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SMOOTHNESS OF THE BOUNDARY VALUES OF FUNCTIONS BOUNDED AND HOLOMORPHIC IN THE DISK

BY

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ABSTRACT. The non-Euclidean counterparts of Hardy-Littlewood's theorems on Lipschitz and mean Lipschitz functions are considered. Let $1 \leqslant p < \infty$ and $0 < \alpha \leqslant 1$. For f holomorphic and bounded, $|f| < 1$, in $|z| < 1$, the condition that

$$f^*(z) \equiv |f'(z)| / \left(1 - |f(z)|^2\right) = O\left((1 - |z|)^{\alpha - 1}\right)$$

is necessary and sufficient for f to be continuous on $|z| \leqslant 1$ with the boundary function $f(e^{it}) \in \sigma\Lambda_\alpha$, the hyperbolic Lipschitz class. Furthermore, the condition that the pth mean of f^* on the circle $|z| = r < 1$ is $O((1 - r)^{\alpha - 1})$ is necessary and sufficient for f to be of the hyperbolic Hardy class H_σ^p and for the radial limits to be of the hyperbolic mean Lipschitz class $\sigma\Lambda_\alpha^p$.

1. Introduction. We shall prove the non-Euclidean counterparts of the following Theorems A and B due to G. H. Hardy and J. E. Littlewood [**2**, Theorem 4, p. 627 and Theorem 3, p. 625] (see [**1**, Theorem 5.1, p. 74 and Theorem 5.4, p. 78]).

Let Φ be the family of complex-valued functions φ defined on the real axis such that φ is periodic with period 2π. We say that $\varphi \in \Phi$ is of Lipschitz class Λ_α $(0 < \alpha \leqslant 1)$ if

$$\sup_{|t-s| \leqslant \tau} |\varphi(t) - \varphi(s)| = O(\tau^\alpha) \quad \text{as } \tau \to +0.$$

Let $D = \{|z| < 1\}$ and let $D^\# = \{|z| \leqslant 1\}$ in the plane.

THEOREM A. *Let f be a function holomorphic in D and let $0 < \alpha \leqslant 1$. Then f is continuous on $D^\#$ and the function $f(e^{it})$ is of class Λ_α if and only if*

$$f'(z) = O\left((1 - |z|)^{\alpha - 1}\right) \quad \textit{as } |z| \to 1 - 0. \tag{1.1}$$

We say that $\varphi \in \Phi$ is of mean Lipschitz class Λ_α^p $(1 \leqslant p < \infty, 0 < \alpha \leqslant 1)$ if the restriction of φ to $[0, 2\pi]$ is of $L^p[0, 2\pi]$ and if

$$\sup_{0 < h \leqslant \tau} \left[\int_0^{2\pi} |\varphi(t + h) - \varphi(t)|^p \, dt\right]^{1/p} = O(\tau^\alpha)$$

as $\tau \to 0$. For $0 \leqslant r < 1$, $0 < p < \infty$, and for v nonnegative and subharmonic in D, we set

$$\mu_p(r, v) = \left[\frac{1}{2\pi} \int_0^{2\pi} v(re^{it})^p \, dt\right]^{1/p};$$

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this is an increasing function of r. The Hardy class H^p $(0 < p < \infty)$ consists of f holomorphic in D such that $\mu_p(r, |f|) = O(1)$ as $r \to 1$, or equivalently, the subharmonic function $|f|^p$ has a harmonic majorant in D. By the boundary value of a complex-valued function g in D at the point e^{it} of the unit circle we mean the radial limit $g(e^{it}) = \lim_{r \to 1} g(re^{it})$. Each function $f \in H^p$ $(0 < p < \infty)$ admits the boundary value $f(e^{it})$ at a.e. point e^{it}, and $f(e^{it}) \in L^p[0, 2\pi]$.

THEOREM B. *Let f be a function holomorphic in D, and let $1 \leqslant p < \infty$, $0 < \alpha \leqslant 1$. Then $f \in H^p$ and the function $f(e^{it})$ is of class Λ_α^p if and only if*

$$\mu_p(r, |f'|) = O\left((1-r)^{\alpha - 1}\right) \quad \text{as } r \to 1. \tag{1.2}$$

In the case $\alpha = 1$, (1.2) says that $f' \in H^p$.

