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NONSEPARABILITY OF QUOTIENT SPACES OF FUNCTION ALGEBRAS ON TOPOLOGICAL SEMIGROUPS

BY

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ABSTRACT. Let S be a topological semigroup, $C(S)$ the space of all bounded real-valued continuous functions on S. We define $WUC(S)$ the subspace of $C(S)$ consisting of all weakly uniformly continuous functions and $WAP(S)$ the space of all weakly almost periodic functions in $C(S)$.

Among other results, for a large class of topological semigroups S, for which noncompact locally compact topological groups are a very special case, we prove that the quotient spaces $WUC(S)/WAP(S)$ and, for nondiscrete S, $C(S)/WUC(S)$ are nonseparable. (The actual setting of these results is more general.) For locally compact topological groups, parts of our results answer affirmatively certain questions raised earlier by Ching Chou and E. E. Granirer.

Introduction. Let S be a (Hausdorff jointly continuous) topological semigroup and $m(S)$ the space of all bounded real-valued functions on S with the usual sup-norm $\| \ \|_S$. For every function f in $m(S)$ and element x of S we define the functions ${}_xf$ and f_x on S by

$$ {}_xf(y) := f(xy) \quad \text{and} \quad f_x(y) := f(yx) $$

for all y in S.

In this paper we shall be concerned with the following closed subspaces of $m(S)$:

$$
\begin{aligned}
C(S) &:= \{f \in m(S) \colon f \text{ is continuous}\},\\
LUC(S) &:= \{f \in C(S) \colon \text{the map } x \to {}_xf \ (x \in S) \text{ is norm continuous}\},\\
RUC(S) &:= \{f \in C(S) \colon \text{the map } x \to f_x \ (x \in S) \text{ is norm continuous}\},\\
UC(S) &:= LUC(S) \cap RUC(S),\\
LWUC(S) &:= \{f \in C(S) \colon \text{the map } x \to {}_xf \ (x \in S) \text{ is weakly continuous}\},\\
RWUC(S) &:= \{f \in C(S) \colon \text{the map } x \to f_x \ (x \in S) \text{ is weakly continuous}\},\\
WUC(S) &:= LWUC(S) \cap RWUC(S),\\
WAP(S) &:= \{f \in C(S) \colon \text{the set } \{{}_xf \colon x \in S\} \text{ is weakly relatively compact}\}.
\end{aligned}
$$

These spaces of functions have appeared before in many publications (see e.g. **[2, 3, 5, 6, 7, 8, 11** and **13]**). In particular for a locally compact topological group G we have that $LUC(G)$ (or $UC(G)$) is the usual space of left uniformly (or uniformly, respectively) continuous functions on G (see e.g. **[10]**).

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Now let G stand for any locally compact topological group for the remainder of this Introduction. Burckel [2] proved that $C(G) = WAP(G)$ if and only if G is compact. In [7] Granirer improved the result to

(1) $UC(G) = WAP(G)$ if and only if G is compact.

(2) If G is noncompact and amenable then the quotient space $UC(G)/WAP(G)$ is nonseparable.

We generalized the result (1) to a very large class of topological semigroups in [5] and kept at hand the possibility of dropping the word "amenable" in (2). The latter problem is also asked in [3] where Ching Chou proved a result parallel to (2): namely that if G is not compact and has property (E), then $UC(G)/WAP(G)$ contains an isometric copy of l^∞ and so is nonseparable. (For the definition of property (E) see §1.) Also in [8] Granirer proved another result, for noncompact amenable G, which is a generalization of (2) and asked whether the word "amenable" could be dropped. We shall show that for all noncompact G the space $UC(G)/WAP(G)$ contains a linear isometric copy of l^∞ as a consequence of a far more general result we prove for a large class of topological semigroups. The related question of Granirer [8] is also answered affirmatively.

Kister [12] showed that $LUC(G) = C(G)$ if and only if G is compact or discrete (see also [11, Theorem 4.1]). In view of the preceding remarks, it is natural to ask whether the quotient space $C(G)/LUC(G)$ is nonseparable whenever G is neither compact nor discrete. We shall answer this question affirmatively as a consequence of a slightly more general result proved for certain topological semigroups.

The plan of this paper is as follows: In §1 we include some further definitions and clarify our notation. In §2 we explicitly state the results proved in this paper. In §3 we prove a key lemma to most of our theorems. §§4, 5 and 6 are devoted to proving the results announced in §2. Finally we present some remarks and mention some open problems in §7.

We are indebted to the referee for the detailed criticisms which enabled us to simplify various parts of this paper.

1. Definitions and notations. Let A, B and C be any subsets of a semigroup S and $x \in S$. We write

$AB := \{ab\colon a \in A \text{ and } b \in B\}$,

$xB := \{x\}B$,

$A^{-1}B := \{y \in S\colon ay \in B \text{ for some } a \in A\}$,

$x^{-1}B := \{x\}^{-1}B$ and $A^{-1}x := A^{-1}\{x\}$.

In a related manner one defines Bx, BA^{-1}, Bx^{-1} and xA^{-1}. The notation $A^{-1}BC^{-1}$ is unambiguous and does not require brackets for $(A^{-1}B)C^{-1} = A^{-1}(BC^{-1})$.

Throughtout this paper a semigroup S is called a *topological semigroup* if S is endowed with a Hausdorff topology with respect to which $(x, y) \to xy$ is a jointly continuous mapping of $S \times S$ into S. For such a semigroup S, we take $M(S)$ to be the set of all bounded real-valued Radon measures on S, and for any x in S, $\bar{x}$ will denote the Dirac measure at x.

