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EXTENSIONS FOR AF C^* ALGEBRAS AND DIMENSION GROUPS

BY

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ABSTRACT. Let A, C be approximately finite dimensional (AF) C^* algebras, with A nonunital and C unital; suppose that either (i) A is the algebra of compact operators, or (ii) both A, C are simple. The classification of extensions of A by C is studied here, by means of Elliott's dimension groups. In case (i), the weak Ext group of C is shown to be $\mathrm{Ext}_{\mathbf{Z}}(K_0(C), \mathbf{Z})$, and the strong Ext group is an extension of a cyclic group by the weak Ext group; conditions under which either Ext group is trivial are determined. In case (ii), there is an unnatural and complicated group structure on the classes of extensions when A has only finitely many pure finite traces (and somewhat more generally).

Our motivating theme is to consider extensions of C^* algebras by other than the algebra of compact operators. Because AF algebras are describable in terms of partially ordered groups, they seem particularly suitable for this extension theory. As the ordered groups arising from simple AF algebras are fairly well understood, it turns out that one can solve completely the problem of classifying simple by simple AF algebras, when the *finite* trace space of the bottom AF algebra has finite dimensional dual. In the course of doing this, we establish formulae for the usual strong and weak Ext groups of AF algebras; our homological approach to this differs from the computational viewpoint of Pimsner and Popa [**14, 15**].

We consider short exact sequences ("extensions") of C^* algebras, $A \to B \to C$, with B an AF algebra. There is a translation to extensions within a class of partially ordered abelian groups and distinguished subset, known as dimension groups with interval, via the functor K_0. This translation is reversible (owing to a recent result of L. Brown that an extension of an AF algebra by an AF algebra is AF), so all C^* extensions are represented as dimension group extensions.

With the appropriate notion of equivalence (extending strong equivalence as defined in [**3**, p. 268], when A is the algebra of compact operators on a separable Hilbert space), the equivalence classes admit a limited additive operation, often forming a disjoint union of groups. A single group results if, for example, both A and C are simple (with A unitless, but not necessarily stable).

§I deals with the appropriate definitions of extensions, dimension groups, equivalence, and the translation between AF algebras and dimension groups. Much of this is well known. In the second section, it is shown that if A is simple stable, and C is

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simple, or if $A = \mathcal{K}$ (the algebra of compacts), then the dimension group extension equivalence classes' group is given by a formula and depends only on the group structures underlying $K_0(C)$ and $K_0(A)$. In particular, if $A = \mathcal{K}$, the weak Ext group of C is simply $\mathrm{Ext}_{\mathbf{Z}}(K_0(C), \mathbf{Z})$, the *group* extensions of $\mathbf{Z}$ by $K_0(C)$. The strong Ext group of C is an extension of a cyclic group by the weak Ext group; roughly speaking, the cyclic group measures the presence of matrix units. When C is simple and A is simple and stable (but different from the compacts), the extension classes form a group with $\mathrm{Ext}_{\mathbf{Z}}(K_0(C), K_0(A))$ appearing as a direct summand.

§§III through VI essentially discuss the situation that C be simple and A be simple and admit a finite trace. In §III, there is given a construction for extensions of the dimension groups; these arise out of closed faces of the state space of the bottom group. Once the face F is fixed, these extensions may be listed as elements of the group of homomorphisms $\mathrm{Hom}_{\mathbf{Z}}(K, \mathrm{Aff}(F))$, where K is the top dimension group.

In §IV, a converse of the construction in §III is shown; namely, if the bottom group has finite dimensional dual space (or more generally, if F above is finite dimensional), then *all* extensions arise out of the construction in §III. Unfortunately, if the finite dimensional hypothesis is dropped, other extensions may arise, and these appear to be beyond classification.

Equivalences (among the dimension group extensions) and how they are implemented among the previously constructed extensions are the main topic of §V. These equivalence classes form a disjoint unit of groups (one for each face of the state space of the bottom group). These classifying groups may be written down.

§VI employs the results of the previous three sections to discuss extensions of dimension groups with interval—corresponding to the original problem in AF algebras. The choice of interval fixes the face of states that is to be lifted, and the possible disjoint union of groups obtained in §V is reduced to a single group. This group (differing somewhat from that in §V, as in this context an equivalence must preserve a fixed element known as an order-unit) is computed. While it is frequently of tremendous size, if for example $K_0(A)$ and $K_0(C)$ are both finitely generated, the classifying group is not intractable. As a specific example, if A is a unitless universal Glimm algebra (not necessarily the stable one), and C is the unital universal Glimm algebra, there is exactly one equivalence class of extensions of A by C.

§VII contains an example of a partially ordered abelian group which is a simple by simple partially ordered group extension of a dimension group by another dimension group, which is not itself a dimension group. This is of some technical interest.

There is an appendix, which contains a somewhat different approach to the result of §II, that for AF algebras, the weak Ext group is $\mathrm{Ext}_{\mathbf{Z}}(K_0(C), \mathbf{Z})$.

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I. K_0-AF connections. The purpose of this section is to relate extensions of C^* algebras to extensions of their corresponding Grothendieck groups. We also introduce some of the definitions and properties useful in studying both AF C^* algebras, and partially ordered groups.

The definition of K_0 for C^* algebras is given in several sources (for example, [**12**, p. 430]) and of course agrees with the usual ring-theoretic version of K_0 [**2**, Chapter 9]. If R is a unital ring, then $K_0(R)$ admits a natural ordering making it into a partially ordered (abelian) group, provided all matrix rings over R satisfy $xy = 1$ implies $yx = 1$ [**10**]. Since we shall be dealing with AF C^* algebras, this latter condition is satisfied in our context, and this ordering on K_0 agrees with that established by Elliott in his fundamental paper on AF algebras [7]. For another approach to K_0 restricted to AF algebras, see [**4**, §3].

When R is a unitless C^* algebra, $K_0(R)$ is defined by forming $\tilde{R}$, the unique C^* unitification of R, and setting $K_0(R)$ to be the kernel of the induced mapping $K_0(\tilde{R}) \to \mathbf{Z} = K_0(\mathbf{C})$, obtained from $\tilde{R} \to \mathbf{C}$.

A *partially ordered group* is a group together with a subset, called the positive cone, that is closed under addition. All of our partially ordered groups are abelian, and if G denotes the group, the positive cone will be indicated by G^+. A *closed interval* in a partially ordered group G will be a subset of G^+ of the form

$$\{g \in G \mid 0 \leqslant g \leqslant x\} = [0, x] \quad \text{for some } x \text{ in } G^+ .$$

An *interval* will be a directed convex subset of G^+, with 0 as least element (*directed* means if a and b are in the interval, there exists c greater than or equal to both a and b in the interval). Intervals are sometimes referred to as hereditary subsets.

A partially ordered group G is called *simplicial* if G is order-isomorphic to the free abelian group $\mathbf{Z}^n$, with the coordinatewise ordering. A partially ordered group is called a *dimension group* if it is a direct limit (over a directed set) of simplicial groups. In some references there is a countability condition imposed on dimension groups, but we shall not require dimension groups to be countable. By [**5**, Theorem 2.2], a partially ordered group G is a dimension group if and only if it satisfies the following three properties.

(i) G is *directed*: That is, $G = G^+ - G^+$.

(ii) G is *unperforated*: If for a positive integer n, and an element x of G, nx belongs to G^+, then x also belongs.

(iii) G satisfies the (Riesz) *interpolation property*: For quadruples of elements of G, a, b, c, d satisfying $a, b \leqslant c, d$, there exists e in G with $a, b \leqslant e \leqslant c, d$.

If A is a C^* algebra, consider the set of projections in A; each projection yields an element of $K_0(A)$, and we refer to the set of such images in $K_0(A)$ as $D(A)$. Then D can be thought of as a set function D: {projections in A} $\to K_0(A)$.

The symbol $\mathcal{K}$ will denote the C^* algebra of compact operators on a separable Hilbert space.

The following theorem summarizes many standard results for AF algebras.

THEOREM I.1 [7]. (i) *Given an AF algebra A, $D(A)$ is a generating interval in $K_0(A)$; it is closed if and only if A is unital.*

(ii) *Given a countable dimension group G, and generating interval I, there exist an AF C^* algebra A and an isomorphism of partially ordered groups φ: $K_0(A) \to G$ so that $\varphi D(A) = I$.*

(iii) *Given two AF algebras, A, A', then $A \simeq A'$ if and only if there is an order-isomorphism φ: $K_0(A) \to K_0(A')$ such that $\varphi D(A) = D(A')$.*

(iv) *Given two AF algebras A, A', and an order-preserving group homomorphism φ: $K_0(A) \to K_0(A')$ with $\varphi D(A) \subset D(A')$, there exists a C^* algebra homomorphism ψ: $A \to A'$ (not necessarily unital) such that $K_0(\psi) = \varphi$; if both $D(A)$ and $D(A')$ are closed intervals, and the interval generated by $\varphi D(A)$ in $K_0(A')$ is all of $D(A')$, then ψ is unital.*

(v) *Two AF algebras A, A' are stably isomorphic (that is, $A \otimes \mathcal{K} \simeq A' \otimes \mathcal{K}$) if and only if $K_0(A) \simeq K_0(A')$ as partially ordered groups.*

An element x of a partially ordered group is called an *order-unit* if $G^+ = \bigcup_{n \in \mathbf{N}}[0, nx]$; if G is directed, this is the same as

for all g in G, there exists a positive integer n such that $g \leqslant nx$.

If A is a unital AF algebra, then the image of 1 in $K_0(A)$, call it $[1_A]$, is an order unit, and $D(A)$ is precisely the closed interval generated by $[1_A]$.

A subset H of a partially ordered group G is said to be an *order-ideal* (or a *convex directed subgroup*) in G if H is a directed subgroup and H^+ is an interval. For groups satisfying the interpolation property, this is the appropriate notion of ideal, and one can easily show that if G is a dimension group, and H is an order-ideal in G, then both H and G/H (endowed with the quotient ordering, $(G/H)^+ = (G^+ + H)/H$) are dimension groups (but *not* conversely, even when H, G, G/H are all real vector spaces, §VII). There is a natural bijection between order-ideals of $K_0(A)$ and closed two-sided ideals of A, when A is an AF algebra. We outline a slightly nonstandard proof of this in the following three results, Theorems I.2, I.3 and I.4.

We may assume A is unital, by forming $\tilde{A}$, in what follows.

Sitting inside the (for the moment, unital) AF C^* algebra A is a limit of semisimple finite dimensional algebras, call it R. Then R is a *unit regular ring*:

for all r in R, there exists an invertible element u such that $rur = r$.

THEOREM I.2 [GLIMM: THESIS]. *The embedding $R \subset A$ induces an isomorphism of partially ordered groups, $K_0(R) \to K_0(A)$, sending $D(R)$ onto $D(A)$.*

THEOREM I.3 [BRATTELI: THESIS]. *Every closed two-sided ideal of A is the closure of its intersection with R; if I is a two-sided ideal of R, then the closure of I (in A), $\bar{I}$, satisfies $\bar{I} \cap R = I$.*

THEOREM I.4 [**8**, 15.20 AND 5.2]. *Let R be a unit regular ring.*

(a) *Given a two-sided ideal I of R, the subgroup H of $K_0(R)$ generated by*

$$\{[eR] \mid e = e^2 \in I\} \subset K_0(R)$$

is an order-ideal in $K_0(R)$.

(b) *Given an order-ideal H of $K_0(R)$, the subset of R defined as*

$$I = \{r \in R \mid [rR] \in H\}$$

is a two-sided ideal of R.

(c) *The assignments $I \mapsto H$, $H \mapsto I$ of* (a), (b) *respectively, are mutually inverse lattice isomorphisms.*

If A is a unitless C^* algebra, and C is a unital C^* algebra, we define an *extension of A by C* to be a unital C^* algebra B together with maps

$$A \overset{\alpha}{\to} B \overset{\pi}{\to} C$$

so that α, π are C^* homomorphisms, with α one-to-one, π onto, $\alpha(A)$ an *essential* ideal of B, $\ker \pi = \operatorname{im} \alpha$, and π unital; loosely speaking, A is an ideal of B, and C is B/A.

An automorphism β of a unital C^* algebra B is *approximately inner* if there is a sequence of inner automorphisms β_n such that for all b in B, $\{\beta_n(b)\}$ converges to $\beta(b)$ in the norm topology. If B is not unital, we may take as definition that the extension of β to $\tilde{B}$ is approximately inner (for AF algebras, this coincides with the usual notion of approximately inner for unitless C^* algebras [**4**, Theorem 3.8]).

Given two extensions $A \to B_1 \to C$, $A \to B_2 \to C$ of A by C, we declare them to be *equivalent* if there exists a (unital) isomorphism of C^* algebras, ψ: $B_1 \to B_2$ such that $\psi\alpha_1(A) = \alpha_2(A)$, and so that the induced automorphisms τ: $A \to A$, $\bar{\psi}$: $C \to C$ are approximately inner; in other words, the diagram

$$\begin{array}{ccccc} A & \overset{\alpha_1}{\to} & B_1 & \overset{\pi_1}{\to} & C \\ \tau\downarrow & & \psi\downarrow & & \bar{\psi}\downarrow \\ A & \underset{\alpha_2}{\to} & B_2 & \underset{\pi_2}{\to} & C \end{array}$$

commutes, where $\tau = \alpha_2^{-1}\psi\alpha_1$ (this makes sense since $\psi\alpha_1(A) = \alpha_2(A)$), and $\bar{\psi}$ is defined as $\bar{\psi}(c) = \pi_2\psi(b)$, where $\pi_1(b) = c$. This notion of equivalence is an enlargement of the usual homological notion of equivalence which allows only the identity down the sides.

Because of the following result (based on a suggestion of George Elliott; the proof below is a modification of the argument in [**6**, Theorem 2.5]), we may assume that $\bar{\psi}$ is the identity map.

THEOREM I.4A. *If B is an AF algebra, A a closed ideal, and $C = B/A$, then approximately inner automorphisms of C can be lifted to approximately inner automorphisms of B.*

PROOF. Adding an identity if necessary, B contains an ascending chain of unital finite dimensional C^* subalgebras, $\{B_k\}$, whose union is dense. Let π: $B \to C$ be the quotient map, and set $C_k = \pi(B_k)$. If α is the automorphism of C, we may find unitaries $\{y_j\}$ in C such that

$$(*) \qquad \left\|(\operatorname{Ad} y_j - \alpha)/C_{i-1}\right\| < 2^{-(j+i+2)} \quad \text{for } j \geqslant i$$

(this is possible, since the unit ball of C_i is contained in the set $\{\Sigma\lambda_s e_s \mid \lambda_s \in \mathbf{C}, |\lambda_s| \leqslant 1\}$, where $\{e_s\}$ is a complete set of matrix units). By approximating by unitaries of $\bigcup C_k$ (easily done), we may suppose that each y_i belongs to $\bigcup C_k$. By refining the index set (for $\{A_k\}$, $\{B_k\}$), we may assume that y_k lies in C_k for all $k \geqslant 2$.

Write $e_k B_k = A \cap B_k$, where e_k is a central projection of B_k. Then π induces a *-isomorphism $(1 - e_k)B_k \to C_k$; in particular, each y_k may be lifted to a unitary z_k in B_k. We observe that $e_k \leqslant e_{k+1}$.

Define unitaries x_k of B_k inductively via

$$x_1 = (1 - e_1)z_1 + e_1,$$

$$x_k = (1 - e_k)z_k + \sum_1^{k-1} (e_{j+1} - e_j)z_j + e_1;$$

then $x_{k+1} = (1 - e_{k+1})z_{k+1} + e_{k+1}x_k$. In particular, $x_k - z_k$ belongs to A.

For b in $e_k B_k$, a simple computation using the centrality of e_i in B_i yields that $\mathrm{Ad}\, x_{k+p}(e_k b) = \mathrm{Ad}\, x_k(e_k b)$ for all p in **N**.

