TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 272, Number 1, July 1982

ANALYSIS OF SPECTRAL VARIATION AND SOME INEQUALITIES

BY

RAJENDRA BHATIA[1]

ABSTRACT. A geometric method, based on a decomposition of the space of complex matrices, is employed to study the variation of the spectrum of a matrix. When adapted to special cases, this leads to some classical inequalities as well as some new ones. As an example of the latter, we show that if U, V are unitary matrices and K is a skew-Hermitian matrix such that $UV^{-1} = \exp K$, then for every unitary-invariant norm the distance between the eigenvalues of U and those of V is bounded by $\|K\|$. This generalises two earlier results which used particular unitary-invariant norms.

1. Introduction. Let $M(n)$ be the space of all $n \times n$ (complex) matrices. An element A of $M(n)$ will also be thought of as a linear operator on the space $\mathbf{C}^n$. A norm $\|\cdot\|$ on $M(n)$ is said to be *unitary-invariant* if $\|A\| = \|UAV\|$ for any two unitary matrices U and V. Two important examples of such norms are the *Banach norm* $\|\cdot\|_B$, which is the usual supremum norm of an operator acting on $\mathbf{C}^n$, and the *Frobenius* norm $\|\cdot\|_F$, defined as $\|A\|_F = (\operatorname{tr} A^*A)^{1/2}$, where, tr denotes the trace of a matrix.

We denote by Eig A the unordered n-tuple consisting of the eigenvalues of A, each counted as many times as its multiplicity. Let $D(A)$ be a diagonal matrix whose diagonal entries are the elements of Eig A. For any norm on $M(n)$ define

$$\|(\operatorname{Eig} A, \operatorname{Eig} B)\| = \min_W \|D(A) - WD(B)W^{-1}\|$$

where the minimum is taken over all permutation matrices W. We can think of Eig A as an element of $\mathbf{C}^n/S_n$, where S_n is the group of permutations on n symbols. Then $\|(\operatorname{Eig} A, \operatorname{Eig} B)\|$ defines a distance between Eig A and Eig B in this space.

A natural question of considerable interest and importance is: If A and B are close to each other in the norm $\|\cdot\|$, then how close are Eig A and Eig B in the above distance?

If A, B are Hermitian matrices, we have for all unitary-invariant norms the inequality

$$\|(\operatorname{Eig} A, \operatorname{Eig} B)\| \leq \|A - B\|. \tag{1}$$

Received by the editors April 6, 1981. Results of this paper were presented in an Invited Talk on Functional Analysis and Topology sponsored by the University Grants Commission, India, and held at Gujarat University, Ahmedabad.

1980 *Mathematics Subject Classification.* Primary 15A42, 15A57, 15A60; Secondary 53A04, 58C05.

Key words and phrases. Unitary-invariant norm, eigenvalue, singular value, submanifold, tangent space, C^1 functions.

[1]Visiting Member, Tata Institute of Fundamental Research, Bombay, 400 005.

0002-9947/81/0000-0385/$03.50

For the Banach norm this is a consequence of the Courant-Fischer-Weyl min-max principle. (See [**17**].) A generalisation of this principle due to Wielandt [**18**] leads to the conclusion that (1) holds for all unitary-invariant norms. This fact is stated explicitly, as such, by Mirsky [**12**].

Hoffman and Wielandt [**11**] proved that the inequality (1) is also satisfied when A and B are any two normal matrices and the norm is the Frobenius norm. It has been conjectured (see, e.g., [**12**]), but not yet established, that this would be true for other unitary-invariant norms as well.

Let U, V be unitary matrices and let K be a skew-Hermitian matrix such that $UV^{-1} = \exp K$. Then from the theorem of Hoffman and Wielandt cited above, it follows that

$$\|(\operatorname{Eig} U, \operatorname{Eig} V)\| \leqslant \|K\|, \tag{2}$$

if the norm is the Frobenius norm. In [**14**] Parthasarathy showed that (2) also holds for the Banach norm.

For arbitrary matrices, results on this question have been obtained by Ostrowski [**13**], Henrici [**10**] and, recently, by Mukherjea, Friedland and this author in [**3**] and [**4**].