For any subsets C, D, F of S we write $C \otimes D := \{CD, C^{-1}D, CD^{-1}\}$ and $C \otimes D \otimes F := (\bigcup\{C \otimes E\colon E \in D \otimes F\}) \cup (\bigcup\{E \otimes F\colon E \in C \otimes D\})$. Hence inductively we can define $A_1 \otimes A_2 \otimes \cdots \otimes A_n$ for any subsets $A_1, \ldots, A_n$ of S. In particular if S is a topological semigroup we have the following definition: A subset B of S is said to be *relatively neo-compact* if B is contained in a (finite) union of sets in $A_1 \otimes A_2 \otimes \cdots \otimes A_n$ for some compact subsets $A_1, A_2, \ldots, A_n$ of S. We urge the reader to note the following observation.

For a topological semigroup S such that $C^{-1}D$ and DC^{-1} are compact whenever C, D are compact subsets of S, we have that $B \subset S$ is relatively neo-compact if and only if B is relatively compact. (*In particular relatively neo-compact subsets of a topological group are precisely the relatively compact subsets.*)

Following **[4]** a topological semigroup S is called a *C-distinguished topological semigroup* if $C(S)$ separates points of S. We shall need the following result noted in **[4]**.

PROPOSITION 1.1. *Let S be a C-distinguished topological semigroup. Then for all f in $C(S)$ and μ in $M(S)$ the maps*

$$x \to \int f(xy)\, d\mu(y) \quad \text{and} \quad x \to \int f(yx)\, d\mu(y)$$

of S into $\mathbf{R}$ are continuous and $M(S)$ admits a convolution operation given by

$$\nu * \mu(f) := \iint f(xy)\, d\nu(x)\, d\mu(y) \qquad (\nu, \mu \in M(S) \text{ and } f \in C(s))$$

with respect to which $M(S)$ is a normed algebra under the usual total variation norm $\| \ \|$.

If M is a (real) normed space, we denote the (real) continuous dual of M by M^*. In particular for a C-distinguished topological semigroup S if ν, $\mu \in M(S)$ and $h \in M(S)^*$ we can define the functionals $\nu \circ h$, $h \circ \nu$ in $M(S)^*$ by

$$\nu \circ h(\mu) := h(\nu * \mu) \quad \text{and} \quad h \circ \nu(\mu) := h(\mu * \nu) \qquad (\nu, \mu \in M(S))$$

and also define $\nu \circ h \circ \mu$ in $m(S)$ by

$$\nu \circ h \circ \mu(x) := (\nu \circ h) \circ \mu(x) = h(\nu * \bar{x} * \mu) \qquad (x \in S).$$

Let S be any topological semigroup. If $\mu \in M(S)$ we take $|\mu|$ to be the measure arising from the total variation of μ and $\text{supp}(\mu) := \{x \in S\colon |\mu|(V) > 0$ for every open neighbourhood V of $x\}$. Let $M_a^l(S) := \{\mu \in M(S)$: the map $x \to |\mu|(x^{-1}C)$ of S into $\mathbf{R}$ is continuous, for all compact $C \subset S\}$, $M_a^r(S) := \{\mu \in M(S)$: the map $x \to |\mu|(Cx^{-1})$ of S into $\mathbf{R}$ is continuous, for all compact $C \subset S\}$ and $M_a(S) := M_a^l(S) \cap M_a^r(S)$.

A measure $\mu \in M(S)$ is said to be *absolutely continuous* if it is in $M_a(S)$. We say S is the *foundation* of $M_a(S)$ if S coincides with the closure of $\bigcup \{\text{supp}(\mu)\colon \mu \in M_a(S)\}$.

For locally compact topological semigroups S the set $M_a(S)$ is well known (see e.g. **[1, 14, 15]** and the references mentioned there) and for a C-distinguished topological semigroup S the set $M_a(S)$ was studied by us in **[4]**. Of course for a locally compact topological group G the set $M_a(G)$ can be identified with $L^1(G)$ (see e.g. **[10]**).

Following [3], we say a noncompact locally compact topological group G has property (E) if there is a subset X of G such that X is not relatively compact and for each neighbourhood U of the identity 1 of G we have that $\bigcap\{x^{-1}Ux\colon x \in X \cup X^{-1}\}$ is a neighbourhood of 1. Examples of groups without property (E) can be found in [3].

2. Statement of results.

MAIN THEOREM 2.1. *Let S be a C-distinguished topological semigroup admitting a nonzero absolutely continuous measure (i.e. $M_a(S)$ is nonzero). Then if S is not relatively neo-compact we have that the quotient space $WUC(S)/WAP(S)$ contains a linear isometric copy of l^∞ and so is nonseparable.*

As a consequence we obtain the following extension (to all locally compact groups) of a result of Granirer [7] and Ching Chou [3].

COROLLARY 2.2. *Let G be a noncompact locally compact topological group. Then the quotient space $UC(G)/WAP(G)$ contains a linear isometric copy of l^∞ so is nonseparable.*

The next theorem is a generalization of [8, Theorem 14] and from it we deduce the latter theorem without assuming the group G to be amenable (see Corollary 2.4)—thus answering affirmatively the question asked in [8, p. 382].