Next for b in $(1 - e_k)B_k$, with $j > k$, we deduce from $x_{j+1} = (1 - e_{j+1})z_{j+1} + e_{j+1}x_j$, that

$$\mathrm{Ad}\, x_{j+1}(b) = \left(1 - e_{j+1}\right)z_{j+1}bz_{j+1}^* + e_{j+1}x_j b x_j^*$$

(as e_{j+1} is central), so $(\mathrm{Ad}\, x_{j+1} - \mathrm{Ad}\, x_j)(b) = (1 - e_{j+1})(z_{j+1}bz_{j+1}^* - x_j b x_j^*)$.

Since π restricts to a *-isomorphism $(1 - e_{j+1})B_{j+1} \to C$, it is an isometry, and thus

$$\begin{aligned}\left\|(\mathrm{Ad}\, x_{j+1} - \mathrm{Ad}\, x_j)(b)\right\| &= \left\|y_{j+1}\pi(b)y_{j+1}^* - y_j\pi(b)y_j^*\right\| \\ &= \left\|(\mathrm{Ad}\, y_{j+1} - \mathrm{Ad}\, y_j)\pi(b)\right\| < 2^{-(j+k)}\|b\|.\end{aligned}$$

(The latter inequality holds, as $\pi(b)$ lies in C_k.)

It follows that for all b in B_k, $\{\mathrm{Ad}\, x_j(b)\}$ converges in norm. Since $\bigcup B_k$ is norm dense, $\{\mathrm{Ad}\, x_k\}$ converges in the point norm topology to an automorphism of B; this lifts α, as $\pi(x_k) = y_k$. □

Returning to the diagram above, lift the inverse of $\bar{\psi}$ to an approximately inner automorphism of B_2 and compose with ψ. We observe that approximately inner automorphisms automatically leave invariant all closed ideals, and their restrictions are also approximately inner [**4**, Theorem 3.8]. There results a diagram in which the vertical map $C \to C$ is the identity.

When A is the algebra of compact operators (and C is AF), this notion of equivalence agrees with Brown-Douglas-Fillmore strong equivalence [**3**, 1.1, Definition], because every automorphism of A is then approximately inner. Also note that the concept of extension defined here agrees, because of our use of *essential* in the definition.

Two extensions of C^* algebras $A \xrightarrow{\alpha_i} B_i \xrightarrow{\pi_i} C$ (B_i, π_i unital) are said to be *stably equivalent* if there is an isomorphism

$$\psi\colon B_1 \otimes \mathcal{K} \to B_2 \otimes \mathcal{K}$$

so that

$$\psi\alpha_1(A \otimes \mathcal{K}) = \alpha_2(A \otimes \mathcal{K}),$$

and so that the induced

$$\tau: A \otimes \mathcal{K} \to A \otimes \mathcal{K}, \qquad \bar{\psi}: C \otimes \mathcal{K} \to C \otimes \mathcal{K}$$

are approximately inner. When $A = \mathcal{K}$, and C is AF, then stable equivalence of extensions agrees with weak equivalence, as is implicit in the proof of II.9 or A-2.

LEMMA I.5. *Let B_i be unital AF C^* algebras, and let $A \to B_i \to C$ be extensions of A by C.*

(a) *Then both A, C are AF algebras, and the functor K_0 induces an extension of dimension groups with interval,*

$$(**) \qquad (K_0(A), D(A)) \to \left(K_0(B_i), \left[1_{B_i}\right]\right) \to (K_0(C), [1_C]).$$

(b) *The two extensions of dimension groups with interval are equivalent if and only if the extensions of C^* algebras are equivalent.*

(c) *Removing the intervals and order-units from* (**) *to obtain an extension of dimension groups, then the extensions of dimension groups are equivalent if and only if the extensions of C^* algebras are stably equivalent.*

PROOF. (a) This is well known, and is implicit in [7].

(b) If the extensions of C^* algebras are equivalent, then since approximately inner automorphisms induce the identity on K_0, the "if" portion of the statement is immediate.

Suppose the extensions of the K_0's are equivalent as dimension groups with interval, say with γ: $(K_0(B_1), [1_{B_1}]) \to (K_0(B_2), [1_{B_2}])$ implementing the equivalence. By [7], there exists a C^* algebra homomorphism δ: $B_1 \to B_2$, that is a unital isomorphism with $K_0(\delta) = \gamma$. Since the diagram of dimension groups commutes, we obtain side mappings $A \to A$, $C \to C$ which induce the identity on K_0; by [**4**, Theorem 3.8], they must be approximately inner.

(c) Tensoring with the compacts destroys the interval, and (b) or the methods involved therein apply. □

A defect in I.5 is that it does not translate extensions of $K_0(A)$ by $K_0(C)$ back to a unique AF C^* algebra B. This is remedied by the following recent theorem of Larry Brown.

THEOREM I.6 [**16**]. *Let A be an AF C^* algebra that is an ideal in a C^* algebra B; set $C = B/A$, Then*

(a) *projections in C lift to projections in B*;

(b) *if C is AF, then so is B.*

That (a) implies (b) is Corollary 3.3 of [**6A**].

Because K_0 with order-interval (or order-unit, in the unital case) is a complete invariant of AF algebras, we deduce from Brown's Theorem that K_0 and the order-unit determine uniquely the extension that arises from the extension of dimension groups with interval. We need only construct an AF algebra B corresponding to the dimension group extension of $K_0(A)$ by $K_0(C)$.

PROPOSITION I.7. *Given an extension of countable dimension groups with interval,*

$$(H, N) \to (G, u) \to (K, w),$$

there exists an extension of AF C^* algebras $A \to B \to C$, together with order-isomorphisms (vertical, in the diagram) so that the diagram

$$\begin{array}{ccccc} (K_0(A), D(A)) & \to & (K_0(B), [1_B]) & \to & (K_0(C), [1_C]) \\ \downarrow & & \downarrow & & \downarrow \\ (H, N) & \to & (G, u) & \to & (K, w) \end{array}$$

commutes.

PROOF. By [7], there exists an AF algebra B' with $K_0(B')$ order-isomorphic to G. Form $B'' = B \otimes \mathcal{K}$; we may find a projection p in B'' such that $D_{B''}(p)$ is sent to u under the isomorphism. Set $B = pB''p$. Since order-ideals correspond to closed two-sided ideals and K_0 behaves well on the quotient C^* algebras, the rest is clear. □

Having made a complete translation of the extension problem from AF C^* algebras to dimension groups with interval, we can now begin our study of the latter, via extensions of dimension groups. In §II, we study (essentially) the case of the interval being all of H^+; this will include the case of the algebra of compacts, and in this case, the BDF-Voiculescu Ext groups of AF algebras can be written down.

The stable equivalence classes behave in a peculiar manner when A is not the compact algebra; if C happens to be simple and AF, then these equivalence classes form a disjoint union of groups (§§III through VI).

Now, there are corresponding definitions of extensions of dimension groups. If H, K, *and* G are dimension groups (and we usually require that G and K possess order-units), then a *dimension group extension of H by K* is a sequence $H \overset{\alpha}{\to} G \overset{\pi}{\to} K$, with

α an order-embedding (meaning α is one-to-one, $\alpha(H^+) \subset G^+$, and $\alpha^{-1}(G^+) = H^+$);

π onto, order-preserving, and $\pi^{-1}(K^+) = G^+ + H$;

$\ker \pi = \operatorname{im} \alpha$;

$\alpha(H)$ an order-ideal in G; and

$\alpha(H) \cap L \neq \{0\}$ for all nonzero order-ideals L of G.

If N is an interval in the dimension group H, and w is an order-unit for K, then we say a sequence $(H, N) \to (G, u) \to (K, w)$ is an *extension of dimension groups with interval* if

$H \to G \to K$ is an extension of dimension groups;

$\pi(u) = w$, and u is an order-unit for G; and

$\alpha(N) = [0, u] \cap \alpha(H)$.

The notion of equivalence, either for dimension group extensions, or for extensions of dimension groups with interval, is the obvious one, with only identity maps allowed down the sides, and intervals, order-units sent to the appropriate objects.

Extensions of dimension groups with interval correspond exactly to extensions of AF algebras. It is much easier to work with extensions of dimension groups first, and then deduce the desired results for extensions with intervals.

II. Lexicographic extensions. A (directed) partially ordered group is *simple*, if it has no order-ideals other than the group itself and $\{0\}$; what amounts to the same thing is that every nonzero positive element be an order-unit.

If (G, u) is a partially ordered group with order-unit u, then we define a *state* (normalized) to be an order-preserving group homomorphism from G to the additive group of the reals, sending u to 1. Sometimes, the notion of state does not require the condition that it send a specific order-unit to 1; this will be clear from the context.

An extension of partially ordered groups $H \to G \to K$ is said to be *lexicographic* if

$$G^+ = \pi^{-1}(K^+ - \{0\}) \cup \alpha(H^+);$$

that is, an element g of G is positive if either $\pi(g)$ is nonzero and positive in K, or if $g = \alpha(h)$ for some h in H^+. This generalizes the usual notion of lexicographic product, in that, in this case we do not require the extension to split as groups.

LEMMA II.1. *Let $H \to G \to K$ be an extension of partially ordered groups, with both H, K dimension groups. If the extension is lexicographic, then G is also a dimension group.*

PROOF. By [5, Theorem 2.2] (see the introduction to §I), we need only check that G is unperforated and satisfies the interpolation property, and this is routine. □

If H is an order-ideal of the dimension group G, and v, u are order-units for H, G respectively, we say *a state f on (H, v) extends to G* if there exists a state p on (G, u) such that $p(H) \neq \{0\}$ and f agrees with $p/p(v)$ on H. Not all states of H extend; for example if the ordering is lexicographic, *no* states extend, and this is frequently a characterization (II.5 below). Since the definition does not depend on a specific order-unit of H, but merely that one exists, we usually say the state f on H extends.

LEMMA II.2 [5, THEOREM 1.4]. *Let G be an unperforated directed group with order-unit. Then an element g of G is positive and an order-unit if and only if $f(g)$ is strictly greater than zero for all pure states f.*

LEMMA II.3. *Let G be a dimension group, and suppose x, y are positive elements of G satisfying*

$$0 \leqslant z \leqslant x, y \quad \textit{implies } z = 0.$$

Then the order-ideals generated by x and y have zero as intersection.

PROOF. Define $I(a) = \{g \in G \mid -na \leqslant g \leqslant na \text{ for some } n \text{ in } \mathbf{N}\}$ for a positive element a of G. Then $I(a)$ is easily seen to be an order-ideal, and is the order-ideal generated by a. We check that the intersection of two order-ideals I, J is itself directed, hence an order-ideal.

Select t in $I \cap J$; we may write $t = a - b = c - d$, with a, b in I^+, and c, d in J^+. Applying Riesz decomposition to $a + d = b + c$, we may write $a = b_1 + c_1$ with $0 \leqslant b_1 \leqslant b$, and $0 \leqslant c_1 \leqslant c$, and so $d = (b - b_1) + (c - c_1)$. Since J is convex, c_1 belongs to J, and the first equation yields that c_1 is a member of the intersection; similarly, $b - b_1$ belongs to $I \cap J$. Thus $t = a - b = (a - b_1) - (b - b_1) = c_1 - (b - b_1)$ represents t as a difference of two elements of $(I \cap J)^+$, so $J \cap I$ is an order-ideal.

The hypothesis clearly implies $I(x)^+ \cap I(y)^+ = \{0\}$, so $I(x) \cap I(y)$ is zero. □

LEMMA II.4. *Let* (G, u) *be a noncyclic dimension group with order-unit. If* v *is an atom of* G^+, *then* $\mathbf{Z}v$ *is an order-ideal. If* $\mathbf{Z}v$ *is essential, then* $nv \leqslant u$ *for all positive integers* n.

PROOF. That $\mathbf{Z}v$ is an order-ideal is straightforward. Suppose $nv \nleqslant u$ for some n. Since u is an order-unit, there exists m so that $v \leqslant mu$; from the atomicity of v and Riesz decomposition, $v \leqslant u$. If k is the largest integer such that $kv \leqslant u$, we see that $0 \leqslant z \leqslant u - kv, v$ implies $z = 0$. By II.3 and essentiality of $\mathbf{Z}v$, $u = kv$; as $\mathbf{Z}v$ is an order-ideal this implies $G = \mathbf{Z}v$, so G is cyclic, a contradiction. □

LEMMA II.5 *Let* $H \to G \to K$ *be an extension of dimension groups, and suppose* H *has an order-unit of its own, as does* G.

(a) *If the extension is lexicographic, no states of* H *extend to* G.

(b) *Suppose either* (i) *or* (ii) *hold.*

(i) $H = \mathbf{Z}$ *or*

(ii) K *is simple, and no states of* H *extend to* G;

then the extension is lexicographic.

(c) *If the extension is lexicographic, then for any order-unit* u *of* G,

$$H^+ = H \cap [0, u].$$

PROOF. If v, u are the selected order-units for H, G respectively, and (a) applies, then $nv \leqslant u$ for all integers n, whence any state of G must send v to 0, and thus annihilate all of H (since $[0, v]$ generates H as an ordered group).

(b) Select an element g such that $g + H$ belongs to $K^+ - \{0\}$. There thus exists an element h in H with $g + h$ in G^+; as v is an order-unit, there exists a positive integer n so that $g + nv$ is positive in G. Let f be a state on (G, u); the hypothesis implies $f(H) = \{0\}$, so f induces a state $\bar{f}$ on $(K, u + H)$; conversely, any state of $(K, u + H)$ induces a state on (G, u). We shall show g is positive in both cases (i) and (ii), by applying II.4 and II.2 respectively.

If (ii) holds, $g + H$ is automatically an order-unit in K, so for all states $\bar{f}$ of K (and we know all such states are induced from G), $\bar{f}(g + H) > 0$; thus $f(g)$ is strictly greater than zero, so from II.2, g is positive.

If (i) holds, set $x = g + (n + 1)v$, and we may as well assume v is the atom. Form $I(x)$, the order-ideal generated by x (as in the proof of II.3). As H is essential and v is an atom generating H, $I(x) \supset H$, and obviously x is an order-unit for the dimension group $I(x)$. By II.4, $mv \leqslant x$ for all positive integers m, so that $x \geqslant (n + 1)v$, so g is positive.

Hence in either case, $G^+ \supset \alpha(H^+) \cup \pi^{-1}(K^+ - \{0\})$. If g is a positive element of G, and $\pi(g) = 0$, then g belongs to H^+. Thus equality holds, so the extension is lexicographic.

(c) This is a tautology. □

If A is a *stable* C^* algebra (that is, $A \simeq A \otimes \mathfrak{K}$), and is AF, then the K_0-sequence derived from an extension of A by any AF algebra C must be

$$\left(K_0(A), K_0(A)^+\right) \to (G, u) \to \left(K_0(C), [1_C]\right)$$

simply because $D(A)$ is all of $K_0(A)^+$ in this case. Hence if either A is the algebra of compacts, or C is simple, the only extensions of the corresponding dimension groups are lexicographic. In particular, this means that the usual Ext theory (with A being the compacts) is particularly straightforward.

Given any two abelian groups X, Z, recall from the standard texts the definition of $\mathrm{Ext}_{\mathbf{Z}}(Z, X)$; this is a group structure on the set of equivalence classes of abelian group sequences $\{0\} \to X \to Y \to Z \to \{0\}$; one definition for the group operation will be given in the course of the proof of II.6. The subscript $\mathbf{Z}$ is employed to remind the reader that only the group structures of X and Z are being considered (cf., the statement of II.6).

If X, Z are abelian groups, then the set of group homomorphisms from Z to X is also an abelian group, and will be denoted either $\mathrm{Hom}_{\mathbf{Z}}(Z, X)$ or $[Z, X]$.