This note has two objects. First, a geometric method for studying this problem is introduced, which is substantially different in approach from the ones hitherto employed. Then this method is used to obtain some inequalities. We show that (2) holds not only for the Banach and the Frobenius norms as stated above, but also for other unitary-invariant norms. We obtain a new proof of (1) for Hermitian matrices as well. This unified approach is likely to lead to some other results. We give, at the end, an adumbration of the possibilities as well as the attendant difficulties.

2. Unitary-invariant norms. Comprehensive surveys of the theory of unitary-invariant norms have been provided by Schatten [**15**], Mirsky [**12**], and Gohberg and Krein [**9**]. Some facts pertinent to our needs are briefly summarised here.

Let $(A^*A)^{1/2}$ denote the positive square root of the positive matrix A^*A. The eigenvalues of this matrix are called the *singular values* of A or the *S-numbers* of A. We write these numbers as

$$s_1(A) \geqslant s_2(A) \geqslant \cdots \geqslant s_n(A) \geqslant 0.$$

It was shown by von Neumann [**16**] that every unitary-invariant norm arises as a "symmetric gauge function" of these numbers. Special examples of such functions are the sums of the first k of these numbers. These lead to the *Ky Fan k-norms*, defined as

$$\|A\|_k = \sum_{j=1}^{k} s_j(A), \qquad k = 1, 2, \ldots, n.$$

In the sequel, a k-norm shall always mean one of these norms. These norms occupy a distinguished position among the unitary-invariant norms. It was shown by Ky Fan [**8**] that an inequality of the type $\|A\| \leqslant \|B\|$, where A and B are two matrices, holds for all unitary-invariant norms if it holds for these special norms. (See [**9**, p. 72].) For $k = 1$, the k-norm is simply the Banach norm $\|\cdot\|_B$.

Another important class of unitary-invariant norms is the class of *Schatten p-norms* defined as

$$\|A\|_p = \left\{ \sum_{j=1}^{n} \left(s_j(A)\right)^p \right\}^{1/p}.$$

For $p = 2$ this just gives the Frobenius norm $\|\cdot\|_F$. We remark that the Frobenius norm is the norm associated with an inner product on $M(n)$ defined as $\langle A, B\rangle = \operatorname{tr} B^*A$. This makes $M(n)$ a Hilbert space. Also, note that if A is the matrix with entries (a_{ij}), $1 \leqslant i, j \leqslant n$, then we have $\|A\|_F = (\sum_{i,j}|a_{ij}|^2)^{1/2}$.

Let $P_1, P_2, \ldots, P_r$ be a complete family of mutually orthogonal orthoprojectors in $\mathbf{C}^n$. Define an operator $\mathcal{C}$ on $M(n)$ as

$$\mathcal{C}(A) = \sum_{i=1}^{r} P_i A P_i.$$

(In [9] this is called the *diagonal-cell operator*. Davis calls it the *pinching* of A by the P_i. In [5] he studies the properties of such operators and in [6] and [7] he obtains, among other things, lower bounds for the distance between the eigenvalues of two Hermitian matrices A and B in terms of the pinching operator corresponding to the spectral subspaces of A.) For any A in $M(n)$, $\mathcal{C}(A)$ is a block diagonal matrix consisting of r diagonal blocks whose sizes are the ranks of the projections P_i. It is the matrix obtained from A by replacing the entries outside these blocks by zeroes. In particular, this means that $\|\mathcal{C}(A)\|_F \leqslant \|A\|_F$. With the Banach norm, $M(n)$ becomes a C^*-algebra and $\mathcal{C}$ is then a completely positive map and, hence, attains its norm at the identity matrix. (See, e.g., [1].) Thus $\|\mathcal{C}\|_B = 1$ and hence $\|\mathcal{C}(A)\|_B \leqslant \|A\|_B$. More generally, we have [9, Theorem 5.1, Chapter II]

$$\sum_{j=1}^{k} s_j(\mathcal{C}(A)) \leqslant \sum_{j=1}^{k} s_j(A), \qquad k = 1, 2, \ldots, n.$$

In other words, the inequality

$$\|\mathcal{C}(A)\| \leqslant \|A\| \tag{3}$$

holds for all k-norms and hence, for all unitary-invariant norms.

