

INTERIOR AND BOUNDARY CONTINUITY OF WEAK SOLUTIONS OF DEGENERATE PARABOLIC EQUATIONS

BY

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ABSTRACT. In this paper we consider degenerate parabolic equations of the form

$$(*) \quad \beta(u)_t - \operatorname{div} A(x, t, u, u_x) + B(x, t, u, u_x) \ni 0$$

where A and B are, respectively, vector and scalar valued Baire functions defined on $U \times \mathbb{R}^1 \times \mathbb{R}^n$, where U is an open subset of $\mathbb{R}^{n+1}(x, t)$. The functions A and B are subject to natural structural inequalities. Sufficiently general conditions are allowed on the relation $\beta \subset \mathbb{R}^1 \times \mathbb{R}^1$ so that the porous medium equation and the model for the two-phase Stefan problem can be considered. The main result of the paper is that weak solutions of $(*)$ are continuous throughout U . In the event that $U = \Omega \times (0, T)$ where Ω is an open set of \mathbb{R}^n , it is also shown that a weak solution is continuous at $(x_0, t_0) \in \partial\Omega \times (0, T)$ provided x_0 is a regular point for the Laplacian on Ω .

0. Introduction. In this paper we consider the problem of continuity of weak solutions of parabolic equations of the form

$$(1) \quad \frac{\partial}{\partial t} \beta(u) - \operatorname{div} A(x, t, u, u_x) + B(x, t, u, u_x) \ni 0$$

where A and B are, respectively, vector and scalar valued Baire functions defined on $U \times \mathbb{R}^1 \times \mathbb{R}^n$, and U is an open subset of $\mathbb{R}^{n+1}(x, t)$. Throughout this paper, U will take the form $\Omega_T = \Omega \times (0, T]$ where Ω is an open, bounded subset of \mathbb{R}^n . The functions A and B are required to satisfy the following structural inequalities.

$$(2) \quad \begin{aligned} |A(x, t, u, p)| &\leq a_0 |p| + a_1(x, t) |u| + a_2(x, t), \\ |B(x, t, u, p)| &\leq b_0 |p|^2 + b_1(x, t) |p| + b_2(x, t) |u| + b_3(x, t), \\ A(x, t, u, p) \cdot p &\geq c_0 |p|^2 - c_1(x, t) |u|^2 - c_2(x, t), \end{aligned}$$

where a_0 , b_0 and c_0 are constants with a_0 and b_0 nonnegative and $c_0 > 0$. The remaining coefficients are nonnegative measurable functions which, for simplicity of exposition, we will assume to be bounded. Thus, we assume there is a nonnegative constant K such that

$$(3) \quad a_i(x, t) \leq K, \quad b_i(x, t) \leq K, \quad c_i(x, t) \leq K.$$

Our results remain valid if the coefficients lie in those appropriate Lebesgue classes which have been considered by previous authors, e.g., [LSU, T]. Throughout, we

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allow sufficiently general conditions on β so that (1) includes both the porous medium equation and a model for the two-phase Stefan problem.

The main objective of this paper is to provide a technique that will establish continuity of weak solutions of (1) at the lateral boundary of Ω_T as well as at points of Ω_T . In the case $A(x, t, u, u_x) = u_x$, $B = 0$ and $n > 1$, continuity of weak solutions was established by Caffarelli and Friedman, [CF], for the porous medium equation, and by Caffarelli and Evans, [CE], for the two-phase Stefan problem. P. Sacks, [S], extended the results of [CE] by considering suitable B of the form $B = B(x, t)$. Recently, DiBenedetto, [D1, D2], established continuity of weak solutions of (1) in essentially the same generality considered in this paper. The approach employed by DiBenedetto is based on a parabolic version of DeGiorgi's technique as presented in [LSU]. The proof that we present here is based primarily on the Moser iteration technique as developed in [T] for quasilinear equations, and is therefore able to avoid much of the technical complexity associated with the DeGiorgi method. Our proof is also able to deal with the question of continuity of the solution at the lateral boundary of a cylindrical domain $\Omega \times (0, T]$. We are able to prove that a solution is continuous at a boundary point (x, t) , where $x \in \partial\Omega$ and $0 < t < T$, provided x is a regular point for the Laplacian on Ω .

The main idea of the proof is to employ the full strength of the Moser technique, which includes the parabolic version of the John-Nirenberg Lemma, to establish that each point of the domain $\Omega \times (0, T]$ is a Lebesgue point of a weak solution. Here, the term "Lebesgue point" refers to a parabolic modification of that classical concept. Then, we appeal to a suitable version of a fundamental lemma of DeGiorgi as presented in [D1, Lemma 3.1] or [CE, Lemma 3.1] which has been adopted to reflect the singularity in the equation. This lemma implies that a weak solution is continuous at each of its Lebesgue points. This same approach is also used to establish continuity at the boundary of a cylindrical domain.

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1. Preliminaries. For Ω a bounded open subset of R^n , the cylindrical domain $\Omega \times (0, T]$ will be denoted by Ω_T and let $\Omega(t) = \Omega \times \{t\}$. We let $x = (x_1, \dots, x_n)$ denote points in R^n and the gradient of a function u will be denoted by u_x or ∇u . Points in R^{n+1} will generally be denoted by $z = (x, t)$. Following common practice, the letter C will denote a constant that may change from line to line in the same proof. We will denote by $W_2^{1,1}(\Omega_T)$, the Sobolev space of functions whose distributional first derivatives belong to $L_2(\Omega_T)$, whereas $W_2^{1,0}(\Omega_T)$ will be the subspace consisting of those functions u for which $\partial u / \partial x_i \in L_2(\Omega_T)$, $i = 1, 2, \dots, n$. The space of those elements of $W_2^{1,1}(\Omega_T)$, whose trace on $\partial\Omega \times (0, T]$ is zero, is denoted by $W_{2,0}^{1,1}(\Omega_T)$. Let $V_2(\Omega_T) \subset W_2^{1,0}(\Omega_T)$ denote the Banach space with norm

$$\|u\|_{V_2(\Omega_T)} = \text{ess sup}_{0 \leq t \leq T} \|u(\cdot, t)\|_{2,\Omega} + \|\nabla u\|_{2,\Omega_T}^2.$$

Finally, let $V_2^{1,0}(\Omega_T) \subset V_2(\Omega_T)$ denote that subspace of elements u for which the map $t \rightarrow u(\cdot, t)$ is continuous relative to the norm on $L_2(\Omega)$.