The non-Euclidean hyperbolic distance between z and w in D is defined by

$$\sigma(z, w) = \frac{1}{2}\log\frac{|1 - \bar{z}w| + |z - w|}{|1 - \bar{z}w| - |z - w|}.$$

We set $\sigma(z) \equiv \sigma(z, 0)$, the hyperbolic counterpart of $|z|$, $z \in D$. We say that $\varphi \in \Phi$ is of class $\sigma\Lambda_\alpha$ $(0 < \alpha \leqslant 1)$ if φ is bounded, $|\varphi| < 1$, and if

$$\sup_{|t-s| \leqslant \tau} \sigma(\varphi(t), \varphi(s)) = O(\tau^\alpha) \quad \text{as } \tau \to +0.$$

Let B be the family of functions f holomorphic and bounded, $|f| < 1$, in D. Then, apparently, $f(e^{it})$ exists a.e. For $f \in B$, the Schwarz-Pick lemma reads

$$f^*(z) \equiv |f'(z)| / \left(1 - |f(z)|^2\right) \leqslant \left(1 - |z|^2\right)^{-1}, \qquad z \in D.$$

We note that $\log f^*$ is subharmonic in D, so that $f^{*p} = \exp(p \log f^*)$ $(0 < p < \infty)$ is subharmonic in D. The hyperbolic analogue of Theorem A is

THEOREM 1. *Let $f \in B$, and let $0 < \alpha \leqslant 1$. Then f is continuous on $D^\#$ and the function $f(e^{it})$ is of class $\sigma\Lambda_\alpha$ if and only if*

$$f^*(z) = O\left((1 - |z|)^{\alpha - 1}\right) \quad \text{as } |z| \to 1 - 0. \tag{1.3}$$

We say that $\varphi \in \Phi$ is of class $\sigma\Lambda_\alpha^p$ $(1 \leqslant p < \infty, 0 < \alpha \leqslant 1)$ if $|\varphi(t)| < 1$ a.e., if the restriction of $\sigma(\varphi)(t) \equiv \sigma(\varphi(t))$ to $[0, 2\pi]$ is of $L^p[0, 2\pi]$, and if

$$\sup_{0 < h \leqslant \tau}\left[\int_0^{2\pi} \sigma(\varphi(t+h), \varphi(t))^p dt\right]^{1/p} = O(\tau^\alpha)$$

as $\tau \to 0$. For $f \in B$ set $\sigma(f)(z) \equiv \sigma(f(z))$, the hyperbolic counterpart of $|f(z)|$ $(z \in D)$. Then $\log \sigma(f)$ is subharmonic in D because $X(x) \equiv \log \sigma(e^x)$ is a convex and increasing function of $x \in (-\infty, 0)$, with $-\infty = X(-\infty) \equiv \lim_{x \to -\infty} X(x)$, and $\log \sigma(f) = X(\log |f|)$. For each $a \in D$, the identity $\sigma(g) = \sigma(f, a)$ holds, where $g = (f - a)/(1 - \bar{a}f) \in B$ for $f \in B$. Therefore $\log \sigma(f, a)$ and $\sigma(f, a)^p = \exp[p \log \sigma(f, a)]$ $(0 < p < \infty)$ are subharmonic in D. Let H_σ^p be the set of all $f \in B$ such that $\mu_p(r, \sigma(f)) = O(1)$ as $r \to 1$, or equivalently, the subharmonic function $\sigma(f)^p$ admits a harmonic majorant in D. The hyperbolic Hardy class H_σ^p $(0 < p < \infty)$ is the counterpart of H^p. We are now ready to propose a hyperbolic analogue of Theorem B.

THEOREM 2. *Let $f \in B$, and let $1 \leqslant p < \infty$, $0 < \alpha \leqslant 1$. Then $f \in H^p_\sigma$ and the function $f(e^{it})$ is of class $\sigma\Lambda^p_\alpha$ if and only if*

$$\mu_p(r, f^*) = O\big((1-r)^{\alpha-1}\big) \quad \text{as } r \to 1. \tag{1.4}$$

In the case $\alpha = 1$ in (1.4), the subharmonic function f^{*p} admits a harmonic majorant.

