

THE METABELIAN p -GROUPS OF MAXIMAL CLASS. II

BY

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This paper gives a classification of the metabelian p -groups of maximal class. A recent idea is used to produce a considerable simplification of an earlier classification scheme for these groups.

The usual notation will be employed here. G is a group, $G_2 = [G, G]$ is the commutator subgroup of G , $G_{i+1} = [G_i, G]$ for $i \geq 2$. G is of maximal class if and only if $|G| = p^n$, $|G_i : G_{i+1}| = p$ for $1 = 2, \dots, n-1$. If G is of maximal class then, by definition, G_1 is the largest subgroup of G such that $[G_1, G_2] \leq G_4$. It is known that G_1 is a characteristic subgroup of index p in G . The basic facts about these groups can be found in [1].

The p -groups of maximal class with G_1 of nilpotence class at most 2 were determined by Leedham-Green in a sequence of three papers [2]. His results cover "most" of the groups here for if G is metabelian and (roughly) $n \geq 2p$ then G_1 is of class 2. However, the main difficulty with the metabelian groups of maximal class, as far as classification is concerned, is that the class of G_1 can be quite large. When $|G| = p^n$ it can be as large as $(n-1)/2$. This fact led to a difficult proof and complicated statements in [3]. This paper gives a way around these problems.

We begin with a description:

THEOREM 1. *Let p be a prime with $p \geq 5$. Let G be a metabelian p -group of maximal class and of order p^n where $n \geq 5$. Then there is a pair of generators $\{x, y\}$ of G , a basis $\{u_2, \dots, u_{n-1}\}$ of G_2 , an integer k , and a set of integers $\{a(k), \dots, a(n-1), w, z\}$ with $a(k) \not\equiv 0 \pmod p$ such that $G_1 = \langle y, G_2 \rangle$ and*

- (i) $u_2 = [y, x]$, $u_{i+1} = [u_i, x]$ for $i \geq 2$,
- (ii) $[u_2, y] = u_k^{a(k)} \cdots u_{n-1}^{a(n-1)}$,
- (iii) $x^p = u_{n-1}^w \cdot y^p u_2^{(z)} \cdots u_p^{(z)} = u_{n-1}^z$,
- (iv) $u_i^{(p)} \cdots u_{i+p-1}^{(p)} = 1$ for $i \geq 2$.

The integer k is an invariant of the group and

$$k \geq \max\{4, n - p + 2\}.$$

Received by the editors June 20, 1980.

1980 *Mathematics Subject Classification.* Primary 20D15; Secondary 20F12.

Key words and phrases. p -groups, metabelian groups.

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 0002-9947/82/0000-0324/\$03.00

The two kinds of Stirling numbers, S_n^m and \mathfrak{S}_n^m , will be needed in what follows. They are defined by

$$(x)_n = \sum_{m=0}^n S_n^m x^m, \quad x^n = \sum_{m=0}^n \mathfrak{S}_n^m(x)_m.$$

By convention $S_n^m = \mathfrak{S}_n^m = 0$ for $m > n$. It will also be convenient to define $\sigma(m, n)$ and $\lambda(m, n)$ by

$$\sigma(m, n) = \frac{m!}{n!} S_n^m \quad \text{and} \quad \lambda(m, n) = \frac{m!}{n!} \mathfrak{S}_n^m.$$

This brings us to

THEOREM 2. *Let G be defined as in Theorem 1 and suppose that $2k - 3 \leq n - 1$. Then there is an x in G such that $G = \langle x, G_1 \rangle$,*

$$[y, x, y] = u_k^{a(k)} \cdots u_{2k-3}^{a(2k-3)} \pmod{G_{2k-2}}$$

and

$$\sum_{i=k}^{2k-3} \lambda(i-2, 2k-5) a(i) = 0.$$

Furthermore, $\langle x, G_2 \rangle$ is a characteristic subgroup of index p in G .

We shall utilize Theorem 2 by replacing (ii) of Theorem 1 by an equivalent defining relation. To this end let

$$s(i, j; q, r) = \sigma(j+1, r+1) \sigma(i-j-1, q-r-1).$$

Next, let

$$u(q, r) = [y, \underset{q-1-r}{x, \dots, x}, \underset{r}{y, \dots, y}].$$

Define $U(i, j)$ for $i = 3, \dots, n - p$ and $j = 1, \dots, i - 2$ and for $i = n - p + 1, \dots, n - 1$ and $j = 0, \dots, i - 2$ by

$$U(i, j) = \prod_{q=i}^{n-1} \prod_{r=j}^{j+q-i} u(q, r)^{s(i, j; q, r)}.$$

The fact that the $U(i, j)$ are well defined for the specified pairs (i, j) is discussed in the paragraphs preceding Lemma 10.