Throughout we will be interested in β of the following two types.

Type I. Here we assume that β is a relation on $R^1 \times R^1$ given by

$$(4) \quad \beta(r) = \begin{cases} r, & r > 0; \\ [-a, 0], & r = 0; \\ r - a, & r < 0; \end{cases}$$

where $a > 0$. As shown in [D1] there is no loss of generality in assuming β is of this simple form. Indeed, if β were assumed to be given by

$$\beta(r) = \begin{cases} \beta_1(r), & r > 0; \\ [-a, 0], & r = 0; \\ \beta_2(r) - a, & r < 0; \end{cases}$$

where $\beta_i, i = 1, 2$, are monotone increasing functions whose derivatives are almost everywhere bounded away from 0 and ∞ , then the question of continuity of weak solutions of (1) easily reduces to the case of β given by (4).

Type II. Let β be a continuous monotone increasing function with $\beta(0) = 0$ satisfying the following.

(i) Assume the upper right (Dini) derivate of β is finite for all $r \neq 0$. Denote this derivate simply by $\beta'(r)$.

(ii) There is a constant α_0 such that

$$\beta'(r) \geq \alpha_0 > 0, \quad \text{for all } r \neq 0.$$

(iii) $\liminf_{r \rightarrow 0} \beta'(r) = \infty$.

(iv) There exists $\delta > 0$ such that $\beta'(s) \leq \beta'(r)$, for $s \notin [-\delta, \delta]$ and $r \in [-\delta, \delta] - \{0\}$, and β' is increasing on $[-\delta, 0)$ and decreasing on $(0, \delta]$.

DEFINITION 1.1. A function $u \in W_2^{1,1}(\Omega_T)$ is called a weak solution of (1) if

$$(5) \quad - \int_{\Omega} w(x, t) \phi(x, t) dx \int_{t_0}^{t_1} + \int_{t_0}^{t_1} \int_{\Omega} w(x, t) \phi_t(x, t) \\ = \int_{t_0}^{t_1} \int_{\Omega} A(x, t, u, u_x) \cdot \nabla \phi + B(x, t, u, u_x) \phi(x, t),$$

for all $\phi \in W_{2,0}^{1,1}(\Omega_T)$ and all intervals $[t_0, t_1] \subset (0, T]$. Here, w is a function such that

$$\{(z, w(z)): z \in \Omega_T\} \subset \{(z, \beta[u(z)]): z \in \Omega_T\},$$

if β is of type I. If β is of type II, then $w = \beta(u)$.

2. Weak solutions and Lebesgue points. Since we will assume that weak solutions of (1) are bounded, it will be convenient to formulate the structure in the following way.

$$(6) \quad |A(x, t, u, p)| \leq a_0 |p| + a_1, \quad |B(x, t, u, p)| \leq b_0 |p|^2 + b_1, \\ A(x, t, u, p) \cdot p \geq |p|^2 - c_1.$$

Choose d such that $d - 1 = a_1 + b_1 + c_1$.

Now consider a function $u \in W_2^{1,1}(\Omega_T)$ that is a weak solution of (1) with structure (5). In this section we will show that any point $z_0 \in \Omega_T$ is a parabolic Lebesgue point for u . For notational simplicity, we will take $z_0 = (0, 0)$. The estimates that are obtained below are set in terms of space-time cylinders, i.e., products of n -balls with time intervals. Thus, we will consider cylinders of the following form:

$$\begin{aligned}
 (7) \quad & R(r) = B(r) \times (\sigma_1 r^2, \sigma_2 r^2), \\
 & R^-(r) = B(r') \times (\tau_1 r^2, \tau_2 r^2), \\
 & R^+(r) = B(r') \times (\tau_3 r^2, \tau_4 r^2), \\
 & R^*(r) = B(r'') \times (\rho_1 r^2, \rho_2 r^2), \\
 & R'(r) = B(r'') \times (\rho_3 r^2, \rho_4 r^2),
 \end{aligned}$$

where $B(r)$ denotes the n -ball of radius r and center 0, and where

$$\begin{aligned}
 & 0 < r'' < r' < r, \\
 & \sigma_1 < \tau_1 < \rho_1 < \rho_2 < \tau_2 < \tau_3 < \rho_3 < 0 < \rho_4 < \tau_4 < \sigma_2.
 \end{aligned}$$

Because we intend to employ both Theorems 2.1 and 3.1 to establish continuity of weak solutions of (1), we impose the following constraints on the geometry of the above cylinders in order that the hypotheses of Theorem 3.1 can be satisfied for a particular cylinder that will be specified later.

First, let $r'' = \frac{1}{2}r$. Let ω be a number such that $0 < \omega < 2^{-1}$, and choose

$$\rho_1 = \frac{1}{4} \left(\frac{\omega}{2} - 1 \right), \quad \rho_2 = -\frac{1}{4} \cdot \frac{\omega}{2}.$$

Now choose $\tau_1 < \rho_1, \rho_2 < \tau_2 < 0$ and select τ_3, τ_4 so that $\tau_2 < \tau_3 < 0, \tau_4 > 0$ and $\tau_4 - \tau_3 = \tau_2 - \tau_1$. Next, choose $\tau_3 < \rho_3 < 0$ and $0 < \rho_4 < \tau_4$. Finally, choose $\sigma_1 < \tau_1, \tau_4 < \sigma_2$ and note that $R((r/2)N) \subset R'(r)$, for some positive integer N . For u defined on Ω_T and $z_0 \in \Omega_T$, let

$$L(z_0) = \operatorname{ess\,lim\,sup}_{z \rightarrow z_0} u(z), \quad l(z_0) = \operatorname{ess\,lim\,inf}_{z \rightarrow z_0} u(z).$$

We can now state the main result in this section.

