SIMPLEXES OF EXTENSIONS OF STATES OF C*-ALGEBRAS

BY

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ABSTRACT. Let $B$ be a C*-subalgebra of a C*-algebra $A$, $F$ a compact face of the state space $S(B)$ of $B$, and $S_F(A)$ the set of all states of $A$ whose restrictions to $B$ lie in $F$. It is shown that $S_F(A)$ is a Choquet simplex if and only if (a) $F$ is a simplex, (b) pure states in $S_F(A)$ restrict to pure states in $F$, and (c) the states of $A$ which restrict to any given pure state in $F$ form a simplex. The properties (b) and (c) are also considered in isolation.

Sets of the form $S_F(A)$ have been considered by various authors in special cases including those where $B$ is a maximal abelian subalgebra of $A$, or $A$ is a C*-crossed product, or the Cuntz algebra $\mathcal{O}_n$.

1. Introduction. Let $A$ be a C*-algebra with quasi-state space $Q(A)$:

$$Q(A) = \{ \phi \in A^*: \phi \geq 0, \| \phi \| \leq 1 \}.$$ 

Let $B$ be a C*-subalgebra of $A$ and $F$ a nonempty (weak*) closed face of $Q(B)$. There are various situations in which one is interested in the structure of extensions of functionals in $F$. Thus one studies the nonempty closed face

$$Q_F(A) = \{ \phi \in Q(A): \phi|_B \in F \}$$

of $Q(A)$. For example, $B$ might be a maximal abelian C*-subalgebra (masa) in $A$, and $F$ consist of a single pure state; a problem of some complexity is to determine whether $Q_F(A)$ also contains only a single (pure) state [1, 2, 3]. Alternatively, $A$ might be (the multiplier algebra of) the crossed product $G \times_\alpha A_0$ of some C*-dynamical system $(A_0, G, \alpha)$, $B$ the C*-subalgebra $C^*(G)$ of $A$, and $F$ consist of the single state $\phi_0$ of $B$ with $\phi_0(u_g) = 1$ $(g \in G)$, where $u$ is the universal representation of $G$ in $C^*(G)$. Then $Q_F(A)$ is isomorphic to the set $Q^G(A_0)$ of $G$-invariant functionals in $Q(A_0)$ [4]. In algebraic models of statistical mechanics, a fundamental question is therefore whether $Q_F(A)$ is a Choquet simplex [7, §4.3].

Here we shall be concerned with the general question of when $Q_F(A)$ is a simplex. A general criterion has been established in [4, 5] for a closed face $K$ of $Q(A)$ to be a simplex. This takes on several forms, the simplest of which is that distinct pure states in $F$ are (unitarily) inequivalent. From this, it will be established in Theorem 4.1 that $Q_F(A)$ is a simplex if and only if the following three conditions are all satisfied:

(a) $F$ is a simplex;
(b) any pure state in $Q_F(A)$ restricts to a pure state of $B$;

Received by the editors March 31, 1981.

1980 Mathematics Subject Classification. Primary 46L30.

Key words and phrases. Pure state, extension, restriction, simplex, face, irreducible representation, invariant state, crossed product.

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0002-9947/81/0000-0377/$03.75
(c) for any pure state $\psi$ in $F$, the extensions of $\psi$ in $Q(A)$ form a simplex.

(Here and throughout the paper, the zero functional is conventionally regarded as a "pure state", so the pure states are the extreme points of $Q(A)$.) Each of these conditions can be considered separately. For (a), this was done in [4, 5]; we shall see in §2 that (b) places a strong restriction on the GNS-representation of pure states in $Q_F(A)$, and in §3 that (b) comes very close to implying (c) in the special case when $F = Q(B)$. (In this special case, (c) is the simplex extension property (SEP) introduced in [6].) The final two sections will be concerned with applications to $C^*$-dynamical systems. Taking $A = G \times_A A_0$ and $B = G \times_B B_0$ where $B_0$ is a $G$-invariant subalgebra of $A_0$, conditions will be obtained which ensure that the system $(A_0, G, \alpha)$ is abelian when it is known that $(B_0, G, \alpha)$ is abelian. Taking $A = G \times_A A_0$ where $A_0$ is commutative and $G$ is discrete, and $B = A_0$, it will be shown that $A_0$ has the (SEP) in $G \times_A A_0$ if and only if the stabiliser in $G$ of each (nonzero) pure state of $A_0$ is abelian. These results include as special cases some known properties of extensions of pure states of masas in $0^\prime$ and $DC < 2 > 0n$ [9, 11].