If $|G| = p^n$ where $n \leq p + 1$ then G_2 is of exponent p so the $U(i, j)$ are defined as above for $i = 2, \dots, n - 1$ and $j = 0, \dots, i - 2$.

The $U(i, j)$ have the property

$$U(i, j) \equiv u(i, j) \pmod{G_{i+1}}.$$

In addition $\{U(n - p + 1, 0), \dots, U(n - 1, 0)\}$ is a basis for G_{n-p+1} . Consequently the equation given in (ii) of Theorem 1 is equivalent to one of the form

$$U(3, 1) = \prod_{i=k}^{n-1} U(i, 0)^{b(i)}$$

where $b(k) \neq 0$. If we suppose we have replaced the equation of Theorem 1 by the one above we shall say that G is defined in terms of the parameters $\{b(k), \dots, b(n-1), w, z\}$.

One other preliminary result needs to be stated:

THEOREM 3. *Suppose that G and \bar{G} are isomorphic metabelian p -groups of maximal class. Let $\theta: \bar{G} \rightarrow G$ be an isomorphism from \bar{G} to G . Define τ from \bar{G} to G by*

$$\bar{x}^\tau = \bar{x}^\theta g, \quad \bar{y}^\tau = \bar{y}^\theta h$$

where g and h are any fixed elements of G_2 . Then τ is an isomorphism from \bar{G} to G .

This brings us to our main result:

THEOREM 4. *Suppose that G is a metabelian p -group of maximal class that is defined in terms of the parameters $\{b(k), \dots, b(n-1), w, z\}$. Thus*

$$U(3, 1) = U(k, 0)^{b(k)} \dots U(n-1, 0)^{b(n-1)}$$

where $b(k) \neq 0$. Suppose also that if $2k-3 \leq n-1$ then $G = \langle x, y \rangle$ where x is the element of Theorem 2. Suppose \bar{G} , defined in terms of $\{\bar{b}(k), \dots, \bar{b}(n-1), w, z\}$ is isomorphic to G under ϑ where

$$\bar{x}^\vartheta = x^\alpha y^\beta, \quad \bar{y}^\vartheta = y^\delta,$$

and $\beta = 0$ if $2k-3 \leq n-1$. Then

$$\delta b(i) = \alpha^{i-2} \bar{b}(i)$$

for $i = k, \dots, n-1$. Furthermore if $2k-3 \leq n-1$ then $b(2k-3) = 0$.

Theorem 4 gives the results that are derived from the commutator structure of the groups. There are two other equations that come out of the power structure:

THEOREM 5. *Let G and \bar{G} be as in Theorem 4. Set $\bar{\psi} = \bar{b}(k)$ if $k = n-p+2$ and $\bar{\psi} = 0$ otherwise. Then*

$$z = \alpha^{n-2} [\bar{z} - 2\bar{\psi} (\beta/\delta)], \quad w = \alpha^{n-3} [\bar{w}\delta - \bar{z}\beta + \bar{\psi} (\beta^2/\delta)].$$

The equations of Theorems 4 and 5 are necessary and sufficient conditions for isomorphism. In addition, if we take $\bar{G} = G$ and $\bar{b}(i) = b(i)$ in Theorem 4 and use the result of Theorem 3 we obtain the automorphism group of G .

Some of the results of [3] are used here. Theorem 3 is Lemma 1.1 of [3]. Theorem 5 is part of Theorem 2 of [3]. The main purpose of this paper is to eliminate the complicated analysis that runs from p. 101 to p. 118 in [3].

1. A function must be defined. Set $F_\alpha(0, 0) = 1$ and $F_\alpha(0, n) = 0$ for $n \geq 1$. For $n \geq 1$ let

$$F_\alpha(m, n) = \sum_x \binom{\alpha}{x_1} \cdots \binom{\alpha}{x_m}$$

where the summation is on those $x = (x_1, \dots, x_m)$ with integral $x_i \geq 1$ for $i = 1, \dots, m$ and $x_1 + \dots + x_m = n$.