THEOREM 2.1. *Let $u \in W_2^{1,1}(\Omega_T)$ be a bounded weak solution of (1) with structure (6), where β is of either type I or type II. Then, for any $z_0 \in \Omega_T$,*

$$\lim_{r \rightarrow 0} r^{-(n+2)} \int_{R^*(r)} |u - L(z_0)| = 0, \quad \text{if } L(z_0) > 0,$$

or

$$\lim_{r \rightarrow 0} r^{-(n+2)} \int_{R^*(r)} |u - l(z_0)| = 0, \quad \text{if } l(z_0) < 0.$$

The proof of this result follows the approach adopted in [T]. Note that the result implies that $L(z_0)$ and $l(z_0)$ cannot have opposite signs.

PROOF. Take $z_0 = (0, 0)$ and assume that $L(z_0) > 0$. For $k > 0$, let $u_k = (u - k)^+$ and define

$$\mu(r) = \text{ess sup}\{u(z) : z \in R(r)\},$$

where the notation of (6) is now being employed. Without loss of generality, we may assume that $\mu(r) > 0$, for all small r , for otherwise we would consider $l(z_0)$. Then, the only interesting case would be $l(z_0) < 0$ and therefore, $\lambda(r) > 0$, for all $r > 0$, where

$$\lambda(r) = -\text{ess inf}\{u(z) : z \in R(r)\}.$$

In what follows, we would then choose $k < 0$, replace $(u - k)^+$ by $(u - k)^-$ and replace $\mu(r)$ by $\lambda(r)$. The proof would remain essentially unchanged in all other aspects.

Now choose $0 < k < L(z_0)$ and define $\mu_k(r) = \mu(r) - k$. For $\varepsilon > 0$ and $\delta(r) = r + dr + (dr)^2$ we define the function

$$v(z) = \begin{cases} \mu_k(r) + \delta(r) - u_k(z), & \text{on } \{u \geq k\}; \\ \mu_k(r) + \delta(r), & \text{on } \{u < k\}; \end{cases}$$

and consider as test functions in Definition 1.1,

$$(8) \quad \phi = \eta^2 \exp - (b_0 u_k(z)) (v^\alpha - [\mu_k(r) + \delta(r)]^\alpha),$$

where $\alpha < 0$ and $\eta \geq 0$, is a smooth cut-off function that will be specified below. For the moment we assume only that $\text{spt } \eta \subset R(r) \subset \Omega_T$. Observe that $\phi = 0$ on $\{u < k\}$. Because $u \in W_2^{1,1}(\Omega_T)$, it is well known that $u(x, \cdot)$ is an absolutely continuous function of t for a.e. $x \in \Omega$. If β is of type II, it follows from classical real analysis that $\beta[(u - k)^+ + k](x, \cdot)$ is also absolutely continuous. Consequently, integration by parts allows identity (5) to be written as

$$(9) \quad \int_{t_0}^{t_1} \int_{\Omega} w_t \phi + A(x, t, u, u_x) \cdot \phi_x + B(x, t, u, u_x) \phi = 0,$$

for all intervals $[t_0, t_1] \subset (0, T)$. In case β is of type I, it is easy to verify that (5) can be written as (9) with w_t replaced by u_t . Thus, in either case, substitution of (8) into (9) yields

$$\begin{aligned} & \int_{t_0}^{t_1} \int_{\Omega} w_t \phi + \int_{t_0}^{t_1} \int_{\Omega} \eta^2 e^{-b_0 u_k} \cdot [|\alpha| v^{\alpha-1} + b_0 (v^\alpha - [\mu_k(r) + \delta(r)]^\alpha)] A \cdot u_x \\ & = - \int_{t_0}^{t_1} \int_{\Omega} 2\eta A \cdot \eta_x e^{-b_0 u_k} (v^\alpha - [\mu_k(r) + \delta(r)]^\alpha) - \int_{t_0}^{t_1} \int_{\Omega} B \phi. \end{aligned}$$

Using $v^\alpha - [\mu_k(r) + \delta(r)]^\alpha \leq v^\alpha$, $v \geq \delta(r)$, the structure (6) and Young's inequality, it follows that there is a constant C_1 depending on the structure (6) such that

$$\begin{aligned} & -|\alpha|^{-1} \int_{t_0}^{t_1} \int_{\Omega} \eta^2 e^{-b_0 u_k} \{v^\alpha - [\mu_k(r) + \delta(r)]^\alpha\} v_t \beta'(u) + \int_{t_0}^{t_1} \int_{\Omega} \eta^2 v^{\alpha-1} |\nabla v|^2 \\ & \leq C_1 (1 + |\alpha|^{-1}) \int_{t_0}^{t_1} \int_{\Omega} [(r^{-1} \eta)^2 + |\nabla \eta|^2] v^{\alpha+1}. \end{aligned}$$

This inequality holds in case β is of type II. In the event β is of type I, the inequality will be the same except $\beta'(u)$ should be replaced by 1. Note that the first term on the left is zero on the set $\{u < k\}$ and therefore $\beta'(u)$ can be replaced by $\beta'(u_k + k)$ in case β is of type II. Let

$$f(v) = \int_0^v e^{-b_0(\mu_k(r) + \delta(r) - s)} \beta'(\mu_k + \delta(r) + k - s) [S^\alpha - (\mu_k + \delta)^\alpha] ds.$$

Thus, the first term on the left side of the above inequality now becomes

$$-|\alpha|^{-1} \int_{t_0}^{t_1} \int_{\Omega} \eta^2 f(v)_t dx dt.$$

From elementary estimates it follows that there is a constant C (depending on k) such that

$$-C^{-1} \left(\frac{\alpha}{\alpha + 1} \right) v^{\alpha+1} \leq f(v) \leq C(\alpha + 1)^{-1} v^{\alpha+1}.$$