THEOREM 2.3. *Let S be a C-distinguished topological semigroup with $M_a(S)$ nonzero. Let $\eta \in M_a(S)$ be a positive measure with $\|\eta\| = 1$ and set $\eta^\perp := \{h \in M_a(S)^*\colon h(\nu * \eta) = 0$ for all $\nu \in M_a(S)\}$. Suppose $WUC(S) \subset \text{norm-cl}(WAP(S) + \eta^\perp + X)$ for some norm separable $X \subset M_a(S)^*$. Then*

(1) *if $\operatorname{supp}(\eta)$ is compact we have that S is relatively neo-compact;*

(2) *if S is a locally compact semigroup with an identity element 1 such that $1 \in \operatorname{supp}(\eta)$, then S is relatively neo-compact.*

COROLLARY 2.4. *Let G be any locally compact group $\eta \in L^1(G)$, $\eta > 0$, with $\|\eta\| = 1$. Let $\eta^\perp := \{f \in L^\infty(G)\colon (f, L^1(G) * \eta) = 0\}$. If $UC(G) \subset \text{norm-cl}(WAP(G) + \eta^\perp + X)$ for some norm separable $X \subset L^\infty(G)$ then G is compact.*

THEOREM 2.5. *Let S be a closed subsemigroup of a locally compact topological group G and suppose that S is neither compact nor discrete. Then the quotient space $C(S)/LWUC(S)$ is nonseparable.*

COROLLARY 2.6. *Let G be a locally compact topological group which is neither compact nor discrete. Then the quotient space $C(G)/LUC(G)$ is nonseparable.*

3. A key lemma. We prove in this section a lemma that forms a key step in our proof of the main theorem and part of our later results. The lemma is a large refinement of [5, Lemma 3.1].

LEMMA. *Let S be any topological semigroup which is not relatively neo-compact, C_0 and D_0 any fixed compact subsets of S. Take $C := C_0 \cup \{1\}$ and $D := D_0 \cup \{1\}$ where 1 is an identity of S (if there is one or an adjoined isolated identity of S). Then* *there exist infinite sequences* $\{x_1, x_2, \dots\}$ *and* $\{y_1, y_2, \dots\}$ *is* S *with*

$$C^{-1}(Cx_n y_m D)D^{-1} \cap C^{-1}(Cx_i y_j D)D^{-1} = \varnothing$$

if any one of the following three conditions holds.

(a) $n \leqslant m$ *and* $i > j$;

(b) $n > m$, $i > j$ *and* $n \neq i$;

(c) $n \leqslant m$, $i \leqslant j$ *and* $m \neq j$.

PROOF. We employ an induction argument. Suppose, by the inductive hypothesis, we have finite sequences $X_p := \{x_1, \dots, x_p\}$ and $Y_p := \{y_1, \dots, y_p\}$ in S such that the lemma holds for n, m, i, j in $\{1, 2, \dots, p\}$. For convenience let

$$L_p := \bigcup_{n=1}^{p} \bigcup_{n \leqslant m \leqslant p} Cx_n y_m D \quad \text{and} \quad R_p := \bigcup_{m=1}^{p-1} \bigcup_{p \geqslant n > m} Cx_n y_m D.$$

In the latter notation, the conclusion of our lemma under item (a) for the finite sequences X_p and Y_p is equivalent to

$$C^{-1}L_p D^{-1} \cap C^{-1}R_p D^{-1} = \varnothing. \tag{1}$$

Now since S is not relatively neo-compact while both

$$T := C^{-1}\big(C\big(C^{-1}L_p D^{-1}\big)D\big)\big(Y_p D\big)^{-1}$$

and

$$T' := C^{-1}\big(C\big(C^{-1}(CX_p Y_p D)D^{-1}\big)D\big)\big(Y_p D\big)^{-1}$$

are relatively neo-compact, we can choose x_{p+1} in $S \setminus (T \cup T')$. Now, $x_{p+1} \notin T$ is equivalent to

$$C^{-1}(Cx_{p+1}Y_p D)D^{-1} \cap C^{-1}L_p D^{-1} = \varnothing; \tag{2}$$

while $x_{p+1} \notin T'$ is equivalent to

$$C^{-1}(Cx_{p+1}Y_p D)D^{-1} \cap C^{-1}(CX_p Y_p D)D^{-1} = \varnothing. \tag{3}$$

Also the subsets

$$Q := \big(CX_{p+1}\big)^{-1}\big(C\big(C^{-1}(CX_p Y_p D)D^{-1}\big)D\big)D^{-1}$$

and

$$Q' := \big(CX_{p+1}\big)^{-1}\big(C\big(C^{-1}\big(R_p \cup Cx_{p+1}Y_p D\big)D^{-1}\big)D\big)D^{-1}$$

are relatively neo-compact. So we can choose y_{p+1} in S such that $y_{p+1} \notin Q$ and $y_{p+1} \notin Q'$. Equivalently, this is such that

$$C^{-1}\big(CX_{p+1}y_{p+1}D\big)D^{-1} \cap C^{-1}\big(CX_p Y_p D\big)D^{-1} = \varnothing \tag{4}$$

and

$$C^{-1}\big(CX_{p+1}y_{p+1}D\big)D^{-1} \cap C^{-1}\big(R_p \cup Cx_{p+1}Y_p D\big)D^{-1} = \varnothing \tag{5}$$

(respectively).