LEMMA 1. Let

$$e(n, i, j) = \binom{\alpha}{i+1} \sum_{t=0}^{j-n} \binom{\beta}{t} F_{\delta}(n+1, j+1-t)$$

and

$$f(m, M, N) = \sum_{a,b} \frac{m!}{a!b!(m-a-b)!} F_{\alpha}(m-a, M) F_{\beta}(m-b, N)$$

where the summation is on all those integers a, b with $a \geq 0, b \geq 0$ and $a + b \leq m$. Set

$$g(m, n, r, s) = \sum_{i=0}^r \sum_{j=n}^s e(n, i, j) f(m, r-i, s-j).$$

Let x and y be elements of a metabelian group and set $u = [y, x]$. Set $x_1 = x^{\alpha}y^{\beta}$, $y_1 = y^{\delta}$, and $u_1 = [y_1, x_1]$. Then

$$[u_1, x_1, \dots, x_1, y_1, \dots, y_1] = \prod_{\substack{r \geq 0, s \geq n \\ r+s \geq m+n}} \prod_{\substack{r \\ s}} [u, x, \dots, x, y, \dots, y]^{g(m,n,r,s)}.$$

PROOF. See [4, Lemma 2.1] for the details.

LEMMA 2. Let $g(m, n, r, s)$ be defined as in Lemma 1. Then

$$g(m, n, r, n) = \delta^{n+1} F_{\alpha}(m+1, r+1)$$

and

$$g(m, n, m-1, n+1) = m\alpha^m \delta^{n+1} \beta.$$

This follows directly from the definitions.

LEMMA 3. Suppose G is defined as in Theorem 1, $x_1 = x^{\alpha}y^{\beta}$, $y_1 = y^{\delta}$ and $u_1 = [y, x]$. Then

$$[u_1, x_1, \dots, x_1] \equiv \prod_{s=0}^1 \prod_{r \geq m-s} [u_{r+2}, y, \dots, y]^{g(m,0,r,s)} \pmod{G_{2k-2}}.$$

PROOF. If G is of maximal class then, in Lemma 1,

$$[u, x, \dots, x] = u_{r+2}.$$

Furthermore if $[u_2, y]$ is defined by (ii) in Theorem 1 then, for $i \geq 2$

$$[u_i, y] \equiv u_{k+i-2}^{a(k)} \pmod{G_{k+i-1}}.$$

Thus if we set aside those commutators corresponding to $s \geq 2$ in the conclusion of Lemma 1 we obtain Lemma 3.

LEMMA 4. Suppose that \bar{G} and G are isomorphic under ϑ where $\bar{x}^{\vartheta} = x^{\alpha}y^{\beta}$ and $\bar{y}^{\vartheta} = y^{\delta}$. Suppose also that

$$[\bar{u}_2, \bar{u}_1] = \prod_{m=k}^{n-1} (\bar{u}_m)^{\bar{a}(m)}.$$

Then

$$[\bar{u}_2, \bar{u}_1]^\vartheta \equiv \prod_{r=k}^{2k-3} u_r^{h(r)} \pmod{G_{2k-2}}$$

where

$$h(r) = \delta \sum_{m=k}^r \bar{a}(m) F_\alpha(m-1, r-1) + \epsilon(r)(k-2)\alpha^{k-2} \delta \bar{a}(k) a(k) \beta$$

with $\epsilon(r) = 0$ for $r \leq 2k - 4$ and $\epsilon(2k - 3) = 1$.

PROOF. This follows from

$$\prod_{m \geq k} (\bar{u}_m^\vartheta)^{\bar{a}(m)} = \prod_{m \geq k} [u_1, x_1, \dots, x_1]_{m-2}^{\bar{a}(m)},$$

Lemma 3, and Lemma 2. The first part of $h(r)$ comes from that part of the product in Lemma 3 corresponding to $s = 0$. The $\epsilon(r)$ term comes from the main part of $[u_{k-1}, y]^\tau$ where $\tau = \bar{a}(k)g(k-2, 0, k-3, 1)$.

LEMMA 5. Suppose the hypotheses of Lemma 4 hold. Then

$$[\bar{u}_2^\vartheta, \bar{u}_1^\vartheta] \equiv \prod_{r=k}^{2k-3} u_r^{i(r)} \pmod{G_{2k-2}}$$

where for $n = k, \dots, 2k - 3$

$$i(r) = \delta^2 \sum_{m=k}^r a(m) F_\alpha(1, r+1-m).$$

PROOF. This follows from Lemma 1. Take $m = 0$ and $n = 1$. Several Stirling number identities are needed at this point.