As in [T] this immediately yields the following inequality:

$$\begin{aligned} \sup_t \int [\eta v(x, t)]^2 dx + \int_{t_0}^{t_1} \int_{\Omega} \eta^2 v^{\alpha-1} |\nabla v|^2 dx dt \\ \leq C(1 + |\alpha|^{-1}) [1 + |\alpha|^{-1} \int_{t_0}^{t_1} \int_{\Omega} [(r^{-1} \eta)^2 + |\nabla \eta|^2 + \eta |\eta_t|] v^{\alpha+1} dx dt, \end{aligned}$$

where C depends on k and is bounded when α is bounded away from 0 and -1 . There is a similar inequality in case $\alpha = -1$. The dependency of the constant on k is immaterial as k is fixed throughout the proof. In case β is of type I, the same inequality results except that C does not depend on k . This inequality is the basis for an iteration procedure involving α which leads to the desired result. Details will not be given, for the argument is similar to that given in [T] or [AS]. For complete details involving the general structure (2), see [GZ2].

We will complete the proof of Theorem 2.1 by indicating the main points of the iteration scheme. For p any extended real number and R a cylinder, let

$$A(p, R) = \left(|R|^{-1} \int_R v^p \right)^{1/p},$$

where $|R|$ denotes the $(n + 1)$ -Lebesgue measure of R . Thus, $A(-\infty, R'(r))$ denotes the infimum of u over the cylinder $R'(r)$. The Sobolev inequality along with (10) and the iteration scheme yields

$$(10) \quad A(-p_0, R^+(r)) \leq CA(-\infty, R'(r)), \quad A(p_1, R^*(r)) \leq CA(p_0, R^-(r)),$$

for any $p_0 > 0$, $p_1 > 1$. In case $\alpha = -1$, the John-Nirenberg lemma implies that there exists $p^* > 0$ such that

$$A(p^*, R^-(r)) \leq CA(-p^*, R^+(r)).$$

Hence, from (10)

$$(11) \quad \left(|R^*(r)|^{-1} \int_{R^*(r)} v^{p_1} \right)^{1/p_1} \leq C \min_{R'(r)} v,$$

or

$$(12) \quad \begin{aligned} |R^*(r)|^{-1} \int_{R^*(r)} [\mu_k(r) + \delta(r) - u_k]^{p_1} \\ \leq C[\mu_k(r) + \delta(r) - \mu_k(r/2^N)]^{p_1}. \end{aligned}$$

Observe that the right side of (12) tends to 0 as $r \downarrow 0$.

$$(13) \quad \begin{aligned} \int_{R^*(r)} [\mu_k(r) + \delta(r) - u_k]^{p_1} \\ = \int_{A^*(r,k)} [\mu(r) + \delta(r) - u]^{p_1} + \int_{B^*(r,k)} [\mu(r) - k + \delta(r)]^{p_1}, \end{aligned}$$

where $A^*(r, k) = R^*(r) \cap \{z: u(z) \geq k\}$ and $B^*(k, r) = R^*(r) - A^*(r, k)$. Since $\mu(r) \rightarrow L(z_0)$ and $0 < k < L(z_0)$, it follows from (12) and (13) that

$$|B^*(r, k)|/|R^*(r)| \rightarrow 0, \quad \text{as } r \downarrow 0,$$

and therefore that

$$(14) \quad |R^*(r)|^{-1} \int_{B^*(r,k)} [\mu(r) + \delta(r) - u]^{p_1} \rightarrow 0, \quad \text{as } r \downarrow 0.$$

Since $\delta(r) \rightarrow 0$, $\mu(r) \rightarrow L(z_0)$ and $|R^*(r)| = cr^{n+2}$, (12), (13) and (14) imply

$$\lim_{r \rightarrow 0} r^{-(n+2)} \int_{R^*(r)} |u - L(z_0)|^{p_1} = 0.$$

3. A maximum principle. In this section we prove a lemma which is fundamental to the theory of regularity as developed by DeGiorgi. Nondegenerate elliptic and parabolic versions of this lemma appear in [LU, p. 83] and [LSU, p. 114]. The lemma was also employed in [GZ1] for the purpose of establishing boundary regularity of weak solutions of nondegenerate elliptic equations. In the case of the two-phase Stefan problem, a version of this lemma is critical to the proof of continuity given by Caffarelli and Evans, [CE, Lemma 3.1]. DiBenedetto, [D1, D2], extended the lemma as it appears in [LSU] to weak solutions of (1) with general structure (2). For the convenience of the reader we will also include a brief proof. For the sake of brevity, we will assume the restricted structure (6) rather than (2).

Throughout this section we let $(x_1, t_1) \in \Omega_T$, $t_1 > 0$ and for $r > 0$ we let $Q(r)$ denote the cylinder

$$Q(r) = B(x_1, r) \times [t_1 - r^2, t_1],$$

where $B(x_1, r)$ denotes the n -ball of radius r and center x_1 . We will consider only those cylinders for which $Q(r) \subset \Omega_T$. In case β is of type II, we will assume that a bounded solution $u \in V_2(Q_T)$ of (1) is the weak $V_2(Q_T)$ limit of a uniformly bounded sequence $\{u_n\}$, where each u_n is a weak solution of (1) in which β is replaced by a $C^\infty(\mathbb{R}^1)$ function β_n . The β_n are chosen so that $\beta_n(0) = 0$,

$$0 < \alpha_0 \leq \beta'_n(r) \leq \beta'(r), \quad r \neq 0, \quad n = 1, 2, \dots,$$

and that $\beta_n \rightarrow \beta$ uniformly on compact subsets of R^1 . This assumption is in accord with results concerning existence of weak solutions of (1), cf. [CD].

