the paths $t^1, \ldots, t^{m-n}$ and t^{m-n}, t^0, t^1 gives $0 = \operatorname{rank}(t^1 \wedge t^{m-n}) = n - 2$, a contradiction. It follows that G_m is a path, say $t^0, \ldots, t^{m-n}$. Since each of the (two) paths of length $m - 1$ in G_m is the graph of a flat of T_m isomorphic to T_{m-1}, we also obtain the equality

$$\operatorname{rank}(t^i \wedge t^j) = \max\{n - j + i, 0\}$$

by induction for all pairs (i, j) other than $(0, m)$. To see the equality in the outstanding case, let $t_{m-1} \subseteq T_m$ be the flat spanned by $t^1, \ldots, t^{m-n}$. Then $t^0 \wedge t_{m-1} = t^0 \wedge t^1$, so that $t^0 \wedge t^{m-n} \subseteq t^1 \wedge t^{m-n} = \hat{0}$. □

We are now in a position to prove that n equals two. Before proceeding, we observe that for $n = 2$, we do obtain a variety. Define the geometries $O_m(q)$ recursively by: $O_2(q)$ is a line with $q + 1$ points; $O_m(q)$ is the union of a copy t_{m-1} of $O_{m-1}(q)$ and a copy t_2 of the line $O_2(q)$ taken in such a way that the intersection $t_{m-1} \wedge t_2$ is a point and t_{m-1} contains a copy of $O_{m-2}(q)$ disjoint from t_2. It is easy to deduce from Lemmas 12 and 13 that this sequence is the only sequence of universal models for which $|T_2| = q + 1$ and T_3 splits.

Another way to describe $O_n(q)$ is to take the Boolean algebra on the point set $\{1, 2, \ldots, n\}$. On each of the lines $\overline{12}, \overline{23}, \ldots, \overline{i(i+1)}, \ldots, \overline{(n-1)n}$, add $q - 1$ points in general position. The resulting geometry is the geometry $O_n(q)$ defined above. The geometries $O_n(q)$ are called the *full origami geometries of order q*. Their subgeometries form a variety called the *variety of origami geometries of order q*. Note that the variety of free geometries is just the variety of origami geometries of order one.

It remains to show that n cannot be greater than two. First we fix some notation. Let $m = |T_{n-1}| + 2n - 1$ and let $T_m = t^0 \cup \cdots \cup t^{m-n}$ (notation as in Lemma 13). Further, for $1 \leq i \leq m - n$, let $t^i_{n-1} = t^{i-1} \cap t^i$; for $1 \leq i \leq m - n - 1$, let $t^i_{n-2} = t^{i-1} \cap t^i \cap t^{i+1}$. Note that t^i_{n-1} is isomorphic to T_{n-1} and t^i_{n-2} is isomorphic to T_{n-2}.

Let us call a point x of t^i_{n-1} an *exceptionable point* if

(e) there is a (possibly empty) set of points (called a set of *foci* for x) $\{z_1,\ldots,z_k\}$ in $T_m \setminus \bigcup_{j=i}^{m-n} t^j$ such that there are n (distinct) copies of T_n containing the image of x in the contraction $T_m/\{z_1,\ldots,z_k\}$.

Note that, in any geometry T in the variety $\mathfrak{T}$, no point can be on more than n distinct copies of T_n. This follows from the fact that, by Lemma 13, any set of n copies of T_n having nonempty intersection must be of the form $\{t^i,\ldots,t^{i+n-1}\}$. For the same reason, a point x is on n copies of T_n in T_m itself if and only if it is the unique point in the intersection of the flats $t^i,\ldots,t^{i+n-1}$ for some i between 0 and $m-2n+1$. We call such exceptionable points x *n-points*.

Assume that $n \geqslant 3$. We shall show that

(f) for $1 \leqslant i \leqslant m-2n+1$, the number of exceptionable points on t^{i+1}_{n-1} is strictly greater than the number of exceptionable points on t^i_{n-1}.

Of course, (f) leads to a contradiction: since t^1_{n-1} contains an exceptionable point (the intersection of $t^0,\ldots,t^{n-1}$), t^{m-2n+2}_{n-1} contains at least $m-2n+2$ exceptionable points. But by our choice of m, t^{m-2n+2}_{n-1} contains only $m-2n+1$ $(=|T_{n-1}|)$ points of any kind.