LEMMA 6. Let $F_\alpha(m, n)$ be defined as above. Then

$$F_\alpha(m, n) = \sum_{t=m}^n \lambda(m, t) \sigma(t, n) \alpha^t.$$

In addition

$$\begin{aligned} \sum_{m=a}^{M-b} \sigma(a, m) \sigma(b, M-m) &= \sigma(a+b, M), \\ \sum_{l=j+c}^N \sigma(j, l-c) \lambda(l, N) &= \lambda(c, N-j), \\ \sum_{m=a}^b \sigma(a, m) \lambda(m, b) &= \delta(a, b) \end{aligned}$$

where $\delta(a, b) = 1$ if $a = b$ and $\delta(a, b) = 0$ if $a \neq b$.

PROOF. See §3 of [4] for the proofs.

LEMMA 7. *Let*

$$h_0(r) = \delta \sum_{m=k}^r \bar{a}(m)F_\alpha(m-1, r-1)$$

and

$$\bar{a}(m) = \sum_{\nu=k}^m \bar{P}(\nu)\sigma(\nu-2, m-2).$$

Let

$$\varphi(r, \nu, \alpha) = \sum_{t=1}^{r+1-\nu} \lambda(1, t)\sigma(t+\nu-2, r-1)\alpha^t.$$

Then

$$h_0(r) = \delta \sum_{\nu=k}^r \alpha^{\nu-2}\bar{P}(\nu)\varphi(r, \nu, \alpha).$$

PROOF. This follows from the identities of Lemma 6.

Incidentally, if $\bar{a}(k), \dots, \bar{a}(m)$ are considered as given, then $\bar{P}(k), \dots, \bar{P}(m)$ are functions of $\bar{a}(k), \dots, \bar{a}(m)$. Since $\sigma(m-2, m-2) = 1$ the number $\bar{P}(m)$ is well defined.

LEMMA 8. *Let*

$$i(r) = \delta^2 \sum_{m=k}^r a(m)F_\alpha(1, r+1-m)$$

and

$$a(m) = \sum_{\nu=k}^m P(\nu)\sigma(\nu-2, m-2).$$

Let $\varphi(r, \nu, \alpha)$ be defined as in Lemma 7. Then

$$i(r) = \delta^2 \sum_{\nu=k}^r P(\nu)\varphi(r, \nu, \alpha).$$

PROOF. Apply Lemma 6.

LEMMA 9. *Suppose that \bar{G} is isomorphic to G under ϑ , $2k-3 \leq n-1$, $\bar{a}(m)$ is defined as in Lemma 7, and $a(m)$ is defined as in Lemma 8. Then*

$$\delta P(r) = \alpha^{r-2}\bar{P}(r) \quad \text{for } r = k, \dots, 2k-4$$

and

$$\delta P(2k-3) = \alpha^{2k-5}\bar{P}(2k-3) + (k-2)\alpha^{k-3}a(k)\bar{a}(k)\beta.$$

PROOF. This follows from Lemmas 4 through 8.

PROOF OF THEOREM 2. First, if $2k-3 \leq n-1$ then $k-2 \not\equiv 0 \pmod p$. For if $k-2 \equiv 0 \pmod p$ then, since $k \geq 4$, $k = 2 + t \cdot p$ where t is a positive integer. Since $2k-3 \leq n-1$ it follows that $n \geq 2p+2$. But if $n \geq 2p+2$ then, since $k \geq n-p+2$, we have $2k-3 \geq n$.

Thus the coefficient of β in the last equation of Lemma 9 is not zero. Consequently if \bar{G} is considered as given we can find an isomorphic image of G with $P(2k - 3) = 0$.

Next, by the definition of the $a(m)$ given in Lemma 2 and by the last identity of Lemma 6

$$\sum_{m=k}^{2k-3} \lambda(m - 2, 2k - 5)a(m) = P(2k - 3).$$

Thus there is an x such that $G = \langle x, G_1 \rangle$ and such that the above sum is zero.