Now for the finite sequences X_{p+1} and Y_{p+1}, item (3) and the inductive hypothesis show that the lemma holds under condition (b), item (4) and the inductive hypothesis show that the lemma holds under condition (c), and to verify the lemma under condition (a) it is sufficient to establish item (1) with $p+1$ in place of p. To the latter end, we note that the inductive hypothesis, items (2) and (5) imply that

$$\begin{aligned} &C^{-1}L_{p+1}D^{-1} \cap C^{-1}R_{p+1}D^{-1} \\ &\qquad = C^{-1}\left(L_p \cup CX_{p+1}y_{p+1}D\right)D^{-1} \cap C^{-1}\left(R_p \cup Cx_{p+1}Y_pD\right)D^{-1} = \varnothing. \end{aligned}$$

Repeating the argument countably many times we get our lemma.

4. The proof of Theorem 2.1 and Corollary 2.2. First we prove a lemma that will form part of our proof of Theorem 2.1.

LEMMA 4.1. *Let S be a C-distinguished topological semigroup, ν and μ nonzero measures in $M(S)$ such that the maps $x \to \nu * \bar{x}$ and $x \to \bar{x} * \mu$ of S into $M(S)$ are weakly continuous. Then*

$$\nu \circ h \circ \mu \in WUC(S) \qquad (h \in M(S)^*).$$

PROOF. We first note that $\nu \circ h \circ \mu \in C(S)$, since $\nu \circ h \circ \mu(x) = \nu \circ h(\bar{x} * \mu)$ and the map $x \to \bar{x} * \mu$ is weakly continuous. To prove that $\nu \circ h \circ \mu$ is in $RWUC(S)$, for example, we take $\zeta \in C(S)^*$ and must show the function $x \to \zeta((\nu \circ h \circ \mu)_x)$ is continuous. But this follows since $\zeta((\nu \circ h \circ \mu)_x) = \zeta(\nu \circ h \circ (\bar{x} * \mu))$ and the functional $\lambda \to \zeta'(\nu \circ h \circ \lambda)$ is in $M(S)^*$, where ζ' is any extension of ζ in $M(S)^*$. Similarly $\nu \circ h \circ \mu$ is in $LWUC(S)$ and our lemma is proved.

PROOF OF THEOREM 2.1. The idea of extracting an isometric linear copy of l^∞ from $WUC(S)/WAP(S)$ employed here is inspired by [3]. Of course it is clear from our definitions that $WAP(S) \subset WUC(S)$.

Theorem 4.1 of [4] implies that we can choose a positive measure $\eta \in M_a(S)$ such that $\|\eta\| = 1$ and $K := \operatorname{supp}(\eta)$ is compact. Now [4, Theorem 4.5] says that we can find $u, v \in S$ such that if $\nu := \eta * \bar{u}$ and $\mu := \bar{v} * \eta$ then the maps $x \to \nu * \bar{x}$ and $x \to \bar{x} * \mu$ of S into $M(S)$ are weakly continuous. We keep these measures fixed for the remainder of our proof and note that both $C_0 := \operatorname{supp}(\nu)$ and $D_0 := \operatorname{supp}(\mu)$ are compact (in fact $C_0 = Ku$ and $D_0 = vK$), and $\|\nu\| = \|\mu\| = 1$. Let $C := C_0 \cup \{1\}$ and $D := D_0 \cup \{1\}$, where 1 is the identity of S (if there is one) or an adjoined isolated identity of S.

Let $A := \{x_1, x_2, \ldots\}$ and $B := \{y_1, y_2, \ldots\}$ be the infinite sequences chosen in our (key) Lemma 3 with respect to the compact sets C and D. We can choose infinite subsequences $A_k := \{x_{k_1}, x_{k_2}, \ldots\}$ of A and $B_k := \{y_{k_1}, y_{k_2}, \ldots\}$ of B such that

(α) $\bigcup_{k=1}^\infty A_k \subset A$ and $\bigcup_{k=1}^\infty B_k \subset B$;

(β) if $n \neq m$ then $A_n \cap A_m = \varnothing$ and $B_n \cap B_m = \varnothing$;

(γ) Lemma 3 remains valid with k_n, k_m, k_i, k_j in place of n, m, i, j, respectively (i.e. when A_k and B_k replace A and B, respectively).

Let $E_k := \bigcup_{i=1}^\infty \bigcup_{i<j} Cx_{k_i}y_{k_j}D$ and $F_k := \bigcup_{j=1}^\infty \bigcup_{i>j} Cx_{k_i}y_{k_j}D$. From ($\beta$), ($\gamma$) and Lemma 3 we note that all the E_k's and F_k's are pairwise disjoint. We now define the function f_k and S by

$$f_k(x) := \nu * \bar{x} * \mu(E_k) - \nu * \bar{x} * \mu(F_k). \tag{1}$$

Let $\{c_k\}$ be any element in l^∞.

We now show that the function $\sum_{k=1}^{\infty} c_k f_k$ is in $WUC(S)$. Since all the E_k's and F_k's are pairwise disjoint we can define the functional $h \in M(S)^*$ by

$$h(\eta) := \sum_{k=1}^{\infty} c_k(\eta(E_k) - \eta(F_k)).$$

Now a simple exercise on our definitions shows that

$$\nu \circ h \circ \mu = \sum_{k=1}^{\infty} c_k f_k$$

and so, by Lemma 4.1, we have that $\sum_{k=1}^{\infty} c_k f_k$ is in $WUC(S)$.