We also assume for simplicity that solutions u of (1) are elements of $W_2^{1,1}(\Omega_T)$. In case β is of type II this is no loss of generality in view of standard techniques involving integral averages. Let $m(r) = \text{ess sup}\{u(z) : z \in Q(r)\}$ and for $k \in R^1$, let

$$A(k, r) = Q(r) \cap \{z : u(z) \geq k\}.$$

LEMMA 3.1. *Let $u \in W_2^{1,1}(\Omega_T)$ be a bounded, weak solution of (1). There exist positive numbers θ, κ and δ depending only on the general structure of (1) such that if $k \in R^1$ and*

$$r_0^{(n/2)\kappa} + k < m(r_0), \quad m(r_0) < k + \delta, \quad \text{and} \quad |A(k, r_0)| < \theta |Q(r_0)|,$$

then

$$|A[2^{-1}(m(r_0) + k), r_0/2]| = 0.$$

PROOF. First, assume that β is of type I and choose a function $w \subset \beta(u)$. Define

$$\sigma(z) = \begin{cases} a, & \text{if } u(z) > 0, \\ w(z) + a, & \text{if } u(z) = 0, \\ 0, & \text{if } u(z) < 0, \end{cases}$$

and note that $w + a = u + \sigma$. Thus, substituting w into (5) yields

$$\begin{aligned} & -\int \sigma(x, t)\phi(x, t) dx \Big|_{t_0}^{t_1} + \int_{t_0}^{t_1} \int_{\Omega} \sigma(x, t)\phi_t(x, t) dx dt \\ & = \int_{t_0}^{t_1} \int_{\Omega} A \cdot \nabla \phi + B\phi + u_t \phi dx dt. \end{aligned}$$

Let $\phi = \eta^2(u - k)^+$ where $k < 0$ and η is a smooth cut-off function that will be specified below. Choose η so that $\eta_t \geq 0$ and η vanishes at t_0 . Then

$$\begin{aligned} -\int \sigma(x, t)\phi(x, t) dx \Big|_{t_0}^{t_1} & = -\int \sigma u^+ \eta^2 dx \Big|_{t_0}^{t_1} + k \int \sigma \eta^2 dx \Big|_{t_0}^{t_1} \\ & \leq -\int \sigma u^+ \eta^2 dx \Big|_{t_0}^{t_1}. \end{aligned}$$

Also,

$$\int_{t_0}^{t_1} \int_{\Omega} \sigma \phi_t dx dt = \int_{t_0}^{t_1} \int_{\Omega} \sigma u_t^+ \eta^2 dx dt + 2 \int_{t_0}^{t_1} \int_{\Omega} \eta \eta_t (u - k)^+ \sigma dx dt,$$

and therefore

$$\int_{t_0}^{t_1} \int_{\Omega} A \cdot \nabla \phi + B\phi + u_t \phi dx dt \leq \int_{t_0}^{t_1} \int_{\Omega} 2\sigma \eta \eta_t (u - k)^+ dx dt.$$

Therefore, from (6) and elementary estimates we have

$$\begin{aligned} & \sup_{t_0 \leq t \leq t_1} \int_{\Omega} [\eta(u - k)^+]^2 dx + \int_{t_0}^{t_1} \int_{\Omega} |\nabla [\eta(u - k)^+]|^2 dx dt \\ & \leq C \left(\int_{t_0}^{t_1} \int_{\Omega} (u - k)^{+2} (|\nabla \eta|^2 + \eta \eta_t) \right. \\ & \quad \left. + \int_{t_0}^{t_1} \int_{\Omega} (u - k)^+ \eta \eta_t dx dt + \int \int \eta^2 \chi(u \geq k) \right), \end{aligned}$$

provided η vanishes at t_0 . Here $\chi(E)$ denotes the characteristic function of a set E . If $k \geq 0$ this estimate is even easier to establish. In case β is of type II, the estimate is easily seen to be valid for the approximating β_n and passage to the limit yields the estimate for β .

Now choose $k \in R^1$ and let $H = m(r_0) - k$. For each positive integer m let

$$\begin{aligned} k_m &= (k + \frac{1}{2}H) - H/2^m, & r_m &= (r_0/2)(1 + 1/2^m), \\ Q_m &= B(x_1, r_m) \times [t_1 - r_m^2, t_1], \end{aligned}$$

and choose smooth cut-off functions η_m such that $\eta_m = 1$ on Q_{m+1} , $\eta_m = 0$ near the parabolic boundary of Q_m , $|\nabla \eta_m| \leq C2^m/r_0$ and $0 \leq \partial \eta_m / \partial t \leq C4^m/r_0^2$. Substituting $k = k_{m+1}$ and $\eta = \eta_m$ into the previous inequality, we have from the parabolic Sobolev inequality, vide [LSU, p. 75],

$$\begin{aligned} & \left(\int_{Q_{m+1}} \int (u - k_{m+1})^{+2\sigma} \right)^{1/\sigma} \\ & \leq \frac{C8^m}{r_0^2} \left[\int_{Q_m} \int (u - k_m)^{+2} + \int_{Q_m} \int (u - k_{m+1})^+ \right] + |Q_m \cap \{u > k_{m+1}\}|, \end{aligned}$$

where $\sigma = (n + 2)/n$. Observe that

$$\begin{aligned} \int_{Q_m} \int (u - k_{m+1})^+ & \leq \frac{C(m + 1)^2}{H} \int_{Q_m} \int (u - k_m)^{+2}, \\ |Q_m \cap \{u > k_{m+1}\}| & \leq \frac{C(m + 1)^4}{H^2} \int_{Q_m} \int (u - k_m)^{+2}, \end{aligned}$$

and

$$\int_{Q_{m+1}} \int (u - k_{m+1})^{+2} \leq \left(\int_{Q_{m+1}} \int (u - k_{m+1})^{+2\sigma} \right)^{1/\sigma} |Q_m \cap \{u > k_{m+1}\}|^{2/(n+2)}.$$

Thus if we let

$$I_m = \int_{Q_m} \int (u - k_m)^{+2},$$

it follows that

$$I_{m+1} \leq \frac{CC_1^m 8^m}{r_0^2} (H^2 + H + r_0^2) \left(\frac{I_m}{H^2} \right)^{(n+4)/(n+2)}.$$