Finally $\langle x, G_2 \rangle$ is a characteristic subgroup of G . To see this apply Lemma 9 with $\bar{x} = x, x^\beta = x^\alpha y^\beta, \bar{a}(m) = a(m),$ and $\bar{P}(2k - 3) = P(2k - 3) = 0$. Then the last equation of Lemma 9 is

$$0 = (k - 2)\alpha^{k-3}a(k)\bar{a}(k)\beta.$$

Thus $\beta = 0$, so $x^\beta = x^\alpha$ and $\langle x, G_2 \rangle$ is a characteristic subgroup of G . It is of index p because G is generated by two elements.

We need to show that the $U(i, j)$ are well defined. We had

$$U(i, j) = \prod_{q=i}^{n-1} \prod_{r=j}^{j+q-i} u(q, r)^{s(i, j, q, r)}$$

for $i = 3, \dots, n - p$ and $j = 1, \dots, i - 2$ and for $i = n - p + 1$ and $j = 0, \dots, i - 2$.

To begin $s(i, j, q, r) = \sigma(j + 1, r + 1)\sigma(i - j - 1, q - r - 1)$. Then, from the definitions

$$|\sigma(a, b)| = \sum_{(x_1, \dots, x_a)} \frac{1}{x_1 \cdots x_a}$$

where the summation is on those integral (x_1, \dots, x_a) with $x_i \geq 1$ and $x_1 + \dots + x_a = b$. Thus $\sigma(a, b)$ is a well-defined residue modulo p if and only if $b - a \leq p - 2$. That is $s(i, j, q, r)$ is a well-defined residue modulo p if and only if $r - j \leq p - 2$ and $q - r - (i - j) \leq p - 2$.

Next if $r - j \geq p - 1$ or $q - r - (i - j) \geq p - 1$ and $u(q, r)$ appears in the definition of $U(i, j)$ then $u(q, r) = 1$. Consider first those cases where $r \geq 1$. Then

$$u(q, r) = [u_{q-2-r}, x, \dots, x, y, \dots, y] = [u_k^{a(k)} \cdots u_{n-1}^{a(n-1)}, x, \dots, x, y, \dots, y].$$

Now if $r - j \geq p - 1$ then $r \geq p - 1$ so $k + r - 1 \geq k + p - 2 \geq n$. If $q - r - (i - j) \geq p - 1$ then $q - r - 2 \geq p - 1$ and $k + q - 2 - r \geq k + p - 1 \geq n$. In either event $u(q, r) = 1$. Consider next the case when $r = 0$. Then, by the definition of $U(i, j), j = 0$. But if $j = 0$ then $i \geq n - p + 1$, and $q - r - (i - j) = q - i \leq n - 1 - (n - p + 1) = p - 2$. Thus the two inequalities mentioned in the last line of the previous paragraph hold.

In the sum, if $u(q, r)$ appears in the definition of $U(i, j)$ then the corresponding exponent is a well-defined residue modulo p .

Finally, by Theorem 1, G_{n-p+1} is of exponent p and $G_k \leq G_{n-p+2}$. Consequently if $r \geq 1$ or if $r = 0$ and $q \geq n - p + 1$ then $u(q, r) \in G_{n-p+1}$. That is, all the $u(q, r)$ appearing in the definition of the $U(i, j)$ are elements of order p .

LEMMA 10. Suppose \bar{G} is isomorphic to G under ϑ where $\bar{x}^\vartheta = x^\alpha$ and $\bar{y}^\vartheta = y^\delta$. Suppose that $2k - 3 \leq n - 1$. Let $U(i, j)$ be defined as in the introduction. Then

$$\bar{U}(i, j)^\vartheta = U(i, j)^{\epsilon(i, j)}$$

where $\epsilon(i, j) = \alpha^{i-j-1}\delta^{j+1}$.

PROOF. This is given in Lemmas 3.2 through 3.4 in [4].

LEMMA 11. Suppose \bar{G} is isomorphic to G under ϑ where $\bar{x}^\vartheta = x^\alpha y^\beta$ and $\bar{y}^\vartheta = y^\delta$. Suppose that $2k - 3 \geq n$. As above, let $\epsilon(i, j) = \alpha^{i-j-1}\delta^{j+1}$. Then

$$\bar{U}(3, 1)^\vartheta = U(3, 1)^{\epsilon(3, 1)}$$

and, for $i = k, \dots, n - 1$

$$\bar{U}(i, 0)^\vartheta = U(i, 0)^{\epsilon(i, 0)}.$$

The proof is similar to the one of Lemma 10, but one uses the fact that $[u_i, y, y] = 1$ for all $i \geq 2$.