If $P_1, \ldots, P_n$ are chosen to be the one-dimensional projections corresponding to the standard orthonormal basis for $\mathbf{C}^n$, then the corresponding pinching operator takes a matrix $A = (a_{ij})$ to the matrix $\operatorname{diag} A$ whose diagonal entries are a_{ii} and the rest of whose entries are zero. So, the inequality (3) yields, in particular,

$$\|\operatorname{diag} A\| \leqslant \|A\| \tag{4}$$

for all unitary-invariant norms.

Let A and B be two commuting matrices, i.e., let $[A, B] = AB - BA = 0$. Then there exists a unitary matrix U such that $UAU^{-1} = T(A)$ and $UBU^{-1} = T(B)$, where $T(A)$ and $T(B)$ are upper triangular matrices. If $D(A)$ and $D(B)$ are the diagonal parts of $T(A)$ and $T(B)$ respectively, then the inequality (4) implies

$\|D(A) - D(B)\| \leqslant \|T(A) - T(B)\|$. Hence, we have for all unitary-invariant norms the inequality

$$\|(\operatorname{Eig} A, \operatorname{Eig} B)\| \leqslant \|A - B\|, \quad \text{if } [A, B] = 0. \tag{5}$$

3. A decomposition of $M(n)$. Some elementary notions of differential geometry will be used in this section. The text we refer to is [**2**].

Let $\mathrm{GL}(n)$ be the multiplicative group of all $n \times n$ invertible matrices. This is a Lie group and has a natural adjoint action on its Lie algebra $M(n)$. This action is defined as $A \to gAg^{-1}$ for $A \in M(n)$, $g \in \mathrm{GL}(n)$. The *orbit* of A under this action is the set

$$\mathbf{O}_A = \{gAg^{-1} \colon g \in \mathrm{GL}(n)\}.$$

In other words, $\mathbf{O}_A$ is the set of all matrices similar to A. This set is a smooth submanifold of the manifold $M(n)$. The tangent space to $\mathbf{O}_A$ at the point A will be denoted by $T_A\mathbf{O}_A$. This is a linear subspace of $M(n)$. Let $\mathbf{Z}(A)$ denote the commutant of A in $M(n)$, i.e., $\mathbf{Z}(A) = \{X \in M(n) \colon [A, X] = 0\}$. The following proposition identifies $T_A\mathbf{O}_A$ and its complement in the space $M(n)$.

PROPOSITION 3.1. *Let $M(n)$ be the Hilbert space of $n \times n$ matrices with the inner product $\langle A, B\rangle = \operatorname{tr} B^*A$. Then, for every $A \in M(n)$, we have*

$$T_A\mathbf{O}_A = \operatorname{span}\{[A, X] \colon X \in M(n)\}, \qquad (T_A\mathbf{O}_A)^{\perp} = \mathbf{Z}(A^*),$$

where $\perp$ denotes the orthogonal complement of a subspace.

PROOF. Every differentiable curve in $\mathbf{O}_A$ which passes through A can be written, locally, as $\lambda(t) = \exp(tX)A\exp(-tX)$ for some $X \in M(n)$. Tangent vectors to $\mathbf{O}_A$ at A are obtained by differentiating such curves at 0:

$$(d/dt)\big|_{t=0}\lambda(t) = XA - AX = [X, A].$$

The space $T_A\mathbf{O}_A$ is precisely the span of these tangent vectors.

To prove the second part, note that $B \in (T_A\mathbf{O}_A)^{\perp}$ if and only if for all $X \in M(n)$ we have

$$\begin{aligned} 0 = \langle [A, X], B\rangle &= \operatorname{tr} B^*(AX - XA) \\ &= \operatorname{tr}(B^*A - AB^*)X = \langle [B^*, A], X^*\rangle. \end{aligned}$$

This is possible if and only if $[B^*, A] = 0$, i.e., if and only if $B \in \mathbf{Z}(A^*)$. □

REMARK. We will be considering $M(n)$ with other unitary-invariant norms too. With any of these norms it is a Banach space. We will write the above decomposition as

$$M(n) = T_A\mathbf{O}_A \oplus \mathbf{Z}(A^*), \tag{6}$$

with the understanding that the symbol $\oplus$ denotes an orthogonal direct sum when we are thinking of $M(n)$ as a Hilbert space with the Frobenius inner product, and it denotes an ordinary vector space direct sum otherwise.