Throughout, $A$ will be a $C^*$-algebra, and $B$ will be a $C^*$-subalgebra of $A$. For a state $\phi$ of $A$ with restriction $\psi$ to $B$, $(\mathbb{C}_\psi, \pi_\psi, \xi_\psi)$ will be the associated cyclic representation of $A$, and $([\pi_\psi(B)\xi_\psi], \pi_\psi|_B, \xi_\psi)$ will be identified with $(\mathbb{C}_\psi, \pi_\psi, \xi_\psi)$. If $\phi$ lies in a closed face $K$ of $Q(A)$, then $p^K_\phi$ will denote the orthogonal projection of $\mathbb{K}_\phi$ onto the closed subspace $\mathbb{K}^K$ of all vectors $\eta$ for which the vector functional $\omega_\phi: a \to \langle \pi_\phi(a)\eta, \eta \rangle$ lies in the cone generated by $K$, and the dimension of $\mathbb{K}^K$ will be called the $K$-multiplicity of $\phi$. Conventionally, the $K$-multiplicity of the zero functional is taken to be 1. It was shown in [4, 5] that $K$ is a simplex if and only if each pure state in $K$ has $K$-multiplicity 1, or, in other words, no two pure states in $K$ are equivalent. This fact will be used repeatedly without further reference.

For a pure state $\psi$ of $B$, let $Q_\psi(A) = Q_F(A)$, where $F = \{\psi\}$, so

$$Q_\psi(A) = \{\phi \in Q(A) : \phi|_B = \psi\}.$$ 

Thus $B$ has the (SEP) in $A$ if and only if $Q_\psi(A)$ is a simplex for each pure state $\psi$ of $B$, or equivalently no two distinct equivalent pure states of $A$ have the same pure restriction to $B$.

In some respects, little would be lost in the following if it was assumed that $A$ and $B$ have a common unit, and that $F$ is a closed (hence compact) face of the state space $S(A)$ of $A$. For one may otherwise adjoin a common unit obtaining $C^*$-algebras $\hat{A}$ and $\hat{B}$ and identify $Q(A)$ with $S(\hat{A})$; nearly all the properties considered here are preserved under the passage between $(A, B)$ and $(\hat{A}, \hat{B})$. However, at certain points it will be necessary to consider ideals and crossed products, so no assumption about the presence of a unit is appropriate.

### 2. Restriction property for faces

The first property of faces which we shall study is described in the following definition.

**Definition 2.1.** A face $K$ of $Q(A)$ has the restriction property (RP) to $B$ if the restriction to $B$ of each pure state in $K$ is a pure state of $B$.

**Lemma 2.2.** Let $K$ be a face of $Q(A)$ with the (RP) to $B$, and $\phi$ and $\phi'$ be equivalent pure states in $K$. Then the restrictions of $\phi$ and $\phi'$ to $B$ are equivalent pure states.
Proof. Suppose that the restrictions \( \psi \) and \( \psi' \) of \( \phi \) and \( \phi' \) are inequivalent. By assumption there is a unit vector \( \eta \) in \( \mathcal{H}_\phi^K \) such that \( \phi' = \omega_\phi^{\eta} \), and the representation \( \pi_\phi \) of \( B \) has inequivalent irreducible restrictions to \( \mathcal{H}_\psi \) and \( \mathcal{H}_{\psi'} \) (\( = [\pi_\phi(B)\eta] \)). Hence \( \mathcal{H}_\psi \) and \( \mathcal{H}_{\psi'} \) are orthogonal. If \( \eta' = 2^{-1/2}(\xi_\phi + \eta) \in \mathcal{H}_\phi^K \), \( \omega_\phi^{\eta'} \) is a pure state of \( A \) lying in \( K \), so its restriction to \( B \) is pure. But \( \omega_\phi^{\eta'} |_B = \frac{1}{2}(\psi + \psi') \), so \( \psi = \psi' \). This is a contradiction.

Theorem 2.3. Let \( F \) be a closed face of \( Q(B) \), \( K = Q_F(A) \), \( \phi \) a (nonzero) pure state in \( K \), and \( \psi = \phi |_B \), and suppose that \( K \) has the (RP) to \( B \), so that \( \psi \) is a pure state of \( B \).

(i) If the \( F \)-multiplicity of \( \psi \) is 1, then

\[
p^K_\phi \pi_\phi(b) p^K_\phi = \psi(b) p^K_\phi \quad (b \in B).
\]

(ii) If the \( F \)-multiplicity of \( \psi \) is greater than 1, then \( \mathcal{H}_\phi^K = \mathcal{H}_\phi^F \).

Proof. (i) Suppose that the \( F \)-multiplicity of \( \psi \) is 1, so that there are no other pure states in \( F \) equivalent to \( \psi \). For any unit vector \( \eta \) in \( \mathcal{H}_\phi^K \), the restriction \( \psi' \) of \( \omega_\phi^{\eta} \) to \( B \) is a pure state in \( F \), and by Lemma 2.2, \( \psi' \) is equivalent to \( \psi \), so \( \psi' = \psi \). Thus

\[
\langle \pi_\phi(b) \eta, \eta \rangle = \psi(b) \| \eta \|^2 \quad (b \in B, \eta \in \mathcal{H}_\phi^K),
\]

so \( p^K_\phi \pi_\phi(b) p^K_\phi = \psi(b) p^K_\phi \).

