CONNECTED ALGEBRAIC MONOIDS

BY

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ABSTRACT. Let $S$ be a connected algebraic monoid with group of units $G$ and lattice of regular $\mathcal{J}$-classes $\mathcal{U}(S)$. The connection between the solvability of $G$ and the semilattice decomposition of $S$ into archimedean semigroups is further elaborated. If $S$ has a zero and if $|\mathcal{U}(S)| \leq 7$, then it is shown that $G$ is solvable if and only if $\mathcal{U}(S)$ is relatively complemented. If $J \in \mathcal{U}(S)$, then we introduce two basic numbers $\theta(J)$ and $\delta(J)$ and study their properties. Crucial to this process is the theorem that for any indempotent $e$ of $S$, the centralizer of $e$ in $G$ is connected. Connected monoids with central idempotents are also studied. A conjecture about their structure is forwarded. It is pointed out that the maximal connected submonoids of $S$ with central idempotents need not be conjugate. However special maximal connected submonoids with central idempotents are conjugate. If $S$ is regular, then $S$ is a Clifford semigroup if and only if for all $f \in E(S)$, the set $\{e \mid e \in E(S), e \geq f\}$ is finite. Finally the maximal semilattice image of any connected monoid is determined.

Introduction. Let $K$ be an algebraically closed field. By a (linear) algebraic semigroup, we mean a Zariski closed subsemigroup of the multiplicative semigroup of all $n \times n$ matrices over a field. An algebraic semigroup is connected if the underlying closed set is an irreducible variety. Connected groups have been well studied [1]. In this paper we continue our study of connected monoids. Our basic model of a connected monoid is the multiplicative semigroup of all $n \times n$ matrices, $\mathcal{M}_n(K)$. Even if one is only interested in invertible matrices, it still becomes necessary to consider noninvertible matrices. In the same way, if a connected group is the group of units of a connected algebraic monoid, then it is only natural to study the monoid and see if its study sheds some light on the structure of the group. A basic concept in $\mathcal{M}_n(K)$ is that of rank $\rho$. The correct generalization of rank to an abstract monoid $S$ is given by the Green's relation $\mathcal{J}$ which is defined as: $a \mathcal{J} b$ if and only if $xay = b$, $sbt = a$ for some $x, y, s, t \in S$. The rank ordering in $\mathcal{M}_n(K)$ generalizes to a partial ordering of the $\mathcal{J}$-classes of $S$. A $\mathcal{J}$-class with an idempotent is called regular. Let $\mathcal{U}(S)$ denote the set of all regular $\mathcal{J}$-classes of $S$. It turns out that $(\mathcal{U}(S), \leq)$ is a lattice if $S$ is a connected monoid. Let us now look at two basic connected monoids. Let $S_1 = \mathcal{M}_n(K)$, $S_2 =$ the monoid of all upper triangular $n \times n$ matrices over $K$. Let $G_i$ denote the group of units of $S_i$ ($i = 1, 2$). $G_2$ is

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solvable while $G_1$ is as far from being solvable as possible. Now let us compare $\mathfrak{U}(S_1)$ and $\mathfrak{U}(S_2)$. $\mathfrak{U}(S_1)$ is a linear chain while $\mathfrak{U}(S_2)$ is a Boolean lattice. The basic conjecture in this direction is the following.

**Conjecture.** Let $S$ be a connected monoid with zero and group of units $G$. Then $G$ is solvable if and only if $\mathfrak{U}(S)$ is a relatively complemented lattice. In this paper, we prove the conjecture when $|\mathfrak{U}(S)| \leq 7$.

Let $S$ be a connected monoid with groups of units $G$. The connection between the solvability of $G$ and the semilattice decomposition of $S$ into archimedean semigroups was pointed out in [9]. This point is further elaborated in the present paper. If $J \in \mathfrak{U}(S)$, then we introduce two basic numbers $\theta(J)$ and $\delta(J)$ and study their properties. Crucial to this process is the theorem that for any idempotent $e$ of $S$, the centralizer of $e$ in $G$ is connected. If $S$ is a connected regular monoid, then the set \{$e : e \in E(S), e \geq f$\} is finite for all $f \in E(S)$, if and only if $S$ is a Clifford semigroup. Connected monoids with central idempotents are also studied. A conjecture about their structure is forwarded. Let $S$ be a connected monoid. Then it is pointed out that the maximal connected submonoids of $S$ with central idempotents need not be conjugate. However special maximal connected submonoids with central idempotents are conjugate.

Let $S$ be a connected monoid with group of units $G$ and let $T$ be a maximal torus of $G$. The maximal semilattice image of $S$ is explicitly determined and is shown to be isomorphic to a subsemilattice of $(\mathfrak{U}(S), \wedge)$ and of $(E(\bar{T}), \cdot)$.

### 1. Preliminaries

Throughout this paper $\mathbb{Z}^+$, $\mathbb{R}$ will denote the sets of positive integers and all reals, respectively. If $X$ is a set then $|X|$ denotes the cardinality of $X$. If $(P, \leq)$ is a partially ordered set, $\alpha, \beta \in P$, then $\beta$ covers $\alpha$ if $\beta > \alpha$ and there is no $\gamma \in P$ such that $\beta > \gamma > \alpha$. A linearly ordered subset of $P$ is a chain. The length of a chain $\{\alpha_1, \ldots, \alpha_k\}$ is $k - 1$. A chain not contained in any other is a maximal chain. If $\xi \in P$, then $\xi$ is a minimal element of $P$ if there is no $\alpha \in P$ such that $\alpha < \xi$. $\xi$ is a minimum element of $P$ if $\xi \leq \alpha$ for all $\alpha \in P$. Let $(L, \vee, \wedge)$ be a lattice. Then $L$ is relatively complemented if for all $\alpha, \beta, \gamma \in L$, $\alpha \beta > \gamma$ implies that there exists $\xi \in L$ such that $\beta \wedge \xi = \gamma$ and $\beta \vee \xi = \alpha$.