Therefore,

$$\frac{I_{m+1}}{r_0^{n+2}H^2} \leq CC_1^m 8^m \left[1 + \frac{1}{H} + \left(\frac{r_0}{H} \right)^2 \right] \left(\frac{I_m}{r_0^{n+2}H^2} \right)^{(n+4)/(n+2)}.$$

With general structure (2), it can be shown that $\kappa < 2/m$. Thus, for small r_0 , we have from hypothesis that $r_0^2 < H^2$ and because u is assumed to be bounded, there is a constant depending on $\sup |u|$ such that $1 < C/H$, and therefore

$$J_{m+1} \leq \frac{C_1^m 8^m}{H} (J_m)^{(n+4)/(n+2)},$$

where $J_m = I_m/r_0^{n+2}H^2$. According to [LU, p. 66], $J_m \rightarrow 0$ as $m \rightarrow \infty$ if J_1 is sufficiently small. However,

$$J_1 \leq C |A(k, r_0)| / r_0^{n+2}$$

and thus, the conclusion now follows.

Continuity of weak solutions of (1) readily follow from Theorem 2.1 and Lemma 3.1. To see this, choose $z_0 \in \Omega_T$ and as in Theorem 2.1, take $z_0 = (0, 0)$ and assume $l(z_0) < 0$. Then it follows from Theorem 2.1 that

$$(15) \quad r^{-(n+2)} \int_{R^*(r)} |u - l(z_0)| \rightarrow 0, \quad \text{as } r \rightarrow 0.$$

Referring to (7) we have that

$$R^*(r) = B\left(\frac{r}{2}\right) \times \left[\left(\frac{\omega - 2}{2} \right) \left(\frac{r}{2} \right)^2, -\frac{\omega}{2} \left(\frac{r}{2} \right)^2 \right],$$

where $0 < \omega < \frac{1}{2}$. Define

$$\begin{aligned} Q\left(\frac{r}{2}\right) &= B\left(\frac{r}{2}\right) \times \left[\left(\frac{\omega - 2}{2} \right) \left(\frac{r}{2} \right)^2, \frac{\omega}{2} \left(\frac{r}{2} \right)^2 \right] \\ &= B\left(\frac{r}{2}\right) \times \left[t_1 - \left(\frac{r}{2} \right)^2, t_1 \right], \end{aligned}$$

where $t_1 = \omega r^2/8$.

Suppose that u is not continuous at z_0 , i.e., that $l(z_0) < L(z_0)$. Select a number k and $\delta^* \leq \delta$ such that

$$L(z_0) - \frac{\delta^*}{2} > k > L(z_0) - \frac{3}{4}\delta^* > l(z_0).$$

Choose r small enough so that

$$(16) \quad m\left(\frac{r}{2}\right) < L(z_0) + \frac{1}{4}\delta^*, \quad \left(\frac{r}{2}\right)^{n/2^k} + k < m\left(\frac{r}{2}\right),$$

and therefore that

$$(17) \quad m(r/2) < k + \delta, \quad m(r/2) + k < 2L(z_0).$$

It follows from (15) that

$$|R^*(r) \cap \{z: u(z) \geq k\}| < (\theta/2) |R^*(r)|$$

for all small r , and therefore

$$(18) \quad |A(k, r/2)| = |Q(r/2) \cap \{z: u(z) \geq k\}| < (\theta/2 + \omega) |Q(r/2)|.$$

Thus, if $2\omega < \min(\theta, 1)$ and if r is small enough to satisfy (16), (17) and (18), we see that the hypotheses of Lemma 3.1 are satisfied with $r_0 = r/2$. Note that

$$Q(r_0/2) = B(r/4) \times [t_1 - (r/4)^2, t_1]$$

and $t_1 - (r/4)^2 = (r^2/16)(2\omega - 1) < 0$. Therefore, $z_0 \in Q(r_0/2)$. However, (17) and Lemma 2.1 imply that $m(r_0/2) < L(z_0)$, a contradiction.

Similar reasoning can be applied if we assume $l(z_0) < L(z_0)$ with $L(z_0) > 0$. However, in this case a statement dual to Lemma 2.1, which involves the measure of $\{z: u(z) \leq k\}$, would have to be employed. Thus, we have shown that $l(z_0) = L(z_0)$ for every $z_0 \in \Omega_T$. Throughout the proof, we have tacitly assumed that $0 < t_0 < T$ where $z_0 = (x_0, t_0)$. The reader can verify that the proof can be easily modified to accommodate the case $t_0 = T$. Therefore we have the following.

THEOREM 3.2. *Suppose $u \in W_2^{1,1}(\Omega_T)$ is a bounded solution of (1) where β is of type I or of type II subject to assumptions of this section. Then, after redefinition on a set of measure zero, u is continuous on Ω_T .*

The author is indebted to the referee who observed that it is possible to obtain a modulus of continuity for weak solutions by performing a more detailed analysis of inequality (12).

4. Continuity at the boundary. In this section, we consider the question of continuity of weak solutions at a boundary point $z_0 \in \partial\Omega \times (0, T)$. We consider a function f defined on all of R^{n+1} such that

$$(19) \quad f \in W_2^{1,1}(R^{n+1}) \quad \text{and} \quad f|_{R^{n+1} - \Omega} \text{ is continuous.}$$

Throughout this section we will consider a bounded weak solution $u \in W_2^{1,1}(\Omega_T)$ of (1) such that

$$(20) \quad u = f \quad \text{weakly on } \partial\Omega_T;$$

that is, $u - f$ is assumed to be in the $W_2^{1,1}(\Omega_T)$ -closure of smooth functions ψ where $\text{spt } \psi \subset \Omega_T$. If β is of type II we assume, as in §3, that a bounded solution u of (1), whose trace is f on the lateral boundary of Ω_T , is the V_2 limit of a uniformly bounded sequence $\{u_n\}$, where each u_n is a weak solution of (1) in which β is replaced by β_n and where the trace of u_n on the lateral boundary is f . Define

$$L(\Omega_T, z_0) = \limsup_{z \rightarrow z_0} u(z), \quad l(\Omega_T, z_0) = \liminf_{z \rightarrow z_0} u(z).$$