(ii) Suppose that the \( F \)-multiplicity of \( \psi \) is greater than 1, and let \( \xi' \) be a unit vector in \( \mathcal{H}_\phi^F \) orthogonal to \( \xi_\phi \), and \( \eta \) any unit vector in \( \mathcal{H}_\phi^K \). The restriction \( \psi' \) of \( \omega_\phi^{\eta} \) to \( B \) is a pure state equivalent to \( \psi \) (Lemma 2.2). Thus there is a unitary operator \( u \) of \( \mathcal{H}_\phi \) onto \( \mathcal{H}_{\psi'} \) (\( = [\pi_\phi(B)\eta] \)) such that

\[
u \pi_\phi(b) \xi = \pi_\phi(b) u \xi' \quad (b \in B, \xi \in \mathcal{H}_\phi).
\]

Now

\[
\langle \pi_\phi(b) u \xi', u \xi' \rangle = \langle u \pi_\phi(b) \xi', u \xi' \rangle = \langle \pi_\phi(b) \xi', \xi' \rangle
\]

so \( \omega_\phi^{u \xi'} |_B = \omega_\phi^{\xi'} |_B = \omega_\phi^{\xi'} \in F \), and therefore \( u \xi' \in \mathcal{H}_\phi^K \). Let \( \eta' = \xi_\phi - u \xi' \), and \( \eta'' = \| \eta' \|^{-1} \eta' \) (note that \( \eta'' \neq 0 \)). Since \( \eta'' \in \mathcal{H}_\phi^K \) and \( K \) has the (RP) to \( B \), the restriction \( \psi'' \) of \( \omega_\phi^{\eta''} \) to \( B \) is a pure state in \( F \).

Since \( \pi_\phi \) is irreducible, Kadison’s Transitivity Theorem shows that there is some \( b_0 \) in \( B \) with

\[
\pi_\phi(b_0) \xi_\phi = \xi_\phi, \quad \pi_\phi(b_0) \xi' = 0.
\]

Then

\[
\pi_\phi(b_0) \eta' = \pi_\phi(b_0) \xi_\phi - u \pi_\phi(b_0) \xi' = \xi_\phi.
\]

Since the restriction to \( B \) of \( \pi_\phi \) is irreducible on \( \mathcal{H}_\psi = [\pi_\phi(B)\eta'] \),

\[
\eta' \in [\pi_\phi(B) \xi_\phi] = \mathcal{H}_\psi.
\]

Hence \( u \xi' = \xi_\phi - \eta' \in \mathcal{H}_\psi \). Since the restriction to \( B \) of \( \pi_\phi \) is irreducible on \( \mathcal{H}_\psi \), which contains \( u \xi' \),

\[
\eta \in [\pi_\phi(B)u \xi'] \subseteq [\pi_\phi(B) \mathcal{H}_\psi] = \mathcal{H}_\psi.
\]
But $\omega^\psi_0 = \omega^\psi_0|_B \in F$, so $\eta \in \mathcal{K}^F_\psi$. Thus $\mathcal{K}^F_\psi \subseteq \mathcal{K}^F_\psi$. The reverse inclusion is immediately verified.

There are various extreme cases of Theorem 2.3. §3 will be devoted to the case when $F = Q(B)$. In the opposite extreme when $F = \{\psi\}$ for a pure state $\psi$ of $B$, $Q_F(A)$ automatically has the (RP) to $B$, and Theorem 2.3(i) is applicable. A slightly less special case occurs when $F$ is generated by two equivalent pure states, and this leads to the following result.

**Corollary 2.4.** Let $\psi$ be a pure state of $B$, and suppose that $Q_\psi(A)$ is not a simplex. Let $\psi'$ be a pure state of $B$ equivalent, but not equal, to $\psi$. Then there is a pure state $\phi$ of $A$ and a real number $\lambda > 0$ such that $\phi|_B$ is not pure, and $\phi(b) \leq \lambda(\psi(b) + \psi'(b))$ $(b \in B)$.

**Proof.** Let $F$ be the smallest face of $Q(B)$ containing $\psi$ and $\psi'$, so that

$$K = Q_F(A) = \{\phi \in Q(A): \phi|_B \leq \lambda(\psi + \psi') \text{ for some } \lambda > 0\}.$$

Suppose that the conclusion of the corollary is false, so that $K$ has the (RP) to $B$. Since $\psi$ and $\psi'$ are distinct equivalent pure states in $F$, the $F$-multiplicity of $\psi$ is greater than 1.

Since $Q_\psi(A)$ is not a simplex, there are distinct equivalent pure states $\phi$ and $\phi'$ in $Q_\psi(A)$. Let $\eta$ be a unit vector in $\mathcal{K}_\phi$ with $\omega^\psi_0 = \phi'$. Then $\omega^\psi_0|_B = \psi \in F$, so $\eta \in \mathcal{K}_\phi = \mathcal{K}_\psi$ (Theorem 2.3(ii)). Now $\omega^\psi_0 = \psi$, and $\pi_\phi$ is irreducible, so $\eta$ is a scalar multiple of $\xi_\psi = \xi_\phi$. But this contradicts the fact that $\phi$ and $\phi'$ are distinct.

