

EXISTENCE OF CHEBYSHEV CENTERS, BEST n -NETS AND BEST COMPACT APPROXIMANTS

BY

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ABSTRACT. In this paper we investigate the existence and continuity of Chebyshev centers, best n -nets and best compact sets. Some of our positive results were obtained using the concept of quasi-uniform convexity. Furthermore, several examples of nonexistence are given, e.g., a sublattice M of $C[0, 1]$, and a bounded subset $B \subset M$ is constructed which has no Chebyshev center, no best n -net and not best compact set approximant.

1. Introduction. It is a familiar situation when one wants to replace a set A of given data by another one C which is more condensed or of a simpler type. If we have some metric on our data space X it is natural to try to replace the set A belonging to the class \mathcal{A} of admissible given data sets by a set C belonging to the class \mathcal{C} of admissible "approximating" sets such that the elements of A will be as close to C as possible. More explicitly, if we denote, in the metric space (X, d) , $B(C, r) = \{x; d(x, C) \leq r\}$ (the closed r -neighbourhood of C), $r(C, A) = \min\{r; A \subset B(C, r)\}$ and $r_{\mathcal{C}}(A) = \inf\{r(C, A); C \in \mathcal{C}\}$, then we are looking for $C_0 \in \mathcal{C}$ with $r(C_0, A) = r_{\mathcal{C}}(A)$. We denote by $Z_{\mathcal{C}}(A)$ the set of all such $C_0 \in \mathcal{C}$.

We shall deal here with the following typical classes of subsets of a Banach space X : $\mathfrak{B}(V)$ — the closed and bounded subsets of a closed $V \subset X$; $\mathfrak{K}(V)$ —the compact subsets of V ; and $\mathfrak{F}_n(V)$ —the class of sets consisting of at most n points of V . $\mathfrak{B}, \mathfrak{K}, \mathfrak{F}_n$ will stand, respectively, for $\mathfrak{B}(X), \mathfrak{K}(X)$ and $\mathfrak{F}_n(X)$, V will stand for $\mathfrak{F}_1(V)$, and $r(A), Z(A)$ —for $r_X(A), Z_X(A)$. $r(A)$ is called the *Chebyshev radius* of A and $Z(A)$ the *Chebyshev center set* of A . Similarly, $r_V(A)$ is called the relative Chebyshev radius of A with respect to V and $Z_V(A)$ the set of relative Chebyshev centers for A in V .

The case $\mathcal{A} = X, \mathcal{C} = V$ is the classical best approximation problem— $r_V(x)$ is the distance $d(x, V)$ and Z_V is the metric projection operator P_V corresponding to $x \in X$ the set of nearest points to x in V . (On the other hand, the general case can be reduced to the classical best approximation problem if we endow $\mathfrak{B}(X)$ with the Hausdorff metric h —then $r_{\mathcal{C}}(A) = h(A, \mathcal{C})$ and $Z_{\mathcal{C}}(A) \cap B(A, r_{\mathcal{C}}(A)) = P_{\mathcal{C}}(A)$, but of course the concrete problem does not become simpler.)

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Elements of $Z_{\mathfrak{F}_n}(A)$ are called *best n -nets* of A .

$r_{\mathfrak{K}}(A)$ is easily seen to be the Kuratowski measure of noncompactness of A , $\alpha(A) = \inf\{r \geq 0; \exists n, F \in \mathfrak{F}_n \text{ with } r(F, A) \leq r\}$.

The concept of Chebyshev centers and best n -nets was introduced by Garkavi [8] and most of the basic results are due to him. Concerning existence problems, he observed the following two existence principles.

PROPOSITION A. *If V carries another topology τ such that $y_\alpha \xrightarrow{\tau} y \Rightarrow d(x, y) \leq \lim d(x, y_\alpha) \forall x \in X$, then a τ -accumulation point of a “minimizing sequence” for the Chebyshev radius is necessarily a Chebyshev center. In particular, if X is a dual normed space, then for every w^* -closed $V \subset X$, $Z_V(A) \neq \emptyset \forall A \in \mathfrak{B}(X)$.*

The same argument applies also to best n -nets.

PROPOSITION B. *If P is a contractive projection of X onto X_0 and $P\mathcal{C} \subset \mathcal{C}$, then $PZ_{\mathcal{C}}(A) \subset Z_{P\mathcal{C}}(A) \forall A \in \mathfrak{B}(X_0)$. In particular, if X is Banach space admitting a norm-one-projection from its bidual (e.g. $X = L_1(\mu)$ for any σ -finite μ), then $Z_{\mathfrak{F}_n}(A) \neq \emptyset \forall A \in \mathfrak{B}(X)$.*

Approximation of bounded sets by compact sets was investigated in [23], in connection with questions of best approximating bounded linear operators by compact ones. M. Feder [7] constructed a bounded subset A of $L_1[0, 1]$ which admits no best compact set approximant. This shows that the analogue of Proposition B does not hold for the class \mathfrak{K} . On the other hand, it follows from the results of Fakhoury [6] and Mach and Ward [23] that every bounded subset of l_1 has a best compact set approximant.

The existence of Chebyshev centers for bounded subsets is established, by Propositions A and B, for the classical spaces $L_p(\mu)$, $1 \leq p < \infty$. For the other classical space, $C(\Omega)$ (Ω compact Hausdorff), the existence of Chebyshev centers for bounded sets was established by Kadec and Zamyatin [12]. There are several ways of proving this:

We first observe that the problem of finding $Z(A)$ in $C(\Omega)$ reduces to the problem of finding $Z_{C(\Omega)}\{(\inf A, \sup A)\}$ in $m(\Omega)$, and this is equivalent to finding $x \in C(\Omega)$ satisfying $\sup A - r(A) \leq x \leq \inf A + r(A)$, which can be guaranteed by the Hahn-Tong interposition theorem [27, p. 100] or by Michael’s selection theorem. Another way is a “successive approximation” argument, which will be exploited in §2.