Let $S$ be a monoid with group of units $G$. If $X, Y \subseteq S$, then $X, Y$ are conjugate if $a^{-1}Xa = Y$ for some $a \in G$. If $X \subseteq S$ and $Y \subseteq G$, then $N_Y(X) = \{y : y \in Y, y^{-1}Xy = X\}$ is the normalizer of $X$ in $Y$. If $X \subseteq S$, $Y \subseteq S$, then $C_Y(X) = \{y : y \in Y, xy = yx \text{ for all } x \in X\}$ is the centralizer of $X$ in $Y$. $C(S) = C_S(S)$ is the center of $S$. Let $S$ be a semigroup. If $a, b \in S$, then $a \mid b$ ($a$ divides $b$) if $xay = b$ for some $x, y \in S^1$. $S$ is archimedean if for all $a, b \in S$, $a \mid b^i$ for some $i \in \mathbb{Z}^+$. Let $S_\alpha (\alpha \in \Omega)$ denote a partition of $S$ into subsemigroups. Then $S$ is a semilattice (union) of $S_\alpha (\alpha \in \Omega)$ if for all $\alpha, \beta \in \Omega$, there exists $\gamma \in \Omega$ such that $S_\alpha S_\beta \cup S_\beta S_\alpha \subseteq S_\gamma$. By the author [4], $S$ is a semilattice of archimedean semigroups if and only if for all $a, b \in S$, $a \mid b$ implies $a^2 \mid b^i$ for some $i \in \mathbb{Z}^+$. A semigroup with no proper semilattice decomposition is said to be $S$-indecomposable. By Tamura [10, 11, 12] every semigroup is a semilattice of $S$-indecomposable semigroups. We let $E(S)$ denote the set of all idempotents of $S$. If $X \subseteq S$, then $E(X) = E(S) \cap X$. If $e, f \in E(S)$, then $e \geq f$ if $ef = fe = f$. We write $E(S)$ for the partially ordered set $(E(S), \leq)$. $\mathcal{J}$, $\mathcal{R}$, $\mathcal{L}$,
\( \mathcal{H}, \mathcal{D} \) will denote the usual Green’s relations on \( S \). If \( a \in S \), then \( H_a, J_a \) will denote the \( \mathcal{H} \)-class and \( \mathcal{D} \)-class of \( a \), respectively. We let \( \mathcal{U}(S) = \{ J \mid J \) is a \( \mathcal{D} \)-class of \( S \) and \( E(J) \neq \emptyset \} \). If \( J_1, J_2 \in \mathcal{U}(S) \), then \( J_1 \succ J_2 \) if \( a_1 \rhd a_2 \) for some (all) \( a_i \in J_i \), \( i = 1, 2 \). We write \( \mathcal{U}(S) \) for the partially ordered set \(( \mathcal{U}(S), \succ) \).

\( K \) will denote a fixed algebraically closed field. If \( n \in \mathbb{Z}^+ \), then \( K^n = K \times \cdots \times K \) is the affine \( n \)-space and \( \mathfrak{M}_n(K) \) the multiplicative monoid of all \( n \times n \) matrices over \( K \). If \( X \subseteq K^n \), then \( X \) is closed if \( X \) is the set of zeros of some polynomials on \( K^n \). If \( X \subseteq K^n \), the \( \overline{X} \) denotes the closure of \( X \). By an algebraic semigroup we mean a closed subset of \( K^n \) along with an associative operation which is a polynomial map. By a homomorphism between algebraic semigroups we mean a semigroup homomorphism which is also a morphism of varieties (= a polynomial map). By a connected semigroup we mean an algebraic semigroup such that the underlying closed set is irreducible (i.e., is not a union of two proper closed subsets). Let \( S \) be an algebraic monoid with group of units \( G \). Then \( S \) is isomorphic to a closed submonoid of \( \mathfrak{M}_n(K) \) for some \( n \in \mathbb{Z}^+ \). A power of each element of \( S \) lies in a subgroup of \( S \), \( J = \mathcal{D} \), and \( \mathcal{U}(S) \) is finite. If \( J \in \mathcal{U}(S) \), then \( J^0 \) is completely \( \mathcal{O} \)-simple. \( G \) is a principal open subset of \( S \) and can therefore be viewed as an algebraic group. \( S \setminus G \) is a closed prime ideal of \( S \). If \( S_1 \) is a closed submonoid of \( S \), then \( G_1 = S_1 \cap G \) is the group of units of \( S_1 \). The identity element \( 1 \) of \( S \) lies in a unique irreducible component \( S^c \) of \( S \). Moreover \( S^c \) is a closed connected submonoid of \( S \) and \( S^c \cap G = G^c \). If \( S_1 \) is a closed submonoid of \( S \), then \( S_1 \cap G = (S_1 \cap G)^c \). If \( S \) is connected, then so is \( G \), and in that case \( \overline{G} = S \). If \( e \in E(S) \), then \( H_e \) is the group of units of \( eS_e \). So \( H_e \) is an algebraic group. If \( S \) is connected, then so is \( H_e \) and \( \overline{H}_e = eSe \).

We will need a number of basic results on (linear) algebraic groups. See [1, 3] for the extensive literature on algebraic groups. Let \( G \) be a connected algebraic group and let \( T \) be a maximal torus of \( G \). We let \( W(G) = N_G(T)/C_G(T) \) denote the Weyl group of \( G \). Then [3, Proposition 24.1A, Corollary 25.2C], \( W(G) \) is a finite group, and \( G \) is solvable if and only if \( |W(G)| = 1 \).