3. The restriction property for algebras.

**Definition 3.1.** A C*-subalgebra $B$ of $A$ has the restriction property (RP) in $A$ if the restriction to $B$ of each pure state of $A$ is pure on $B$.

Thus $B$ has the (RP) in $A$ if and only if $Q(A)$ has the (RP) to $B$. Any (closed two-sided) ideal has the (RP) in $A$; an abelian C*-subalgebra has the (RP) in $A$ if and only if it is contained in the centre of $A$; a C*-subalgebra $B$ has the (RP) in $A$ if $A$ coincides with the C*-subalgebra $(B : A)$ generated by operators of the form $zb$, where $b \in B$ and $z$ is a central multiplier of $A$.

The next two results follow from Theorem 2.3 and Corollary 2.4 on taking $F = Q(B)$, so that $K = Q(A)$, $\mathcal{K}^F_\phi = \mathcal{K}_\phi$, $\mathcal{K}^K_\phi = \mathcal{K}_\phi$, and the $F$-multiplicity of $\psi$ is 1 if and only if $\psi$ is multiplicative. The proof of Theorem 2.3 can be made very short in this case.

**Proposition 3.2.** Suppose $B$ has the (RP) in $A$, and let $\psi$ be a pure state of $B$ and $\phi$ a pure state of $A$ extending $\psi$.

(i) If $\psi$ is multiplicative, then $\tau_\phi(b) = \psi(b)1$ $(b \in B)$.

(ii) If $\psi$ is not multiplicative, then $\mathcal{K}_\phi = \mathcal{K}_\psi$.

**Corollary 3.3.** Suppose $B$ has the (RP) in $A$, and $\psi$ is a nonmultiplicative pure state of $B$. Then $Q_\psi(A)$ is a simplex.

It is to be expected that the (RP) will be related to properties of restrictions of irreducible representations. The precise extent of this connection will now be discussed.
DEFINITION 3.4. A C*-subalgebra $B$ of $A$ has the irreducible representation property (IRP) in $A$ if, for each irreducible representation $\pi$ of $A$, $\pi(B)$ is either irreducible or zero.

It is immediate from Definition 3.4 that a C*-subalgebra which is rich (in the sense of [10, §11.1.1]) has the (IRP); any ideal has the (IRP) in $A$; if $A = (B : A)$, then $B$ has the (IRP) in $A$. If $\Omega$ is a locally compact Hausdorff space, $\{ A_\omega : \omega \in \Omega \}$ is a family of C*-algebras, $\Gamma_1$ and $\Gamma_2$ are continuous vector fields over this family with $\Gamma_2 \subseteq \Gamma_1$, and $A$ and $B$ are the corresponding C*-algebras, then $B$ has the (IRP) in $A$ [10, Théorème 10.4.3].

PROPOSITION 3.5. Let $J_0$ be the ideal in $A$ generated by the commutators $\{ ab - ba : a \in A, b \in B \}$. The following are equivalent:

(i) $B$ has the (RP) in $A$;

(ii) $B \cap J_0$ has the (IRP) in $J_0$;

(iii) there is an ideal $J$ in $A$ such that $(B + J)/J$ is contained in the centre of $A/J$, and $B \cap J$ has the (IRP) in $J$.

PROOF. (i) $\Rightarrow$ (ii) Let $(\mathcal{H}, \pi)$ be an irreducible representation of $J_0$, $(\mathcal{H}, \widehat{\pi})$ its unique extension to $A$, $\xi$ a unit vector in $\mathcal{H}$ and $\phi$ the vector state: $\phi(a) = \langle \pi(a)\xi, \xi \rangle$. Since $\pi$ does not vanish on $J_0$, it follows from Proposition 3.2 that $\phi$ is not multiplicative on $B$, and hence that $\mathcal{H} = \{ \#(\#)\xi \}$. Thus $(\mathcal{H}, \widehat{\pi}|_B)$ is the GNS-representation of the pure state $\phi|_B$, and is therefore irreducible. Hence $\pi(B \cap J_0)$ is zero or irreducible.

(ii) $\Rightarrow$ (iii) This is immediate on taking $J = J_0$.

(iii) $\Rightarrow$ (i) Let $\phi$ be a pure state of $A$. If $\phi$ vanishes on $J$, then $\phi$ induces a pure state $\phi$ of $A/J$, which is therefore multiplicative on $(B + J)/J$. Hence $\phi$ is multiplicative and therefore pure on $B$.

If $\phi$ does not vanish on $J$, then $\pi_\phi(J)$ is irreducible on $\mathcal{H}_\phi$. By assumption, $\pi_\phi(B \cap J)$ is either zero or irreducible. If $\pi_\phi(B \cap J)$ is irreducible, then $\phi|_B$ is the unique extension of the pure state $\phi|_{B \cap J}$ to a state of $B$, so $\phi$ is pure on $B$.