Garkavi [8] also gave the first example of nonexistence of Chebyshev centers: He showed that if X is the nonproximal hyperplane $\{x \in C[-1, 1]; \int_{-1}^0 x(t) dt = \int_0^1 x(t) dt\}$ of $C[-1, 1]$, then $Z(A) = \emptyset$ for some $A \in \mathfrak{F}_3$. Garkavi showed later [9] that if X is a finite-codimensional subspace of $C(\Omega)$, then $Z_X(A) \neq \emptyset \forall A \in \mathfrak{K}(C(\Omega))$ iff X is proximal. Garkavi and Zamyatin [10] showed that for such a proximal X , $Z_X(A) \neq \emptyset \forall A \in \mathfrak{B}(C(\Omega))$ iff the support of every $\mu \in X^\perp$ is “extremally disconnected with respect to Ω ” (i.e. $\overline{G} \cap \text{spt } \mu$ is open in $\text{spt } \mu$ for every open $G \subset \Omega$). Some remarks concerning these examples are made in §3.

It is now natural to investigate the existence of Chebyshev centers for bounded and compact sets in the “semiclassical” Banach spaces. As such can be considered

the Lindenstrauss spaces (preduals of L_1 -spaces) and the spaces $C(\Omega, X)$, $L_p(\mu, X)$ of vector-valued functions, where X is "classical". The case of Lindenstrauss spaces is studied in §4, and the case of vector-valued functions in §5.

2. Quasi-uniform convexity and the successive approximation argument. We say that the normed space X is *quasi-uniformly convex (q.u.c.) with respect to its subspace Y* if $\forall \epsilon > 0 \exists 0 < \tilde{\delta} < 1$ such that $\forall y \in Y \exists z \in Y$ with $\|z\| \leq \epsilon$, and such that if $u \in X$, $\|u\| \leq 1$ and $\|u - y\| \leq 1 - \tilde{\delta}$, then also $\|u - z\| \leq 1 - \tilde{\delta}$. In the case $X = Y$ we say that X is quasi-uniformly convex. X is said to be *strictly convex with respect to Y* if $y - z \in Y$, $\|y\| = \|z\| = \|\frac{1}{2}(y + z)\| \rightarrow y = z$, and *uniformly convex with respect to Y* if $\forall \epsilon \exists \delta$ such that $y - z \in Y$, $\|y\| = \|z\| = 1$, $\|y - z\| \geq \epsilon \Rightarrow \|\frac{1}{2}(y + z)\| \leq 1 - \delta$ [3].

The concept of quasi-uniform convexity is due to Calder, Coleman and Harris [4].

2.1. LEMMA. *If $\epsilon < \frac{1}{2}$ and $\tilde{\delta} = \tilde{\delta}(\epsilon)$ satisfies the q.u.c. condition, so does every $\eta \in (0, \tilde{\delta})$.*

PROOF. By homogeneity, η satisfies the q.u.c. conditions with respect to $(1 - \eta) \cdot (1 - \tilde{\delta})^{-1}\epsilon \leq 1$. Therefore, given $y \in Y$ there is $u \in Y$ with $\|u\| \leq 1$ and $B(y, 1 - \eta) \cap B(0, 1) \subset B(u, 1 - \eta)$. There is $z \in Y$ with $\|z\| \leq \epsilon$ such that $B(u, 1 - \tilde{\delta}) \cap B(0, 1) \subset B(z, 1 - \tilde{\delta})$, and then

$$u + (1 - \tilde{\delta})(1 - \eta)^{-1}[B(y, 1 - \eta) \cap B(0, 1) - u] \subset B(u, 1 - \tilde{\delta}) \cap B(0, 1) \subset B(z, 1 - \tilde{\delta}).$$

Let $w = u + (1 - \eta)(1 - \tilde{\delta})^{-1}(z - u)$. Then $B(y, 1 - \eta) \cap B(0, 1) \subset B(w, 1 - \eta)$, and since $z \in [u, w]$, also $B(y, 1 - \eta) \cap B(0, 1) \subset B(z, 1 - \eta)$.

2.2. PROPOSITION. *The following are equivalent:*

- (a) X is uniformly convex with respect to Y .
- (b) X is q.u.c. with respect to Y and strictly convex with respect to Y .
- (c) X is q.u.c. with respect to Y and we can take $z \in [0, y]$.

PROOF. If X is uniformly convex with respect to Y then it is clearly strictly convex with respect to Y . Given $y \in Y$, $\epsilon > 0$, let

$$z = \begin{cases} y & \text{if } \|y\| \leq \epsilon, \\ \epsilon y / \|y\| & \text{if } \|y\| \geq \epsilon. \end{cases}$$

If $\|u\| \leq 1$ and $\|u - y\| \leq 1 - \delta(\epsilon)$, then $\|u - \frac{1}{2}y\| \leq 1 - \frac{1}{2}\delta(\|y\|)$. If $\|y\| \geq \epsilon$ then

$$\begin{aligned} \|u - z\| &= \left\| u - \frac{\epsilon}{\|y\|} y \right\| \leq \frac{2\epsilon}{\|y\|} \left\| u - \frac{y}{2} \right\| + \left(1 - \frac{2\epsilon}{\|y\|} \right) \|u\| \\ &\leq \frac{2\epsilon}{\|y\|} \left(1 - \frac{1}{2}\delta(\|y\|) \right) + 1 - \frac{2\epsilon}{\|y\|} = 1 - \frac{\epsilon}{\|y\|} \delta(\|y\|) \leq 1 - \delta(\epsilon) \end{aligned}$$

(since $\delta(t)/t$ increases with t). If $\|y\| \leq \epsilon$ then $\|u - z\| = \|u - y\| \leq 1 - \delta(\epsilon)$. Thus $\tilde{\delta}(\epsilon) = \delta(\epsilon)$. Thus (a) \Rightarrow (b), (c).