Recall that a matrix A is normal if and only if $\mathbf{Z}(A) = \mathbf{Z}(A^*)$. In this case we can write

$$M(n) = T_A\mathbf{O}_A \oplus \mathbf{Z}(A) \quad \text{if } A \text{ is normal}. \tag{7}$$

Since similar matrices have identical spectra, we have

$$\|(\operatorname{Eig} A, \operatorname{Eig} B)\| = 0 \quad \text{if } B \in \mathbf{O}_A. \tag{8}$$

Relations (5), (7) and (8) suggest that the variation of the spectrum of a normal matrix can be estimated componentwise along two complementary directions. To make this idea precise we use the following lemma.

LEMMA 3.2. *Let X be a Banach space and let φ be a real-valued function of class C^1 on X. Let γ: $[0, 1] \to X$ be a piecewise C^1 curve. Suppose the following conditions are satisfied*:

(i) $\gamma(0) = x_0$, $\gamma(1) = x_1$ *and* $\varphi(x_0) = 0$.

(ii) *For every t in $[0, 1]$, the space X (which is also the tangent space $T_{\gamma(t)}X$ in our notation) splits into a direct sum $X = T^{(1)}_{\gamma(t)} \oplus T^{(2)}_{\gamma(t)}$ in such a way that*

$$v^{(1)}\varphi = 0 \quad \textit{for all } v^{(1)} \in T^{(1)}_{\gamma(t)},$$
$$v^{(2)}\varphi \leqslant \|v^{(2)}\| \quad \textit{for all } v^{(2)} \in T^{(2)}_{\gamma(t)}.$$

(Here $v^{(1)}\varphi$ and $v^{(2)}\varphi$ are thought of as the directional derivatives of φ in these two directions.)

Let $P^{(1)}_t$, $P^{(2)}_t$ denote the complementary projections in X onto the spaces $T^{(1)}_{\gamma(t)}$ and $T^{(2)}_{\gamma(t)}$ respectively. Let $\gamma'(t)$ denote the derivative of γ at t. Then we have,

$$\varphi(x_1) \leqslant \int_0^1 \|P^{(2)}_t \gamma'(t)\| \, dt.$$

PROOF. We have

$$\begin{aligned}\varphi(x_1) &= \int_0^1 \gamma'(t)(\varphi)\, dt = \int_0^1 \left(P^{(1)}_t\gamma'(t)\right)(\varphi)\, dt + \int_0^1 \left(P^{(2)}_t\gamma'(t)\right)(\varphi)\, dt \\ &\leqslant 0 + \int_0^1 \|P^{(2)}_t\gamma'(t)\|\, dt\end{aligned}$$

by the hypothesis (ii). □

REMARK. The statement of this lemma remains valid if φ is C^1 on a dense open subset G of X and γ is a piecewise C^1 curve which intersects the complement of G only at a finite number of points. In such a case we say that φ is *generically* C^1 and γ is a curve *adapted* to φ.

Let $(a_1, \ldots, a_n)$ be a fixed point in $\mathbf{C}^n$ and let $(x_1, \ldots, x_n)$ be any arbitrary point in $\mathbf{C}^n$. Let σ be an element of the permutation group S_n. Arrange the numbers $|x_i - a_{\sigma(i)}|$ in a descending order of magnitude and let this new enumeration of these numbers be $|x'_i - a'_{\sigma(i)}|$, $i = 1, 2, \ldots, n$. Let

$$f_k(x_1, \ldots, x_n) = \min_{\sigma \in S_n} \sum_{i=1}^{k} |x'_i - a'_{\sigma(i)}|.$$

For $k = 1, 2, \ldots, n$, these are well-defined functions on $\mathbf{C}^n$. These functions are invariant under the action of S_n on $\mathbf{C}^n$ and hence they are well defined on the quotient space $\mathbf{C}^n/S_n$. These functions are differentiable except at the set F of points which satisfy either of the two conditions:

(1) There exists a permutation σ such that the numbers $|x_i - a_{\sigma(i)}|$ are not all distinct.

(2) The minimum in the definition of f_k is attained at two different permutations.

It is clear that the set F is a nowhere dense closed subset of $\mathbf{C}^n$. So the functions f_k are generically C^1.