If $\phi$ vanishes on $B \cap J$, then $\phi$ induces a pure state of $B/(B \cap J)$ which is isomorphic to $(B + J)/J$, so by the Hahn-Banach theorem, there is a pure state $\phi'$ of $A$ coinciding with $\phi$ on $B$ and vanishing on $J$. As in the first part of this proof, $\phi'|_B$ is pure, so $\phi|_B$ is pure.

COROLLARY 3.6. If $B$ has no multiplicative states, or if $A$ is simple and $B$ is nontrivial, then the (RP) and the (IRP) are equivalent.

If $A$ is type I and $B$ has the (IRP) in $A$, then there is a composition series of ideals $(I_\rho)_{0 \leq \rho < \rho_0}$ for $A$ such that for each ordinal $\rho < \rho_0$, either $(B \cap I_{\rho + 1} + I_\rho)/I_\rho$ is contained in the centre of $I_{\rho + 1}/I_\rho$, or $((B \cap I_{\rho + 1} + I_\rho)/I_\rho : I_{\rho + 1}/I_\rho) = I_{\rho + 1}/I_\rho$. However, the converse of this is not valid. For example, let $\mathcal{K}$ be the C*-algebra of compact operators on some Hilbert space $\mathcal{H}$, $B$ a (type I) C*-algebra of operators on $\mathcal{H}$ with $B \cap \mathcal{K} = \{ 0 \}$, $A = B + \mathcal{K}$, $I_0 = \{ 0 \}$, $I_1 = \mathcal{K}$, $I_2 = A$. The identity representation is irreducible on $A$ but not on $B$. 

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4. $Q_F(A)$ as a simplex. It has already been seen in Corollaries 2.4 and 3.3 that the (RP) is related to whether extension faces $Q^\psi(A)$ are simplexes. The following result strengthens this point.

**Theorem 4.1.** Let $F$ be a closed face of $Q(B)$. Then $Q_F(A)$ is a simplex if and only if the following three conditions are all satisfied:

(a) $F$ is a simplex;
(b) $Q_F(A)$ has the (RP) to $B$;
(c) for each pure state $\psi$ of $B$ in $F$, $Q^\psi(A)$ is a simplex.

**Proof.** Suppose first that $Q_F(A)$ is a simplex. For any pure state $\psi$ in $F$, $Q^\psi(A)$ is a face of $Q_F(A)$ and is therefore a simplex.

Let $\psi$ and $\psi'$ be equivalent pure states in $F$, and $\phi$ be any pure state in $Q^\psi(A)$. There is a vector $\eta$ in $\mathcal{K}_\psi (\subseteq \mathcal{K}_\phi)$ such that $\psi' = \omega^\eta_\psi$. Thus $\omega^\eta_\phi$ belongs to the simplex $Q_F(A)$, so $\phi = \omega^\eta_\phi$ and $\psi = \psi'$. Hence $F$ is a simplex.

Let $\phi$ be a pure state in $Q_F(A)$, $\phi = \phi|_B$, and suppose that $\phi = \frac{1}{2}(\psi_1 + \psi_2)$ for some $\psi_1$ and $\psi_2$ in $F$. Since $\psi_j \leq 2\psi$ ($j = 1, 2$), there are vectors $\eta_j$ in $\mathcal{K}_\psi (\subseteq \mathcal{K}_\phi)$ such that $\psi_j = \omega^\eta_\psi$. Let $\phi_j = \omega^\eta_\phi$, so that $\phi_j|_B = \psi_j$. Then $\phi_1$ and $\phi_2$ are equivalent pure states of $A$ belonging to the simplex $Q_F(A)$. Hence $\phi_1 = \phi_2$, so $\psi_1 = \psi_2$. Thus $\psi$ is pure.

Conversely, suppose that conditions (a)–(c) are satisfied. Let $\phi$ and $\phi'$ be equivalent pure states in $Q_F(A)$. It follows from (b) and Lemma 2.2 that $\phi|_B$ and $\phi'|_B$ are equivalent pure states of $B$. It now follows from (a) that $\phi|_B = \phi'|_B$, so $\phi$ and $\phi'$ are equivalent pure states in $Q^\psi(A)$ for some pure state $\psi$ in $F$. Finally, condition (c) shows that $\phi = \phi'$, so $Q_F(A)$ is a simplex.

**Corollary 4.2.** Suppose $B$ has the (RP) in $A$, and $F$ is a closed face of $Q(B)$ containing no multiplicative functionals (in particular, $0 \notin F$). Then $Q_F(A)$ is a simplex if and only if $F$ is a simplex.