If X is q.u.c. with respect to Y and we can always take $z \in [0, y]$, let $d = \delta(\epsilon/3)$. If $\|u\| = \|v\| = 1$, $\|u - v\| \geq \epsilon$ and $u - v \in Y$, there is $z \in [v, (1 - d)u]$ with $\|z - v\| \leq \epsilon/3$ and $\|z\| \leq 1 - d$. Let $w = (1 - d)(2 - d)^{-1}(u + v)$. Then $w \in [(1 - d)u, v]$ and

$$\|w - v\| = (2 - d)^{-1}\|(1 - d)u - v\| > \frac{1}{2}\|(1 - d)u - v\| \geq \epsilon/3;$$

$w \in [(1 - d)u, z]$ and $\|w\| \leq 1 - d$. Therefore

$$\|\frac{1}{2}(u + v)\| = \frac{1}{2}(2 - d)(1 - d)^{-1}\|w\| \leq 1 - \frac{1}{2}d,$$

so that $\delta(\epsilon) \geq \frac{1}{2}\tilde{\delta}(\epsilon/3)$. Thus (c) \Rightarrow (a).

Finally, if X is q.u.c. with respect to Y and strictly convex with respect to Y , and $\|u\| = \|v\| = 1$ satisfy $u - v \in Y$, $\|u - v\| \geq 4\epsilon$ and if $\|\frac{1}{2}(u + v)\| = 1 - d$ where $d \leq \tilde{\delta}(\epsilon)$, then by Lemma 2.1 there is $z \in Y$ with $\|z\| \leq \epsilon$ and $B(\frac{1}{2}(u - v), 1 - d) \cap B(0, 1) \subset B(z, 1 - d)$. Since $\|u - (u - v)/2\| = \|-v - (u - v)/2\| = 1 - d$, we have $\|u - z\| \leq 1 - d$, $\|v + z\| \leq 1 - d$ and $\|\frac{1}{2}(u + v)\| \leq 1 - d$. Equality implies, by strict convexity, $u - z = v + z$, i.e. $z = \frac{1}{2}(u - v)$ and we get the contradiction $\epsilon > \|z\| \geq 2\epsilon$. Therefore $\delta(4\epsilon) \geq \tilde{\delta}(\epsilon)$.

REMARK. Proposition 2.2 is, essentially, due to Calder, Coleman and Harris [4].

Examples of nonstrictly convex spaces which are q.u.c. are the $C(\Omega)$ spaces and some of their subspaces, as shown in Proposition 4.4.

2.3. PROPOSITION. *If X is q.u.c. and Y is the range of a norm-one projection P in X , then Y is also q.u.c. and $\tilde{\delta}_Y(\epsilon) \geq \tilde{\delta}_X(\epsilon)$.*

PROOF. Given $y \in Y$, $\epsilon > 0$, let $\|z\| \leq \epsilon$, $z \in X$ satisfy

$$B(z, 1 - \tilde{\delta}_X(\epsilon)) \supset B(y, 1 - \tilde{\delta}_X(\epsilon)) \cap B(0, 1).$$

Then $\|Pz\| \leq \epsilon$ and

$$B(Pz, 1 - \tilde{\delta}_X(\epsilon)) \supset B(y, 1 - \tilde{\delta}_X(\epsilon)) \cap B(0, 1) \cap Y.$$

2.4. PROPOSITION. *If a Banach space X is q.u.c. with respect to a closed subspace Y , then:*

(a) *Every bounded $A \subset X$ has a nonempty Chebyshev center set in Y , $Z_Y(A)$, and $A \rightarrow Z_Y(A)$ is uniformly continuous on $\{A; r_Y(A) \leq R\}$ for every $R > 0$.*

(b) *Every bounded set $A \subset X$ has a nonempty set of best compact set approximants in Y , $Z_{\mathfrak{K}(Y)}(A)$, and the mapping $A \rightarrow Z_{\mathfrak{K}(Y)}(A)$ is uniformly continuous on $\{A; r_Y(A) \leq R\}$ for every $R > 0$.*

PROOF. (a) For a $\gamma > 0$ let y_n satisfy

$$r(y_n, A) \leq \frac{r_Y(A)}{\prod_{k=n}^{\infty} (1 - \tilde{\delta}(2^{-k}\gamma))}$$

(since $\tilde{\delta}(\eta) \leq \eta$, the product converges). Let $z_1 = y_1 \cdot z_{n+1} \in Y$ can be chosen by the q.u.c. property so that

$$\|z_{n+1} - z_n\| \leq 2^{-n}\gamma r(y_n, A) \quad \text{and} \quad r(z_{n+1}, A) \leq \frac{r_Y(A)}{\prod_{k=n+1}^{\infty} (1 - \tilde{\delta}(2^{-k}\gamma))}.$$

The sequence (z_n) is Cauchy and converges to a Chebyshev center $z \in Z_Y(A)$, and $\|z - y_1\| \leq \gamma r(y_1, A)$.

To prove the uniform continuity of Z_Y let $\varepsilon > 0$ be given. Let $\gamma = \varepsilon/2R$, and $h(A, A') = \eta r_Y(A)$ with $\eta \leq \frac{1}{2}\tilde{\delta}(\gamma/2)$. Then $|r_Y(A) - r_Y(A')| \leq \eta r_Y(A)$ and for $z \in Z_Y(A)$, $r(z, A') \leq (1 + \eta)r_Y(A)$. Furthermore,

$$(1 + \eta)r_Y(A) \leq \frac{(1 - \eta)r_Y(A)}{1 - \tilde{\delta}(\gamma/2)} \leq \frac{r_Y(A')}{\prod_{k=1}^{\infty}(1 - \tilde{\delta}(2^{-k}\gamma))}$$

and, therefore, by the first part of the proof, there is $z' \in Z_Y(A')$ satisfying $\|z - z'\| \leq \gamma(1 + \eta)r_Y(A)$. Thus $h(Z_Y(A), Z_Y(A')) \leq \varepsilon$.