Now let A_0 be a fixed matrix with eigenvalues $a_1, \ldots, a_n$ and let

$$\varphi_k(A) = \|(\text{Eig}\, A_0, \text{Eig}\, A)\|_k.$$

By the definition of the k-norms and by the above comments these functions are generically C^1. (Matrices whose eigenvalues constitute n-tuples which belong to the set F mentioned above form a closed nowhere dense subset of $M(n)$. Outside this set the φ_k are C^1 functions.) With this knowledge, we can prove

THEOREM 3.3. *Let $M(n)$ be the space of matrices with any of the k-norms $\|\cdot\|$. Let A: $[0,1] \to M(n)$ be a piecewise C^1 curve with the following properties*:

(i) *$A(t)$ is normal for all* $0 \leqslant t \leqslant 1$,

(ii) $A(0) = A_0$, $A(1) = A_1$,

(iii) *$A(t)$ is adapted to the generically C^1 function* $\varphi(A) = \|(\text{Eig}\, A_0, \text{Eig}\, A)\|$.

Let $P_t^{(1)}$ and $P_t^{(2)}$ denote the complementary projection operators in $M(n)$ corresponding to the direct sum decomposition $M(n) = T_{A(t)}\mathbf{O}_{A(t)} \oplus \mathbf{Z}(A(t))$. Then

$$\|(\text{Eig}\, A_0, \text{Eig}\, A_1)\| \leqslant \int_0^1 \|P_t^{(2)} A'(t)\|\, dt \leqslant \int_0^1 \|A'(t)\|\, dt, \tag{9}$$

where $A'(t)$ denotes the derivative of $A(t)$.

PROOF. We apply Lemma 3.2 to the Banach space $M(n)$, the function $\varphi(A)$ and the curve $A(t)$. Let $T_{A(t)}^{(1)} = T_{A(t)}\mathbf{O}_{A(t)}$, $T_{A(t)}^{(2)} = \mathbf{Z}(A(t))$. Choose and fix a point s in $[0, 1]$. For every $B \in \mathbf{O}_{A(s)}$ we have $\varphi(B) = \varphi(A(s))$. Hence, the derivative of φ in the direction of $\mathbf{O}_{A(s)}$ is zero, i.e.,

$$v^{(1)}\varphi = 0 \quad \text{for all } v^{(1)} \in T_{A(s)}^{(1)}.$$

For $A \in M(n)$, define $\psi(A) = \|(\text{Eig}\, A(s), \text{Eig}\, A)\|$, and put

$$h(A) = \varphi(A(s)) + \psi(A) = \|(\text{Eig}\, A_0, \text{Eig}\, A(s))\| + \|(\text{Eig}\, A(s), \text{Eig}\, A)\|.$$

Note that $\varphi(A(s)) = h(A(s))$ and $\varphi(A) \leqslant h(A)$ for all A in $M(n)$. Hence,

$$v^{(2)}\varphi \leqslant v^{(2)}h \quad \text{for all } v^{(2)} \in T_{A(s)}^{(2)}.$$

(In fact, this last inequality holds for the derivative in any direction and so, in particular, for the direction $T_{A(s)}^{(2)}$.) But since, for a fixed s, $\varphi(A(s))$ is constant, we have $v^{(2)}h = v^{(2)}\psi$ for all $v^{(2)} \in T_{A(s)}^{(2)}$. Now recall that $T_{A(s)}^{(2)} = \mathbf{Z}(A(s))$ and hence we have, from the inequality (5), that

$$v^{(2)}\psi \leqslant \|v^{(2)}\| \quad \text{for all } v^{(2)} \in T_{A(s)}^{(2)}.$$

So we have $v^{(2)}\varphi \leqslant \|v^{(2)}\|$ for all $v^{(2)} \in T_{A(s)}^{(2)}$. Since s was any arbitrary point in $[0, 1]$, we obtain, from Lemma 3.2, the inequality

$$\varphi(A_1) \leqslant \int_0^1 \|P_t^{(2)} A'(t)\|\, dt.$$

This proves the first inequality in (9). To prove the second one we claim that $\|P_t^{(2)}B\| \leq \|B\|$, for all $B \in M(n)$. Indeed, $P_t^{(2)}$ is the projection on $\mathbf{Z}(A(t))$ along the complementary space $T_{A(t)}\mathbf{O}_{A(t)}$. Since $A(t)$ is normal and $U\mathbf{Z}(A(t))U^{-1} = \mathbf{Z}(UA(t)U^{-1})$ for every unitary U, we can assume, without loss of generality, that $A(t)$ is diagonal. Then $\mathbf{Z}(A(t))$ consists of block-diagonal matrices and $P_t^{(2)}B$ is just the pinching of B by the spectral projections of $A(t)$. Our claim, therefore, follows from the inequality (3). This proves the theorem completely. □