THEOREM II.6. *Let H be a simple dimension group, and (K, w) a dimension group with order-unit. Suppose that either H is cyclic, or K is simple. Then the equivalence classes of extensions of dimension groups with interval of (H, H^+) by (K, w) form a group isomorphic to an abelian group extension of*

$$H/([K, H](w)) \quad \text{by } \mathrm{Ext}_{\mathbf{Z}}(K, H),$$

where $[K, H](w) = \{t(w) \mid t \in \mathrm{Hom}_{\mathbf{Z}}(K, H)\}$.

PROOF. Given a group extension G (that is, a representative of an element of $\mathrm{Ext}_{\mathbf{Z}}(K, H)$), we can construct an extension as follows. Select any element u of G such that $\pi(u) = w$, and impose the lexicographic ordering on G; then u is an order-unit. By II.1, this yields an extension of dimension groups with interval

$$(H, H^+) \to (G, u) \to (K, w).$$

It follows from II.5(b), (c) that every extension must be of this form. We thus have freedom in choosing from $\mathrm{Ext}_{\mathbf{Z}}(K, H)$, and then from $\pi^{-1}(w)$ (of course, π: $G \to K$ comes as part of the choice of G).

To determine the group structure on the equivalence classes, we first recall the group operation on $\mathrm{Ext}_{\mathbf{Z}}(K, H)$. If

$$H \xrightarrow{\alpha_i} G_i \xrightarrow{\pi_i} K, \qquad i = 1, 2,$$

are two group extensions, first form the pullback of the mappings π_i: $G_i \to K$, i.e., we obtain $D = \{(g_1, g_2) \in G_1 \oplus G_2 \mid \pi_1(g_1) = \pi_2(g_2)\}$. Then let d: $H \oplus H \to H$ be the mapping $d(h) = (h, -h)$. As $d(H)$ is contained in D, we may form the group $G = D/(d(H))$. Let α: $H \to G$ be the map $H \to H \oplus H$ given by $h \mapsto (h, 0)$, composed with d and the quotient map $D \to G$. The mapping π: $G \to K$ is the obvious one. This constructs an extension of groups $H \to G \to K$, and this defines the addition operation in $\mathrm{Ext}_{\mathbf{Z}}(K, H)$.

On the G_i of the preceding paragraph, impose the lexicographic ordering, and suppose u_i are corresponding order-units, so that $\pi_i(u_i) = w$. Then (u_1, u_2) lies in D, and we may add (G_1, u_1) and (G_2, u_2) by setting the sum equal to $(G, (u_1, u_2) + d(H))$.

Two extensions of this form are equivalent precisely when there exists an order-isomorphism φ: $(G, u) \to (G', u)$ inducing the identity on H and K; in particular, $G = G'$ (as group extensions). Let us show that any equivalence of the extensions within the category of abelian group extensions, that sends u to u', actually implements an equivalence in the category of dimension groups with interval.

Given φ: $G \to G$ such that the diagram

$$(***) \qquad \begin{array}{ccccc} H & \overset{\alpha}{\to} & G & \overset{\pi}{\to} & K \\ & \alpha\searrow & \downarrow\varphi & \nearrow\pi & \\ & & G & & \end{array}$$

commutes, we see that φ must be a group isomorphism. Since the ordering on G is lexicographic, it follows that $g \in \alpha(H^+)$ if and only if $\varphi(g) \in \alpha(H^+)$, and that $\pi(g) \in K^+$ if and only if $\pi\varphi(g) \in K^+$. Hence φ must be an order-isomorphism, and so implements an equivalence of extensions of dimension groups.

Next, the group automorphisms of G that permit (***) to commute are in bijection with $\mathrm{Hom}_{\mathbf{Z}}(K, H)$. Given the automorphism φ, form $t = \varphi - \mathrm{id}$, and observe that $t(H) = 0$ (more accurately, $t\alpha = 0$). Hence t induces a homomorphism $\bar{t}$: $K \to G$, satisfying $\bar{t}\pi = t$. Since $\pi\varphi = \pi$, $\pi t = 0$; hence $\bar{t}(K) \subset H$ (more accurately, $\bar{t}(K) \subset \alpha(H)$). Conversely, given s: $K \to H$, define t: $G \to H \subset G$, via $t = s\pi$. Observe that if $\varphi = t + \mathrm{id}$, (***) commutes, and furthermore, $\varphi^{-1} = -t + \mathrm{id}$, as $t^2 = 0$.

Hence two pairs of the form discussed above (G_1, u_1), (G_2, u_2), are equivalent if and only if $G_1 = G_2$ and there exists s: $K \to H$ such that $s(w) = u_2 - u_1$. It is routinely verified that this equivalence relation is compatible with the operation defined above on the extensions, and the induced operation on the equivalence classes is associative and commutative. There is an obvious semigroup homomorphism onto $\mathrm{Ext}_{\mathbf{Z}}(K, H)$ (sending the class of (G, u) to the class of the extension determined by G in $\mathrm{Ext}_{\mathbf{Z}}(K, H)$), and it is routine to verify that the whole semigroup is cancellative.

The kernel of this mapping to $\mathrm{Ext}_{\mathbf{Z}}(K, H)$ consists of pairs of the form $(H \oplus K, (h, w))$ (or rather, their equivalence classes). The operation restricted to this set simply adds the H-coordinates, and two choices for the H-coordinate will yield equivalent extensions if and only if the difference belongs to $[K, H](w)$. Thus the semigroup is cancellative abelian, and is exhibited as an extension of two groups, $H/([K, H](w))$ by $\mathrm{Ext}_{\mathbf{Z}}(K, H)$. It follows that the semigroup is actually a group. □

COROLLARY II.7. *Let C be a unital AF C^* algebra. Then the strong* Ext *group of C is an abelian group extension of*

$$\mathbf{Z}/\left([K_0(C), \mathbf{Z}]([1_C])\right) \quad by \ \mathrm{Ext}_{\mathbf{Z}}(K_0(C), \mathbf{Z}).$$

PROOF. The comments followng I.4A show that the equivalence used in II.6 agrees with strong equivalence, and now one checks that the addition of the equivalence classes of extensions used in the proof of II.6 agrees with that induced by the addition in Ext(C) (viz., [**3**, p. 270]) (see also the proof of A-3). □

COROLLARY II.8. *Let C be a simple unital AF algebra, and A a stable simple AF algebra. Then the equivalence classes of extensions of A by C are completely classified by the elements of a group extension of*

$$K_0(A)/\left([K_0(C), K_0(A)]([1_C])\right) \quad \text{by } \operatorname{Ext}_{\mathbf{Z}}(K_0(C), K_0(A)).$$

There is a more complete and explicit description available for the classifying groups in II.6, II.7 and II.8. Given an extension

$$\begin{array}{ccccc} H & \rightarrow & (G,u) & \overset{\pi}{\rightarrow} & (K,w) \\ \downarrow & & \downarrow & & \downarrow \rho \\ H & \rightarrow & X & \overset{\pi}{\rightarrow} & K/\mathbf{Z}w \end{array}$$

we can create a second line by defining X to be $G/\mathbf{Z}u$. The kernel of the map $X \to K/\mathbf{Z}w$ is a natural copy of H, so we obtain an extension of H by $K/\mathbf{Z}w$. Conversely, given a group extension

$$H \to X \to K/\mathbf{Z}w$$

a group extension of H by (K, w) with a specified element u sent to w is determined uniquely by the pullback of the maps $X \to K/\mathbf{Z}w$, $K \to K/\mathbf{Z}w$, $G = \{(x,k) \mid \pi(x) = \rho(k)\}$, with $u = (0, w)$. One checks in a straightforward fashion that this identification is a group isomorphism between the group described in II.6 and $\operatorname{Ext}_{\mathbf{Z}}(K/\mathbf{Z}w, H)$.

Thus II.6 gives the complete classification of the dimension group extensions as the group, $\operatorname{Ext}_{\mathbf{Z}}(K/\mathbf{Z}w, \mathbf{Z})$; II.7 yields that the strong Ext group of C is $\operatorname{Ext}_{\mathbf{Z}}(K_0(C)/\mathbf{Z}\cdot[1_C], \mathbf{Z})$ (but see the comment below), and II.7 results in the classification being given by

$$\operatorname{Ext}_{\mathbf{Z}}(K_0(C)/\mathbf{Z}\cdot[1_C], K_0(A)).$$

I should point out that this formulation was suggested by D. Voiculescu in an address at Queen's University in August 1980, in pointing out that Pimsner and Popa had (effectively) computed the strong Ext group as $\operatorname{Ext}_{\mathbf{Z}}(K_0(C)/\mathbf{Z}\cdot[1_C], \mathbf{Z})$ [**14**].

If $C = M_n\mathbf{C}$, then $(K_0(C)\cdot[1_C]) = (\mathbf{Z}, n)$, so the strong Ext group is $\mathbf{Z}/n\mathbf{Z}$. If C is a finite direct sum of $M_{n(i)}\mathbf{C}$, $i = 1, 2, \ldots, k$, then $(K_0(C), [1_C]) = (\mathbf{Z}^k, (n(1), \ldots, n(k)))$. Let $m = \gcd n(i)$; then $[K_0(C), \mathbf{Z}]([1_C]) = m\mathbf{Z}$, so by II.7 (as stated), the strong Ext group is $\mathbf{Z}/m\mathbf{Z}$. Both of these results are well known.

In case C is the unital p^∞ UHF algebra, then $(K_0(C), [1_0]) = (\mathbf{Z}[1/p], 1)$ (with the ordering as a subring of the real numbers). Then using $\mathbf{Z}[1/p]/\mathbf{Z}1 \simeq \mathbf{Z}_{p^\infty}$ (the Prüfer group), the strong Ext group is $\operatorname{Ext}_{\mathbf{Z}}(\mathbf{Z}_{p^\infty}, \mathbf{Z})$, which is known to be the completion of the integers localized at p, in the p-adic topology. (This example appears in unpublished notes of George Elliott, and can also be computed as proj lim $\mathbf{Z}/p^n\mathbf{Z}$; essentially this had been obtained in [**15**].)

On the other hand, if C is, for example, the AF algebra defined by the single matrix $\left[\begin{smallmatrix}1 & 1\\ 2 & 1\end{smallmatrix}\right]$ repeated infinitely often, then $K_0(C) = \mathbf{Z} + 2^{1/2}\mathbf{Z}$ (as a subgroup of the reals) is a simple dimension group free of rank 2, so that the group extensions are all trivial. If we select $w = [1_C]$, then w can be extended to a basis for $K_0(C)$, and thus $[K_0(C), \mathbf{Z}](w)$ is all of $\mathbf{Z}$. Hence there is only one strong equivalence class of extensions of this algebra (or of any simple unital AF C^* algebra whose dimension group is free, and such that the algebra itself is not a matrix ring of any size over another ring).

To describe the weak equivalence classes of extensions of AF algebras, we show that the cyclic portion of II.7 becomes zero, as the dependence on the choice of order-unit is eliminated.

LEMMA II.9. *Let C be a unital AF algebra.*

(a) *Two extensions of C by the compacts are weakly equivalent if and only if there is an order-isomorphism φ: $K_0(B_1) \to K_0(B_2)$ so that the diagram*

$$\begin{array}{ccccc} \mathbf{Z} & = K_0(\mathcal{K}) & \to & K_0(B_1) & \to & K_0(C) \\ \| & & & \downarrow \varphi & & \| \\ \mathbf{Z} & & \to & K_0(B_2) & \to & K_0(C) \end{array}$$

commutes.

(b) *Given a dimension group extension $\mathbf{Z} \to G \to K_0(C)$, we may find an AF C^* algebra B, and an extension of $\mathcal{K}$ by C, $\mathcal{K} \to B \to C$ so that $K_0(\mathcal{K} \to B \to C)$ is isomorphic to $\mathbf{Z} \to G \to K_0(C)$.*

PROOF. (a) Two weakly equivalent extensions, after tensoring with the compact algebra, become strongly equivalent (with the appropriate notion of strong equivalence for unitless extensions), and the only if portion of the statement is straightforward.

On the other hand, any order-isomorphism lifts to an isomorphism of the stable C^* algebras, and this may be restricted to a weak equivalence.

(b) G must have the lexicographic ordering by II.5. Pick any element of G in the pre-image of $[1_C]$, and call it u. We obtain an extension of dimension groups with interval

$$(\mathbf{Z}, \mathbf{N}) \to (G, u) \to (K_0(C), [1_C]);$$

now I.7 applies. □

COROLLARY II.10. *For C a unital AF algebra, the weak equivalence classes of extensions of C by the compacts are classified completely by* $\mathrm{Ext}_{\mathbf{Z}}(K_0(C), \mathbf{Z})$, *and this is isomorphic to* $\mathrm{Ext}_w(C)$.

PROOF. The change of order-units is measured by the cyclic portion of II.7 in the case of strong equivalence. However, the extensions with the same element of $\mathrm{Ext}_{\mathbf{Z}}(K_0(C), \mathbf{Z})$ but corresponding to different order-units are all weakly equivalent by II.9; so the cyclic contribution disappears, and it is routine to check that this is precisely the kernel of the map $\mathrm{Ext}_s(C) \to \mathrm{Ext}_w(C)$. □

Without the hypothesis of II.6 (that either $H = \mathbf{Z}$ or K is simple), the extensions of (H, H^+) by (K, w) can still often be listed but rather than having a group structure, they may be listed as a disjoint union of groups. An example illustrating this phenomenon is exhibited below. This may be translated back to AF algebras, and it follows that the classification of simple stable AF algebras (other than the compacts) by nonsimple AF algebra extensions is not given by a group, but a disjoint union of them. This renders complicated the possible prolongation of the theory of extensions to incorporate simple algebras unequal to the compacts, on the left.

EXAMPLE. This is an extension of dimension groups with interval of the form

$$(H, H^+) \to (G, u) \to (K, w),$$

where H is simple, but the extension is not lexicographic.

Set $G = \mathbf{R}^3$, with

$$G^+ = \{(x, y, z)\colon z > 0, \textit{or} \text{ all of } x > 0, y \geqslant 0, z = 0\} \cup \{(0,0,0)\}.$$

Set $H = \{(x, 0, 0)\}$, and $H^+ = G^+ \cap H$; then $H = \mathbf{R}$ with the usual ordering. With $u = (0, 0, 1)$, we see that no states of H can extend to G as $n(1, 0, 0) < u$ for all n. However $(0, 1, 0)$ becomes positive modulo H, yet is not positive in G—hence the extension is not lexicographic. To obtain a countable example (of relevance for AF algebras), simply replace the reals by the rationals.

This is a simplification, due to the referee, of my original example.

From II.7 and II.10, we can determine which AF algebras have trival weak or strong Ext groups (II.12).

LEMMA II.11. *If C is a unital AF algebra, and C is an $n \times n$ matrix ring, then*

$$[K_0(C), \mathbf{Z}]([1_C]) \subset n\mathbf{Z}.$$

PROOF. If C is an $n \times n$ matrix ring, then $[1_C] = np$ for some p in $K_0(C)$, so that for all f in $\mathrm{Hom}_{\mathbf{Z}}(K_0(C), \mathbf{Z})$, $f([1_C]) = nf(p) \in n\mathbf{Z}$. □

(The converse is certainly not true: Take for $K_0(C)$ any torsion-free indecomposable rank 2 abelian group embedded in $\mathbf{R}$.)

COROLLARY II.12. *Let C be a unital AF algebra. Then*

(a) $\mathrm{Ext}_w(C) = \{0\}$ *if and only if $K_0(C)$ is free as an abelian group*;

(b) $\mathrm{Ext}_s(C) = \{0\}$ *if and only if $K_0(C)$ is free and C is not an $n \times n$ matrix ring for any n greater than* 1.