It remains to show the (clearly linear) map

$$\{c_k\} \to \sum_{k=1}^{\infty} c_k f_k + WAP(S)$$

is an isometry of l^∞ into the quotient space $WUC(S)/WAP(S)$. This will be achieved when we demonstrate

$$\|f + g\|_S \geqslant \|\{c_k\}\|_\infty, \tag{2}$$

where $f := \sum_{k=1}^{\infty} c_k f_k$ and $g \in WAP(S)$. Suppose on the contrary there exist a g in $WAP(S)$ and $\varepsilon > 0$ such that

$$\|f + g\|_S < \|\{c_k\}\|_\infty - 2\varepsilon.$$

We can choose a positive integer k' such that $\|\{c_k\}\|_\infty - \varepsilon < |c_{k'}|$ and hence get $\|f + g\|_S < |c_{k'}| - \varepsilon$. We may assume that $c_{k'}$ is nonnegative and thus write

$$\|f + g\|_S < c_{k'} - \varepsilon. \tag{3}$$

From the definition of the f_k's we have that

$$\sum_{k=1}^{\infty} c_k f_k(x_{k'_i} y_{k'_j}) = c_{k'} f_{k'}(x_{k'_i} y_{k'_j}) = \begin{cases} c_{k'} & \text{if } i \leqslant j, \\ -c_{k'} & \text{if } i > j. \end{cases} \tag{4}$$

We now have (from (3) and (4)) that

$$\begin{cases} i \leqslant j \text{ implies } g(x_{k'_i} y_{k'_j}) \leqslant -c_{k'} + |c_{k'} + g(x_{k'_i} y_{k'_j})| < -\varepsilon, \\ i > j \text{ implies } g(x_{k'_i} y_{k'_j}) \geqslant c_{k'} - |-c_{k'} + g(x_{k'_i} y_{k'_j})| > \varepsilon. \end{cases} \tag{5}$$

From (5) and [**9**, Theorem 6] we have $g \notin WAP(S)$, which contradicts our choice of g. Hence (2) holds.

Now noting that

$$\left\| \sum_{k=1}^{\infty} c_k f_k \right\|_S = \|\{c_k\}\|_\infty$$

and recalling (2) we have that

$$\left\| \sum_{k=1}^{\infty} c_k f_k + WAP(S) \right\|_{WUC(S)/WAP(S)} := \inf\left\{ \left\| \sum_{k=1}^{\infty} c_k f_k + g \right\|_S : g \in WAP(S) \right\}$$
$$= \|\{c_k\}\|_\infty.$$

Consequently the mapping from l^∞ into $WUC(S)/WAP(S)$ defined earlier is an isometry. This completes our proof of the theorem.

PROOF OF COROLLARY 2.2. Theorem 7 of Mitchell [13] says that $WUC(G) = UC(G)$ and so our corollary follows from Theorem 2.1 and the definition of relatively neo-compact subsets of G.

5. The proof of Theorem 2.3 and Corollary 2.4. As the reader can easily note, our proof here is extracted from our studies in §4. First we prove Theorem 2.3.

To prove item (1), suppose S is not relatively neo-compact and note that $K := \operatorname{supp}(\eta)$ is compact. In the proof of Theorem 2.1 take

$$C := (C_0 \cup \{1\})(K \cup \{1\}) \quad \text{and} \quad D := (K \cup \{1\})(D_0 \cup \{1\}),$$

where 1 denotes the identity of S (if there is one) or an adjoined isolated identity of S; let the functions f_k be as constructed there (with respect to the above C and D). Taking $\mathcal{Q} := \text{norm-cl}(\eta \circ WAP(S) \circ \eta)$ it is easy to deduce from §4 that the maps

$$\{c_k\} \to \sum_{k=1}^{\infty} c_k f_k + WAP(S) \qquad (\{c_k\} \in l^\infty), \tag{1}$$

$$\{c_k\} \to \sum_{k=1}^{\infty} c_k(\eta \circ f_k \circ \eta) + \mathcal{Q} \qquad (\{c_k\} \in l^\infty) \tag{2}$$

of l^∞ into $WUC(S)/WAP(S)$ and $WUC(S)/\mathcal{Q}$, respectively, are linear isometries. (Here $\eta \circ WAP(S) \circ \eta := \{\eta \circ g \circ \eta \colon g \in WAP(S)\}$ and as is well known (see e.g. [2]) $\eta \circ WAP(S) \circ \eta \subset WAP(S)$.)

Now fix any $\{c_k\} \in l^\infty$ and consider the function $f := \sum_{k=1}^{\infty} c_k f_k$. Given $\varepsilon > 0$, we can find $g \in WAP(S)$, $h \in \eta^\perp$ and $t \in X$ such that

$$\|f - (g + h + t)\|_S < \varepsilon. \tag{3}$$

Now $\eta \circ h \circ \eta(x) := h((\eta * \bar{x}) * \eta) = 0$ $(x \in S)$ since $\eta * \bar{x} \in M_a(S)$ (see e.g. [4, Theorem 4.8]). So from (3) we get

$$\|\eta \circ f \circ \eta - \eta \circ g \circ \eta - \eta \circ t \circ \eta\|_S \leqslant \|f - (g + h + t)\|_S < \varepsilon. \tag{4}$$

Taking $\widetilde{\eta \circ f \circ \eta} := \eta \circ f \circ \eta + \mathcal{Q}$ and $\widetilde{\eta \circ t \circ \eta} := \eta \circ t \circ \eta + \mathcal{Q}$ we see that (4) implies

$$\|\widetilde{\eta \circ f \circ \eta} - \widetilde{\eta \circ t \circ \eta}\|_{WUC(S)/\mathcal{Q}} < \varepsilon. \tag{5}$$

Since X is separable, (2) and (5) imply that l^∞ is separable—conflict! By this conflict S must be relatively neo-compact. This completes our proof for item (1).