We deduce some explicit estimates as corollaries. First note that inequalities of the type (1) and (2) would be valid for all matrices if they hold on a dense subset. By perturbing the matrix A by a small amount, if necessary, we may assume that A lies in the set on which φ is C^1. In the next few paragraphs we will make this assumption without mentioning it. In the same way, for the sake of brevity, a curve passing through A_0 will mean a curve adapted to the function $\varphi(A) = \|(\operatorname{Eig} A_0, \operatorname{Eig} A)\|$.

COROLLARY 3.4. *Let A_0, A_1 be Hermitian matrices. Then for all unitary-invariant norms, we have*

$$\|(\operatorname{Eig} A_0, \operatorname{Eig} A_1)\| \leq \|A_0 - A_1\|. \tag{10}$$

PROOF. The curve $A(t) = A_1 + t(A_1 - A_0)$ satisfies the conditions of the theorem. Note that $A'(t) = A_1 - A_0$. So, the inequality (10) holds for all the k-norms and hence, it holds for all unitary-invariant norms. □

COROLLARY 3.5. *Let U_0, U_1 be unitary matrices and let K be a skew-Hermitian matrix such that $U_1U_0^{-1} = \exp K$. Then for all unitary-invariant norms, we have*

$$\|(\operatorname{Eig} U_0, \operatorname{Eig} U_1)\| \leq \|K\|. \tag{11}$$

PROOF. The curve $U(t) = (\exp tK)U_0$ joins U_0 and U_1 and satisfies the conditions of the theorem. We have $U'(t) = K(\exp tK)U_0$ and hence $\|U'(t)\| = \|K\|$ for every unitary-invariant norm. As before, the conclusion follows. □

REMARK. The last inequality in (9) is strict whenever the Lebesgue measure of the set $\{t\colon \|P_t^{(2)}A'(t)\| < \|A'(t)\|\}$ is positive. When the Frobenius norm is being used this condition is equivalent to saying that $P_t^{(1)}A'(t) \neq 0$ on a set of positive Lebesgue measure. Thus, for the Frobenius norm, the inequality (10) is strict whenever $[A_0, A_1] \neq 0$ and the inequality (11) is strict whenever $[U_0, U_1] \neq 0$.

COROLLARY 3.6. *Let A_0 and A_1 be normal matrices such that $[A_0^*, A_1]$ is skew-Hermitian, i.e. $A_1 - A_0$ is also normal. Then for all unitary-invariant norms, we have*

$$\|(\operatorname{Eig} A_0, \operatorname{Eig} A_1)\| \leq \|A_0 - A_1\|.$$

PROOF. Under the hypothesis, it is easy to see that the path $A_0 + t(A_1 - A_0)$ lies entirely within the set of normal matrices. The proof is then the same as that of Corollary 3.4. □

REMARKS. The above corollaries give some old and some new inequalities. However, an answer to whether the inequality of Corollary 3.6 is true for all normal matrices still eludes us. The problem is that of finding a "good" path linking two normal matrices. The "obvious" path does not quite work. Nevertheless, the following calculation is instructive. Let A_0 and A_1 be normal matrices. Then we can write $A_i = U_i D_i U_i^{-1}$, where U_i are unitary and D_i are diagonal matrices for $i = 0, 1$. Again, let K be a skew-Hermitian matrix such that $U_1 U_0^{-1} = \exp K$, and let $U(t) = \exp(tK)U_0$. Let $D(t) = D_1 + t(D_1 - D_0)$. Then the path $A(t) = U(t)D(t)U(t)^{-1}$ connects A_0 and A_1. Differentiation leads to the equation $A'(t) = [K, A(t)] + U(t)(D_1 - D_0)U(t)^{-1}$. It is interesting to note that the first component lies in the subspace $T_{A(t)}\mathbf{O}_{A(t)}$. Hence,

$$\| P_t^{(2)} A'(t) \| \leqslant \| U(t)(D_1 - D_0)U(t)^{-1} \| = \| D_1 - D_0 \|.$$

This, however, merely leads to the tautological inequality $\|(\operatorname{Eig} A, \operatorname{Eig} A_0)\| \leqslant \| D_1 - D_0 \|$.