The proof of Theorem 1 is not difficult and depends on Theorem A; we need comparisons of the non-Euclidean distance and the Euclidean distance. The proof of the "if" part of Theorem 2 is, in a sense, routine. Not easy is the proof of the "only if" part of Theorem 2. There is no relation between $\sigma(f)$ and f^* like that between $|f|$ and $|f'|$, namely, one cannot assert that $\sigma(f') = f^*$ even if $|f'| < 1$.

2. Proof of Theorem 1. Consider the two inequalities

$$|z - w| \leqslant \sigma(z, w), \qquad z, w \in D, \tag{2.1}$$

$$\sigma(z, w) \leqslant 2|z - w|/|1 - \bar{z}w| \tag{2.2}$$

for $z, w \in D$ with $|z - w|/|1 - \bar{z}w| \leqslant 1/\sqrt{2}$. The inclusion formula $\sigma\Lambda_\alpha \subset \Lambda_\alpha$ follows from (2.1). If $\varphi \in \Lambda_\alpha$ and if $|\varphi(t)| < 1$ for all $t \in (-\infty, \infty)$, then $\varphi \in \sigma\Lambda_\alpha$. To observe this we set $\max|\varphi(t)| = M < 1$ because φ is continuous. Then there exist two positive constants K and δ such that

$$K\delta^\alpha \leqslant (1 - M^2)/\sqrt{2} \quad \text{and} \quad |\varphi(t) - \varphi(s)| \leqslant K\tau^\alpha$$

for all τ, $0 < \tau < \delta$, and for all t, s with $|t - s| \leqslant \tau$. Since

$$|\varphi(t) - \varphi(s)| \leqslant (1 - M^2)/\sqrt{2},$$

it follows that

$$|\varphi(t) - \varphi(s)|/|1 - \overline{\varphi(t)}\varphi(s)| \leqslant 1/\sqrt{2},$$

whence, by (2.2),

$$\sigma(\varphi(t), \varphi(s)) \leqslant [2/(1 - M^2)]\,|\varphi(t) - \varphi(s)| \leqslant K_1\tau^\alpha$$

for all t, s with $|t - s| \leqslant \tau < \delta$ ($K_1 = 2K/(1 - M^2)$). Therefore $\varphi \in \sigma\Lambda_\alpha$.

To prove the "only if" part of Theorem 1, we notice first that $f(e^{it}) \in \Lambda_\alpha$. Since $|f(e^{it})| < 1$ for all t, it follows from the maximum modulus principle that $A = \max\{|f(z)|;\ z \in D^\#\} < 1$. Since $f^* \leqslant |f'|/(1 - A^2)$, the conclusion (1.3) follows from (1.1).

To prove the "if" part of Theorem 1 we first note that (1.1) holds by $|f'| \leqslant f^*$. By Theorem A, f is continuous on $D^\#$ and $f(e^{it}) \in \Lambda_\alpha$. Now, if $|f(e^{it})| = 1$ for a certain t, then

$$\infty = \lim_{r\to 1}\sigma\big(f(re^{it}), f(0)\big) \leqslant \lim_{r \to 1}\int_0^r f^*(\rho e^{it})\,d\rho < \infty$$

by (1.3); this is a contradiction. Therefore $\max|f(e^{it})| < 1$, which, together with $f(e^{it}) \in \Lambda_\alpha$, shows that $f(e^{it}) \in \sigma\Lambda_\alpha$.

3. Proof of Theorem 2. For the proof of the "if" part we assume that

$$\mu_p(r, f^*) \leqslant K(1-r)^{\alpha-1} \quad \text{for } 0 < r < 1, \tag{3.1}$$

where $K > 0$ is a constant. To prove that $f \in H_\sigma^p$ we apply the continuous form of the Minkowski inequality (see [**3**, (7), p. 20]) to

$$\sigma(f(re^{it}), f(0)) \leqslant \int_0^r f^*(\rho e^{it})\, d\rho$$

for $0 \leqslant t \leqslant 2\pi$ $(0 < r < 1)$. Then

$$\mu_p(r, \sigma(f, f(0))) \leqslant \int_0^r \mu_p(\rho, f^*)\, d\rho \leqslant K/\alpha < \infty$$

by (3.1). Since $\sigma(f) \leqslant \sigma(f, f(0)) + \sigma(f(0), 0)$, the Minkowski inequality in the usual form yields that $\mu_p(r, \sigma(f)) = O(1)$, or $f \in H_\sigma^p$. Since $\mu_p(r, \sigma(f))$ is bounded for $0 < r < 1$, the Fatou lemma shows that $|f(e^{it})| < 1$ a.e. and $\sigma(f)(e^{it}) \in L^p[0, 2\pi]$.