To prove the last equation, for example, start from the definition of the $U(i, j)$ to get

$$\bar{U}(i, 0) = \prod_{q=i}^{n-1} \prod_{r=0}^{q-i} [\bar{u}_2, \bar{x}, \dots, \bar{x}, \bar{y}, \dots, \bar{y}]^{s(i, 0, q, r)}.$$

Since $[u_i, y, y] = 1$ those commutators above where $r \geq 2$ can be dropped. Furthermore for $r = 1$ one has

$$[\bar{u}_2, \bar{x}, \dots, \bar{x}, \bar{y}] = [\bar{u}_{q-1}, \bar{y}] \in \bar{G}_{q-1+k-2}.$$

Since $q + k - 3 \geq 2k - 3 \geq n$ the last group is $\langle 1 \rangle$. Thus we have

$$\bar{U}(i, 0) = \prod_{q=i}^{n-1} [\bar{u}_2, \bar{x}, \dots, \bar{x}]^{s(i, 0, q, 0)}.$$

To continue, apply ϑ to $\bar{U}(i, 0)$ and apply Lemma 1. This produces a multiple product of commutators. Next, use the considerations that appeared in the previous paragraph to set aside those commutators in this multiple product that have one or more “ y ”. One then has

$$\bar{U}(i, 0)^\vartheta = \prod_{q \geq i} [u_2, x, \dots, x]^{m(i, q)}$$

where

$$m(i, q) = \sum_{t=i}^q g(t - 2, 0, q - 2, 0) s(i, 0, t, 0).$$

Now, by definition, $s(i, 0, t, 0) = \sigma(i - 1, t - 1)$. So, by Lemma 2,

$$m(i, q) = \delta \sum_{t=i}^q F_\alpha(t - 1, q - 1) \sigma(i - 1, t - 1).$$

Then, by Lemma 6,

$$m(i, q) = \delta \alpha^{i-1} \sigma(i-1, q-1) = \epsilon(i, 0) s(i, 0, q, 0).$$

This proves that $\bar{U}(i, 0)^\Phi = U(i, 0)^{\epsilon(i, 0)}$.

The $\delta b(i) = \alpha^{i-2} \bar{b}(0)$ equation of Theorem 4 now follows from Lemmas 10 and 11.

We shall now prove that $b(2k-3) = 0$. By part (ii) of Theorem 1 and the definition of $U(3, 1)$,

$$U(3, 1) \equiv \prod_{q \geq k} u_q^{A(q)} \pmod{G_{2k-2}}$$

where

$$A(q) = \sum_{i=k}^q a(i) \sigma(1, q+1-i).$$

Next, if $i > k$ then

$$U(i, 0) \equiv \prod_{q \geq i} [u, x, \dots, x]_{q-2}^{s(i, 0, q, 0)} \pmod{G_{2k-2}}.$$

To see this, rearrange the double product that defines $U(i, 0)$ as

$$U(i, 0) = \prod_{r \geq 0} \prod_{q \geq i+r} [u, x, \dots, x, y, \dots, y]_{q-2-r}^{s(i, j, q, r)}.$$

Consider those commutators here where $r \geq 1$. We have

$$[u, y, x, \dots, x, y, \dots, y]_{q-2-r}^{r-1} \in G_{k+q-3}.$$

Since $q \geq i+r \geq i+1 \geq k+1$ we have $G_{k+q-3} \leq G_{2k-2}$. This proves the first assertion of this paragraph.

Consequently,

$$\prod_{i \geq k} (U(i, 0))^{b(i)} \equiv \prod_{q \geq k} u_q^{B(q)} \pmod{G_{2k-2}}$$

where

$$B(q) = \sum_{i=k}^q b(i) \sigma(i-1, q-1).$$

Next, multiply the equation $A(q) = B(q)$ by $\lambda(q-1, m-1)$ and sum from $q = k$ to $q = m$ to obtain, with the aid of Lemma 6,

$$b(m) = \sum_{i=k}^m \lambda(i-2, m-2) a(i).$$

To finish take $m = 2k - 3$. Then, by the choice of the $a(i)$,

$$b(2k - 3) = \sum_{i=k}^{2k-3} \lambda(i - 2, 2k - 5)a(i) = 0.$$

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