**Lemma 1.1.** Let \( S \) be an algebraic monoid, \( S_1, S_2 \) closed submonoids of \( S \). Then \((S_1^c \cap S_2^c)^c = (S_1^c \cap S_2^c)^c = (S_1 \cap S_2)^c \).

**Proof.** \( S_1^c \cap S_2^c \subseteq S_1^c \cap S_2 \subseteq S_1 \cap S_2 \). Hence \((S_1^c \cap S_2^c)^c \subseteq (S_1^c \cap S_2^c)^c \subseteq (S_1 \cap S_2)^c \subseteq S_1^c \cap S_2^c \). This yields the result.

**Lemma 1.2.** Let \( S \) be an algebraic monoid with group of units \( G \). Then

1. If \( X, Y \subseteq S \), then \( \overline{XY} = \overline{XY} \).
2. If \( X \subseteq S, a, b \in G \), then \( a\overline{Xb} = \overline{aXb} \).
3. If \( X \subseteq S, Y \subseteq G \), then \( N_Y(X) \subseteq N_Y(\overline{X}) \).
4. If \( X, Y \subseteq S \), then \( C_Y(X) = C_Y(\overline{X}) \).

**Proof.** (1) The set \( \{ x \mid x \in S, xY \subseteq \overline{XY} \} \) is closed and contains \( X \). Hence \( \overline{XY} \subseteq \overline{XY} \). Similarly \( \overline{XY} \subseteq \overline{XY} \). Since \( XY \subseteq \overline{XY} \), \( XY = \overline{XY} \).

(2) Since \( a, b \in G \), \( a\overline{Xb} \) is closed. Since \( a\overline{Xb} \subseteq a\overline{Xb}, a\overline{Xb} \subseteq a\overline{Xb} \). Similarly, since \( X \subseteq a^{-1}a\overline{Xb}^{-1} \), we see that \( \overline{X} \subseteq a^{-1}a\overline{Xb}^{-1} \).
(3) Let \( a \in N(Y(X)) \). Then \( a^{-1}Xa = X \). By (2), \( a^{-1}Xa = \overline{X} \).

(4) Let \( y \in C_Y(X) \). Then \( C_\gamma(y) \) is a closed set containing \( X \). Hence \( \overline{X} \subseteq C_\gamma(y) \). So \( y \in C_Y(X) \).

**Lemma 1.3.** Let \( S \) be a connected monoid with group of units \( G \). Let \( T \) be a maximal torus of \( G \), \( e \in E(\overline{T}) \). Let \( S_1 = \{ a \mid a \in S, ae = ea = e \} \), \( T_1 = \{ a \mid a \in T, ae = ea = e \} \), \( S_2 = S_1^\prime \), \( T_2 = T_1^\prime \), \( G_2 \) the group of units of \( S_2 \). Then

1. for all \( f \in E(S) \), \( f \geq e \) implies \( f \in S_2 \);
2. \( T_2 \) is a maximal torus of \( G_2 \);
3. if \( e \) is a minimal idempotent of \( S \), then \( E(T) = E(T_2) \).

**Proof.** (1) By [7, Lemma 1.1], there exists a closed connected submonoid of \( S' \) of \( S \) such that \( f \) is the zero of \( S' \). Then \( S' \subseteq S_1 \). Hence \( S' \subseteq S_1^\prime = S_2 \). So \( f \in S_2 \).

(2) Let \( \Gamma = \{ 1 > e_1 > \cdots > e_k = e > e_{k+1} > \cdots > e_m \} \) be a maximal chain in \( E(\overline{T}) \). Then by [9, Lemma 17], \( \Gamma \) is a maximal chain in \( E(S) \). Let \( \Gamma_0 = \{ 1 > e_1 > \cdots > e_k = e \} \). Then by (1) (applied to \( \overline{T} \)), \( \Gamma_0 \subseteq E(\overline{T}_2) \). Clearly \( \Gamma_0 \) is a maximal chain in \( E(S_2) \). Since \( e \) is the zero of \( \overline{T}_2 \), we see by [6, Theorem 3.17] that \( \dim T_2 = k \). Let \( T_2 \subseteq T_3 \) where \( T_3 \) is a maximal torus of \( G_2 \). Then clearly \( \Gamma_0 \) is a maximal chain in \( E(\overline{T}_3) \) and \( e \) is the zero of \( \overline{T}_3 \). So again by [6, Theorem 3.17], \( \dim T_3 = k \). So \( T_2 = T_3 \).

(3) If \( e \) is a minimal idempotent of \( S \), then clearly \( e \) is the minimum idempotent of \( \overline{T} \) and so by (1) (applied to \( \overline{T} \)), \( E(\overline{T}) \subseteq E(\overline{T}_2) \).

**Lemma 1.4.** Let \( S \) be a connected monoid with group of units \( G \). Let \( T \) be a maximal torus of \( G \), \( e \in E(\overline{T}) \). Then \( eT \) is a maximal torus of \( He \).

**Proof.** Clearly \( T \subseteq C_\gamma(e)^\circ \). Consider the homomorphism, \( \phi: C_\gamma(e)^\circ \rightarrow H_e \) given by \( \phi(a) = ea \). By [8, Theorem 4], \( \phi \) is surjective. Hence by [3, Corollary 21.3C], \( \phi(T) = eT \) is a maximal torus of \( H_e \).

**Lemma 1.5.** Let \( S \) be a connected monoid with group of units \( G \). Let \( T \) be a maximal torus of \( G \). Let \( e, f \in E(T) \) such that \( e \geq f \). Then the following conditions are equivalent:

1. \( e \) covers \( f \) in \( E(S) \).
2. \( e \) covers \( f \) in \( E(\overline{T}) \).
3. \( J_e \) covers \( J_f \) in \( \mathfrak{H}(S) \).