PROOF. (a) By II.10, $\mathrm{Ext}_w(C) = \{0\}$ if and only if $\mathrm{Ext}_{\mathbf{Z}}(K_0(C), \mathbf{Z})$ is zero; since $K_0(C)$ is countable, this is equivalent to $K_0(C)$ being free (see any text on infinite abelian groups, under the Whitehead problem).

(b) If $\mathrm{Ext}_s(C)$ is zero, by II.7, $\mathrm{Ext}_{\mathbf{Z}}(K_0(C), \mathbf{Z}) = \{0\}$ (whence $K_0(C)$ is free), and $[K_0(C), \mathbf{Z}]([1_C]) = \mathbf{Z}$; by II.11, C is not an $n \times n$ matrix ring.

On the other hand, $K_0(C)$ being free implies $\mathrm{Ext}_{\mathbf{Z}}(K_0(C), \mathbf{Z})$ is zero, and the matrix ring hypothesis translates to

$$[1_C] \neq np \quad \text{for all } n \text{ in } \mathbf{Z} - \{+1\}, \text{ all } p \text{ in } K_0(C).$$

Since $K_0(C)$ is free, it easily follows that $\{[1_C]\}$ can be completed to a $\mathbf{Z}$-basis for $K_0(C)$, and so $[K_0(C), \mathbf{Z}]([1_C]) = \mathbf{Z}$, whence by II.7 $\mathrm{Ext}_s(C) = \{0\}$. □

It also follows from II.7 and II.12, that $\mathrm{Ext}_s(C) = \mathbf{Z}$ is impossible for any unital AF algebra C; other impossible groups include all noncyclic countable abelian groups.

III. Simple by simple extensions of dimension groups. For suitable dimension groups H, K, we classify the dimension group extensions of H by K. This will be used in subsequent sections to determine the extensions of dimension groups with interval, to yield results about the corresponding extensions of AF algebras.

In the treatment of lexicographic extensions, we saw there was a group structure imposed on the extensions. In this section, we see that there is a structure of some sort that can be imposed on the extensions, but rather than being a group, it is a disjoint union of groups. When we specialize these results to extensions of dimension groups with interval, we will be selecting one of the groups appearing in the disjoint union, adding an extension and factoring out a subgroup.

Here H, K will be simple dimension groups. What turns out to be the important factor in this classification is which face of states extends from H, and how it extends.

On a first reading of this section, it might be practical to ignore the extra complication of the group obstruction to splitting, $\mathrm{Ext}_{\mathbf{Z}}(K, H)$, and simply assume that all the extensions of dimension groups split *as groups*; then the mysterious Lemma III.4 has a particularly transparent form.

LEMMA III.1. *Let H, K be simple dimension groups, and $H \xrightarrow{\alpha} G \xrightarrow{\pi} K$ an extension of dimension groups. Then*

(i) $G^+ = \alpha(H^+) \cup \{$*order-units of* $G\}$.

(ii)

$$G^+ = \alpha(H^+) \cup \left\{ g \in G \;\middle|\; \begin{array}{l} \pi(g) \in K^+ - \{0\}, \text{ and } f(g) > 0 \text{ for all states } f \\ \text{of } G \text{ that are extensions of states of } H. \end{array} \right\}$$

PROOF. (i) Certainly the inclusion $\supset$ holds, as $\alpha(H)$ is an order-ideal of G. Select g in $G^+ - \alpha(H)$; if g were contained in a proper order-ideal J, we would have $\alpha(H) + J = G$ (since $\alpha(H)$ is a maximal order-ideal); as H is simple, $\alpha(H) \cap J = \{0\}$, contradicting the essentialness of $\alpha(H)$. Thus g must generate an improper order-ideal, and so g is an order-unit.

(ii) Since K is a simple dimension group, by Lemma II.2, $K^+ - \{0\}$ is the set of elements of K that are strictly positive at all states of K; since a state of G either annihilates $\alpha(H)$ (in which case it induces a state of K) or restricts to a state of H, the elements of the form indicated that are not in $\alpha(H^+)$, are strictly positive at all states of G. By Lemma II.2, these constitute exactly the set of order-units of G, and (i) applies. □

For the rest of this section, continuing our tacit policy, we regard H as an order-ideal of G (and so suppress the α, except when convenient), and K as the quotient group G/H, with the quotient ordering.

Since knowledge about K, H and how the states of H extend determines the ordering on G completely (by III.1), it is crucial to show that the restriction to H of a pure state of G, is either zero or pure.

THEOREM III.2 [**11**, THEOREM 3.1]. *Let (G, u) be any dimension group with order-unit, and suppose f is a state of (G, u). Then f is extremal if and only if for all $\varepsilon > 0$, and for all x, y in G^+, there exists z in G such that $0 \leqslant z \leqslant x, y$ and $f(z) + \varepsilon > \min\{f(x), f(y)\}$.*

COROLLARY III.3. *Let G be a dimension group with order-unit, and let H be an order-ideal which possesses an order-unit of its own. If f is a pure state of G, and $f(H) \neq \{0\}$, then after renormalization, the restriction of f is a pure state of H.*

PROOF. Let v be an order-unit for H. Then $f(v) \neq 0$, so $t = f/f(v)$ is a (normalized) state for (H, v). Now III.2 is a two-sided criterion, and it easily follows that t is pure. $\square$

From now on, H will be assumed a simple dimension group, with a fixed order-unit v. We shall also assume H is not cyclic, since the extension problem for $H = \mathbf{Z}$ has already been solved (II.6).

Let $S(H, v)$ denote the state space of H, computed with respect to the order-unit v. Let F be a closed face of $S(H, v)$; we wish to extend the states of F to states on a larger group G, so that G/H is a given dimension group, and $H \to G \to G/H$ is an extension of dimension groups, with the only extendible states of H arising from F. We will obtain a general construction (III.5), and it will be shown that in many cases this is the most general construction possible.

LEMMA III.4. *Let $L \to M \xrightarrow{\pi} N$ be a short exact sequence of torsion-free abelian groups, and let A be a torsion-free divisible group. Let $\alpha\colon L \to A$ be a group homomorphism. Then, the collection of group homomorphisms $\gamma\colon M \to A$ such that*

$$\begin{array}{ccc} L & \longrightarrow & M \\ & \searrow^{\alpha} & \downarrow \gamma \\ & & A \end{array}$$

commutes is nonempty, and has an unnatural group structure isomorphic to that of $[N, A]$.

PROOF. Since A is divisible, and is thus injective (as a $\mathbf{Z}$-module), at least one such extension γ exists. Fix a specific extension γ_0. We can impose a group structure on the extensions γ, by means of

$$\gamma_1 \boxplus \gamma_2 = \gamma_1 + \gamma_2 - \gamma_0;$$

then γ_0 is the zero element.

Then observing that $(\gamma - \gamma_0)(H) = \{0\}$, the mapping

$$\gamma \mapsto \overline{(\gamma - \gamma_0)}$$

(where $\overline{(\gamma - \gamma_0)} \cdot \pi = (\gamma - \gamma_0)$) is readily verified to be a group embedding from the group of extension maps to $[N, A]$. We check the map is onto.

Find a subgroup V of M such that $L \cap V = \{0\}$, and $M/(L \oplus V)$ is torsion. If λ: $N \to A$ is an element of $[N, A]$, define γ_1: $L + V \to A$ via

$$\gamma_1(s + v) = \gamma_0(s) + \lambda\pi(v) \qquad (s \text{ in } L).$$

Then γ_1 extends α and as A is divisible, γ_1 can be extended to γ: $M \to A$.

Now $\overline{(\gamma - \gamma_0)}$ agrees with λ on $\pi(V)$. Observe that

$$\frac{M}{L+V} \simeq \frac{(M/L)}{(L+V)/L} = \frac{N}{\pi(V)},$$

so $N/\pi(V)$ is a torsion group. Since $\overline{(\gamma - \gamma_0)} - \lambda$ takes values in a torsion-free group (A), it must be zero, so $\gamma \mapsto \lambda$, and thus the map is onto. □

If L is a compact convex set, denote by Aff(L), the collection of continuous convex-linear ("affine") real-valued functions on L. Equipped with the pointwise ordering, Aff(L) is an archimedean partially ordered group. If L is a Choquet simplex, and we impose the *strict* ordering on Aff(L), namely g in Aff(L) is positive if either g is zero or g is bounded below by a positive scalar, then Aff(L) becomes a simple dimension group [**5**, Lemmas 3.1 and 3.2]. We write $g \gg g'$ if $g - g'$ is not zero but is strictly positive on L.

With (H, v) a noncyclic simple dimension group having v as order-unit, $S(H, v)$ is a Choquet simplex. Let F be a closed face of $S(H, v)$. Define the natural mappings

$\Theta \equiv \Theta_H$: $H \to \operatorname{Aff} S(H, v)$,

$h \mapsto \hat{h}$, $\hat{h}(f) = f(h)$,

$\theta \equiv \theta_F$: $H \to \operatorname{Aff}(F)$, the restriction of Θ to F.

Now select any abelian group extension of H by K (a representative of an element of $\operatorname{Ext}_{\mathbf{Z}}(K, H)$), $H \overset{\iota}{\to} G \overset{\pi}{\to} K$. Applying III.4 to the diagram

$$\begin{array}{ccccc} H & \overset{\iota}{\to} & G & \overset{\pi}{\to} & K \\ & \theta\searrow & & & \\ & & \operatorname{Aff}(F) & & \end{array}$$

(of course, Aff(F) is a real vector space, so is a divisible abelian group), we obtain a family of maps γ: $G \to \operatorname{Aff}(F)$, extending γ, indexed unnaturally by $[K, \operatorname{Aff}(F)]$. Notice that all group homomorphisms, not just order-preserving ones, from K to Aff(F) are admissible.

Define a candidate partial ordering for each choice of γ, by declaring

$$(1) \qquad G^+ = \iota(H^+) \cup \{g \in G \mid \gamma(g) \gg 0 \text{ and } \pi(g) \in K^+ - \{0\}\}.$$

It will be shown that G, with this ordering, is a dimension group, $\iota(H)$ is the only order-ideal, $H \to G \to K$ is an extension of dimension groups (so the given ordering on K agrees with the quotient ordering), and the states of H that extend to G are exactly the points of F. In subsequent sections, we shall show that (for example) if F is finite dimensional, all dimension group extensions of H by K with F the face of liftable states arise in this manner, and then determine which of the extensions are equivalent.

We shall refer to G with its potential ordering as described in (1), as the extension group determined by the data

$$\{H, F; K, \gamma; G\}.$$

Sometimes the G will be suppressed. We fix v as the order-unit of H with respect to which its states are computed. To avoid excess notation, we shall usually refer to H as a subgroup of G, and K as the quotient group.

With the face F and the group extension G fixed, the possible choices for γ form a real vector space

$$\operatorname{Hom}_{\mathbf{Z}}(K, \operatorname{Aff}(F)).$$

Each choice for γ will be shown to yield a dimension group extension, so our construction yields such extensions in bijection with the disjoint union of groups

$$\operatorname{Ext}_{\mathbf{Z}}(K, H) \times \dot{\bigcup_{\{F \text{ closed face of } S(H, v)\}}} \operatorname{Hom}_{\mathbf{Z}}(K, \operatorname{Aff}(F)).$$

For those confused by the construction, let us see what happens if we pick G to be the trivial group extension $G = H \oplus K$ (as groups), F finite dimensional, with the maps $H \to G \to K$ the obvious ones. If we select $\partial_e F = \{f_i\}_{i=1}^s$ in $\partial_e S(H)$, then define γ_0 to be the projection onto H followed by $(f_1, f_2, \ldots, f_s)$: $H \to \mathbf{R}^s$. Selecting the same number of x's, x_i: $K \to \mathbf{R}$, γ is defined in the obvious way,

$$\gamma(h, k) = (f_1(h) + x_1(k), f_2(h) + x_2(k), \ldots, f_s(h) + x_s(k));$$

and the positive cone is relatively easy to visualize.

THEOREM III.5. *Let H, K be simple dimension groups, with H not cyclic. Let F be a (possibly empty) closed face of $S(H, v)$, and suppose that G is an abelian group extension of H by K, and let γ be a group homomorphism from G to* $\operatorname{Aff}(F)$.

Then the potential ordering on G determined by the data (viz. (1)) $\{H, F; K, \gamma; G\}$ *is a partial ordering making G into a dimension group, and causing $H \to G \xrightarrow{\pi} K$ to be an extension of dimension groups. An order-unit u may be selected from $\pi^{-1}(w)$.*

The states of H that extend to G are exactly the points of F, and their image (under the map $f \mapsto \delta_f \gamma / \delta_f \gamma(u)$, where δ_f is evaluation at f) is a closed face, which is the complement to $H^\perp$. In particular, the extreme points of $S(G, u)$ are precisely

(i) *the preimages of the extremal states of (K, w) and*

(ii) *the renormalized extensions of extremal points of F.*

The proof of this theorem will occupy the rest of this section.

Until further notice, we assume K is not cyclic.

LEMMA III.6. *Let H be a noncyclic simple dimension group, and let F be a closed face of the state space of H. Then the mapping*

$$\theta: H \to \operatorname{Aff}(F),$$
$$a \mapsto \tilde{a}, \qquad \tilde{a}(f) = f(a),$$

has dense image. Furthermore, if Γ belongs to $\operatorname{Aff}(F)$, *and* $\Gamma \gg 0$, *then for all* $\varepsilon > 0$, *there exists h in H^+ so that $\Gamma \gg \theta(h)$ and $\|\Gamma - \theta(h)\| < \varepsilon$.*

PROOF. The second statement would imply the first, and we restrict our attention to that one. We may assume ε is sufficiently small so that $\Gamma \gg 2\varepsilon$. The mapping (via restriction) $\mathrm{Aff}(S(H, v)) \to \mathrm{Aff}(F)$ is just the quotient mapping with kernel $F^{\perp}$ [**1**, II.5.19], so there exists h_1 in $\mathrm{Aff}(S(H, v))$ with $h_1 \geqslant 2\varepsilon$, and $h_1/F = \Gamma$. Consider the open neighbourhood in $\mathrm{Aff}(S(H, v))$,

$$U = \left\{h_0 \in \mathrm{Aff}(S(H, v)) | h_1 - \varepsilon/2 \gg h_0 \gg h_1 - 3\varepsilon/2\right\}.$$

From the density of $\Theta(H)$ [**11**, Corollary 4.10], there exists h in H so that $\Theta(h)$ lies in U, and so $h_1 \gg \Theta(h) \gg \varepsilon/2$; in particular, by II.2, h lies in H^+. Thus $\Gamma \gg \Theta(h)/F$, but since $\Theta(h) \gg h_1 - 3\varepsilon/2$, $\|\theta(h) - \Gamma\| < 3\varepsilon/2$. □

Now let our group G be endowed with the potential ordering described in (1). It is clear that $G^+ + G^+ \subseteq G^+$, and $G^+ \cap (-G^+) = \{0\}$, so G^+ is a cone for a translation invariant partial ordering. Since H, and K are unperforated, it follows easily that G is as well in this ordering.

Next we show that the combined mapping

$$(\gamma, \Theta_K \pi) \colon G \to \mathrm{Aff}(F) \oplus \mathrm{Aff}(S(K, w))$$

has dense image (in the max-sup norm). From [**11**, Corollary 4.10], $\Theta_K(K)$ is dense in $\mathrm{Aff}(S(K, w))$ so $P = \Theta_K \pi$ maps G densely to $\mathrm{Aff}(S(K, w))$, and $H \subset \ker P$. Since $\gamma/H = \theta$, $\gamma(H)$ is dense in $\mathrm{Aff}(F)$, and it easily follows that $(\gamma, P)(G)$ is dense.