Because we are assuming that $u = f$ weakly on $\partial\Omega_T$, it will be convenient to extend u to all of R^{n+1} by setting

$$u = \begin{cases} u, & \text{on } \Omega_T, \\ f, & \text{on } R^{n+1} - \Omega_T. \end{cases}$$

Thus, u can now be considered as an element of $W_2^{1,1}(R^{n+1})$ and therefore

$$(21) \quad \lim_{r \rightarrow 0} r^{-(n+2)} \int_{W(z_0, r)} |u(z) - u(z_0)| dz = 0,$$

for all $z_0 = (x_0, t_0) \in R^{n+1}$, except perhaps for a set of Newtonian capacity zero, see [DZ]. Here $W(z_0, r)$ denotes the parabolic cylinder

$$B(x_0, r) \times [t_1 - r^2, t_1],$$

where $t_1 = t_0 + r^2/8$.

In order to facilitate the proof of boundary regularity we need

DEFINITION 4.1. For an arbitrary set $E \subset R^{n+1}$, we define a capacity by

$$(22) \quad \Gamma(E) = \inf \left\{ \text{ess sup}_t \int u(x, t)^2 dx + \int \int |\nabla u|^2 dx dt \right\},$$

where the infimum is taken over all functions $u \in W_2^{1,1}(R^{n+1})$ for which

$$E \subset \text{int}\{z: u(z) \geq 1\}.$$

Recall that the Newtonian capacity of a set $A \subset R^n$ can be defined as

$$\gamma(A) = \inf \left\{ \int |\nabla u|^2 \right\},$$

where the infimum is taken over all $u \in W_2^1(R^n)$ such that $A \subset \text{int}\{x: u(x) \geq 1\}$. It is well known that $\gamma[B(r)] = Cr^{n-2}$. As an immediate consequence of the definitions, we have the estimate

$$(23) \quad \Gamma(E) \geq \int_{-\infty}^{\infty} \gamma[E(t)] dt + \text{ess sup}_t |E(t)|.$$

Now suppose that $E = A \times (0, T]$, where $A \subset B(x_0, a)$ for some $x_0 \in R^n$ and $a \in R^1$. It follows from elementary estimates that there are constants C_1, C_2 such that

$$(24) \quad \Gamma(E) \leq C_1 T \gamma(A) + C_2 a^2 \gamma(A).$$

In particular, if E is taken as a parabolic cylinder $R(r)$ (see (7)), then (23) and (24) imply that $\Gamma[R(r)] \sim r^n$.

It will be shown below that a weak solution of (1) is continuous at z_0 if

$$(25) \quad \int_0^1 \frac{\Gamma[R^*(r) - \Omega_T]}{\Gamma[R^*(r)]} \frac{dr}{r} = \infty.$$

Because $\Omega_T = \Omega \times (0, T]$, (23) and (24) imply

$$\Gamma[R^*(r) - \Omega_T] \leq C_1 r^2 \gamma[B(x_0, r) - \Omega] + C_2 r^2 \gamma[B(x_0, r) - \Omega]$$

and

$$\Gamma[R^*(r) - \Omega_T] \geq (r/2)^2 \gamma[B(x_0, r/2) - \Omega].$$

Since $\Gamma[R^*(r)] \sim r^n$ and $\gamma[B(x_0, r)] \sim r^{n-2}$, (25) holds if and only if

$$(26) \quad \int_0^1 \frac{\gamma[B(x_0, r) - \Omega]}{\gamma[B(x_0, r)]} \frac{dr}{r} = \infty,$$

which is the Wiener criterion for regularity at $x_0 \in \partial\Omega$.

We now consider parabolic cylinders

$$Q(r) = B(x_1, r) \times [t_1 - r^2, t_1]$$

where $(x_1, t_1) \in \partial\Omega \times (0, T)$ and let

$$m(\Omega_T, r) = \sup\{u(z) : z \in \Omega_T \cap Q(r)\}.$$

The proof of the following is almost identical to that of Lemma 2.1.

LEMMA 4.1. *Let $u \in W_2^{1,1}(\Omega_T)$ be a bounded weak solution of (1) such that $u = f$ weakly on $\partial\Omega_T$. There exist positive numbers θ, κ and δ depending only on the general structure of (1) such that if*

$$k \geq \sup\{f(z) : z \in Q(r_0) - \Omega_T\}, \quad r_0^{(n/2)\kappa} + k < m(\Omega_T, r_0), \\ m(\Omega_T, r_0) < k + \delta \quad \text{and} \quad |A(k, r) \cap \Omega_T| < \theta |Q(r_0)|,$$

then

$$|A[2^{-1}(m(\Omega_T, r_0) + k), r_0/2] \cap \Omega_T| = 0.$$

LEMMA 4.2. *Suppose $z_0 = (x_0, t_0) \in \partial\Omega \times (0, T)$ is such that*

$$\liminf_{r \rightarrow 0} \frac{|W(z_0, r) \cap \Omega_T|}{|W(z_0, r)|} = 0.$$

Then

$$\lim_{\substack{z \rightarrow z_0 \\ z \in \Omega_T}} u(z) = f(z_0).$$

PROOF. It suffices to prove $L(\Omega_T, z_0) \leq f(z_0)$ for the proof that $f(z_0) \leq l(\Omega_T, z_0)$ is similar. Consider parabolic cylinders of the form

$$Q(r) = W(z_0, r), \quad |Q(r) \cap \Omega_T| < \theta |Q(r)|.$$

Suppose $L(\Omega_T, z_0) > f(z_0)$ and choose k and $\delta^* \leq \delta$, such that $L(\Omega_T, z_0) - \delta^*/2 > k > L(\Omega_T, z_0) - (3/4)\delta^* > f(z_0)$. Now choose an r_0 so that

$$k \geq \sup\{f(z) : z \in Q(r_0) - \Omega_T\}, \\ m(\Omega_T, r_0) < L(\Omega_T, z_0) + \frac{1}{4}\delta^*, \quad r_0^{(n/2)\kappa} + k < m(\Omega_T, r_0),$$

and therefore that

$$m(\Omega_T, r_0) < k + \delta^*, \quad m(\Omega_T, r_0) + k < 2L(\Omega_T, z_0).$$

However, $z_0 \in Q(r_0/2)$ and therefore Lemma 4.1 implies

$$L(\Omega_T, z_0) < \frac{1}{2}[m(\Omega_T, r_0) + k],$$

a contradiction.