**Proof.** This is immediate from Theorem 4.1 and Corollary 3.3.

5. C*-dynamical systems. Let $G \times_\alpha A$ be the C*-crossed product of a C*-dynamical system $(A, G, \alpha)$; let $u: G \to M(G \times_\alpha A)$ be the universal representation of $G$ in the multiplier algebra of $G \times_\alpha A$, and regard $A$ as embedded as a C*-subalgebra of $M(G \times_\alpha A)$. Let

$$F_G(A) = \{ \phi \in Q(G \times_\alpha A): \phi(u_g) = \|\phi\|(g \in G) \}.$$  

Then $F_G(A)$ is a closed face of $Q(G \times_\alpha A)$, and the restriction map is an affine homeomorphism of $F_G(A)$ onto the set $Q^G(A)$ of $G$-invariant functionals in $Q(A)$ [4, Theorem 4.2]. It will be convenient to refer to the extreme points of $Q^G(A)$ (including 0) as "$G$-ergodic states". The system $(A, G, \alpha)$ is said to be abelian if $Q^G(A)$ is a simplex. (See [4, 5] for equivalent definitions.)
Theorem 5.1. Let $B$ be a $G$-invariant $C^*$-subalgebra of $A$, and suppose that the following three conditions are all satisfied:

(a) $(B, G, \alpha)$ is abelian;
(b) every $G$-ergodic state of $A$ restricts to a $G$-ergodic state of $B$,
(c) For any $G$-ergodic state $\psi$ of $B$, the set $Q^G(A) = \{\phi \in Q^G(A): \phi|_B = \psi\}$ is a simplex (possibly empty).

Then $(A, G, \alpha)$ is abelian.

Conversely, if $(A, G, \alpha)$ is abelian, then (b) and (c) hold. If $G$ is amenable and $(A, G, \alpha)$ is abelian, then (a) also holds.

Proof. Let $A_1 = G \times_\alpha A$, $\Phi: G \times_\alpha B \to A_1$ be the canonical $*$-homomorphism acting as the identity on $B$ and on $u_G$, $B_1 = \Phi(G \times_\alpha B)$, and

$$F = \{\phi|_{B_1}: \phi \in F^G(A)\}, \quad F' = \{\phi \circ \Phi: \phi \in F^G(A)\}.$$  

Then $F$ and $F'$ are affinely homeomorphic closed faces of $S(B_1)$ and $F^G(B)$, respectively, so (a) implies that $F$ is a simplex. Since $Q^G(A_1) = F^G(A)$, (b) and (c) reduce to the corresponding conditions of Theorem 4.1 for the $C^*$-algebra $A_1$ and subalgebra $B_1$. If $G$ is amenable, then $\Phi$ is isometric, and $F' = F^G(B)$, so (a) is equivalent to $F$ being a simplex. Thus the results follow from Theorem 4.1.

Corollary 5.2. Let $B$ be a $G$-invariant $C^*$-subalgebra of $A$, and suppose that $\alpha(G)$ includes all the inner automorphisms of $A$ implemented by unitaries in $B$. Then $(A, G, \alpha)$ is abelian if and only if, for any $G$-ergodic state $\psi$ of $B$, the set $Q^G(A)$ is a simplex.

Proof. Any $C^*$-dynamical system containing all the inner automorphisms is abelian, so Theorem 5.1(a) is satisfied. It has been shown in [12, Lemma 3] that (b) is satisfied.

Recall that a $C^*$-subalgebra $B$ has the simplex extension property (SEP) in $A$ if $Q^G(A)$ is a simplex for each pure state of $B$ [6]; a functional $\phi$ in $A^*$ is $B$-central if $\phi(ab) = \phi(ba)$ ($a \in A$, $b \in B$). The relationship between $B$-central states and extensions has been studied in [3] in the case when $B$ is a masa in $A$. It was shown there that an abelian $C^*$-subalgebra $B$ has the (SEP) in a type I $C^*$-algebra $A$ if $\pi(B)$ is a masa of $\pi(A)$ for each irreducible representation $\pi$ of $A$, and that under these circumstances the $B$-central functionals form a simplex. This latter fact is a special case of the following result.

Corollary 5.3. Let $B$ be an abelian $C^*$-subalgebra of $A$. Then $B$ has the (SEP) in $A$ if and only if the $B$-central functionals in $Q(A)$ form a simplex.

Proof. Let $G$ be the unitary group of $B$ acting as inner automorphisms of $A$. Then $Q^G(A)$ is the set of $B$-central functionals in $Q(A)$, and the $G$-ergodic states are precisely those pure states of $A$ which are multiplicative on $B$ [3, 12]. The result now follows immediately from Corollary 5.2.
Corollary 5.4. Let \((A, G, \alpha)\) be an abelian \(C^*\)-dynamical system, where \(G\) is abelian, and let \((G \times_\alpha A, G, \tilde{\alpha})\) be the system defined by
\[
\tilde{\alpha}_g(x) = u_g x u_g^* \quad (x \in G \times_\alpha A, g \in G).
\]
Then \((G \times_\alpha A, G, \tilde{\alpha})\) is abelian.

Proof. Let \(B\) be the \(C^*\)-subalgebra of \(G \times_\alpha A\) generated by \(u_G\), so \(B\) is (isomorphic to) the group \(C^*\)-algebra of \(G\), whose pure state space is the dual group \(\hat{G}\). The \(G\)-invariant functionals on \(G \times_\alpha A\) are precisely those which are \(B\)-central, so (a slight extension of) Corollary 5.3 shows that it is sufficient to prove that \(Q_y(G \times_\alpha A)\) is a simplex for each \(y\) in \(\hat{G}\). Let \((G \times_\alpha A, \hat{G}, \tilde{\alpha})\) be the dual system of \((A, G, \alpha)\) [13, §7.9]. Then \(\phi \to \phi \circ \tilde{\alpha}_g\) is an affine homeomorphism of \(Q_y(G \times_\alpha A)\) onto \(F_G(A)\), which, by assumption, is a simplex.