The following is a restatement of (1).

$$(2) \qquad G^+ = H^+ \cup \{g \in G \,|\, (\gamma, P)(g) \gg 0 \text{ in } \mathrm{Aff}(F) \oplus \mathrm{Aff}(S(K, w))\}.$$

From the density of $(\gamma, P)(G)$, we may find u_0 in G with $(\gamma, P)(u_0) \gg 1$ (the constant function), so u_0 is an order-unit by (2). Thus G is directed, and in fact every element of $G^+ - H^+$ is an order-unit. It follows immediately that H is an order-ideal in G.

Now we show that G satisfies the interpolation property. Given a, c, y in G with a, $c \geqslant y, 0$, we must find e in G so that $a, c \geqslant e \geqslant y, 0$. There are a number of cases.

I. *Either a or c belongs to H*. We may assume a belongs to H, and that at least one of c, y does not (if all belong to H, we are reduced to interpolation within H). We may also assume $a \neq 0$, $a \neq y$, $c \neq 0$, and $c \neq y$.

I(i). $c \notin H$, $y \in H$. We have $(\gamma(c), P(c)) \gg 0$, $\gamma(c) \gg \gamma(y)$; and $\Theta(a) \gg 0$, $\Theta(y)$. Since $\Theta(H)$ is dense in $\mathrm{Aff}(S(H, v))$, and $\gamma(c)$ is strictly positive we may find h in H^+ with $2\varepsilon \gg \gamma(c) - \gamma(h) \gg \varepsilon$ for some ε greater than zero, such that $\gamma(c) \gg 2\varepsilon$, $\gamma(y) + 2\varepsilon$ (III.6).

Now allow $\mathrm{Aff}(S(H, v))$, $\mathrm{Aff}(F)$ to be equipped with their usual pointwise ordering, and observe that the mapping $\mathrm{Aff}(S(H, v)) \to \mathrm{Aff}(F)$ is just the quotient mapping for a suitable order-ideal, L. Then $\Theta(h) + L \gg 0$, $\Theta(y) + L$. We may thus find h' in $\mathrm{Aff}(S(H, v))$ with $h' \gg 0$, $\Theta(y)$ and $h' - \Theta(h) \in L$. Then we may find h'' in H^+ so that $\Theta(h'') \gg h'$ and $\|h' - \Theta(h'')\| < \varepsilon/2$. As $h'/F = \Theta(h)/F$, we have

$$\begin{aligned}\|\gamma(h'') - \gamma(h)\| &= \|\theta(h'' - h)\|_F \leqslant \|h' - \Theta(h)\|_F + \|h' - \Theta(h'')\|_F \\ &\leqslant 0 + \varepsilon/2;\end{aligned}$$

as h'' belongs to H^+, $\Theta(h'') \gg 0$. Also, $\Theta(h'') \gg \Theta(y)$, and thus

$$\Theta(h''), \Theta(a) \gg 0, \Theta(y).$$

From the density of $\Theta(h)$, we may find e in H so that

$$\Theta(h''), \Theta(a) \gg \Theta(e) \gg 0, \Theta(y).$$

Hence, h'', $a \geqslant e \geqslant 0$, y in H. Now $P(c) \gg 0 = P(e)$ (as e lies in H), and $\gamma(c) \gg \gamma(h) + \varepsilon \gg \gamma(h'') + \varepsilon/2 = \theta(h'') + \varepsilon/2 \gg \theta(e) + \varepsilon/2 = \gamma(e) + \varepsilon$, so $(\gamma, P)(c) \gg (\gamma, P)(e)$. Thus $c \geqslant e$.

I(ii). $c \in H$, $y \notin H$. Reduce to case I(i), by observing that $a - y, a \geqslant 0, a - c$, and a, $a - c$ both belong to H, but $a - y$ does not. If $a - y, a \gg b \gg 0, a - c$, then $a, c \gg a - b \gg 0, y$.

I(iii). $c \notin H$, $y \notin H$. Since $P(a) = 0$, and $P(a) \gg P(y)$, we have $P(y) \ll 0$. Thus $c - y$ does not belong to H (else $P(y) = P(c) \gg 0$), so $\gamma(c) \gg \gamma(y), 0$. We also have $\gamma(a) = \theta(a) \gg 0, \gamma(y)$. The density of $\theta(H)$ yields an element h_0 in H so that

$$\gamma(c), \gamma(a) \gg \theta(h_0) = \gamma(h_0) \gg 0, \gamma(y).$$

As in the proof of I(i), there exists h' in $\mathrm{Aff}(S(H, v))$ so that $h'/F = \theta(h_0)$ and $\Theta(a) \gg h' \gg 0$. We may find $\varepsilon > 0$ so that $\Theta(a) - \varepsilon \gg h' \gg \varepsilon$ and $\gamma(c) - \varepsilon, \gamma(a) - \varepsilon \gg \gamma(h_0) \gg \varepsilon, \gamma(y) + \varepsilon$, and then pick e in H so that $\|\Theta(e) - h'\| < \varepsilon/2$. Then $\|\theta(e) - h'/F\| < \varepsilon/2$, and so

$$\gamma(c), \gamma(a) \gg \gamma(e) \gg 0, \gamma(y).$$

As $P(c) \gg 0 = P(e) \gg P(y)$, we have $(\gamma, P)(c) \gg (\gamma, P)(e) \gg (\gamma, P)(y), 0$; so $c \geqslant e \geqslant y$; also $\Theta(e) \gg \varepsilon/2$ so $e \geqslant 0$. On the other hand, $\Theta(a) - \Theta(e) \gg h' + \varepsilon - \Theta(e) \gg \varepsilon/2$, whence $a \geqslant e$.

II. *Neither a nor c lies in H.* If either $a - y$ or $c - y$ belongs to H, we reduce to case I by subtracting y from all the terms. So we may assume neither occurs, and thus

$$(\gamma, P)(a), (\gamma, P)(c) \gg (\gamma, P)(y), 0.$$

From the density of $(\gamma, P)(G)$ and interpolation in $\mathrm{Aff}(F) \oplus \mathrm{Aff}(S(H, v))$ there exists e in G so that

$$(\gamma, P)(a), (\gamma, P)(c) \gg (\gamma, P)(e) \gg (\gamma, P)(y), 0$$

whence $a, c \geqslant e \geqslant y, 0$.

Since any question of interpolating A, $C \geqslant Y$, Z can be reduced to interpolation of $A - Z$, $C - Z \geqslant Y - Z, 0$, cases I and II constitute a complete proof that interpolation holds. By [**5**, Theorem 2.2], G is a dimension group.

Since the positive elements of G not in H are order-units, any proper order-ideal of G must be contained in H; since H is simple, H is thus the only order-ideal of G.

Now we are in a position to show that the quotient ordering on $K = G/H$ agrees with its original ordering.

Lift a fixed order-unit w of K (with respect to which the states of K, and the natural map Θ_K: $K \to \mathrm{Aff}(S(K, w))$ have been computed) to an element x of G. From the density of $\theta(H)$ in $\mathrm{Aff}(F)$, we may find y in H so that for $u = x + y$, $\gamma(u) \gg 0$, and so u is an order-unit for G. Of course, $\pi(u) = w$.

If g lies in $G^+ - H$, certainly $P(g) \gg 0$, so $\Theta_K(g + H) > 0$, whence $g + H \in K^+$, by II.2. On the other hand, if g is an element of G, and $g + H$ lies in $K^+ - \{0\}$, certainly $P(g) \gg 0$. Since $\gamma(H)$ is dense in $\gamma(G)$, we may find z in H so that $\gamma(g + z) \gg 0$, whence $g + z \geqslant 0$ in G, and thus g belongs to $G^+ + H$. Therefore the two orderings on K are equal.

It follows that $H \to G \to K$ is an extension of dimension groups, and that the pure states of (G, u) that annihilate H are precisely the pre-images of the pure states of (K, w). Now we can determine among other things, the rest of the pure states. To begin with we show that only members of F extend to states of G.

Let t be a state of (G, u) with $t(H) \neq \{0\}$. Let $\bar{t}$ be the restriction of t to H; we show that $f = \bar{t}/t(v)$ belongs to F.

Certainly f is a state of (H, v). If f does not lie in F, consider

$$F^\perp = \{m \in \operatorname{Aff}(S(H, v)) | m/F = 0\}.$$

By [**1**, II.622], $F = \{j \in S(H, v) \,|\, m(j) = 0 \text{ for all } m \text{ in } F^\perp\}$ (that is, $F = F^{\perp\perp}$). Since f does not lie in F, there exists m in $F^\perp$ with $m(f) > 0$. We may thus find m in $\operatorname{Aff}(S(H, v))$ so that $m/F = 0$, but $m(f) > 3$. Subtracting $3/2$ from m, we find m' so that $m'(f) > 1$, but $m'/F \ll -1$. From the density of $\Theta(H)$ in $\operatorname{Aff}(S(H, v))$, we can find h in H so that $\Theta(h)$ approximates m', and so $f(h) > 1$, but $\gamma(h) = \theta(h) \ll -1$. Select a positive integer n such that $t(v) > 1/n$. From the density of $(\gamma, P)(G)$, there exists a in $G^+ - H$ with $na \leqslant u$. Set $z = a - h$. Then $P(z) = P(a) \gg 0$, and $\gamma(z) = \gamma(a) - \gamma(h) \gg 0$, so z is positive in G. However, $t(z) = t(a) - t(h) \leqslant 1/n - t(v)f(h) < 0$, a contradiction, since t is supposed to be a state of G.

On the other hand, any point in F extends to a state of G (after renormalization): Simply observe that $\gamma(G)$ is a subgroup of $\operatorname{Aff}(F)$, so the function $\delta_f\gamma \colon G \to \mathbf{R}$ (δ_f is the evaluation at f) makes sense, is positive, and since γ extends θ, $\delta_f\gamma/H = f$.

We shall now check that the collection of normalized states of (G, u), $F_1 = \{\delta_f\gamma/\delta_f\gamma(u) \,|\, f \in F\}$ is a closed face and is also the complementary face to $H^\perp$ in $S(G, u)$.

Given an unperforated group with order-unit, (L, u), the mapping

$$\Theta_L \colon (L, u) \to (\operatorname{Aff}(S(L, u)), 1),$$
$$b \mapsto \hat{b}, \qquad \hat{b}(p) = p(b), \qquad p \in S(L, u),$$

is referred to as the canonical map, and induces a (pseudo-) norm on L,

$$\|b\| = \sup\{|\hat{b}(p)|, p \in S(L, u)\}.$$

The image of L under Θ is denoted $\hat{L}$.

LEMMA III.7. *Let (L, u) be a dimension group with order-unit, and suppose H is an order-ideal.*

(a) *If p is any state on (L, u), p extends uniquely to a state on* $\operatorname{Aff}(S(L, u))$, *and the extension is continuous and real linear.*

(b) *If q is a (pseudo-metric) bounded continuous additive mapping $q \colon L \to \mathbf{R}$, q is a difference of (unnormalized) states on L.*

(c) *If q is as in* (b), *and additionally $q(H) = \{0\}$, then q is a difference of unnormalized states each annihilating H.*

PROOF. (a) This is well known.

(b) Since q is continuous, q must annihilate the closure of $\{0\}$ in L, namely ker Θ. Hence q induces a continuous mapping on the normed subgroup of Aff($S(L, u)$), $\hat{L}$. Now the norm on Aff($S(L, u)$) agrees with that on L by (a); since q obviously extends to $\hat{L} \cdot \mathbf{Q}$, the rational vector space generated by $\hat{L}$ inside Aff($S(L, u)$), it does so in a continuous manner, and thus extends to a continuous group homomorphism from the closure of $\hat{L} \cdot \mathbf{Q}$ to $\mathbf{R}$. Since the rationals are dense in the reals, $\hat{L} \cdot \mathbf{Q}$ is dense in Aff($S(L, u)$) [**1**, §1], so q extends to a continuous, hence linear, homomorphism q_0: Aff($S(L, u)$) $\to \mathbf{R}$—that is, q_0 is an element of the dual space of Aff($S(L, u)$); but the dual space is lattice-ordered since $S(L, u)$ is a Choquet simplex [**1**, p. 84]; and (b) follows immediately.

(c) Since (as in (b)), the dual of Aff($S(L, u)$) is a lattice, we may write the q_0 obtained there as the difference of two unnormalized states $a - b$, with $a \wedge b = 0$. Certainly, if for some h in $H^+ - \{0\}$, $a(h) \neq 0$, then $a(h) = b(h) \neq 0$, and similarly a and b agree on the closed interval $[0, h]$. But the ordering on the dual space, as described, for example, in [**11**, Proposition 2.6] precludes this, so that $a(H^+) = b(H^+) = 0$; as H is directed, $a(H) = b(H) = \{0\}$. □

Now a sequence of short arguments (a̲) through (e̲), completes the proof of III.5.

(a̲) $H^\perp \cap \text{Face}(\delta_f\gamma/\delta_f\gamma(u)) = \varnothing$ for all f in F. Pick x in the intersection. Then $x(H) = \{0\}$, but there is a positive real number r so that $x \leqslant r\delta_f\gamma$ [**11**, Proposition 2.5]. Given $\varepsilon > 0$, find h in H^+ so that $\gamma(u) \gg \theta(h)$ but $\|\gamma(u) - \theta(h)\| < \varepsilon\delta_f\gamma(u)$ (III.6). As $P(u) \gg 0$, $u \geqslant h$, and $h \geqslant 0$ by construction. Thus $(\delta_f\gamma/\delta_f\gamma(u))(u - h) < \varepsilon$ but $x(u - h) = x(u) = 1$. Thus $1 \leqslant r'\varepsilon$ $(r' = \delta_f(u)r)$ for all $\varepsilon > 0$, a contradiction.

(b̲) $F_1 \subset \bigcup\{$faces of $S(G, u)$ disjoint from $H^\perp\}$. This is an immediate consequence of (a̲).

Let F_2 denote the complementary face of $H^\perp$; F_2 is the union of the faces disjoint from $H^\perp$ [**1**, II.6.22], so $F_1 \subset F_2$.

(c̲) $F_1 = F_2$. Select t in F_2. Then t does not belong to $H^\perp$, so $t/H = df$ for some f in F, some positive real number d. Setting $r = d\delta_f\gamma(u)$, we have

$$\left(t - r\delta_f\gamma/\delta_f\gamma(u)\right)(H) = \{0\},$$

so by III.7, there exist a, b, unnormalized states of G, with $a(H) = b(H) = \{0\}$ and $t - r\delta_f\gamma/\delta_f\gamma(u) = a - b$. Hence $t + b = r\delta_f\gamma/\delta_f\gamma(u) + a$. Since F_2 is the complementary face to $H^\perp$, the decomposition of $t + b$ is unique. Thus $t = r\delta_f\gamma/\delta_f\gamma(u)$; applying u, $r = 1$, so t belongs to F_1.

(d̲) F_2 is closed. Pick t in the closure of F_2. Decompose t with respect to the facial decomposition of $S(G, u)$ obtained from $H^\perp$ and F_2,

$$t = \frac{d\delta_f\gamma}{\delta_f\gamma(u)} + (1 - d)a, \qquad 0 \leqslant d \leqslant 1, a \in H^\perp, \text{ using (c̲)}.$$

By III.6, given $\varepsilon > 0$, there exists h in H^+ so that $\gamma(u) \gg \theta(h)$ (and thus $u \geqslant h$, as $P(h) = 0$) and if $k = \text{Inf}\{\delta_{f'}\gamma(u) \mid f' \in F\}$,

$$\|\gamma(u) - \gamma(h)\| < \varepsilon \cdot k \qquad (k > 0 \text{ since } \gamma(u) \gg 0).$$

Then

$$t(h) = d\delta_f\gamma(h)/\delta_f\gamma(u) = df(h)/\delta_f\gamma(u)$$
$$\leqslant \frac{d(\delta_f\gamma(u) + \varepsilon \cdot k)}{\delta_f\gamma(u)} \leqslant d(1+\varepsilon).$$

But for s in F_1, $\delta_s\gamma(h)/\delta_s\gamma(u) = s(h)/\delta_s\gamma(u) \geqslant 1 - \varepsilon$. Since t lies in the closure of F_2, by (c), $t(h) \geqslant 1 - \varepsilon$, and thus $1 - \varepsilon \leqslant d(1+\varepsilon)$ for all $\varepsilon > 0$. Hence $d \geqslant 1$, so $d = 1$, whence $t = \delta_f\gamma/\delta_f\gamma(u)$.