Now, let $0 < h \leqslant \tau < 1/2$, and set $s = t + h$ for $t \in (-\infty, \infty)$. Let $(h <)\ 1 - h < r < 1$, and set $\rho = r - h$. Then

$$\sigma(f(re^{is}), f(re^{it})) \leqslant \int_\rho^r f^*(\lambda e^{it})\, d\lambda + \int_\rho^r f^*(\lambda e^{is})\, d\lambda + \int_t^s f^*(\rho e^{ix})\rho\, dx.$$

The third term in the right-hand side is not greater than $Kh(1-\rho)^{\alpha-1}$ by (3.1). Applying the Minkowski inequality first in the usual and then in the continuous form we obtain

$$\left[\frac{1}{2\pi}\int_0^{2\pi} \sigma(f(re^{i(t+h)}), f(re^{it}))^p\, dt\right]^{1/p} \leqslant 2\int_\rho^r \mu_p(\lambda, f^*)\, d\lambda + Kh(1-\rho)^{\alpha-1}. \tag{3.2}$$

The first term in the right-hand side is not greater than $(2K/\alpha)h^\alpha$ by (3.1), together with $(1-\rho)^\alpha \leqslant (1-r)^\alpha + h^\alpha$, while the second term is not greater than $K(1-\rho)^\alpha \leqslant 2^\alpha K h^\alpha$. Therefore the left-hand side of (3.2) is not greater than $K_1\tau^\alpha$, where $K_1 > 0$ is a constant. Letting $r \to 1$ and considering the Fatou lemma one finds that

$$\left[\frac{1}{2\pi}\int_0^{2\pi} \sigma(f(e^{i(t+h)}), f(e^{it}))^p\, dt\right]^{1/p} \leqslant K_1\tau^\alpha,$$

which completes the proof of $f(e^{it}) \in \Lambda_\alpha^p$.

For the proof of the "only if" part in the case $0 < \alpha < 1$ we remember [**1**, p. 74] that

$$\int_{-\pi}^{\pi} \frac{|t|^\alpha\, dt}{1 - 2r\cos t + r^2} = O((1-r)^{\alpha-1}). \tag{3.3}$$

Fix $z = re^\theta \neq 0$ in D for a moment, and set

$$g(w) = (f(w) - f(z))/(1 - \overline{f(z)}f(w)), \qquad w \in D. \tag{3.4}$$

Since $g \in B$, the Cauchy integral formula of $g - g(e^{i\theta})$ yields

$$g'(z) = \frac{1}{2\pi i}\int_{|\zeta|=1} \frac{g(\zeta) - g(e^{i\theta})}{(\zeta - z)^2}\,d\zeta,$$

whence

$$f^*(z) = |g'(z)| \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{|g(e^{i(t+\theta)}) - g(e^{i\theta})|}{1 - 2r\cos t + r^2}\,dt. \tag{3.5}$$

Since

$$\begin{aligned}|g(e^{i(t+\theta)}) - g(e^{i\theta})| &\leqslant \sigma\big(g(e^{i(t+\theta)}), g(e^{i\theta})\big)\\ &= \sigma\big(f(e^{i(t+\theta)}), f(e^{i\theta})\big),\end{aligned}$$

it follows from (3.5) that

$$f^*(re^{i\theta}) \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\sigma\big(f(e^{i(t+\theta)}), f(e^{i\theta})\big)}{1 - 2r\cos t + r^2}\,dt. \tag{3.6}$$