Finally, we remark that it is conceivable, though not yet clear, that this method could be applied to nonnormal matrices as well. Of course, in this case, the decomposition (7) is no longer valid. But we could go back to (6) and split $\mathbf{Z}(A^*)$ into two further components, $\mathbf{Z}(A) \cap \mathbf{Z}(A^*)$ and its complement. In the first of these components the estimate (5) is still applicable; in the second, one has weaker but explicit estimates derived in [3] and [4]. How to combine these is a problem that needs further investigation.

ACKNOWLEDGEMENT. The author is indebted to Kalyan Mukherjea for several conversations which led to the idea of this analysis, and to Christopher Croke, A. Ramanathan and V. S. Sunder for helpful comments.

NOTE ADDED IN PROOF. Consider the following example. Let

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then $\|(\operatorname{Eig} A, \operatorname{Eig} B)\| \geqslant \| A - B \|$, in all Schatten p-norms for $1 \leqslant p < 2$. Thus in these norms the inequality (1) breaks down as one steps beyond Hermitian to normal, or even to unitary, matrices. In view of this, Corollaries 3.5 and 3.6 of this paper assume added significance. They might be the best results to expect if all unitary invariant norms are simultaneously involved.

REFERENCES

1. W. B. Arveson, *Subalgebras of C*-algebras*, Acta Math. **123** (1969), 141–224.
2. L. Auslander and R. MacKenzie, *Introduction to differentiable manifolds*, McGraw-Hill, New York, 1963.
3. R. Bhatia and S. Friedland, *Variation of Grassman powers and spectra*, Linear Algebra Appl. **40** (1981), 1–18.
4. R. Bhatia and K. Mukherjea, *On the rate of change of spectra of operators*, Linear Algebra Appl. **27** (1979), 147–157.
5. Ch. Davis, *Various averaging operations onto subalgebras*, Illinois J. Math. **3** (1959), 538–553.
6. ______, *The rotation of eigenvectors by a perturbation*, J. Math. Anal. Appl. **6** (1963), 159–173.
7. ______, *The rotation of eigenvectors by a perturbation*. II, J. Math. Anal. Appl. **11** (1965), 20–27.
8. Ky Fan, *Maximal properties and inequalities for the eigenvalues of completely continuous operators*, Proc. Nat. Acad. Sci. U.S.A. **37** (1951), 760–766.
9. I. C. Gohberg and M. G. Kreĭn, *Introduction to the theory of linear nonselfadjoint operators*, Transl. Math. Monographs, vol. 18, Amer. Math. Soc., Providence, R. I., 1969.
10. P. Henrici, *Bounds for iterates, inverses, spectral variation and fields of values of nonnormal matrices*, Numer. Math. **4** (1962), 24–39.

11. A. J. Hoffman and H. W. Wielandt, *The variation of the spectrum of a normal matrix*, Duke Math. J. **20** (1953), 37–39.

12. L. Mirsky, *Symmetric gauge functions and unitarily invariant norms*, Quart. J. Math. Oxford Ser. (2) **11** (1960), 50–59.

13. A. M. Ostrowski, *Über die Stetigkeit von charakteristischen Wurzeln in Abhangigkeit von den Matrizenelementen*, Jahresber. Deutsch Math.-Verein. **60** (1957), 40–42.

14. K. R. Parthasarathy, *Eigenvalues of matrix-valued analytic maps*, J. Austral. Math. Soc. Ser. A **26** (1978), 179–197.

15. R. Schatten, *Norm ideals of completely continuous operators*, Springer-Verlag, Berlin, 1960.

16. J. von Neumann, *Some matrix inequalities and metrization of matric space*, Tomsk. Gos. Univ. Učen. Zap. **1** (1937), 286–299.

17. H. Weyl, *Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen*, Math. Ann. **71** (1912), 441–479.

18. H. Wielandt, *An extremum property of sums of eigenvalues*, Proc. Amer. Math. Soc. **6** (1955), 106–110.

School of Mathematics, Tata Institute of Fundamental Research, Bombay 400 005, India