We now deduce item (2) from item (1). Since for any $\mu \in M(S)$ with $\mu \ll \eta$ we have $\mu \in M_a(S)$ (see e.g. [4, Theorem 4.1]) and $1 \in \operatorname{supp}(\eta)$, we can find a positive measure $\nu \in M_a(S)$ with

$$\operatorname{supp}(\nu) \text{ compact}, \quad 1 \in \operatorname{supp}(\nu) \quad \text{and} \quad \|\nu\| = 1.$$

We claim that $\eta^\perp \subset \nu^\perp$. To prove this claim let $h \in \eta^\perp$. We may assume that h is a positive functional (i.e. $\rho \geqslant 0$ implies $h(\rho) \geqslant 0$ $(\rho \in M_a(S))$). For any $y \in S$ we know that $\nu * \bar{y} \in M_a(S)$ and so

$$0 = h(\nu * \bar{y} * \eta) = \int h(\nu * \bar{y} * \bar{x})\, d\eta(x) \tag{6}$$

by [**1**, Lemma 2.2]. Now the map $x \to h(\nu * \bar{y} * \bar{x})$ of S into $[0, \infty)$ is continuous and $1 \in \operatorname{supp}(\eta)$, so equation (6) gives $h(\eta * \bar{y}) = 0$ $(y \in S)$. Hence, again using [**1**, Lemma 2.2],

$$\int h(\bar{x} * \bar{y} * \nu)\, d\nu(x) = h(\nu * \bar{y} * \nu) = \int h(\nu * \bar{z})\, d\bar{y} * \nu(z) = 0 \tag{7}$$

for all $y \in S$. So the continuity of the map $x \to h(\bar{x} * \bar{y} * \nu)$ of S into $[0, \infty)$, equation (7) and the fact that $1 \in \operatorname{supp}(\nu)$ implies that

$$h(\bar{y} * \nu) = 0 \qquad (y \in S).$$

So for any $\mu \in M_a(S)$ we have that

$$h(\mu * \nu) = \int h(\bar{y} * \nu)\, d\mu(y) = 0.$$

Thus $h \in \nu^{\perp}$. This completes the proof for the inclusion $\eta^{\perp} \subset \nu^{\perp}$, claimed above. So we have

$$WUC(S) \subset \text{norm-cl}(WAP(S) + \nu^{\perp} + X). \tag{8}$$

Since $\operatorname{supp}(\nu)$ is compact it follows from (8) and item (1) that S must be relatively neo-compact.

PROOF OF COROLLARY 2.4. As remarked before $M_a(G)$ can be identified with $L^1(G)$ and Mitchell [**13**] showed that $WUC(G) = UC(G)$. Now if η is as stated in the corollary and $x \in \operatorname{supp}(\eta)$, we observe that the identity 1 of G is such that $1 \in \operatorname{supp}(\bar{y} * \eta)$, where $y = x^{-1}$. Further if $f \in \eta^{\perp}$ then

$$\left(f, L^1(G) * (\bar{y} * \eta)\right) = \left(f, \left(L^1(G) * \bar{y}\right) * \eta\right) = \left(f, L^1(G) * \eta\right) = 0.$$

So we may assume that $1 \in \operatorname{supp}(\eta)$, without losing generality. Recalling item (2) of Theorem 2.3 our corollary follows.

6. The proof of Theorem 2.5 and Corollary 2.6. Since for a locally compact topological group G we have $LWUC(G)$ equal to $LUC(G)$, Corollary 2.6 is an immediate consequence of Theorem 2.5. So we shall only give a formal proof for the latter. Also the idea of our proof is inspired by the studies of [**3**].

First we mention the following simple lemma whose proof is standard (and we omit it).

LEMMA 6.1. *Let* $(N, \|\ \|)$ *and* $(M, \|\ \|')$ *be normed linear spaces,* $c > 0$ *a constant and* $\tau\colon N \to M$ *an injective linear map such that* $c\|\tau(x)\|' \geqslant \|x\|$ $(x \in N)$. *If* $(N, \|\ \|)$ *is nonseparable then* $(M, \|\ \|')$ *is nonseparable.*

There exist an infinite compact subset K of S and a point z in K which is not isolated in K. We can find compact neighbourhoods O, U and V in G such that

$$O \text{ is a neighbourhood of the identity of } G,\ K \subset U \text{ and } OU \subset V.$$

Since S is a noncompact closed subset of G we can find an infinite sequence $A := \{x_1, x_2, \ldots\}$ in S such that

$$i \neq j \text{ implies } Vx_i \cap Vx_j = \varnothing \qquad (i, j \in \mathbf{N}). \tag{1}$$

We can choose a sequence $\{U_n\}$ of compact neighbourhoods of z in G and an infinite sequence $\{y_n\} \subset K$ such that

(2) $$U_1 = U, \quad U_{n+1} \subset \operatorname{int}(U_n) \quad \text{and} \quad y_n \in U_n \setminus U_{n+1}.$$

We can find $g_n \in C(G)$ such that

$$g_n(x) = 1 \quad (x \in U_{n+1}) \qquad \text{and} \qquad g_n(x) = 0 \quad (x \in S \setminus U_n).$$

We can choose infinite subsequences $A_k := \{x_{k_1}, x_{k_2}, \dots\}$ of A such that

(3) $$\bigcup_{k=1}^{\infty} A_k = A \text{ and if } k \neq k' \text{ then } A_k \cap A_{k'} = \varnothing.$$

We define f_k on G by $f_k := \sum_{k=1}^{\infty} (g_{k_i})_{x_{k_i}^{-1}}$.