(e) The mapping $F_2 \to F$, $\delta_f\gamma/\delta_f\gamma(u) \mapsto f$ is a homeomorphism inducing a bijection $\partial_e F_2 \to \partial_e F$. This mapping is just the restriction map renormalized. Since F_2 is compact, and the mapping is one-to-one (c), onto (only states in F extend), and continuous, the mapping is a homeomorphism. By III.3, $\partial_e F_2$ is mapped to $\partial_e F$ (F_2, F are both faces). On the other hand, if t belongs to F_2 and maps to an extreme point of F, then an easy application of III.7 yields that t is itself extremal, so belongs to $\partial_e F_2$.

The equivalence relation obtained between the two faces F_2 and F in (e) is the best possible, in the sense that it is equivalent to $\mathrm{Aff}(F_2)$ being order-isomorphic to $\mathrm{Aff}(F)$.

Now suppose $K = \mathbf{Z}$. Then $(\gamma, P)(G)$ is dense in $\mathrm{Aff}(F) \oplus \mathbf{Z}$; this density allows the same processes to be carried out (and note that in this case, $\mathrm{Ext}_{\mathbf{Z}}(K, H) = \{0\}$, so there is no interference from the group extensions).

This completes the proof of Theorem III.5.

IV. The converse. We wish to show that Theorem III.5 has a valid converse; when F is, for example, finite dimensional, the construction of III.5 yields all equivalence classes of extensions. This essentially amounts to showing that the states of an order-ideal in a dimension group that extend form a face of the states of the order-ideal, when the latter has an order-unit (IV.2)

To establish this, we require results on extensions of states in the style of [**10**]. Recall therefrom that given a partially ordered group (G, u) and a subgroup H of G *that contains* u, a formula was established for extending states of (H, u) to states of (G, u), where extend is used in the strict sense. Our present problem revolves around u not belonging to H (with a corresponding relaxation of the notion of "extend"—we do not require that the extended state be automatically normalized). However, if we can extend the state to $(H + \mathbf{Z} \cdot u, u)$, then we can extend all the way to (G, u) by [**10**]. The following extracts the idea of the argument in [**10**, Lemma 3.1].

LEMMA IV.1. *Let (M, u) be a partially ordered abelian group with order-unit u, and let L be an order-ideal of M. Let f: $L \to \mathbf{R}$ be an order-preserving group homomorphism. Then f may be extended to a state of M (after possible renormalization) if and only if the member of $\mathbf{R}^+ \cup \infty$ defined by*

$$p(f) = \sup\left\{ \frac{f(h)}{n} \,\middle|\, h \in L, n \in \mathbf{N}, h \leqslant nu \right\}$$

is finite.

PROOF. If $p(f)$ is finite, define g: $L + \mathbf{Z} \cdot u \to \mathbf{R}$ via $g(u) = p(f)$ and $g/L = f$; this extends to a group homomorphism, since $L \cap \mathbf{Z} \cdot u = \{0\}$ (else L would contain an order-unit of M). The method of [**10**, Lemma 3.1] shows that g is an unnormalized state of $L + \mathbf{Z} \cdot u$, so $g/g(u)$ extends to a normalized state of (M, u) by [**10**, Theorem 3.2]. Hence, after renormalization, f may be extended.

On the other hand, if f extends to a state g of (M, u), then $g/L = df$ for d a positive real number. Then $h \leqslant nu$ implies $n \geqslant g(h) = df(h)$, so $f(h) \leqslant n/d$, whence $p(f) \leqslant 1/d$. □

COROLLARY IV.2. *Let (M, u) be a dimension group with order-unit. Let L be an order-ideal of M, and suppose v is an order-unit for L. Then the collection of (normalized) states of (L, v) which extend to M is a face of $S(L, v)$.*

PROOF. By IV.1, $F = \{f \in S(L, v) \mid p(f) < \infty\}$ is precisely the set of states that extend. Clearly convex combinations of extendible states are extendible, so F is convex. By [**11**, 2.5], to show F is a face, we need only establish that

$$f' \in S(L, v), \quad f \in F, \quad d \in \mathbf{R}^+, \quad \text{and} \quad f' \leqslant df$$

imply $f' \in F$.

But $f' \leqslant df$ implies that whenever $h \leqslant nu$, $f'(h) \leqslant df(h) \leqslant ndp(f)$ so that $p(f') \leqslant dp(f)$; thus $f' \in F$. □

COROLLARY IV.3. *Let $H \to G \to K$ be an extension of dimension groups with both H and K simple, but H not cyclic. Then G has an order-unit, call it u, and if the complementary face to $H^\perp$ in $S(G, u)$ is closed, the extension arises as described in* III.5, *that is, it is equivalent to the extension given by the data $\{H, F; K, \gamma; G\}$ for some F, γ.*

In particular, if only finitely many pure states of H extend, the extension is as described above.

REMARK. Thus IV.3 and III.5 are mutually converse.

PROOF. Select an order-unit v for H. Let w be an order-unit for K, and lift it back to a positive u' in G. Set $u = u' + v$; one checks routinely that u is an order-unit (this process will work for any extension with both ends possessing an order-unit).

Let F_1 be the complementary face to $H^\perp$ in $S(G, u)$. Considering v as an element of G, its image acting on $S(G, u)$, $\hat{v}$, cannot vanish on F_1 (for if $f(v) = 0$, then $f(H) = 0$, whence f lies in $H^\perp$). Since F_1 is closed, $\hat{v}$ is bounded below, and the map

$$\begin{aligned} \varphi: \quad F_1 &\to S(H, v), \\ f &\mapsto \frac{f}{f(v)}, \end{aligned}$$

is one-to-one and continuous. Since F_1 is compact, the image is also; on the other hand, the image is obviously the face of extendible states of (H, v) (since any state of (G, u), f, decomposes as $\alpha f_1 + (1 - \alpha)f_2$, with f_1 in F_1, and f_2 in $H^\perp$). So the face of extendible states of (H, v), F, is closed, and φ is a homeomorphism from F_1 to F.

Define a group homomorphism $\gamma\colon G \to \mathrm{Aff}(F)$ as follows. As F_1 is compact, $\hat{v}$ is bounded below on it (since $f(v) \neq 0$ for all f in F, and so the same holds for all $f(\hat{v})$); thus $v_1 = \hat{v}/F_1$ is an order-unit for $\mathrm{Aff}(F_1)$. There is a natural affine homeomorphism between the state space of $(\mathrm{Aff}(F_1), v_1)$ and the face of $S(H, v)$, F. This yields an order-isomorphism $\rho\colon \mathrm{Aff}(F) \to \mathrm{Aff}(F_1)$ (under which the image of the constant function 1 need not be constant). Define $\gamma\colon G \to \mathrm{Aff}(F)$ via

$$G \to \mathrm{Aff}(S(G, u)) \to \mathrm{Aff}(F_1) \xrightarrow{\rho} \mathrm{Aff}(F).$$

Observe that $\hat{g}/F_1 \gg 0$ if and only if $\gamma(g) \gg 0$.

In view of III.5, we need only check that (1) below holds.

$$G^+ = H^+ \cup \{g \in G \mid \widehat{\pi(g)} \gg 0, \gamma(g) \gg 0\}. \tag{1}$$

If g belongs to $G^+ - H^+$, then $\pi(g)$ lies in $K^+ - \{0\}$, so $\widehat{\pi(g)} \gg 0$; since H is the only proper order ideal of G) $\hat{g}/F_1 \gg 0$, and thus $\gamma(g) \gg 0$. On the other hand, the reverse inequality is straightforward, using II.2 and that $S(G, u)$ is the convex hull of $H^\perp$ and F_1.

Now suppose that F has only finitely many pure states, and let F_1 be as above. From II.3, pure states of F_1 map to pure states of F; but $F_1 \to F$ is one-to-one, so F has only finitely many pure states. Being a complementary face, F_1 is dense in the closed convex hull of its pure states. Thus F_1 is finite dimensional, so is closed, and the previous paragraph applies. □

Professor E. G. Effros has given an example of a simplex K with the property that $G = \mathrm{Aff}(K)$ (usual ordering) has a closed order-ideal H with an order-unit of its own, without $H^\perp$ having closed complementary face. One can now construct a simple by simple dimension group extension which does not arise from the construction of §III.

Let G_1 be the collection of bounded sequences of real numbers $(a_1, a_2, \ldots)$ such that $\lim a_n = (a_1 + a_2)/2$; with the pointwise ordering G_1 becomes an order-unit space, archimedean, and satisfying the Riesz interpolation property. Let H_1 be $\ker f_1$ (f_i is the valuation at the ith component); then H_1 is a norm closed order-ideal and $(0, 1, 1, 1/2, 1/2, \ldots)$ is an order-unit for H_1. Define the simple group H to be H_1 with the strict ordering, and let G be G_1 with the ordering given by

$$G^+ = H^+ \cup \{(a_i) \mid a_i > 0 \text{ for all } i\}.$$

Then G is a dimension group, H is an order-ideal, and G/H is order-isomorphic to the reals, so G is simple by simple. However the complementary face to $H^\perp$ has as its extremal points $\{f_2, f_3, \ldots\} = S$ so the face cannot be closed, as $(1/2)(f_2 + f_3)$ is a limit point of S. A countable example may also be constructed, by taking suitable countable dense subgroups of H and G.

V. Equivalence. Given the data $\{H, F; K, \gamma; G\}$ and $\{H, F'; K, \gamma'; G'\}$, we decide when the corresponding dimension group extensions are equivalent, and how these equivalences may be implemented. Certainly for equivalence to hold, we must have G equivalent to G' as group extensions of H by K. Thus we may assume $G = G'$ (as groups). Now the obvious theorem holds.

THEOREM V.1. *Let H, K be simple dimension groups, and suppose H is not cyclic. Fix an order-unit v of H with respect to which states and faces are computed. Then the extensions of H by K given by the data*

$$\{H, F; K, \gamma; G\} \qquad \text{and} \qquad \{H, F'; K, \gamma'; G'\}$$

are equivalent if and only if $F = F'$, and there exists a group homomorphism X: $K \to H$ so that

$$\overline{\gamma' - \gamma} = \theta X.$$

In particular, the equivalence classes of dimension group extensions of H by K are in bijective correspondence with a disjoint union of groups,

$$\operatorname{Ext}_{\mathbf{Z}}(K, H) \times \dot{\bigcup_{\{F \text{ closed faces of } S(H, v)\}}} [K, \operatorname{Aff}(F)]/\theta_F[K, H].$$

REMARK. The quotient group measures how far $\overline{\gamma' - \gamma}$: $K \to \operatorname{Aff}(F)$ is from factoring through θ.

PROOF. (a) Suppose the extensions given by the data are equivalent. Then there is an order-automorphism φ so that

$$\begin{array}{ccccccc} H & \xrightarrow{\alpha} & G & \xrightarrow{\pi} & K \text{ data:} & & F, \gamma; \\ \| & & \downarrow \varphi & & \| & & \\ H & \xrightarrow{\alpha} & G & \xrightarrow{\pi} & K & & F', \gamma'; \end{array}$$

commutes. As $\varphi^{-1}\alpha = \alpha$, precisely the same states of H must lift in the first extension as in the second (since φ is order-preserving, and sends states to unnormalized states) so it follows from the last paragraph of III.5, that $F = F'$.

Consider $\varphi^{-1} - \mathrm{id}$: $G \to G$; since $\varphi^{-1}\alpha = \alpha$, $(\varphi^{-1} - \mathrm{id})\alpha = 0$, and thus there exists ψ: $K \to G$ so that $\psi(g + \alpha(H)) = (\varphi^{-1} - \mathrm{id})(g)$; that is, $\psi\pi = \varphi^{-1} - \mathrm{id}$. Since $\pi = \pi\varphi^{-1}$, $\pi\varphi\pi = 0$; since $\alpha(H) = \ker \pi$, $\psi\pi(G) \subset \alpha(H)$. Thus $X = \alpha^{-1}\psi$ makes sense and is a group homomorphism from K to H.

As φ is an order-isomorphism, it sends (for example) pure states to (unnormalized) pure states. Since φ induces the identity on K, for all q in $(\alpha(H))^{\perp}$, $q = q\varphi^{-1}$, so $P = P'$ (cf., the discussion in §III). As $\varphi\alpha = \alpha$, $f\varphi^{-1}/\alpha(H) = f$ for all f in F. Hence, there exists a positive real number $d \equiv d_f$ so that $(\delta_f\gamma)\varphi^{-1} = d(\delta_f\gamma')$. Applying this to our fixed order-unit v, we obtain $d = 1$, and so in fact $\gamma\varphi^{-1} = \gamma'$. Hence

$$\gamma(\varphi^{-1} - \mathrm{id}) = \gamma' - \gamma.$$

But $(\varphi^{-1} - \mathrm{id})(G) \subset H$, and $\gamma/H = \theta$. Thus from the definition of X, $\theta X = \overline{(\gamma' - \gamma)}$.

(b) Now suppose such an X exists. Define the group homomorphism φ: $G \to G$ as $\varphi = \mathrm{id} - \alpha X\pi$; then φ is invertible with $\varphi^{-1} = \mathrm{id} + \alpha X\pi$. From $\Theta X = \overline{\gamma' - \gamma}$ and the definition of φ, it follows that $\gamma = \gamma'\varphi$, and it is also clear that $\pi = \pi\varphi$ and $\varphi\alpha = \alpha$. Since the orderings are determined by $\theta(H)$, γ, γ', and K, φ and φ^{-1} are both order-preserving, hence are order-automorphisms, and yield an equivalence between the two extensions.

We have shown that for each closed face F of $S(H, v)$, for each representative G of an element of $\mathrm{Ext}_{\mathbf{Z}}(K, H)$, equivalence of the two extensions is determined by whether or not $\overline{\gamma' - \gamma}: K \to \mathrm{Aff}(F)$ factors through θ. Since the γ's may be added and the equivalence relation is compatible with this operation, the equivalence classes are classified by the quotient group, as displayed. □

In particular, if F is finite dimensional, say with s pure states, then $[K, \mathbf{R}^s]$ is a real vector space of dimension equalling $s \cdot (\mathrm{rank}\ K)$, and $[K, H]$ tends to be fairly small (especially if K and H are both countable, as occurs if they emanate from AF algebras).

EXAMPLES V.2. (a) If $K = \mathbf{Q} = H$, $\mathrm{Ext}_{\mathbf{Z}}(K, H)$ consists only of the split group extension (since H is divisible), and as $\mathbf{Q}$ has just one state, only two possibilities arise for F: Either $F = \varnothing$, in which case the extension is lexicographic and has been dealt with in §II; or $F = \{\mathrm{id}\}$, so that $\theta = \mathrm{id}$, and $[K, H] = \mathbf{Q}$, and so these extensions are classified by $\mathbf{R}/\mathbf{Q}$.

(b) If $K = \mathbf{Z} + 2^{1/2}\mathbf{Z}$ (with the total ordering as a subgroup of the reals), and $H = \mathbf{Q}$, then since $\mathrm{Ext}_{\mathbf{Z}}(K, H) = \{0\}$, only the split group extension occurs. If F is not empty, from the formula, the classification is $\mathbf{R}^2/\Delta(\mathbf{Q})$, where Δ is the diagonal embedding.