Consider a point $z_0 \in \partial\Omega \times (0, T)$ at which (21) holds. In view of Lemma 4.2 it follows that

$$(27) \quad L(\Omega_T, z_0) \geq f(z_0) \geq l(\Omega_T, z_0),$$

for if not, then $L(\Omega_T, z_0) < f(z_0) - \epsilon$ for some $\epsilon > 0$. Consequently, $u(z) < f(z_0) - \epsilon/2$ for $z \in \Omega_T \cap W(z_0, r)$ and all small r . But (21) implies that

$$\lim_{r \rightarrow 0} \frac{|W(z_0, r) \cap \Omega_T|}{|W(z_0, r)|} = 0,$$

thus contradicting Lemma 4.2. A similar argument shows that $f(z_0) \geq l(\Omega_T, z_0)$. Thus, (27) holds for all $z_0 \in \partial\Omega \times (0, T)$ except for a set S of Newtonian capacity zero. Let

$$(28) \quad A = S \cap \{z: W(z, r) - \Omega_T \subset S, \text{ for some } r > 0\}, \quad \text{and} \\ B = S - A.$$

Observe $W(z, r) - \Omega_T$ is a set of the form $W(z, r) - \Omega_T = (B(x, r) - \Omega) \times I$ where I is an interval. As S has zero Newtonian capacity in R^{n+1} , it follows that $\gamma[B(x, r) - \Omega] = 0$. Thus, from (25) and (26) we have the

$$(29) \quad \int_0^1 \frac{\Gamma[R^*(r) - \Omega_T]}{\Gamma[R^*(r)]} \frac{dr}{r} = 0 < \infty,$$

for all $z \in A$. Note that points of B are limits of points at which (27) holds because S is of Newtonian capacity zero, and therefore has Hausdorff dimension at most $n - 1$ in R^{n+1} . As f is continuous on $R^{n+1} - \Omega_T$, it follows that (27) holds at all points of B . Thus, B is empty and

$$(30) \quad L(\Omega_T, z_0) \geq f(z_0) \geq l(\Omega_T, z_0)$$

at all $z_0 \in \partial\Omega \times (0, T)$, except possibly at those where (29) holds.

To complete the proof of regularity at $z_0 \in (\partial\Omega) \times (0, T)$, we first prove a result similar to Theorem 2.1. To this end let z_0 be a point at which (30) holds, assume $u = f$ weakly and let $f(z_0) \geq 0$. For $k > f(z_0)$ let

$$u_k(z) = \begin{cases} (u(z) - k)^+, & z \in \Omega_T, \\ 0, & z \notin \Omega_T; \end{cases} \\ \mu_k(r) = \sup\{u_k(z): z \in R(r)\};$$

and define

$$v(z) = \begin{cases} \mu_k(r) + \delta(r) - u_k(z), & z \in R(r) \cap \Omega_T, \\ \mu_k(r) + \delta(r), & z \in R(r) - \Omega_T. \end{cases}$$

Proceeding exactly as in the proof of Theorem 2.1, we have

$$(31) \quad \left(r^{-(n+2)} \int_{R^*(r)} v^{p_1} \right)^{1/p_1} \leq C \min_{R'(r)} v,$$

for some $p_1 > 1$, and therefore

$$(32) \quad r^{-(n+2)} \int_{R^*(r) \cap \Omega_T} |u - L(\Omega_T, z_0)|^{p_1} \rightarrow 0.$$

Observe that a similar argument yields

$$(33) \quad r^{-(n+2)} \int_{R^*(r) \cap \Omega_T} |u - l(\Omega_T, z_0)|^{p_1} \rightarrow 0, \quad \text{as } r \downarrow 0,$$

if $l(\Omega_T, z_0) < f(z_0) \leq 0$.

If $f(z_0) \geq 0$, condition (30) allows us to treat (1) as though it were a nondegenerate equation whenever the test function ϕ in (8) involves $(u - k)^+$ for $k > f(z_0)$ or $(u - k)^-$ for $k < f(z_0) \leq 0$. For this reason the results of [Z] carry over to the present situation with essentially no change and thus yield the following.

THEOREM 4.3. *Suppose (25) holds at $z_0 \in \partial\Omega \times (0, T)$. Then $L(\Omega_T, z_0) = f(z_0)$ if $f(z_0) \geq 0$ and $l(\Omega_T, z_0) = f(z_0)$ if $f(z_0) \leq 0$.*

Observe that the conclusions of Theorem 4.3 follow immediately from (31) if a stronger condition than (25) is assumed to hold at $z_0 = (x_0, t_0)$, namely, if $R^n - \Omega$ is assumed to have positive upper metric density at x_0 , i.e., if

$$\limsup_{r \rightarrow 0} \frac{|B(x_0, r) \cap (R^n - \Omega)|}{|B(x_0, r)|} > 0.$$

Now by employing Lemma 4.1, Theorem 4.3 and the argument used to establish Theorem 3.2, we have

THEOREM 4.4. *Let $u \in W_2^{1,1}(\Omega_T)$ be a bounded weak solution of (1) such that $u = f$ weakly on $\partial\Omega_T$. We assume β to be of type I or of type II. If (25) holds at $z_0 \in \partial\Omega \times (0, T)$, then*

$$\lim_{\substack{z \rightarrow z_0 \\ z \in \Omega_T}} u(z) = f(z_0).$$

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