(b) Let $F_n \in \mathfrak{F}_n(Y)$ satisfy

$$r(F_n, A) \leq \frac{r_{\mathfrak{F}_n(Y)}(A)}{\prod_{k=n}^{\infty}(1 - \tilde{\delta}(2^{-k}\varepsilon))}.$$

Let $G_1 = F_1$. Having G_{n-1} , replacing each point x of G_{n-1} by an n -tuple of points in $B(x, \varepsilon/2^n) \cap Y$, we get G_n of cardinality $n!$ satisfying $r(G_n, A) \leq r(F_n, A)$. $G = \bigcup G_n$ is compact and $r(G, A) = r_{\mathfrak{F}(Y)}(A)$. The proof of the uniform continuity is analogous to part (a).

2.5. EXAMPLE. *A 3-dimensional space which is not quasi-uniformly convex:* Let B be the symmetric convex hull of the circle $\{\cos te_1 + \sin te_2; 0 \leq t \leq 2\pi\}$ and the segment $\{se_1 + e_3; -1 \leq s \leq 1\}$, and $\|\cdot\|$ the norm whose unit ball is B , i.e.

$$\|(x, y, z)\| = \begin{cases} \sqrt{y^2 + (|x| - |z|)^2} + |z| & \text{if } |x| \geq |z|, \\ |y| + |z| & \text{if } |x| \leq |z|. \end{cases}$$

Since $Z(-e_1, e_1) = [-e_3, e_3]$ and $Z(-e_1 - \eta e_2, e_1 + \eta e_2) = 0$, it is clear that $A \rightarrow Z(A)$ is not continuous, so that the space is not quasi-uniformly convex. Indeed, let

$$\begin{aligned} x &= (1 - \eta)e_3, \\ u &= (1 - \frac{1}{2}\eta)e_1 + \frac{1}{2}\sqrt{3}e_2 + (1 - \eta)e_3, \\ v &= -(1 - \frac{1}{2}\eta)e_1 - \frac{1}{2}\sqrt{3}e_2 + (1 - \eta)e_3. \end{aligned}$$

Then $r(x, [u, v]) = (1 - \eta + \eta^2)^{1/2} < r(z, [u, v])$ for every z with $\|z\| < 1$. This shows that $\tilde{\delta}(\varepsilon) = 0 \forall \varepsilon < 1$.

2.6. PROPOSITION. *If (Ω, μ) contains $2n + 1$ disjoint sets of finite positive measure, then $\tilde{\delta}_{L_1(\mu)}(\varepsilon) \leq \varepsilon/(2n + 1) \forall \varepsilon < 1$.*

PROOF. Let A_1, \dots, A_{2n+1} be disjoint of finite positive measure. Let

$$x = \frac{1}{2n + 1} \sum_{i=1}^{2n+1} \frac{A_i}{\mu(A_i)}$$

and suppose z satisfies the q.u.c. condition for x, ε and $\varepsilon/(2n + 1)$. Denote $\alpha_i = \int_{A_i} |z_i| d\mu$. We may assume $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_{2n+1}$. Let

$$u = \frac{1}{n + 1} \sum_{i=n+1}^{2n+1} \frac{\chi_{A_i}}{\mu(A_i)}.$$

Clearly $\|u\| = 1$,

$$\|u - x\| = \sum_{i=1}^n \frac{1}{2n+1} + \sum_{i=n+1}^{2n+1} \left(\frac{1}{n+1} - \frac{1}{2n+1} \right) = 1 - \frac{1}{2n+1},$$

while

$$\begin{aligned} \|u - z\| &\geq \sum_{i=1}^n \alpha_i + \sum_{i=n+1}^{2n+1} \frac{1}{n+1} - \sum_{i=n+1}^{2n+1} \alpha_i = 1 + 2 \sum_{i=1}^n \alpha_i - \sum_{i=1}^{2n+1} \alpha_i \\ &\geq 1 + \left(\frac{2n}{2n+1} - 1 \right) \sum_{i=1}^{2n+1} \alpha_i \geq 1 - \frac{1}{2n+1} \varepsilon. \end{aligned}$$

2.7. COROLLARY. *Infinite-dimensional $L_1(\mu)$ -spaces are not quasi-uniformly convex.*

REMARK. The case $L_1[0, 1]$ follows also from Proposition 2.4 and results of Feder [7]. In this case we have the stronger result which will be used in §5:

2.8. PROPOSITION. *In $L_1[0, 1]$, if $x = 1$, no $\|z\| < 1$ and $d \in (0, 1)$ satisfy*

$$B(x, 1 - d) \cap B(0, 1) \subset B(z, 1 - d).$$

PROOF. Take A with $\mu(A) = (1 - d)/2$ and $\int_A |z| d\mu$ maximal, and let $u = 2(1 - \chi_A)/(1 + d)$. Then $\|u\| = 1$,

$$\|u - x\| = \frac{1 - d}{1 + d} \frac{1 + d}{2} + \frac{1 - d}{2} = 1 - d,$$

while

$$\begin{aligned} \|u - z\| &\geq \int_A |z| d\mu + \int_{A^c} u d\mu - \int_A |z| d\mu = 2 \int_A |z| d\mu - \|z\| + 1 \\ &\geq 1 - \|z\| + 2 \cdot \frac{1}{2} (1 - d) \|z\| = 1 - \|z\| d. \end{aligned}$$

REMARK. The q.u.c. property is the simultaneous approximation version of a simpler property which was studied, in several variants, by Holmes [11], Lau [14] and others:

Call a subspace Y of X “ U -proximal” if $\forall \varepsilon > 0 \exists \delta > 0$ such that

$$B(0, 1) \cap (Y + B(0, 1 - \delta)) \subset (B(0, \varepsilon) \cap Y) + B(0, 1 - \delta),$$

i.e. such that if $\|u\| \leq 1, y \in Y, \|u - y\| \leq 1 - \delta$, then $\|u - z\| \leq 1 - \delta$ for some $z \in Y, \|z\| \leq \varepsilon$.