(1) By [8, Theorem 10], there exists \( e_1 \in E(S) \) such that \( e_1 \geq f_1 \) and \( e \geq f_1 \). By Lemma 1.3, \( eT \) is a maximal torus of \( H_e \). Since \( H_e \) is the group of units of \( eSe \) and since \( f_1 \in eSe \), we see [7, Theorem 1.8] that there exists \( f' \in E(\overline{T}_2) \) such that \( f_1 \geq f' \). So \( f' \in E(\overline{T}) \), \( e \geq f' \).

(2) By [8, Theorem 10], there exists \( e_1 \in E(S) \) such that \( e_1 \geq f_1 \) and \( e \geq f_1 \). Let \( S_1 = \{ a \mid a \in S, af = fa = f \} \), \( T_1 = \{ a \mid a \in T, af = f \} \), \( S_2 = S_1^\prime \), \( T_2 = T_1^\prime \), \( G_2 \) the group of units of \( S_2 \). Then by Lemma 1.3, \( f_1 \geq f' \). Then \( e' \geq f', e' \in E(\overline{T}) \).

**Lemma 1.6.** Let \( S \) be a connected monoid with group of units \( G \). Let \( T \) be a maximal torus of \( G \) and let \( e, f \in E(\overline{T}) \), \( e \geq f \). Then the following conditions are equivalent:

1. \( e \) covers \( f \) in \( E(S) \).
2. \( e \) covers \( f \) in \( E(\overline{T}) \).
3. \( J_e \) covers \( J_f \) in \( \mathfrak{H}(S) \).
Proof. That $(1) \Rightarrow (2)$ is obvious. So assume $(2)$. Suppose $J_e$ does not cover $J_f$. Then there exists $e_1 \in E(S)$ such that $e_1 \not\in J_e$, $e_1 \notin J_f$ and $e_1 | f$. By [6, Lemma 1.3], there exists $e_2 \in E(S)$ such that $e > e_2$, $e_2 \neq e_1$. So $e_2$, $f \in eS_e$. By Lemma 1.4, $eT$ is a maximal torus of $H_e$. By [6, Lemma 1.7], $e_2$ $f$ in $eS_e$. So by Lemma 1.5, there exists $e_3 \in E(S)$ such that $e_3 \geq e_2$ and $e_3 > f$. Then $e > e_3 > f$, $e_3 \in E(T)$. This contradiction shows that $(2) \Rightarrow (3)$. Next assume $(3)$. Suppose $e > e_1 > f$ for some $e_1 \in E(S)$. Then $J_e > J_{e_1} > J_f$, a contradiction. Hence $(3) \Rightarrow (1)$.

Lemma 1.7. Let $S$ be a connected monoid with group of units $G$ and let $T$ be a maximal torus of $G$.

(1) If $J \in \mathcal{U}(S)$, $e \in J \cap E(T)$, then $J \cap E(T) = \{x^{-1}e | x \in N_G(T)\}$.

(2) If $J_1$, $J_2 \in \mathcal{U}(S)$, then $J_1 > J_2$ if and only if there exist $e_i \in J_i \cap E(T)$, $i = 1, 2$, such that $e_1 > e_2$.

Proof. (1) Let $f \in E(T) \cap J$. Then by [8, Theorem 9], $x^{-1}e = f$ for some $x \in G$. Since $T \subseteq C_G(f)$, $xT \subseteq C_G(e)$. So $T$, $xT^{-1}$ are maximal tori of $C_G(e)$. Hence there exists $y \in C_G(e)^\circ$ such that $y^{-1}Ty = xT^{-1}$. Then $x^{-1}y^{-1}Ty = T$. So $yx \in N_G(T)$. Since $y \in C_G(e)$, $(yx)^{-1}eyx = x^{-1}e = f$. This proves (1).

(2) By [7, Theorem 1.8], there exists $e_1 \in J_1 \cap E(T)$. If $J_1 > J_2$, then by Lemma 1.5, there exists $e_2 \in J_2 \cap E(T)$ such that $e_1 > e_2$. The converse is trivial.

Lemma 1.8. Let $S$ be a connected monoid, $e \in E(S)$. Then $\mathcal{V} = \{J \in \mathcal{U}(S) | J \leq J_e\}$ is a sublattice of $\mathcal{U}(S)$ and $\mathcal{U}(eS_e) = \{J \cap eS_e | J \in \mathcal{V}\} = \mathcal{V}$.

Proof. Let $J \in \mathcal{V}$, $f \in E(J)$. Then $e | f$. By [6, Lemma 1.3], $e \geq f'$ for some $f' \in E(J)$. So $f' \in J \cap eS_e$. It follows by [6, Lemma 1.7] that $J \cap eS_e \in \mathcal{U}(eS_e)$. The rest follows from [6, Lemma 1.7].

Corollary 1.9. Let $S$ be a connected monoid, $e \in E(S)$. If $\mathcal{U}(S)$ is a relatively complemented lattice, then so is $\mathcal{U}(eS_e)$.

Lemma 1.10. Let $S$ be a connected monoid, $e \in E(S)$. Suppose $eS_e$ is a semilattice of archimedean semigroups. Let $f_1$, $f_2 \in E(S)$ such that $e \geq f_1$, $e \geq f_2$, $f_1 \neq f_2$ and $f_1f_2 = f_2f_1$. Then $f_1 = f_2$.

Proof. Let $f_1$, $f_2 \in J \in \mathcal{U}(S)$. Then by Lemma 1.8, $f_1$, $f_2 \in J \cap eS_e \in \mathcal{U}(eS_e)$. By [9, Proposition 21] $J \cap eS_e$ is completely simple. Since $f_1f_2 = f_2f_1$, we see that $f_1 = f_2$.

Lemma 1.11. Let $S$ be a closed connected subsemigroup of $\mathcal{M}_n(K)$ and let $J \in \mathcal{U}(S)$. Suppose that for all $e, f \in E(J)$, any eigenvalue of $ef$ is either 0 or 1. Then $J^2 \subseteq J$.