Now let $\{c_k\}$ be any element in l^∞ and consider the function

$$f := \sum_{k=1}^{\infty} c_k f_k.$$

To see that $f \in C(G)$ fix x in G and note that $O^{-1}x$ is a neighbourhood of x. Suppose for some $u \in O^{-1}x$ and positive integer k_j' we have $g_{k_j'}(ux_{k_j'}^{-1}) \neq 0$. Then $u \in U_{k_j'} x_{k_j'}$ and so

$$u \in OU_{k_j'} x_{k_j'} \subset V x_{k_j'}.$$

Consequently if m is any positive integer with $m \neq k_j'$ then

$$(g_m)_{x_m^{-1}}(u) = 0 \quad \text{and so} \quad f(u) = c_{k'} (g_{k_j'})_{x_{k_j'}^{-1}}(u) \qquad (u \in O^{-1}x).$$

Hence f is continuous. Further we have that $\| f_k \|_G \leqslant 1$ and with the help of (1) and (3) we easily note that

$$k \neq k' \text{ and } f_{k'}(x) \neq 0 \quad \text{implies } f_k(x) = 0.$$

Hence $\| f \|_G \leqslant \| \{c_k\} \|_\infty$ and so f is bounded. Consequently f is in $C(G)$.

Now for every g in $m(G)$ taking $g_{|S}$ to be the restriction of g to S, we thus get $f_{|S} := \sum_{k=1}^{\infty} c_k f_{k|S}$ in $C(S)$. Let $f' := f_{|S}$. Observe that if $p \in \{k_1', k_2', \dots\}$ then

(4) $$_{y_n}f'(x_p) := c_{k'} g_p\left(y_n x_p x_p^{-1}\right) = \begin{cases} c_{k'} & \text{if } n > p, \\ 0 & \text{if } n < p. \end{cases}$$

(Here we have used the fact that if $f_{k'}(x) \neq 0$ then $f_k(x) = 0$ for all $k \neq k'$ and note that $g_{k'}(y_n x_p x_{x'}^{-1}) = 0$ for $k_i' \neq p$, by (1), and then finally use (2) to evaluate $g_p(y_n)$.)

Next we consider the function $h := 2f' - \| \{c_k\} \|_\infty$ and note

(5) $$h(y_n x_{k_i'}) = \begin{cases} 2c_{k'} - \| \{c_k\} \|_\infty & \text{if } n > k_i', \\ -\| \{c_k\} \| & \text{if } n < k_i'. \end{cases}$$

Now using (4) and (5) and recalling the definition of $\| \{c_k\} \|_\infty$, an easy adjustment (which we omit) of the argument used to justify item (2) in proof of Theorem 2.1, shows that

$$\| 2f' - \| \{c_k\} \|_\infty + g \|_S = \| h + g \|_S \geqslant \| \{c_k\} \|_\infty \qquad (g \in LWUC(S)).$$

Since $LWUC(S)$ contains the constant functions we thus have

$$2\| f' + g \|_S \geqslant \| \{c_k\} \|_\infty \qquad (g \in LWUC(S)).$$

To summarise the preceding argument, we have shown that the (clearly injective linear map) $\{c_k\} \to \sum_{k=1}^{\infty} c_k f_k + LWUC(S)$ of l^∞ into $C(S)/LWUC(S)$ satisfies the condition

$$2\left\| \sum_{k=1}^{\infty} c_k f_k + LWUC(S)\right\|_{C(S)/LWUC(S)} \geqslant \|\{c_k\}\|_\infty .$$

Since l^∞ is nonseparable, Lemma 6.1 now teaches us that $C(S)/LWUC(S)$ is nonseparable and our proof is complete.

7. Some remarks.

7.1. If S is a C-distinguished topological semigroup with $M_a(S)$ nonzero we then have that $WUC(S) = WAP(S)$ implies that S is relatively neo-compact, by Theorem 2.1. The converse does not always hold. Indeed if S is any topological semigroup with a nonisolated identity element and S_d denotes the semigroup S with discrete topology, then taking $\mathcal{Q} := M(S)$ in [**5**, Lemma 4.2] we can find sequences $\{x_n\}$, $\{y_m\}$ in S such that if $H_1 := \{x_n y_m\colon n < m\}$ and $H_2 := \{x_n y_m\colon n > m\}$ then $H_1 \cap H_2 = \varnothing$. So, as in the proof of [**5**, Theorem 3.5] we note that the characteristic function of H_1 is in $WUC(S_d) \setminus WAP(S_d)$, so that $WUC(S_d) \neq WAP(S_d)$. Now if S is the unit interval [0, 1) with maximum operation then S_d is relatively neo-compact since $S_d = \{1\}1^{-1}$. More generally the reader (who is also familiar with our results in [**5**]) can easily show that if S is a topological semigroup with a nonisolated identity element then $WUC(S_d)/WAP(S_d)$ contains an isometric linear copy of l^∞.

7.2. Let G be a nondiscrete locally compact topological group. G. L. Itzkowitz [**11**] showed that if G is nonunimodular then $LUC(G) \setminus RUC(G) \neq \varnothing$. In particular for such G we thus have $UC(G) \subsetneq LUC(G)$. So the following conjecture seems reasonable.

Conjecture. Let G be a locally compact topological group. If G is nondiscrete and nonunimodular the quotient space $LUC(G)/UC(G)$ is nonseparable.