Now, it is an easy exercise to observe that

$$\int_0^{2\pi} \sigma\big(f(e^{i(t+\theta)}), f(e^{i\theta})\big)^p d\theta \leqslant K_2 |t|^{p\alpha}$$

for all t, $|t| < \pi$, where $K_2 > 0$ is a constant. The Minkowski inequality, together with (3.3), asserts from (3.6) that, for $0 < r < 1$,

$$\mu_p(r, f^*) = O\big((1 - r)^{\alpha - 1}\big).$$

To prove that $\mu_p(r, f^*) = O(1)$ if $f \in H^p_\sigma$ and if $f(e^{it}) \in \sigma\Lambda^p_1$ we need some properties of $F \in H^p_\sigma$ with $F(e^{it}) \in \sigma\Lambda^p_1$. Since $\sigma\Lambda^p_1 \subset \sigma\Lambda^1_1 \subset \Lambda^1_1$, $F(e^{it})$ is equal a.e. to a function of bounded variation on $[0, 2\pi]$ (see [**1**, Lemma 1, p. 72]). Since $F \in B \subset H^1$, $F(e^{it})$ can be considered as an absolutely continuous function on $[0, 2\pi]$ by [**1**, Theorem 3.10, p. 42]. Furthermore, by [**1**, Theorem 3.11, p. 42],

$$F'_*(e^{it}) \equiv \frac{d}{dt}F(e^{it}) = ie^{it}\lim_{r\to 1} F'(re^{it}) = e^{it}F'(e^{it})$$

exists a.e. on $[0, 2\pi]$; this derivative $F'_*(e^{it})$ is of class $L^1[0, 2\pi]$. The principal point we need is the fact that

$$F^*(e^{it}) \equiv |F'_*(e^{it})| / \big(1 - |F(e^{it})|^2\big)$$

for $t \in [0, 2\pi]$ is of class $L^p[0, 2\pi]$. In effect, since $F(e^{it}) \in \sigma\Lambda^p_1$, there exist constants $K_3 > 0$ and $\delta > 0$ such that

$$\int_0^{2\pi} \left[\frac{\sigma\big(F(e^{i(t+h)}), F(e^{it})\big)}{|h|}\right]^p dt \leqslant K_3$$

for all h with $0 < |h| < \delta$. Letting $h \to 0$ and considering the Fatou lemma, one obtains that

$$\int_0^{2\pi} F^*(e^{it})^p dt \leqslant K_3.$$

Now, consider g of (3.4). Since $f \in H^p_\sigma$ and $f(e^{it}) \in \sigma\Lambda^p_1$, if follows that $g \in H^p_\sigma$ and $g(e^{it}) \in \sigma\Lambda^p_1$. Therefore g is absolutely continuous and $g'_*(e^{it})$ is of $L^1[0, 2\pi]$. Differentiating the Poisson integral

$$g(w) = \frac{1}{2\pi}\int_0^{2\pi} P(R, s-t) g(e^{it})\, dt$$

with respect to s, where $w = Re^{is} \neq 0$, and $P(R, s-t) = (1 - R^2)/|e^{it} - Re^{is}|^2$, one observes that

$$\begin{aligned} iwg'(w) &= \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial}{\partial s} P(R, s-t) g(e^{it})\, dt \\ &= -\frac{1}{2\pi}\int_0^{2\pi} \left[\frac{\partial}{\partial t} P(R, s-t)\right] g(e^{it})\, dt \qquad (3.7) \\ &= \frac{1}{2\pi}\int_0^{2\pi} P(R, s-t) g'_*(e^{it})\, dt. \end{aligned}$$

On the other hand,

$$|g'_*(e^{it})| = \frac{|f'_*(e^{it})|\left(1 - |f(z)|^2\right)}{|1 - \overline{f(z)} f(e^{it})|^2} \leq f^*(e^{it}).$$

It then follows from (3.7), together with $f^*(e^{it}) \in L^p[0, 2\pi]$ that

$$|w|^p |g'(w)|^p \leq \frac{1}{2\pi}\int_0^{2\pi} P(R, s-t) f^*(e^{it})^p\, dt.$$

On setting $w = z = re^{i\theta}$, one obtains that

$$|z|^p f^*(z)^p \leq \frac{1}{2\pi}\int_0^{2\pi} P(r, \theta - t) f^*(e^{it})^p\, dt,$$

so that $\mu_p(r, f^*) = O(1)$.

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