Proof. By the proof of [9, Proposition 23], $E(J)^2 \subseteq E(J)$. Since $J^0$ is completely 0-simple, it follows easily that $J^2 \subseteq J$.

Lemma 1.12. Let $S$ be a connected semigroup, $J \in \mathcal{U}(S)$. If $x, y \in J$, then there exist $e_1, e_2 \in E(J)$ such that $xR e_1\subseteq e_2R y$.

Proof. $J^0$ is completely 0-simple, and hence admits a Rees representation $J^0 = (\Gamma \times G \times \Lambda) \cup \{0\}$ with sandwich map $P: \Lambda \times \Gamma \to G^0$ where $G$ is a group.
Multiplication in $J^0$ is given by

$$(a, a, \beta)(a, a, b, \beta) = \begin{cases} (a, aP(\beta, \gamma)b, \beta) & \text{if } P(\beta, \gamma) \neq 0, \\ 0 & \text{if } P(\beta, \gamma) = 0. \end{cases}$$

Let $x = (a_1, a_2, \beta_1), y = (a_2, a_2, \beta_2)$. By [5, Theorem 2.16] there exists $\mu \in \Lambda$ such that $P(\mu, a_1) \neq 0, P(\mu, a_2) \neq 0$. Let $e_1 = (a_1, P(\mu, a_1)^{-1}, \mu), e_2 = (a_2, P(\mu, a_2)^{-1}, \mu).$ Then $e_1, e_2 \in E(J)$ and $x \otimes e_1 \otimes e_2 \otimes y$.

**Lemma 1.13.** Let $S$ be a commutative connected monoid, $e \in E(S)$. If $\Omega$ is a finite group of automorphisms of $S$ having $e$ as a common fixed point, then there exists a closed connected submonoid $S_1$ of $S$ such that $e \in S_1$ and $\phi(a) = a$ for all $\phi \in \Omega$, $a \in S_1$.

**Proof.** Let $\Omega = \{ \phi_1, \ldots, \phi_k \}$ and define $\psi: S \to S$ as $\psi(x) = \phi_1(x) \cdots \phi_k(x)$. Since $S$ is commutative, $\psi$ is a homomorphism. Clearly $\psi(1) = 1$ and $\psi(e) = e$. Let $\phi \in \Omega$. Then since $\phi \Omega \Omega = \Omega, \phi \psi(x) = \psi(x)$ for all $x \in S$. Let $S_1 = \psi(S)$. Then $S_1$ is a closed connected submonoid of $S$ and $e \in S_1$. Clearly $\phi(a) = a$ for all $\phi \in \Omega$ and $a \in S_1$.

**Proposition 1.14.** Let $S$ be a connected monoid with group of units $G$ and let $T$ be a maximal torus of $G$. Let $\mathcal{X} = \{ J | J \in \mathfrak{U}(S), G \text{ covers } J \}, \mathcal{Y} = \{ J | J \in \mathfrak{U}(S), J_1 \text{ covers } J \text{ for some } J_1 \in \mathcal{X} \}, X = \{ e | e \in E(T), 1 \text{ covers } e \}, Y = \{ f | f \in E(T), e \text{ covers } f \text{ for some } e \in X \}$. Then

1. $X = \bigcup_{J \in \mathcal{X}} E(T) \cap J$.
2. $Y = \bigcup_{J \in \mathcal{Y}} E(T) \cap J$.
3. If $J_1 \in \mathcal{X}, J_2 \in \mathcal{Y}$ and if $G > J_1 > J_2$, then $J_1 \in \mathcal{X}$ and $J_1$ covers $J_2$.
4. If $e \in E(T), f \in Y$, and if $1 > e > f$, then $e$ covers $f$ and $e \in X$.
5. If $J' \in \mathcal{Y}$, then $| \{ J | J \in \mathcal{X}, J > J' \} | \leqslant 2$.
6. If $f \in Y$, then $| \{ e | e \in X, e > f \} | \leqslant 2$.
7. If $\mathfrak{U}(S)$ is relatively complemented, if $f \in Y, e_1, e_2 \in X, e_1 > f, e_2 > f$, and if $e_1 \otimes e_2$, then $e_1 = e_2$.

**Proof.** (1), (2) follow from Lemmas 1.5 and 1.6. Next let $J_2 \in \mathcal{Y}, J_1 \in \mathfrak{U}(S)$ such that $G > J_1 > J_2$. There exists $J \in \mathcal{X}$ such that $J$ covers $J_1$. So we have a maximal chain $\Gamma = \{ G > J > J_i > J_1 > i \}$ in $\mathfrak{U}(S)$. Then $\Lambda = \{ G > J_1 > J_2 > J_3 > \cdots \}$ is a chain in $\mathfrak{U}(S)$. Since by [7, Theorem 1.9] all maximal chains in $\mathfrak{U}(S)$ have the same length, $\Lambda$ is also a maximal chain in $\mathfrak{U}(S)$. So $G$ covers $J_1$ and $J_1$ covers $J_2$. This proves (3). Now (4) follows from Lemma 1.6.

We now prove (5). Let $J_1, J_2, J_3$ be distinct elements of $\mathcal{X}, J' \in \mathcal{Y}$ such that $J_i > J_i', i = 1, 2, 3$. Let $f \in J' \cap E(T)$. By Lemma 1.5, there exists $e_i \in J_i \cap E(T), i = 1, 2, 3$, such that $e_i > f, i = 1, 2, 3$. Then by (1), (2), $e_1, e_2, e_3 \in X$ and $f \in Y$. Let $T_1 = \{ a | a \in T, af = f \}, T_2 = T_2'$. Then by Lemma 1.3, $e_1, e_2, e_3, f \in T_2$. Clearly $f$ is the zero of $T_2$. By (4), $\{ 1 > e_1 > f \}$ is a maximal chain in $E(T_2)$. By [6, Theorem 3.17], $\dim T_2 = 2$. So by [6, Theorem 3.18], $| E(T_2) | \leqslant 4$. However, $1, e_1, e_2, e_3, f$ are distinct elements of $E(T_2)$. This contradiction proves (5). We now obtain (6) by applying (5) to $T$. 