7.3. Following G. L. Sleijpen (cf. [**14, 15**]) we say a locally compact topological semigroup S with an identity element 1 is a stip if for all compact neighbourhoods V of 1 we have

(a) $x \in \operatorname{int}(V^{-1}(Vx) \cap (xV)V^{-1})$ for each $x \in s$, and

(b) $1 \in \operatorname{int}(V^{-1}v \cap uV^{-1})$ for some $u, v \in V$.

It is shown in [**14**] that if S is a locally compact topological semigroup with an identity element and such that the closure of $\bigcup\{\operatorname{supp}(\nu)\colon \nu \in M_a(S)\} = S$, then S is a stip. However it is still open whether every stip S is such that $M_a(S)$ is nonzero (cf. [**14**]). The properties of a stip S (see [**14, 15**]) clearly show that S behaves as if $M_a(S)$ were nonzero. So the following conjecture seems reasonable.

Conjecture. Let S be a stip which is not relatively neo-compact. Then the quotient space $WUC(S)/WAP(S)$ is nonseparable.

Here we are able to prove the following weaker form.

THEOREM. *Let S be a stip which is not relatively neo-compact. Then $C(S)/WAP(S)$ contains an isometric linear copy of l^∞ so is nonseparable.*

PROOF. We briefly outline how one can use the key lemma and our studies in §4 to obtain the result.

From [14, Theorem 2.7] we can choose compact neighbourhoods V, W of 1 and $w \in W$ such that

$$(1)\qquad \begin{cases} V \subset wW^{-1} \cap W^{-1}w \cap W, \\ x \in \operatorname{int}((xW)w^{-1}) \qquad (x \in S). \end{cases}$$

In our key lemma, take $C := WV$ and $D := VW$. Now property (a) in our definition of a stip enables us to find a compact neighbourhood $O_{n,m}$ of $x_n y_m$ with

$$(2)\qquad O_{n,m} \subset V^{-1}(Vx_n y_m V)V^{-1} \qquad (n, m \in \mathbf{N}).$$

Consider any $x \in S$. By (1) we have that $(xW)w^{-1}$ is a neighbourhood of x. Suppose $(xW)w^{-1} \cap V^{-1}(Vx_n y_m V)V^{-1} \neq \varnothing$, then $xWw^{-1} \cap w^{-1}(WVx_n y_m VW)w^{-1} \neq \varnothing$ by (1). An easy calculation now gives

$$x \in (WV)^{-1}(WVx_n y_m VW)(VW)^{-1} = C^{-1}(Cx_n y_m D)D^{-1}.$$

Summarising this we have the following: If i, j, n, m are as stated in Lemma 3 then

$$(3)\qquad \begin{cases} (xW)w^{-1} \cap V^{-1}(Vx_n y_m V)V^{-1} \neq \varnothing \text{ implies} \\ (xW)w^{-1} \cap V^{-1}(Vx_i y_j V)V^{-1} = \varnothing. \end{cases}$$

Now we can find positive functions $h_{n,m}$ in $C(S)$ such that

$$(4)\qquad h_{n,m}(x_n y_m) = 1, \quad \|h_{n,m}\| = 1 \quad \text{and} \quad h_{n,m} = 0 \text{ outside } O_{n,m}.$$

Taking all sequences of elements of S constructed in §4 as done there, we define functions f_k by

$$f_k := \sum_{i=1}^{\infty} \sum_{i \leqslant j}^{\infty} h_{k_i,k_j} - \sum_{j=1}^{\infty} \sum_{j<i}^{\infty} h_{k_i,k_j}.$$

Using (2) and (3) one can easily get that $f_k \in C(S) \setminus WAP(S)$. The reader who is familiar with our studies in §4 can now easily complete the proof.

Thus the main difficulty in proving the above conjecture lies in the lack of a technique for constructing functions in $WUC(S) \setminus WAP(S)$. We suspect the notion of functions with separable orbits used to characterize certain members of $UC(S)$ in [6] might be of use in solving the above conjecture.

7.4. In the first draft of this paper we raised the following conjecture.

Conjecture. If S is a stip, then $WUC(S) = UC(S)$ and hence $WAP(S) \subseteq UC(S)$.

This conjecture has subsequently been solved by us in the paper: *Weak and norm continuity of semigroup actions on normed linear spaces*, Quart. J. Math. Oxford Ser. (2) (to appear).

7.5. We wish to point out one very significant contribution made in §4. In [3] the notion of groups G with property (E) is used to construct functions in $UC(G) \setminus WAP(G)$ (see [3, Lemma 4.5 and the proof of Theorem 5.1]). This is done by considering certain translates of a function with compact support. Such a device is not available in the semigroup case (as one can easily check by considering the additive semigroup $S := [0, \infty)$ with the usual topology) and does not seem to be available for general locally compact groups (i.e. without necessarily assuming property (E)). In §4 this difficulty is avoided by using $M_a(G)$ to yield functions in $UC(G) \setminus WAP(G)$ for all noncompact locally compact groups G.

7.6. A more recent collection of results on various subspaces of $C(S)$, for other types of semigroups S, can be found in the book of J. F. Berglund, H. D. Junghenn and P. Milnes, *Compact right topological semigroups and generalizations of almost periodicity*, Lecture Notes in Math., Vol. 663, Springer-Verlag, Berlin and New York, 1978.

7.7. It is not difficult to observe that results analogous to ours will hold for complex-valued functions.

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