This property is weaker than q.u.c. and is enjoyed, e.g., by “semi- M -ideals” Y in any Banach space X (with $\delta = \varepsilon$).

Y is called “locally U -proximal” in X if it is U -proximal in $\text{span}\{Y, x\} \forall x \in X$.

Local U -proximality implies proximality in a way analogous to Proposition 2.4, and in the U -proximal case the metric projection is continuous.

3. Nonexistence of centers and best compact approximants in hyperplanes of $C(\Omega)$ and isomorphs of $C(\Omega)$. In the particular case $\Omega = [a, b]$ (or, more generally, any metric compact), the Garkavi-Zamyatin result shows that the finite codimensional

subspace $X = [\mu_1, \dots, \mu_n]_{\perp}$ of $C(\Omega)$ admits relative centers for bounded sets in $C(\Omega)$ iff each of the annihilating measures has a finite support [31]. In fact, if this fails we get more nonexistence results.

3.1. EXAMPLE. Let $X = \{x \in C[0, 1]; \int_0^1 x(t) dt = \frac{1}{4}\}$, $A = \{x \in X; |x(t)| \leq \frac{1}{2}\chi_{[0, 1/2]}(t) + \chi_{(1/2, 1]}(t)\}$. Then A has no best n -net for any $n \geq 1$ and no best compact approximant in X .

PROOF. For any $n \in \mathbf{N}$,

$$x_n(t) = \frac{1}{2}\chi_{[0, 1/2 - 1/2n]}(t) + \frac{2n^2(1 - 2t) + 1}{2(2n + 1)}\chi_{(1/2 - 1/2n, 1/2]}(t) + (1/2(2n + 1))\chi_{(1/2, 1]}(t)$$

satisfies $x_n \in X$ and $r(x_n, A) = 1 + 1/2(2n + 1) \rightarrow 1$.

Suppose now $K \subset X$ is compact; then it is equicontinuous and therefore there is $0 < \delta < \frac{1}{10}$ such that $|s - t| < \delta$, $y \in K$, implies $|y(s) - y(t)| < \frac{1}{10}$. Let $x \in A$ oscillate between the upper and lower bounds faster than η , where $0 < \eta < \frac{1}{100}$ is such that $|s - t| < \eta$, $y \in K$, implies $|y(s) - y(t)| < \delta/10$, and suppose $y \in K$ satisfies $\|x - y\| \leq 1$. Then y satisfies $|y| \leq \delta/10$ in $[\frac{1}{2}, 1]$. We also must have $y \leq \frac{1}{2} + \delta/10$ on $[0, \frac{1}{2}]$; hence

$$\int_0^s y(t) dt \leq s\left(\frac{1}{2} + \frac{1}{10}\delta\right) \quad \forall s \in \left[0, \frac{1}{2}\right],$$

and for some $t \in [s, \frac{1}{2}]$ we must have

$$y(t) \geq \left(\frac{1}{4} - \frac{1}{20}\delta - s\left(\frac{1}{2} + \frac{1}{10}\delta\right)\right) / \left(\frac{1}{2} - s\right).$$

For this t we have

$$y(t) - y\left(\frac{1}{2}\right) \geq \frac{1}{2} - \left(\frac{\delta/10 + 2s\delta/10}{1 - 2s} + \frac{\delta}{10}\right) = \frac{1}{2} - \frac{2\delta/10}{1 - 2s}.$$

Taking $s = (1 - \delta)/2$, we get $y(t) - y(\frac{1}{2}) = \frac{1}{2} - \frac{2}{10} > \frac{1}{10}$, which contradicts the choice of δ . Thus $r(K, A) > 1 \forall K \in \mathfrak{K}$.

REMARKS. The fact that the sets of type of A fail to have centers in X was observed by Smith and Ward [28].

The next example is similar, only that this time we have nonexistence in a strictly convex isomorph of $C[-1, 1]$.

3.2. EXAMPLE. Let X be $C[-1, 1]$ with the norm $\|x\| = \|x\|_{\infty} + \|x\|_2$, $A = \{x \in C[-1, 1]; -2 \leq x \leq 0 \text{ on } [-1, 0), 0 \leq x \leq 2 \text{ on } (0, 1]\}$. Then A has no Chebyshev center in X .