Next we prove (7). Suppose \( \mathcal{U}(S) \) is relatively complemented, \( e_1, e_2 \in X \), \( e_1 \neq e_2 \), \( e_1 > f, e_2 > f \). Let \( J_1 \) denote the \( f \)-class of \( e_1 \), \( J' \) the \( f \)-class of \( f \). Then \( J_1 \cap J = J' \), \( J_1 \cup J = G \) for some \( J \in \mathcal{U}(S) \). By (1), (2), (3), \( J_1, J \in \mathcal{X} \) and \( J' \in \mathcal{Y} \). By Lemma 1.5, there exists \( e_3 \in J \cap E(\widehat{T}) \) such that \( e_3 > f \). By (1), \( e_3 \in X \). So \( e_i > f, i = 1, 2, 3 \). This contradicts (6), proving (7).

**Lemma 1.15.** Let \( S \) be a connected monoid such that the group of units \( T \) of \( S \) is a torus. Let \( X = \{ e \mid e \in E(S), 1 \text{ covers } e \}, Y = \{ f \mid f \in E(S), e \text{ covers } f \text{ for some } e \in X \} \). Then for any \( e, e' \in X \) with \( e \neq e' \), there exist distinct \( f_1, \ldots, f_k \in Y \) and distinct \( e_1, \ldots, e_{k+1} \in X \) such that \( e = e_1, e' = e_{k+1} \) and \( e_i > f_i, e_{i+1} > f_i, i = 1, \ldots, k \).

**Proof.** By Lemma 1.3, we can assume that \( S \) has a zero. If \( P \) is a polytope in \( \mathbb{R}^n \), then let \( \mathcal{T}(P) \) denote the lattice of all faces of \( P \) (see [2]). By [7, Theorem 3.6], \( E(S) \) is isomorphic to \( \mathcal{T}(P) \) for some \( \mathbb{R}^n \), \( P \). Let \( P^* \) denote the dual polytope of \( P \) [2, §3.4, p. 46]. Then \( \mathcal{T}(P) \) is anti-isomorphic to \( \mathcal{T}(P^*) \). Let \( \phi: E(S) \to \mathcal{T}(P^*) \) denote the corresponding map. Then \( \phi(X) \) consists of all vertices of \( P^* \) and \( \phi(Y) \) consists of all edges of \( P \). By a result of Balinski (see [2, §11.3, Theorem 2]) there exist vertices \( u_1, \ldots, u_{k+1} \) and edges \( v_1, \ldots, v_k \) such that \( \phi(e) = u_1, \phi(e') = u_{k+1} \) and \( u_i, u_{i+1} \) are the vertices of \( v_i, i = 1, \ldots, k \). Let \( e_i = \phi^{-1}(u_i), i = 1, \ldots, k + 1 \). Then \( e_i > f_i, e_{i+1} > f_i, i = 1, \ldots, k \). If we choose \( k \) to be minimal, then clearly, \( e_1, \ldots, e_{k+1} \) and \( f_1, \ldots, f_k \) will be distinct.

**Lemma 1.16.** Let \( G \) be a connected group, \( H \) a closed connected normal subgroup of \( G \). Then \( G/H \) is a Borel subgroup of \( G \) such that \( B_0 \) is a Borel subgroup of \( G \). Then \( |W(G)| = |W(H)| \).

**Proof.** Since \( H < G \) and since all Borel subgroups of \( G \) are conjugate, \( BH = G \) for all Borel subgroups \( B \) of \( G \). Let \( U \) be a Borel subgroup of \( H \). Then \( \mathfrak{C}(P) \) is a Borel subgroup of \( H \). By [3, Corollary 23.1A] \( U \cap H = U \) is a Borel subgroup of \( H \). Since \( H < G \), \( B \cap H \) is a Borel subgroup of \( H \) for any Borel subgroup \( B \) of \( G \). Let \( T \) be a maximal torus of \( H \) and let \( \mathfrak{C} = \{ B \mid B \text{ is a Borel subgroup of } G, T_1 \subseteq B \}, \mathfrak{C}_1 = \{ B \cap H \mid B \subseteq \mathfrak{C} \}. \) Then by [3, Proposition 24.1A], \( |\mathfrak{C}| = |W(H)| \). Let \( B_1, B_2 \in \mathfrak{C} \) such that \( B_1 \cap H = B_2 \cap H \). Now \( x^{h^{-1}B_1 x} = B_2 \) for some \( x \in G = B_1 H \). So \( x = bh \) for some \( b \in B_1 \), \( h \in H \). So \( h^{-1}B_1 h = B_2 \). Thus \( h^{-1}B_1 \cap H = B_2 \cap H = B_1 \cap H \). So \( h \in N_H(B_1 \cap H) \). By [3, Theorem 23.1], \( N_H(B_1 \cap H) = B_1 \cap H \). Thus \( h \in B_1 \cap H \subseteq B_1 \), and \( B_1 = B_2 \). It follows that \( |\mathfrak{C}| = |\mathfrak{C}_1| < \infty \). Hence \( T_1 \) is a regular torus of \( G \) (see [3, Chapter IX]). Hence [3, Proposition 24.2], \( |\mathfrak{C}| = |W(G)| \).

**Corollary 1.17.** Let \( G \) be a connected group, \( H \) a closed normal subgroup of \( G \), \( B \) a Borel subgroup of \( G \) such that \( BH = G \). Then \( |W(G)| = |W(H^c)| \).