(c) If $K = \mathbf{Q}$ and $H = \mathbf{Z} + 2^{1/2}\mathbf{Z}$, then $[K, H] = \{0\}$, so we obtain that the nonlexicographic extensions are classified by

$$\mathrm{Ext}_{\mathbf{Z}}(\mathbf{Q}, \mathbf{Z})^2 \times \mathbf{R}.$$

(d) If $[K, H]$ are both totally ordered and free of rank 2 subgroups of the reals, then $\mathrm{Ext}_{\mathbf{Z}}(K, H) = \{0\}$, and $[K, H]$ maps to all of H, so the extensions are classified by a torus $\mathbf{R}^2/\mathbf{Z}^2$, with $\mathbf{Z}^2$ embedded as a lattice.

VI. Extensions of dimension groups with interval. Here we use the results of the previous two sections to classify the simple by simple extensions of the title, and hence by §I, the simple by simple extensions of AF algebras.

Given a simple dimension group with interval (H, N), and a simple dimension group with order-unit (K, w), we find that only certain of the dimension group extensions $H \to G \to K$ are extensions of dimension groups with interval $(H, N) \to (G, u) \to (K, w)$. In particular, we see that once the N is chosen, only one of the possible closed faces of $S(H)$ will be allowed to extend. Thus the disjoint groups of §III are reduced to a single group.

We summarize several results implicit in the proofs of previous results.

LEMMA VI.1. *Let $(H, N) \to (G, u) \to (K, w)$ be an extension of dimension groups with interval, with H and K simple, and H not cyclic. Define a subset of $S(H)$ (with respect to some fixed order-unit, v),*

$$F = \{f \in S(H) \mid \sup\{f(h) \mid h \in N\} < \infty\}.$$

Then

(a) *the members of F are precisely the states of H that extend to G, and F is a face of $S(H)$;*

(b) *if the extension is as constructed in* §III *and* $\gamma\colon G \to \mathrm{Aff}(F)$ *is the function extending* θ (*viz.* IV.3, III.5 *and* III.4), *then for all f in F*,

$$\delta_f \gamma(u) = \sup\{f(h) \mid h \in N\};$$

further $\gamma(u) \gg \theta(h)$ *for all h in N, and there exists a sequence* $\{h_n\} \subset N$ *so that* $\gamma(u) = \lim \theta(h_n)$ *uniformly*.

PROOF. (a) Since $N = H \cap [0, u]$, F is precisely the set of states of $S(H, v)$ that extend IV.1, and by IV.2, F is a face,

(b) Since $u \geqslant h$ for all h in N, and $P(h) = 0$ by III.5 and IV.3, $\gamma(u) \gg \theta(h)$. On the other hand, by III.6 we may find, for arbitrarily small ε, h in H^+ so that $\gamma(u) \gg \theta(h)$ but $\|\gamma(u) - \theta(h)\| < \varepsilon$. Since $P(h) = 0$, $(\gamma, P)(u) \gg (\gamma, P)(h)$, so $u \geqslant h$, whence $h \in N$. Now (b) follows immediately. □

Thus, specifying the interval actually determines which face of $S(H)$ is to be lifted, and somewhat more. To motivate VI.2, let us consider the example $H = K = \mathbf{Q}$, with N varying over all possible intervals of H; here we select w the order-unit of K, to be 1.

For each real number greater than zero, r, define an interval N_r of H,

$$N_r = \{q \in H^+ \mid 0 \leqslant q < r\}.$$

The extensions must be of the form $\mathbf{Q} \oplus \mathbf{Q}$, with two pure states,

$$(a, b) \overset{P}{\to} b \in \mathbf{R},$$

$$(a, b) \overset{\gamma}{\to} a + x(b) \in \mathbf{R}, \quad \text{for some } x\colon K \to \mathbf{R}.$$

As the values of r vary, only particular functions x are admissible.

(a) r is rational: We must be able to find an element $(a, 1)$ so that $a + x(1) = r$ (in order that $(a, 1)$ will play the role of u in VI.1); since a is rational, this means that $x(1)$ rational is necessary (and sufficient).

(b) r is irrational: Here we must be able to solve $a + x(1) = r$ so $x(1) \equiv r \bmod(\mathbf{Q})$.

We also see that once x is determined (and it must satisfy $x(1) \in r + \mathbf{Q}$), the choice for the order-unit $(a, 1) = u$ is uniquely determined. So given two extensions of (H, N_r) by $(\mathbf{Q}, 1)$, the choices for x, x' are

$$x(1) = r - a \qquad \text{and} \qquad x'(1) = r - a';$$

and now define $X\colon K \to H$, by $X(b) = (a' - a)b$. Then the map

$$\varphi\colon \{H, \{\mathrm{id}\}, N_r; K, x\} \to \{H, \{\mathrm{id}\}, N_r; K, x'\},$$
$$\varphi(h, k) = (h + X(k), k)$$

defines an order-isomorphism. So we have established, for every order-interval N of $H = \mathbf{Q}$, and for every order-unit w of $K = \mathbf{Q}$, there is exactly one equivalence class of extensions of dimension groups with interval of (H, N) by (K, w) (this includes the case $N = H^+$ discussed in §II). As a corollary, we deduce that there is exactly one extension of a unitless universal Glimm algebra by the unital universal Glimm algebra.

An interval N of a dimension group H is *open* if there exists an extension of dimension groups with interval $(H, N) \to (G, u) \to (K, w)$, for some (K, w), with the complementary face to $H^\perp$ closed.

LEMMA VI.2. *Let (H, v) be a noncyclic simple dimension group with order-unit, and let N be a subset of H. The following are equivalent.*

(a) *There exist a closed face F of $S(H, v)$, and a strictly positive Γ in* $\mathrm{Aff}(F)$, *so that with θ the restriction of Θ to F,*

$$N = \{h \in H \mid 0 \leq h \text{ and } \Gamma \gg \theta(h)\}, \qquad \text{“}N \equiv N(F, \Gamma)\text{”}.$$

(b) *N is an open interval of H.*

Additionally, when (a) *holds, the F and Γ are uniquely determined.*

PROOF. (a) $\Rightarrow$ (b) We construct an extension of (H, N) by $(K, w) = (\mathbf{Q}, 1)$. Form the direct sum of groups $G = H \oplus \mathbf{Q}$, and define $\gamma\colon G \to \mathrm{Aff}(F)$ via

$$\gamma(h, q) = h + q\Gamma \quad \text{for } q \in \mathbf{Q}.$$

Form the extension of dimension groups given by the data $\{H, F; \mathbf{Q}, \gamma; H \oplus \mathbf{Q}\}$. In this case, the element $u = (0, 1)$ has value Γ at γ, and 1 at the one state from K, so is an order-unit; clearly, $H \cap [0, u] = N$.

(b) $\Rightarrow$ (a) This is Lemma VI.1.

Finally, F is simply the face of extendible states, so is uniquely determined by the extension, and uniqueness of Γ follows from the final portion of VI.1. □

THEOREM VI.3. *Let H, K be simple dimension groups with H not cyclic. Let N be an (open) interval, and affiliated to N, the face F of $S(H, v)$ and the element Γ of* $\mathrm{Aff}(F)^{++}$. *Let w be an order-unit for K. Then, if F is finite dimensional (or more generally, the complementary face to $H^\perp$ in the state space of the extension group is closed; this cannot be expressed a priori), every extension of dimension groups with interval, (H, N) by (K, w), is obtainable from the following process, and all outcomes of the process are such extensions.*

(i) *Select a representative of an element of* $\mathrm{Ext}_{\mathbf{Z}}(K, H)$,

$$H \to G \to K;$$

(ii) *select u in $\pi^{-1}w$;*

(iii) *select $\gamma\colon G \to$* $\mathrm{Aff}(F)$, *a group homomorphism, so that*

$$\gamma/H = \theta, \qquad \gamma(u) = \Gamma;$$

(iv) *define the ordering on G from the data* $\{H, F; K, \gamma; G\}$ (§III);

(v) *then $(H, N) \to (G, u) \to (K, w)$ is an extension of dimension groups with interval.*

These constructions may be indexed by the elements of the group

$$\mathrm{Ext}_{\mathbf{Z}}(K, H) \times \mathrm{Hom}_{\mathbf{Z}}(K/\mathbf{Z} \cdot w, \mathrm{Aff}(F)) \times H.$$

PROOF. Let us first show (i) through (iv) yield the desired extension. By III.5, $H \to G \to K$ is an extension of dimension groups. Since $\gamma(u) = \Gamma \gg 0$ and $\gamma(u) = w$, u is an order-unit for G, and by VI.1, $H \cap [0, u] = N$.

Conversely, suppose an extension of dimension groups as displayed in (v) is given, with the complementary face to $H^{\perp}$ closed. Certainly (i) occurs; by IV.3, we know such a γ exists to determine the ordering as in (iv). By VI.1(b), $\gamma(u) = \Gamma$; thus (iii) holds.

To describe the (noncanonical) group structure hinted at in the final statement, pick a fixed u_0 in $\pi^{-1}w$. Select a fixed γ_0: $G \to \mathrm{Aff}(F)$, extending θ_F: $H \to \mathrm{Aff}(F)$, such that $\gamma_0(u_0) = \Gamma$.

Define

$$Y = \{(\gamma, h) \mid \gamma: G \to \mathrm{Aff}(F), \gamma/H = \theta, \gamma(u_0) = \Gamma - \theta(h)\}.$$

We can define a group structure on Y,

$$(\gamma, h) + (\gamma', h') = (\gamma + \gamma' - \gamma_0, h + h').$$

Clearly the projection on the H-component of Y is a group homomorphism; it will be shown to be onto.

Given h in H, define $\beta: H \oplus \mathbf{Z}u_0 \to \mathrm{Aff}(F)$, a group homomorphism with $\beta/H = \theta$, and $\beta(u) = \Gamma - \theta(h)$. As $\mathrm{Aff}(F)$ is divisible, we may extend β to an element γ: $K \to \mathrm{Aff}(F)$; then (γ, h) belongs to Y, so the projection, call it p, is onto.

Now

$$\ker p = \{(\gamma, 0) \mid \gamma: G \to \mathrm{Aff}(F), \gamma/H = \theta, \gamma(u_0) = \Gamma\}.$$

The function q: $\ker p \to \mathrm{Hom}_{\mathbf{Z}}(K/\mathbf{Z}w, \mathrm{Aff}(F))$ defined by

$$(\gamma, 0) \mapsto \bar{\gamma}, \quad \text{where } \bar{\gamma}((g + H) + \mathbf{Z}w) = (\gamma - \gamma_0)(g)$$

is a well-defined group isomorphism (onto, from the divisibility of $\mathrm{Aff}(F)$). So we have exhibited Y itself as an extension of groups

$$\{0\} \to \mathrm{Hom}_{\mathbf{Z}}(K/\mathbf{Z}w, \mathrm{Aff}(F)) \to Y \to H \to \{0\};$$

since the left side is a real vector space, it is divisible, and so the extension splits. The elements of Y describe completely the extensions obtained in (i) through (iv), since if u belongs to $\pi^{-1}w$, $u - u_0$ lies in H. □

We can now describe when two extensions as described above are equivalent as dimension groups with interval. We maintain the notation of the proof of VI.3.

For two equivalent extensions given by γ, γ', and order-units u, u', by V.1, there must exist a group homomorphism X: $K \to H$ so that

$$\overline{\gamma - \gamma'} = \theta X \qquad \text{and} \qquad (\mathrm{id} - \alpha X \pi)(u) = u'.$$

The second condition translates to $X(w) = u - u'$. If we view u as $u_0 + h$, u' as $u_0 + h'$ (with h, h' in H necessarily), then it becomes clear that equivalence is compatible with the group operation as defined in the proof of VI.3. Now the identity element of the group Y defined there is $(\gamma_0, 0)$. The subgroup of Y consisting of those extensions equivalent to that defined by $(\gamma_0, 0)$ is

$$Z = \{(\gamma_0 + \theta X \pi, X(w)) \mid X: K \to H\};$$

thus the complete classification of equivalence classes is given by

$$\mathrm{Ext}_{\mathbf{Z}}(K, H) \times Y/Z.$$

In particular, $X(K) \subset \ker\theta$ if and only if $(\gamma_0, X(w)) \in Z$. If $[K, H] = \{0\}$, then obviously $Z = \{0\}$, and thus all the extensions obtained in VI.3 are inequivalent. On the other hand, if $[K, H](w) = H$, then we obtain an exact sequence of groups

$$\{0\} \to Z \cap \ker p \to Z \to H \to \{0\},$$

so that in this case, $Y/Z \simeq [K/\mathbf{Z}w, \mathrm{Aff}(F)]/\theta\{X\colon K \to H \mid X(w) = 0\}$. The bottom group tends to be quite a bit smaller than the top.

EXAMPLES VI.4. (a) $H = \mathbf{Z} + b\mathbf{Z}$, $K = \mathbf{Z} + a\mathbf{Z}$, with a, b in $\mathbf{R} - \mathbf{Q}$, with the total orderings inherited from the reals. Both groups being free, $\mathrm{Ext}_{\mathbf{Z}}(K, H) = 0$, so as groups we obtain only the split extension. Suppose $N = \{h \in H \mid 0 \leqslant h < 1\}$, and suppose $w = 1$; then $\gamma(u) = 1$. Set $G = H \oplus K$ (as groups); let γ_0 be defined via $\gamma_0(h, k) = \frac{1}{2}h + \frac{1}{2}k \in \mathbf{R}$; the admissible homomorphisms (VI.3) are listed, $[a\mathbf{Z}, \mathbf{R}] \simeq \mathbf{R}$. Let $u_0 = (1, 1)$, so $\gamma_0(u_0) = 1$. Clearly $\theta\colon H \to \mathbf{R}$ is the identity. Since K is free and w $(= 1)$ may be extended to a basis, $[K, H](w) = H$, so

$$Y/Z \simeq [a\mathbf{Z}, \mathbf{R}]/\theta[a\mathbf{Z}, H] = \mathbf{R}/(\mathbf{Z} + b\mathbf{Z}),$$

that is, the reals modulo a free dense subgroup of rank 2. The dependence on b is illusory: for any b' in $\mathbf{R} - \mathbf{Q}$,

$$\mathbf{R}/(\mathbf{Z} + b\mathbf{Z}) \simeq (\mathbf{Q}/\mathbf{Z})^2 \oplus \mathbf{R} \simeq \mathbf{R}/(\mathbf{Z} + b'\mathbf{Z})$$

as abelian groups.

(b) $K = H = \mathbf{Q}$, $N \neq \mathbf{Q}^+$. Then the group extensions are trivial, and $K/\mathbf{Z}w$ is torsion, so $Y = Z = H$, and there thus is just one equivalence class of extensions of dimension groups with interval. This verifies our previous computation. Similarly, if K is any rank 1 dimension group, $[K/\mathbf{Z}w, \mathbf{R}] = \{0\}$, and the classification simplifies to $\mathrm{Ext}_{\mathbf{Z}}(K, H) \times H/[K, H](w)$.

By I.5, I.6 and I.7, the classification of extensions of AF algebras by AF algebras is equivalent to the classification of the corresponding dimension groups with intervals $(K_0(A), D(A))$ by $(K_0(C), [1_C])$. So when A is simple, and C is simple and unital, VI.3 completely classifies a significant class of extensions.

If instead of considering the equivalence classes of extensions (either of AF algebras, or of dimension groups with intervals), one were interested in merely the *isomorphism* classes of the extensions, then some information is readily obtainable, and in many cases this will be complete. To obtain the isomorphism classes, widen the notion of equivalence to allow *arbitrary* automorphisms down the sides. In general, one does not obtain any sort of group of semigroup structure on the resulting equivalence classes. However, if for example, neither $K_0(C)$ admits any order-automorphisms fixing $[1_C]$ nor does $K_0(A)$ allow any order-automorphisms fixing $D(A)$ globally, then the equivalence classes of extensions, and the isomorphism classes of extensions, coincide.