Let, then, $1 \leqslant i \leqslant m-2n+1$. Observe that there is a unique n-point, x_{i+1} say, which is in t^{i+1}_{n-1} but not in t^i_{n-2}. This point is the intersection of $t^i,\ldots,t^{i+n-1}$. We will have proved (f) if we can construct an injection from the set of exceptionable points in $t^i_{n-1} \setminus t^i_{n-2}$ to the set of exceptionable points which are not n-points in $t^{i+1}_{n-1} \setminus (t^i_{n-2} \cup \{x_{i+1}\})$. To this end, choose a point z in $t^i \setminus (t^i_{n-1} \cup t^{i+1}_{n-1})$. Such a point exists since T_n does not split. Consider the mapping $x \mapsto x := (x \vee z) \wedge t^{i+1}_{n-1}$; that is, x is mapped onto the intersection x' of the line $\overline{xz}$ and the modular flat t^{i+1}_{n-1}. This map is clearly an injection from $t^i_{n-1} \setminus t^i_{n-2}$ into $t^{i+1}_{n-1} \setminus t^i_{n-2}$. Let x be an exceptionable point in $t^i_{n-1} \setminus t^i_{n-2}$ and suppose that x' is on r copies of T_n: these must be the flats $t^i,\ldots,t^{i+r-1}$. Suppose that $\{z_1,\ldots,z_k\}$ is a set of foci for x. Then, in the contraction $T_m/\{z_1,\ldots,z_k, z\}$, the image of x' (which is also the image of x) is on $n+r-2$ copies of T_n. (For the image of x in $T_m/\{z_1,\ldots,z_k\}$ is on n copies of T_n and the image of x' in $T_m/\{z_1,\ldots,z_k\}$ is still on r copies of T_n—remember that the foci are points in $T_m \setminus \bigcup_{j=i}^{m-n} t^j$—and contraction by z destroys one copy each from the two collections of copies of T_n.) Since x' is in the intersection $t^i \wedge t^{i+1}$, $r \geqslant 2$. Hence, as $n+r-2 \leqslant n$, $r=2$. This implies that x' is indeed an exceptionable point (with a set of foci $\{z_1,\ldots,z_k, z\}$) and that x' is not an n-point (in particular, $x' \neq x_{i+1}$). Thus, $x \mapsto x'$ is an injection satisfying the required properties and we have proved (f). This completes the proof that $n = 2$.

9. The main theorem. We have now proved the following theorem.

THEOREM 14. *Let $\mathfrak{T}$ be a variety of geometries with a sequence of universal models (T_n). Then $\mathfrak{T}$ is one of the following collections*:

A. *the variety of free geometries,*

B. *the variety of matchstick geometries of order q* (§4),

C. *the variety of origami geometries of order q* (§8),

D. *the variety of geometries coordinatizable over the finite field $GF(q)$,*

E. *the variety of voltage-graphic geometries with voltages in a fixed finite group A* (§7).

The corresponding universal models T_n *are*

A. *the Boolean algebra* B_n,

B. *the full matchstick geometry* $M_n(q)$ *of order* q,

C. *the full origami geometry* $O_n(q)$ *of order* q,

D. *the projective geometry* $P_n(q)$ *over the finite field* $GF(q)$,

E. *the Dowling geometry* $Q_n(A)$ *based on the finite group* A.

One can easily deduce further results from the theorem. For example, the only varieties of binary geometries are the free geometries, the matchstick and origami geometries of order two, the graphic geometries and all binary geometries.

A more interesting result which seems difficult to prove independently is

COROLLARY 15. *The only varieties which are closed under orthogonal duality are the varieties of free geometries and the varieties of geometries coordinatizable over a fixed finite field.*

The proof consists of checking that the other varieties are not closed under orthogonal duality. This is obvious for the matchstick and origami geometries (with $q > 1$). For voltage-graphic geometries, it is easy to show by a counting argument that the orthogonal dual of $Q_4(A)$ cannot be an A-labelled voltage-graphic geometry.

Our theorem says that in some sense the only nondegenerate examples of well-structured hereditary classes are the geometries coordinatizable over a finite field and voltage-graphic geometries. While the study of projective geometry over a finite field is a classical subject, the study of voltage-graphic geometries has just been initiated by Zaslavsky in [**14** and **15**]. Our present work would have been unimaginably more difficult had voltage-graphic geometries not been discovered.

Voltage-graphic geometries, like the coordinatizable geometries, can be described in an "economical" fashion. More precisely, to describe a voltage-graphic geometry, one need only specify a multigraph whose edges are labelled with the group elements, just as to describe a coordinatizable geometry, one need only specify a finite set of n-tuples of field elements. These descriptions are in general more compact (require less storage space in a computer) and easier to manipulate than purely set-theoretic descriptions such as a listing of all the circuits or bases. Thus, an important practical consequence of our theorem is that geometries belonging to a variety have compact descriptions.

What happens if we drop the restriction that our geometries be finite? Two unpleasant phenomena arise. The first is that (as we have defined it) a sequence of universal models need not be unique. For example, if k is a field which is isomorphic to a subfield of itself, the projective geometry $P_n(k)$ is embeddable in the affine geometry $A_n(k)$. The second, even harder to deal with, is that there are other varieties of geometries. Apart from the existence of varieties of geometries related to the origami geometries, we also encounter varieties of algebraic geometries as in [**9**]; these are geometries in which the dependence relation is given by algebraic dependence over a field. Over fields of positive characteristic, it is known [**13**, p. 185] that these varieties are different from the varieties of coordinatizable geometries. Moreover, the natural universal models are not even supersolvable.

We conclude with a question:

Can varieties of finite combinatorial geometries be characterised by a finite list of excluded minors?

That is to say, for any variety $\mathfrak{T}$ of geometries, does there exist a finite number of geometries $E_1, \ldots, E_n$ such that G is in the variety $\mathfrak{T}$ if and only if none of the geometries $E_1, \ldots, E_n$ occurs as a minor of G?

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