**Proof.** Since \( H < G \), \( H^c < G \). So \( BH^c \) is a subgroup of \( G \). Since \( BH^c \) is the image of the product map from \( B \times H^c \) into \( G \), \( BH^c \) is a constructible subgroup of \( G \). Hence [3, Proposition 7.4A], \( BH^c \) is a closed subgroup of \( G \). By [3, Proposition 7.3], \( H = H^c h_1 \cup \cdots \cup H^c h_k \) for some \( h_1, \ldots, h_k \in H \). Hence \( G = BH = BH^c h_1 \cup \cdots \cup BH^c h_k \). Since \( G \) is connected, \( G = BH^c h_i \) for some \( i \). So \( h_i \in BH^c \) and \( G = BH^c \). We are now done by Lemma 1.16.
Corollary 1.18. Let \( \phi: G \to G' \) be a surjective homomorphism of connected groups such that \( G' \) is solvable and let \( H \) denote the kernel of \( \phi \). Then \( |W(G)| = |W(H^c)| \).

Proof. Let \( B \) be a Borel subgroup of \( G \). Then by [3, Corollary 21.3C], \( \phi(B) = G' \). So \( G = BH \). We are done by Corollary 1.17.

Lemma 1.19. Let \( \phi: G \to G' \) be a surjective homomorphism of connected groups and let \( H \) denote the kernel of \( \phi \). Then \( |W(G)| = |W(H^c)| \).

Proof. Let \( B \) be a Borel subgroup of \( G \). Then by [3, Chapter IX], \( \phi(B) = G' \). So \( G = BH \). We are done by Corollary 1.17.

Corollary 1.20. Let \( S \) be a connected monoid with group of units \( G \), \( e \in E(C(S)) \). Let \( G_1 = \{ a | a \in G, ae = e \} \). Then \( |W(G)| = |W(G_1^c)| \).

Proof. The homomorphism \( \phi: G \to H_e \) is given by \( \phi(x) = ex \) is surjective by [8, Lemma 3]. Clearly \( G_1 \) is the kernel of \( \phi \). We are done by Lemma 1.19.

Lemma 1.21. Let \( S \) be an \( \mathcal{S} \)-indecomposable semigroup, \( J \in \mathcal{U}(S) \). Then \( J^2 \subseteq J \) if and only if \( J \) is the kernel of \( S \).

Proof. By Tamura [10], \( S^1JS^1 \), being an ideal of \( S \), is \( \mathcal{S} \)-indecomposable. If \( J \) is not the kernel of \( S \), then \( \{ J, S^1JS^1 \} \) is a proper semilattice decomposition of \( S^1JS^1 \), a contradiction.

2. Main section. Fundamental in the theory of connected algebraic monoids is the following theorem (see [7, Theorems 1.8, 3.6, Corollary 1.6]; [6, Theorem 3.20]).

Theorem 2.1 [7]. Let \( S \) be a connected monoid with group of units \( G \) and let \( T \) be a maximal torus of \( G \). Then

1. \( \mathcal{U}(S) \) is a finite lattice.
2. \( E(\overline{T}) \) is a finite, relatively complemented lattice (in fact the lattice of all faces of a rational polytope in some \( \mathbb{R}^n \)).
3. Any maximal chain in \( \mathcal{U}(S) \) has the same length as any maximal chain in \( E(\overline{T}) \) or \( E(S) \).
4. \( E(S) = \bigcup_{x \in G} x^{-1}E(\overline{T})x \).

See [8, Theorem 3] for the following theorem.
THEOREM 2.2 [8]. Let $S$ be a connected monoid with group of units $G$. If $J \subseteq \mathcal{U}(S)$, and if $a, b \in J$, then $xay = b$ for some $x, y \in G$. If $e, f \in E(J)$, then $x^{-1}ex = f$ for some $x \in G$.

A special case of the following basic result was proved in [9, Theorem 14].

THEOREM 2.3. Let $S$ be a connected monoid with group of units $G$. Then for any $e \in E(S)$, $C_G(e)$ is a connected group.

PROOF. By Theorem 2.1, $e \in \overline{T}$ for some maximal torus $T$ of $G$. First let $u \in C_G(e) \cap N_G(T)$. Since $N_G(T)/C_G(T)$ is a finite group [3, Chapter IX], $u^i \in C_G(T)$ for some $i \in \mathbb{Z}^+$. Consider the automorphism $\phi: \overline{T} \to \overline{T}$ given by $\phi(x) = u^{-1}xu$. Since $u \in C_G(e)$, $\phi(e) = e$. By Lemma 1.13 there exists a subtorus $T_1$ of $T$ such that $e \in T_1$ and $\phi(a) = a$ for $a \in T_1$. So $u \in C_G(T_1)$. By [3, Theorem 22.3], $C_G(T_1)$ is connected. On the other hand, since $e \in T_1$, $C_G(T_1) = C_G(\overline{T_1}) \subseteq C_G(e)$. So $C_G(T_1) \subseteq C_G(e)^c$. Therefore,

$$C_G(e) \cap N_G(T) \subseteq C_G(e)^c.$$

Now let $a \in C_G(e)$. Then $T, a^{-1}Ta$ are maximal tori of $C_G(e)^c$. So $b^{-1}a^{-1}Tab = T$ for some $b \in C_G(e)^c$. Thus $ab \in C_G(e) \cap N_G(T) \subseteq C_G(e)^c$. Since $b \in C_G(e)^c$, $a \in C_G(e)^c$.

DEFINITION. Let $S$ be a connected monoid with group of units $G$ and let $T$ be a maximal torus of $G$.

1. $\alpha(S) = |E(\overline{T})|$.
2. $\beta(S) = |\mathcal{U}(S)|$.
3. If $W$ is the Weyl group of $G$, then $\gamma(S) = |W|$.
4. If $J \in \mathcal{U}(S)$ and if $e \in E(J)$, then $\theta(e) = \theta(J) = |J \cap E(\overline{T})|$ and $\delta(e) = \delta(J) = \gamma(C_G(e))$.