Corollary 5.4 may fail if \(G\) is not abelian. For example, one may take \(G\) to be the alternating group on 7 letters, \(A\) to be the group \(C^*\)-algebra of \(G\), and \(\alpha\) to be the action by conjugation.

6. Topological dynamical systems. In this final section, let \((C, G, \alpha)\) be a \(C^*\)-dynamical system, where \(C\) is abelian and \(G\) is discrete. (Some of the discussion can be modified for locally compact groups.) Let \(\Omega\) be the pure state space of \(C\), so that \(C \cong C_0(\Omega)\), and there is an action of \(G\) on \(\Omega\) such that \(\omega(\alpha_g(x)) = (g^{-1} \cdot \omega)(x)\) \((x \in C, \omega \in \Omega, g \in G)\). The alternative notation \(C^*(\Omega, G)\) will be used for the \(C^*\)-crossed product \(G \times_\alpha C\). Now \(C\) is a subalgebra of \(C^*(\Omega, G)\), and Proposition 6.1 will identify the faces \(Q_\omega(C^*(\Omega, G))\). Let \(P: C^*(\Omega, G) \to C\) be the canonical projection, so that
\[
P(x u_g) = 0 \quad (x \in C, g \in G, g \neq e),
P(x) = x \quad (x \in C).
\]
States \(\phi\) of \(C^*(\Omega, G)\) will be identified with normalised positive-definite functions \(\Phi: G \to C^*\) given by
\[
\Phi(g)(x) = \phi(x u_g).
\]
In particular, there are no states of \(C^*(\Omega, G)\) which vanish on \(C\).

Proposition 6.1. Let \(\omega\) be a point of \(\Omega\). Then \(Q_\omega(C^*(\Omega, G))\) is affinely homeomorphic to the state space of the group \(C^*\)-algebra \(C^*(G_\omega)\) of the stabilizer \(G_\omega\) of \(\omega\) in \(G\). Furthermore \(\omega \circ P\) is pure if and only if \(G_\omega\) is trivial.

Proof. Let \(\Psi: G \to C\) be a normalised positive-definite function, and define \(\Phi: G \to C^*\) by
\[
\Phi(g) = \Psi(g) \omega \quad (g \in G_\omega),
\]
\[
= 0 \quad (g \in G \setminus G_\omega).
\]
For \( g_i \) in \( G \) and \( x_i \) in \( C \) \((i = 1, \ldots, n)\),

\[
\sum_{i, j=1}^{n} \Phi \left( g_i^{-1}g_j \right) \left( \alpha_{g_i^{-1}}(x_i^*x_j) \right) = \sum_{g_i^{-1}g_j \in G_{\omega}} \overline{(g_i \cdot \omega)(x_i)} (g_i \cdot \omega)(x_j) \Psi \left( g_i^{-1}g_j \right)
\]

\[
= \sum_{\omega \in \Omega} \sum_{i, j=1}^{n} \lambda_i(\omega') \lambda_j(\omega') \Psi \left( g_i^{-1}g_j \right) \geq 0,
\]

where \( \lambda_i(\omega \cdot \omega') = (g_i \cdot \omega)(x_i), \lambda_i(\omega') = 0 \) \((\omega' \neq g_i \cdot \omega)\). Thus \( \Phi \) is positive-definite. Furthermore the corresponding state \( \phi \) of \( C^*(\Omega, G) \) satisfies

\[
\phi(x) = \Phi(e)(x) = \omega(x) \quad (x \in C)
\]

so \( \phi \) belongs to \( Q_{\omega}(C^*(\Omega, G)) \).

Conversely, for \( \phi \) in \( Q_{\omega}(C^*(\Omega, G)) \), \( x \) in \( C \) and \( g \) in \( G \),

\[
\omega(x)\phi(u_g) = \phi xu_g = \phi(u_g^*x^*) = \phi(\alpha_{g^{-1}}(x^*)u_g^*) = (g \cdot \omega)(x)\phi(u_g).
\]

Thus \( \phi(u_g) = 0 = \phi(xu_g) \) \((g \in G \setminus G_{\omega})\) and \( \phi(xu_g) = \omega(x)\phi(u_g) \) \((g \in G_{\omega})\), so \( \phi \) is of the above form. It is clear that \( \Psi \to \Phi \) is an affine homeomorphism.

If \( G_{\omega} \) is trivial, then \( \omega \circ P \) is the unique state extension of the pure state \( \omega \) to \( C^*(\Omega, G) \), and is therefore pure.