PROOF. For any $n \in \mathbf{N}$,

$$x_n = \chi_{[-1, -1/n]} + nt\chi_{[-1/n, 1/n]} + \chi_{[1/n, 1]}$$

satisfies

$$r(x_n, A) = 2 + \left[2\left(1 - \frac{1}{n}\right) + 2\int_0^{1/n}(2 - nt)^2 dt\right]^{1/2} = 2 + \sqrt{2 + 8/3n} \rightarrow 2 + \sqrt{2}.$$

Suppose $z \in Z(A)$. Let $*$ denote the involution $x^*(t) = -x(-t)$ in X , and let $\hat{x} = \frac{1}{2}(x + x^*)$. Since $A = A^* = \hat{A}$, we have $r(\hat{y}, A) = r(y, A) \forall y$, so that we may assume that the Chebyshev center z is an odd function. Define

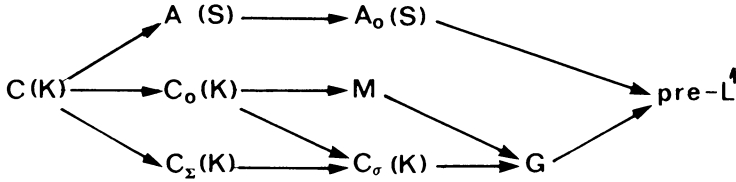
$$f(t) = 2 - \frac{1}{6}t + \sqrt{2} \left[\frac{1}{2}t\left(\frac{7}{4}\right)^2 + \left(1 - \frac{3}{2}t\right)\left(1 + \frac{1}{6}t\right)^2 \right]^{1/2}.$$

Let $0 < \delta < 1$ be such that $|s - t| \leq \delta \Rightarrow |z(s) - z(t)| < \frac{1}{4}$ and such that $f(\delta) > 2 + \sqrt{2}$. Further, let $0 < \eta < \delta/2$ be such that $|s - t| \leq \eta \Rightarrow |z(s) - z(t)| < \delta/6$.

Let x be odd and satisfy $\mu\{t \in [0, 1]; |x(t) - 1| \neq 1\} < \delta/2$, $x(t) = 2$ if $t \in [\eta, 1]$ and $z(t) \leq 1 - \delta/6$, and $x(t) = 0$ if $t \in [\eta, 1]$ and $z(t) \geq 1 + \delta/6$. Then

$$\begin{aligned} \|x - z\| &\geq |(x - z)(\eta)| + \sqrt{2} \left(\int_0^\delta |x(t) - z(t)|^2 dt + \int_\delta^1 |x(t) - z(t)|^2 dt \right)^{1/2} \\ &\geq f(\delta) > 2 + \sqrt{2}. \end{aligned}$$

4. Lindenstrauss spaces. Recall the Lindenstrauss-Wulbert diagram of pre- L^1 spaces:



We want to find out in which of these spaces Chebyshev centers and best compact set approximants exist.

4.1. PROPOSITION. *If compact sets in the Banach space X have relative Chebyshev centers in the subspace Y , then they also have best n -nets in Y for all $n \in \mathbb{N}$.*

PROOF. Let $K \in \mathfrak{K}(X)$, and let \hat{K} be its canonical embedding in X^{**} . K has a best n -net in $Y^{\perp\perp}$, y_1, \dots, y_n (by Proposition B). Let $r = r_{\mathfrak{F}_n(Y)}(K)$, $\hat{r} = r_{\mathfrak{F}_n(Y^{\perp\perp})}(\hat{K})$, $K_i = K \cap B(y_i, \hat{r})$, $z_i \in Z_Y(K_i)$, $i = 1, \dots, n$. By a result of Lima [16, Corollary 1.3], it follows that $r_Y(E) = r_{Y^{\perp\perp}}(\hat{E})$ for all $E \in \mathfrak{K}(X)$, hence $r(z_i, K_i) = r_{Y^{\perp\perp}}(\hat{K}_i) \leq \hat{r} \leq r$, and $\{z_1, \dots, z_n\} \in Z_{\mathfrak{F}_n(Y)}(K)$.

REMARKS. (1) The converse is obvious—if $Z_{\mathfrak{F}_{n+1}(Y)}(K) \neq \emptyset$ for all $K \in \mathfrak{K}(X)$, then also $Z_{\mathfrak{F}_n(Y)}(K) \neq \emptyset$ for all $K \in \mathfrak{K}(X)$ —just add a remote point $y \in Y$ to K .

(2) If we assume E only bounded, we may have $r_Y(E) \neq r_{Y^{\perp\perp}}(E)$, e.g., $X = Y = C[-1, 1]$, $E = \{x, -1 \leq x \leq 0 \text{ on } [-1, 0], 0 \leq x \leq 1 \text{ on } [0, 1]\}$.

4.2. PROPOSITION [19, THEOREM 2.2]. *If Y is an M -ideal in an L_1 -predual space X , then $Z_Y(K) \neq \emptyset$ for all $K \in \mathfrak{K}(X)$.*

4.3. COROLLARY. *If Y is an M -ideal in an L_1 -predual X , then $Z_{\mathfrak{F}_n(Y)}(K) \neq \emptyset$ for all $K \in \mathfrak{K}(X)$.*

4.4. PROPOSITION. *If X is the linear sublattice of $C(\Omega)$ (Ω compact Hausdorff) determined by the full set of conditions: $x(s_\alpha) = \lambda_\alpha x(t_\alpha)$, $s_\alpha, t_\alpha \in \Omega$, $\lambda_\alpha \geq 0$, and if $\inf\{\lambda_\alpha; \lambda_\alpha > 0\} > \delta_0 > 0$, then $C(\Omega)$ is quasi-uniformly convex with respect to X , with $\tilde{\delta}(\epsilon) \geq \delta_0 \epsilon$.*

PROOF. Let $N = \{t \in \Omega; x(t) = 0 \forall x \in X\}$. If $t \notin N$, there is $y_t \in X$ with $\|y_t\| \leq 1$ and $y_t(t) > \delta_0$. Let $V_t = \{s; y_t(s) > \delta_0\}$. Given $x \in X$, let $K_0 = \{t; |x(t)| \geq \delta_0 \epsilon\}$. By compactness there are t_1, \dots, t_m such that $\cup_{i=1}^m V_{t_i} \supset K_0$. Let $y = \max_{1 \leq i \leq m} y_{t_i}$ and let $z = \epsilon y \wedge x \vee (-\epsilon y)$. Clearly $\|z\| \leq \epsilon$, and if $u \in C(\Omega)$, $\|u - x\| \leq 1 - \delta_0 \epsilon$ and $\|u\| \leq 1$, consider for $t \in \Omega$ the three possible cases (we may assume $x(t) \geq 0$):

- (i) $x(t) \leq \epsilon y(t)$; then $|(u - z)(t)| = |u(t) - x(t)| \leq 1 - \delta_0 \epsilon$.
- (ii) $x(t) \geq \epsilon y(t) \geq u(t)$; then $|(u - z)(t)| \leq |u(t) - x(t)| \leq 1 - \delta_0 \epsilon$.
- (iii) $x(t) \geq \epsilon y(t)$, $u(t) \geq \epsilon y(t)$; then $|(u - z)(t)| = u(t) - \epsilon y(t) \leq 1 - \delta_0 \epsilon$.