Having few or no automorphisms is a fairly frequent occurrence. For example, no totally ordered subgroup of the reals can have any (nonidentity) order-automorphisms with a fixed point; more generally, this is true of any dense rank $n + 1$ free subgroup of $\mathbf{R}^n$ [**13**] (such groups, with the strict ordering, are simple dimension groups)—in fact almost all of these have no nontrivial order-automorphisms whatever.

On the other hand, some classes of dimension groups allow an unpleasantly large collection of order-automorphisms, with plenty of fixed points, fixed order-units, etcetera. Let K be any noncyclic simple dimension group such that Θ_K is an embedding, and let M be any torsion-free abelian group. Form the direct sum, $N = K \oplus M$, with the ordering given by

$$N^+ = \{(k, m) \mid k \in K^+ - \{0\}, m \in M\} \cup \{(0,0)\};$$

the N is a simple dimension group [**5**, 3.1 and 3.2], and its automorphisms are of the form $\varphi(k, m) = (\gamma(k), \delta(k) + \tau(m))$, where γ: $K \to K$ is an order-automorphism, δ: $K \to M$ is any group homomorphism, and τ: $M \to M$ is a group automorphism. Even if K has no nontrivial order-automorphisms, the order-automorphism group of N is a semidirect product of $[K, M]$ by the (possibly) huge group $\mathrm{GL}(M)$. In this case, $K \oplus \{0\}$ is pointwise fixed.

Other types of automorphisms move the states around. Let L be any real (i.e., totally real) finite dimensional Galois extension field of the rationals, and define L^+ to be the set of sums of squares. Then viewing L as a subfield of $\mathbf{R}$ (via a fixed embedding), the Galois automorphisms are the pure states of L (viewing them as maps $L \to L \subset \mathbf{R}$). By the Artin-Schreier Theorem, L^+ is exactly the set of elements that are strictly positive at every pure state (in fact, because the states are multiplicative, their kernels are zero), so that the ordering on L is given by that of the strict ordering on a dense subgroup of $\mathbf{R}^n$ (density follows from L being a rational vector space). Viewing Galois automorphisms as automorphisms of L, we see that they act transitively on the states. These automorphisms are precisely the order-automorphisms of L that have fixed points (others include multiplication by positive elements of L), and the fixed subgroup under them all is of course $\mathbf{Q}$.

VII. A nondimension group extension of dimension groups. Throughout this paper, we have required, when dealing with extensions of dimension groups, that the middle group be a dimension group—this does not follow from the two ends being dimension groups. An easy example, with both ends cyclic, is $H = \mathbf{Z} = K$, and $G = H \oplus K$ with $G^+ = \{(n,0) \mid n \geqslant 0\} \cup \{(a, b) \mid a > 0, b > 0\}$. But this example is slightly fraudulent, in that having a cyclic group at the left is a special case (cf. II.5(b)(i), and the characterization that simple dimension groups are either cyclic or dense). We present a simple by simple example, with both ends real vector spaces, and the middle not a dimension group. This is also an example to show that the extremal criterion of III.2 does not apply when the group is not an interpolation group.

Set $H = \mathbf{R}^2$ with the strict ordering, $K = \mathbf{R}$ with the usual ordering. Then the extremal states of H are (up to normalization) $\{e_1, e_2\}$ the coordinate projections. Set $f = e_1 + e_2$, form $G = H \oplus K$, and define G^+ so that f extends, as follows,

$$G^+ = \{(a, b, 0) \mid a > 0, b > 0\} \cup \{(a, b, c) \mid a + b > 0, c > 0\} \cup \{(0,0,0)\}.$$

Certainly G^+ is a bona fide partial ordering, G is directed in this ordering, $H \oplus \{0\}$ is an order-ideal (the only one), and the quotient is order-isomorphic to K. Note also that f extends to a state on G, namely $f \oplus 0$.

We observe that

$$(0,0,0), (2, -2, 0) \leqslant (3, 1, 0), (1, 0, 1)$$

cannot be interpolated.

It is also possible to prove that $f \oplus 0$ and the projection on the third coordinate are the only pure states of G (with respect to the order-unit $u = \frac{1}{2}(1, 1, 2)$); however, the former does not restrict to a pure state of H. This shows that the 'only if' portion of the criterion in III.2 fails if interpolation is deleted from the hypotheses. Of course, **R** may be replaced by **Q** if a countable example is desired.

Appendix. Weak equivalence and K_0. If one is only interested in computing the weak equivalence classes of extensions of $\mathcal{K}$ by an AF algebra, there is a general approach available using the (abelian group) functor, K_0.

A (unital) C^* algebra C is *stably finite* if for all n, for all X, Y in M_nC,

$$XY = 1_n \quad \text{implies } YX = 1_n.$$

For general C^* algebras (i.e., not necessarily unital), C is stably finite, if in the stable C^* algebra, $C \otimes \mathcal{K}$, if p is a proper subprojection of q, then q is not equivalent to p. The two definitions are easily seen to be consistent. All AF algebras, commutative algebras, and many others are stably finite.

Let C be a C^* algebra, and let B_1, B_2 represent two extensions, i.e., $\mathcal{K}_1 \to B_1 \subset B(H_1)$, $\mathcal{K}_2 \to B_2 \subset B(H_2)$, and there are *-isomorphisms φ_i: $B_i/K_i \to C$. One forms the sum of the two extensions [**3**, p. 271], by defining a subalgebra D of $B(H_1) \oplus B(H_2) \subset B(H_1 \oplus H_2)$,

$$D = \{T_1 \oplus T_2 \mid T_i \in B_i, \text{and } \varphi_1(T_1) = \varphi_2(T_2)\},$$

and then setting $B = D + \mathcal{K}(H_1 \oplus H_2)$. (Note that D is isomorphic to the ring-theoretic pullback of the maps φ_i: $B_i \to C$.) The natural and obvious map φ: $D \to C$ extends to the map $\varphi: B \to C$, with kernel $\mathcal{K}(H_1 \oplus H_2)$.

Then $\mathrm{Ext}_w(C)$ is obtained by imposing the weak equivalence relation [**3**, p. 270] and the addition defined above is compatible with it, so $\mathrm{Ext}_w(C)$ becomes an additive semigroup, and in many cases (e.g., C is nuclear), $\mathrm{Ext}_w(C)$ is a group.

Let us assume C is stably finite (and unital, although this is not really crucial), and $\mathrm{Ext}_w(C)$ is a group. Given an extension $\mathcal{K} \xrightarrow{i} B \xrightarrow{\varphi} C$, of $\mathcal{K}$ by C, there is an induced map on K_0,

$$K_0(\mathcal{K}) = \mathbf{Z} \xrightarrow{K_0(i)} K_0(B) \xrightarrow{K_0(\varphi)} K_0(C).$$

Now projections lift modulo the compacts, so $K_0(\varphi)$ is onto. Next, the sequence is exact at the middle term, by [**2**, p. 448]. Finally, the kernel of $K_0(i)$ is of the form $n\mathbf{Z}$ for some unique nonnegative integer n, and we thus obtain the short exact sequence

$$\{0\} \to \mathbf{Z}/n\mathbf{Z} \to K_0(B) \to K_0(C) \to \{0\}.$$

We would next like to show that weakly equivalent extensions, B_1, B_2 yield equal sequences when K_0 is applied. If $B_i \subset B(H_i)$, and if V is the partial isometry from H_1 to H_2 with finite dimensional kernel and cokernel that implements the weak equivalence, then with $VV^* = p$, $V^*V = q$, we see that p is a cofinite dimensional projection in $B(H_2)$, and thus p belongs to B_2; similarly, q belongs to B_1, and V induces a strong equivalence between the corner algebras, qB_1q and pB_2p, acting on their respective cut-down Hilbert spaces. Now strong equivalences are easily seen to induce equivalence of the corresponding extensions of the abelian groups arising from K_0. On the other hand p, q are full projections in their C^* algebras, i.e., $BpB = B$, so, by the usual Morita equivalence arguments, cutting down by p (respectively, q) does not affect the K_0-induced sequence. Thus the two sequences $K_0(\mathcal{K}_i) \to K_0(B_i) \to K_0(C)$ are equivalent.

With this in mind, we define a subset of $\mathrm{Ext}_{w}(C)$, for each nonnegative integer n,

$$E_n(C) = \{\text{equivalence classes of extensions } \mathcal{K} \to B \to C, \text{ with } \operatorname{Ker} K_0(i) = n\mathbf{Z}\}.$$

LEMMA A-1. $E_0(C) = \{$*equivalence classes of extensions* $\mathcal{K} \to B \to C$ *with* B *stably finite*$\}$.

PROOF. Suppose the equivalence class of the extension $\mathcal{K} \to B \to C$ lies in $E_0(C)$. Since C is stably finite, we may tensor with another copy of the compacts without changing the hypotheses. Suppose $p \leqslant q$, both are projections in B, and p is equivalent to q. Then the stable finiteness of C ensures that $q - p$ must lie in the kernel, $\mathcal{K}$, and so must be a finite dimensional projection. From the relation $q = (q - p) + p$, we see that the image of $(q - p)$ in $K_0(\mathcal{K})$ is in the kernel of $K_0(i)$, a contradiction unless $q = p$, whence B must be stably finite.

Conversely, if the intermediate term in the extension, B, is stably finite, if some finite rank projection goes to zero, we have $a + p \sim p$ for a, p orthogonal projections in B, with nonzero a of finite dimensional range, contradicting stable finiteness. Hence no finite rank projection may go to zero in $K_0(B)$, and since the elements of $K_0(\mathcal{K})$ are either $\pm$ the image of a projection, we see that $\operatorname{Ker} K_0(i)$ must be zero. □

LEMMA A-2. $E_0(C)$ *is a subgroup of* $\mathrm{Ext}_{w}(C)$.

PROOF. We have to check $E_0(C)$ is closed under addition and subtraction. To check addition, in view of the lemma above, we need only check that the sum of stably finite extensions is still stably finite. If as in the definition earlier on, B_1, B_2 are stably finite, so is the pullback D. Now if T belongs to $D + \mathcal{K}(H_1 \oplus H_2)$, and T is right invertible, but not left invertible, then its index is negative. A compact perturbation will yield an element $T_1 \oplus T_2$ of D, with the same index, so both T_i are Fredholm on their respective Hilbert spaces, and at least one has nonzero index; a compact perturbation will yield a one-sided invertible element with negative index in one of B_i, a contradiction. Thus B is directly finite, and since we can go up to matrix rings and apply the same argument, the sum B is stably finite.

If $\mathcal{K} \to B_1 \to C$ lies in $E_0(C)$ (or more precisely, its equivalence class does), and $\mathcal{K} \to B_2 \to C$ lies in $E_n(C)$ for some n greater than zero, we must show the sum is not stably finite. (For then $E_0(C)$ will be closed under additive inverses.) Going up to matrix rings and cutting down by projections if necessary, we may assume B_2 is not finite, so there exists T_2 in B_2 with $T_2T_2^* = 1$, but $T_2^*T_2 \neq 1$. Select any element T_1 of B_1 with $\varphi_1(T_1) = \varphi_2(T_2)$ (such exist, since φ_i are onto). Then T_1 is Fredholm, but as in the previous argument, its index must be zero, so the index of the direct sum $T_1 \oplus T_2$ is not zero, and a compact perturbation will yield a one-sided invertible element, so B is not stably finite. □

LEMMA A-3. *The set map*

$$E_0(C) \to \mathrm{Ext}_{\mathbf{Z}}(K_0(C), \mathbf{Z}),$$
$$\mathcal{K} \to B \to C \mapsto (\{0\} \to \mathbf{Z} \to K_0(B) \to K_0(C) \to \{0\})$$

is a group homomorphism.

PROOF. We have already seen that the assignment indicated above is well defined. We need only check that it preserves addition.

With B_1, B_2 extensions of $\mathcal{K}$ by C, form $D + \mathcal{K}(H_1 \oplus H_2) = B$ as described earlier. Since the maps on K_0 are onto $(K_0(B_i) \to K_0(C))$, it is straightforward to check that $K_0(D)$ is the pullback of the maps $K_0(B_i) \to K_0(C)$. Identifying $K_0(\mathcal{K}_i)$ with $\mathbf{Z}$, and regarding $K_0(D)$ as a subgroup of $K_0(B_1) \oplus K_0(B_2)$, one easily checks that

$$K_0(B) = \frac{K_0(D) + \{(n_1, n_2)\}}{\{(n, -n)\}}$$

and the corresponding extension is thus

$$\{0\} \to \mathbf{Z} \to \frac{K_0(D) + \{(n_1, n_2)\}}{\{(n, -n)\}} \to K_0(C) \to \{0\}, \qquad n \mapsto (n, 0).$$

But this is exactly the sum of the two extensions $\{0\} \to \mathbf{Z} \to K_0(B_i) \to K_0(C) \to \{0\}$ as computed in $\mathrm{Ext}_{\mathbf{Z}}(K_0(C), \mathbf{Z})$. □

For C an AF algebra, all extensions of $\mathcal{K}$ by C are AF, hence are stably finite, and thus $E_0(C) = \mathrm{Ext}_{w}(C)$; we thus obtain a homomorphism

$$\mathrm{Ext}_{w}(C) \to \mathrm{Ext}_{\mathbf{Z}}(K_0(C), \mathbf{Z}).$$

This is onto (as indicated in §II); given an extension of abelian groups $\{0\} \to \mathbf{Z} \to G \to K_0(C) \to \{0\}$, impose the lexicographic ordering on G. Then G is a dimension group, and there corresponds an AF algebra, B_0 such that B_0 has an ideal isomorphic to $\mathcal{K}$ with the quotient Morita equivalent to C. Then tensor B_0 with another copy of $\mathcal{K}$, and one may find a projection P such that if $p = P + \mathcal{K}$, $p(B_0/\mathcal{K} \otimes \mathcal{K})p \simeq C$; then $P(B_0 \otimes \mathcal{K})P$ is the right choice for B. On the other hand the map from $\mathrm{Ext}_{w}(C)$ is one-to-one, since in view of II.2 and II.3, the ordering on any intermediate $K_0(B)$ must be lexicographic, and so the ordering is determined by the ordering on $K_0(C)$ and the group extension.

In general, $E_0(C)$ may be trivial with $\mathrm{Ext}_{w}(C)$ not trivial; the simplest example occurs in the commutative case, with $C = C(T)$. Then each $E_n(C)$, for $n \geq 1$, contains exactly two equivalence classes, and $E_0(C)$ contains only the trivial extension. It would be interesting to decide if the map from $E_0(C)$ to the group extensions is always one-to-one, or onto.

To try to use K_0 to study the elements of $E_n(C)$ (n fixed) for n greater than 0 is pointless when $K_0(C)$ is torsion-free: The extension B lies in $E_n(C)$; we obtain the short exact sequence

$$\{0\} \to \mathbf{Z}/n\mathbf{Z} \to K_0(B) \to K_0(C) \to \{0\}.$$

Since $K_0(C)$ is assumed torsion-free, and the left side is finite, the sequence splits, so $K_0(B) = K_0(C) \oplus \mathbf{Z}/n\mathbf{Z}$. In other words, K_0 does not distinguish elements of $E_n(C)$ from each other if $n \neq 0$. The subset $E_1(C)$ appears to be particularly interesting.

One property of stably finite C^* algebras that will guarantee that $E_0(C)$ is all of $\mathrm{Ext}_w(C)$, is that $\mathrm{Hom}_{\mathbf{Z}}(K_1C, \mathbf{Z}) = \{0\}$, (here, K_1 refers to the *topological* K-group K_1). In particular, if the stable unitary group of C is connected (so $K_1C = \{0\}$), we obtain a natural group homomorphism, $\mathrm{Ext}_w(C) \to \mathrm{Ext}_{\mathbf{Z}}(K_0(C), \mathbf{Z})$. Presumably, this need not be either one-to-one or onto, but I have no examples.

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