In general, the GNS-representation of \( \omega \circ P \) may be identified with the induced representation \( \pi \times \lambda \) of \( C^*(\Omega, G) \) on \( l^2(G) \), where \( \lambda \) is the left regular representation of \( G \), and

\[
(\pi(x)\xi)(h) = (h \cdot \omega)(x)\xi(h) \quad (x \in C, \xi \in l^2(G), h \in G).
\]

Let \( \rho \) be the right regular representation of \( G \) on \( l^2(G) \), so that \( \rho_G \subseteq \lambda_G \). For \( g \) in \( G_{\omega} \),

\[
(\pi(x)\rho_g \xi)(h) = (h \cdot \omega)(x)\xi(hg) = (hg \cdot \omega)(x)\xi(hg) = (\rho_g\pi(x)\xi)(h)
\]

so \( \rho_g \in (\pi \times \lambda)(C^*(\Omega, G))' \). Thus, if \( \omega \circ P \) is pure, \( \rho_g \) is a scalar, so \( g = e \).

A \( C^* \)-subalgebra \( B \) of a \( C^* \)-algebra \( A \) is said to have the extension property (EP) in \( A \) if \( Q_{\omega}(A) \) contains a unique functional for each pure state \( \psi \) of \( B \).

**Corollary 6.2.** Let \( G \) be a discrete group acting on a locally compact Hausdorff space \( \Omega \), and let \( C = C_0(\Omega) \). The abelian \( C^* \)-subalgebra \( C \) has the (EP) in \( C^*(\Omega, G) \) if and only if \( G \) acts freely on \( \Omega \); \( C \) has the (SEP) in \( C^*(\Omega, G) \) if and only if the stabiliser of each point in \( \Omega \) is abelian.

Let \( n \geq 2 \) be a fixed integer, \( \mathbb{Z}_n \) be the group of integers mod \( n \), \( \Omega_- = \bigoplus_{i=-\infty}^{-1} \mathbb{Z}_n \) (in the discrete topology), \( \Omega_+ = \bigoplus_{i=0}^{\infty} \mathbb{Z}_n \) (in the product topology), \( \Omega = \Omega_- \times \Omega_+ \) and \( C = C_0(\Omega) \). Let \( G_0 = \bigoplus_{i=-\infty}^{\infty} \mathbb{Z}_n \), and \( G \) be the semidirect product \( G_0 \times \lambda \mathbb{Z} \) of \( G_0 \) by the shift \( \lambda \) to the right. The discrete group \( G \) acts on \( \Omega \) by

\[
((r_i), m) \cdot (s_i) = (r_i + s_i - m) \quad ((r_i) \in G_0, m \in \mathbb{Z}, (s_i) \in \Omega).
\]

For \( \omega = (\omega_-, \omega_+) \in \Omega \), \( G_0 \) is either trivial or isomorphic to \( \mathbb{Z} \), so by Proposition 6.1, \( Q_{\omega}(C^*(\Omega, G)) \) either contains \( \omega \circ P \) only, or is isomorphic to the space of Radon probability measures on the unit circle, the latter case occurring when the sequence
\( \omega_+ = (r_t)_{t \geq 0} \) eventually becomes periodic. If \( \omega_- = 0 \), then \( \omega \circ P \) is the state \( \phi_{\{x_t\}} \) considered in [11, Theorem 3.4], which result is therefore a special case of Proposition 6.1.

It was shown in [11, Proposition 3.3] (see also [5, §2.1]) that \( C^*(\Omega, G) \) is isomorphic to the unique \( C^* \)-tensor product \( \mathcal{K} \otimes \Theta_n \) of the algebra \( \mathcal{K} \) of compact operators on a separable Hilbert space and the Cuntz algebra \( \Theta_n \) generated by isometries \( S_1, \ldots, S_n \) satisfying

\[
S_i^* S_j = 1 = \sum_{j=1}^n S_j S_j^*.
\]

Let \( \mathfrak{D} \) be the masa of all operators in \( \mathcal{K} \) which are diagonal with respect to some basis of \( \mathcal{K} \), \( q \) a minimal projection in \( \mathfrak{D} \), \( p \) in \( C \) the characteristic function of the subset \( \{0\} \times \Omega_+ \) of \( \Omega \), and \( \mathfrak{D}_n \) the masa in \( \Theta_n \) generated by words of the form \( S_{i_1} \cdots S_{i_k} S_{k}^* \cdots S_1^* \). The isomorphism may be chosen so that \( C \) corresponds to \( \mathfrak{D} \otimes \mathfrak{D}_n \), and \( p \) to \( q \otimes 1 \). Thus there is an induced isomorphism between \( pC^*(\Omega, G)p \) and \( \mathfrak{D}_n \), taking \( pC = C(\Omega_+) \) onto \( \mathfrak{D}_n \). If \( \omega_- = 0 \), the description of \( Q_\omega(C^*(\Omega, G)) = Q_{\omega_+}(pC^*(\Omega, G)p) \) obtained above reduces to the result of [9, Proposition 3.1].

Acknowledgements. The author is grateful to J. Cuntz, D. E. Evans, and especially R. J. Archbold for helpful discussions and providing preprints.

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