4.5. COROLLARY. *If $X \subset C(\Omega)$ is the M -space determined by the full set of relations $x(s_\alpha) = \lambda_\alpha x(t_\alpha)$, $s_\alpha, t_\alpha \in \Omega$, $\lambda_\alpha \geq 0$, and if $\delta_0 = \inf\{\lambda_\alpha; \lambda_\alpha \neq 0\} > 0$, then every bounded subset A of $C(\Omega)$ has a nonempty set of Chebyshev centers in X , $Z_X(A)$, and a nonempty set of best compact approximants in X , $Z_{\mathfrak{K}(X)}(A)$, and the mappings $A \rightarrow Z_X(A)$, $A \rightarrow Z_{\mathfrak{K}(X)}(A)$ are Lipschitz continuous in the Hausdorff metric with constant $2/\delta_0$.*

PROOF. The existence is immediate from Propositions 4.4 and 2.4. As to the continuity statement, given bounded sets A, A' , and $z \in Z_Y(A)$ define $\epsilon = 2h(A, A')/(\delta_0(r_Y(A') + 2h(A, A')))$. Since $r(z, A') \leq r_Y(A') + 2h(A, A')$, we also have

$$r(z, A') \leq r_Y(A') / (1 - \delta_0 \epsilon)$$

and therefore the proof of Proposition 2.4 shows the existence of $z' \in Z_Y(A')$ such that

$$\|z - z'\| \leq \epsilon r(z, A') \leq 2/\delta_0 h(A, A').$$

This proves $2/\delta_0$ to be a Lipschitz constant for the Chebyshev center map. The proof for the map $A \rightarrow Z_{\mathfrak{K}(X)}(A)$ is similar.

REMARK. Observe that Corollary 4.5 applies to the following particular cases:

(a) $X = C(\Omega)$ (here we have only the trivial relations $x(t_\alpha) = x(t_\alpha)$, and we have $\tilde{\delta}(\epsilon) = \epsilon \forall \epsilon > 0$)—the fact that $Z(A) \neq \emptyset$ was proved first in [12].

(b) X is a closed subalgebra of $C(\Omega)$ (here, by the Stone-Weierstrass theorem, the determining relations are $x(t) = x(s)$ when $t \sim s$ under the induced equivalence relation, and $x(t) = 0$ for $t \in \cap_{x \in X} x^{-1}(0)$, and again $\tilde{\delta}(\epsilon) = \epsilon$). The fact that $Z_X(A) \neq \emptyset$ for bounded $A \subset X$ and that it depends continuously on A was observed first in [29].

4.6. COROLLARY. *If $X \subset C(\Omega)$ is the G -space determined by the full set of relations $x(s_\alpha) = \mu_\alpha x(t_\alpha)$, $s_\alpha, t_\alpha \in \Omega$, and if $\delta_0 = \inf\{|\mu_\alpha|; \mu_\alpha \neq 0\} > 0$, then every bounded subset of X has a nonempty Chebyshev center in X and a best compact set approximant in X , and the mappings $A \rightarrow Z_X(A)$, $A \rightarrow Z_{\mathfrak{K}(X)}(A)$ are Lipschitz continuous in the Hausdorff metric with constant $2/\delta_0$.*

PROOF. By [18, THEOREM 3] there is a norm-1 projection from $\{x \in C(\Omega \times \Omega); x(s_\alpha^1) = \mu_\alpha x(t_\alpha^1) \text{ if } \mu_\alpha \geq 0, x(s_\alpha^2) = -\mu_\alpha x(t_\alpha^2) \text{ if } \mu_\alpha < 0\}$ onto X , and we apply Proposition 2.4 and Corollary 4.5.

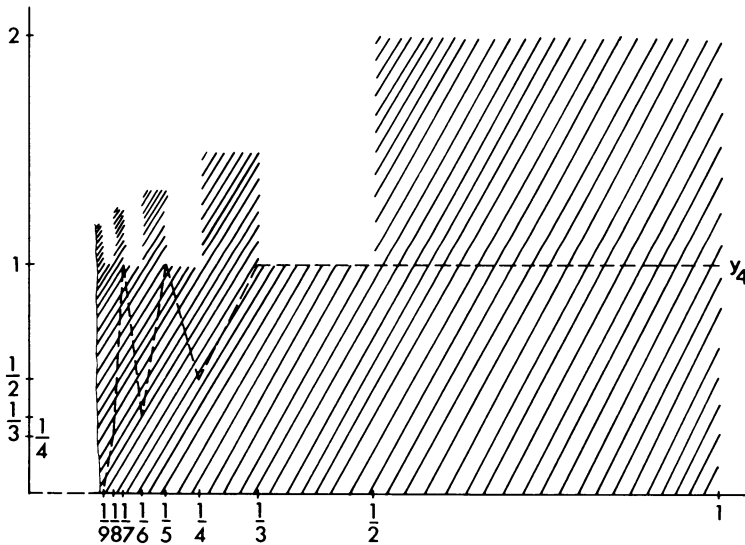
REMARK. Corollary 4.6 contains the particular case when $X = C_\sigma(\Omega)$ for some involutive homeomorphism σ of Ω onto Ω (here the nontrivial relations are $x(t) = -x(\sigma t)$, $t \in \Omega$, and again $\delta(\epsilon) = \epsilon \forall \epsilon > 0$).

4.7. EXAMPLE. An M -space X and a bounded $A \subset X$ which has no Chebyshev center and no best compact set approximant. Let

$$X = \left\{ x \in C[0, 1]; x\left(\frac{1}{2n}\right) = \frac{1}{n}x\left(\frac{1}{2n-1}\right), n = 1, 2, \dots \right\},$$

$$A = \left\{ x \in X; 0 \leq x \leq \sum_{n=1}^{\infty} \left(\chi_{[1/(2n+1), 1/2n]} + \left(1 + \frac{1}{n}\right) \chi_{(1/2n, 1/(2n-1))} \right) \right\}.$$

Let $y_n(t)$ be the piecewise linear function with the vertices $(0, 0)$, $(1/(2k - 1), 1)$ and $(1/2k, 1/k)$ for $k = 1, \dots, n$, and $(1/(2n + 1), 0)$. Then $r(y_n, A) = 1 + 1/(n + 1) \rightarrow 1$. If $K \subset X$ is compact and $r(K, A) \leq 1$, take n so that $y(1/(2n - 1)) < \frac{1}{2} \forall y \in K$ and then take $0 < \delta < 1/2n$ such that $|s - t| \leq \delta \Rightarrow |y(s) - y(t)| < 1/2n \forall y \in K$. Consider the positive linear function x with the vertices $(0, 0)$, $(1/2n + \delta, 1 + 1/n)$, $(1/(2n - 1), 0)$, $(1, 0)$. $x \in A$ and therefore, for some $y_0 \in K$, $\|x - y_0\| < 1$. In particular, $y_0(1/2n + \delta) \geq 1/n$; hence $y_0(1/2n) \geq 1/2n$ and $y_0(1/(2n - 1)) \geq \frac{1}{2}$, which contradicts the choice of n .



4.8. COROLLARY. The space $A(S)$ of continuous affine functions on a compact Choquet simplex need not have Chebyshev centers or best compact set approximants for bounded sets.

PROOF. By a result of Lazar and Lindenstrauss [15, Theorem 5.5], every separable infinite-dimensional pre- L_1 Banach space is isometric to a subspace of some $A(S)$ space on which there is a norm-1 projection from $A(S)$. Thus by Example 4.7 and Proposition B, there is a set A in $A(S)$ which has not best n -net for any $n \geq 1$ and no best compact set approximant.

5. Continuous vector-valued function spaces. Observe that the quasi-uniform convexity condition for uniformly convex spaces and for $C(\Omega)$ spaces (and the more general class discussed in Proposition 4.2) can be satisfied with $z = z(x, \epsilon, \tilde{\delta})$ depending continuously on x (here ϵ and $\tilde{\delta}$ are fixed). More precisely, a space X is called continuously q.u.c. (c.q.u.c.) if for all $\epsilon > 0$ and the corresponding $\tilde{\delta}(\epsilon)$ there is a continuous selection $z(x, \epsilon, \tilde{\delta})$ for the point z in the q.u.c. definition at the beginning of §2.

5.1. PROPOSITION. *If X is c.q.u.c. then, for every topological space Ω , $C(\Omega, X)$ is q.u.c. with $\tilde{\delta}_{C(\Omega, X)} = \tilde{\delta}_X$.*

PROOF. Given $f \in C(\Omega, X)$, let $g(t) = z(f(t), \epsilon, \tilde{\delta})$. $g \in C(\Omega, X)$ by our assumption and clearly satisfies the q.u.c. condition.

5.2. COROLLARY [2]. *If X is uniformly convex, then $C(\Omega, X)$ admits Chebyshev centers for bounded sets and the Chebyshev center map is uniformly continuous with respect to the Hausdorff metric on bounded families of sets.*

If X is not q.u.c., $C(\Omega, X)$ may fail to admit centers:

5.3. EXAMPLE. *A bounded subset of $c(L_1[0, 1]) = C(\omega, L_1[0, 1])$ with no Chebyshev center:* Let

$$A = \{y \in c(L_1); y(n) = (-1)^n \chi_E / (1 - n^{-1})\mu(E), \mu(E) = (n + 1)/2n\}.$$

As in Proposition 2.8, $Z_{L_1}(\{y(n); y \in A\}) = (-1)^n / (1 - n^{-1})$ and the radius is 1. Let

$$z_m(n) = \begin{cases} (-1)^n / (1 - n^{-1}), & n \leq m, \\ 0, & n > m. \end{cases}$$

Then $r(z_m, A) = 1 / (1 - m^{-1}) \xrightarrow{m \rightarrow \infty} 1$. Clearly, the radius 1 cannot be realized.

5.4. EXAMPLE. *A bounded subset of $c(X) = C(\omega, X)$, X 3-dimensional, without a Chebyshev center:* Let X be as in Example 2.5,

$$A = \{y \in c(X); y(n) \in (1 + n^{-2})^{-1/2} [(-1)^n e_3 \pm (e_1 + n^{-1}e_2)]\}.$$

As in Example 5.3, we have

$$Z_X(\{y(n); y \in A\}) = (-1)^n (1 + n^{-2})^{-1/2} e_3, \quad r_X(\{y(n); y \in A\}) = 1.$$

$r(0, \{y(n); y \in A\}) = (1 + n^{-1})(1 + n^{-2})^{-1/2}$ shows that $r_{c(X)}(A) = 1$ cannot be realized.

REMARK. Similar examples can be constructed in $c(l_1)$ and other $c(X)$ spaces, whenever the Chebyshev center map in X is not upper semicontinuous.

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