We write $\delta_3, \delta_5$ if we want to specify $S$.

REMARK 2.4. Since all maximal tori in $G$ are conjugate, since all idempotents of $J$ are conjugate and since $x^{-1}Jx = J$ for all $x \in G$, we see that the above definitions are independent of the choice of $T$ and the particular idempotent $e$ in $J$.

REMARK 2.5. If $S$ is a connected monoid and if $S_1$ is a closed connected submonoid of $S$, then $\alpha(S_1) \leq \alpha(S)$. However, it can very well happen that $\beta(S_1)$ is larger than $\beta(S)$. If $e \in E(S_1)$, then $\theta_3(e) \leq \theta_5(e)$.

THEOREM 2.6. Let $S$ be a connected monoid. Then

1. For all $J \in \mathcal{U}(S)$, $\gamma(S) = \theta(J)\delta(J)$. In particular, $1 \leq \theta(J) \leq \gamma(S)$.
2. $\alpha(S) = \sum_{J \in \mathcal{U}(S)} \theta(J)$.
3. $\beta(S) \leq \alpha(S) \leq \beta(S)\gamma(S)$.
4. If $J_1, J_2 \in \mathcal{U}_S$, then $\theta(J_1 \wedge J_2) \leq \theta(J_1)\theta(J_2)$.

PROOF. Let $J \in \mathcal{U}(S)$, $T$ a maximal torus of $G$, the group of units of $S$. Let $A = E(\overline{T}) \cap J$. Then $\theta(J) = |A|$. Let $e \in A$. If $x \in N_G(T)$ and $f \in A$, then let $f^x = x^{-1}fx \in A$. Clearly, $(f^x)^y = f^{xy}$ for all $x, y \in N_G(T)$. So the group $N_G(T)$ acts on $A$. By Lemma 1.7, $N_G(T)$ acts transitively on $A$. The stabilizer of $e$ is $V = C_G(e) \cap N_G(T)$. Thus $\theta(J) = |A| = |N_G(T)/V|$. Clearly $T \subseteq C_G(T) \subseteq V$. So
Now \( V \) is the normalizer of \( T \) in \( C_G(e) \) and \( C_G(T) \) is the centralizer of \( T \) in \( C_G(e) \). So \( \delta(e) = |V/C_G(T)| \). Clearly \( \gamma(S) = |N_G(T)/C_G(T)| \). This proves (1). Clearly \( \alpha(S) = |E(\bar{T})| = \sum_{T \in \mathfrak{U}(S)} |J \cap E(\bar{T})| \). This yields (2). Now (3) follows from (1) and (2). Finally we prove (4). Let \( h \in (J_1 \cap J_2) \cap E(\bar{T}) \). Then by Lemma 1.5, there exists \( e_i \in J_i \cap E(\bar{T}) \) such that \( e_i \geq h \), \( i = 1, 2 \). So \( e_1e_2 \geq h \). By Lemma 1.7, \( e_1e_2 = h \). So \( (J_1 \cap J_2) \cap E(\bar{T}) \subseteq (J_1 \cap E(\bar{T}))(J_2 \cap E(\bar{T})) \). Hence \( \theta(J_1 \cap J_2) = \theta(J_1)\theta(J_2) \).

**Corollary 2.7.** Let \( S \) be a connected monoid with group of units \( G \) and let \( T \) be a maximal torus of \( G \). Then the following conditions are equivalent.

1. \( \alpha(S) = \beta(S) \).
2. \( \mathfrak{U}(S) = \mathfrak{U}(S) \) for all \( J \in \mathfrak{U}(S) \).
3. \( \mathfrak{U}(S) \cong E(\bar{T}) \).

**Proof.** That (1) \( \Rightarrow \) (2) follows from Theorem 2.6. Next assume (2). Define \( \phi: \mathfrak{U}(S) \to E(\bar{T}) \) as \( \phi(J) = e \) if \( J \cap E(\bar{T}) = \{e\} \). Then \( \phi \) is clearly a bijection. That it is an isomorphism follows from Lemma 1.7. Thus (2) \( \Rightarrow \) (3). That (3) \( \Rightarrow \) (1) is obvious.

**Theorem 2.8.** Let \( S \) be a connected monoid with group of units \( G \). Let \( e \) be a minimal idempotent of \( S \). Let \( S_1 = \{x \in S, ex = xe = e\} \) and let \( S_2 = S \). Then

1. \( \gamma(S) = \gamma(S_2) \gamma(eSe) \).
2. \( \mathfrak{U}(S) \cong \mathfrak{U}(S_2) \), \( \alpha(S) = \alpha(S_2) \) and \( \beta(S) = \beta(S_2) \).

**Proof.** (1) By [8, Theorem 1], \( eSe \subseteq C_G(e) \). By Corollary 1.20, \( \gamma(C_G(e)) = \gamma(S_2) \gamma(eSe) \). Since \( e \) lies in the kernel of \( S \), \( \theta(e) = 1 \). By Theorem 2.6, \( \gamma(S) = \delta(e) \). This proves (1).

(2) Let \( G, G_2 \) denote the groups of units of \( S \) and \( S_2 \) respectively. Let \( T \) be a maximal torus of \( G \) with \( e \subseteq f \). By Lemma 1.3, \( T \) is a maximal torus of \( G_2 \) and \( E(\bar{T}) = E(\bar{T}) \). Let \( f \in E(\bar{T}) \) and set \( S_3 = C_G(f) \). By [8, Theorem 1], \( eSe \subseteq fSf \subseteq S \). Then by Lemma 1.1, Theorem 2.6 and (1), \( \gamma(S) = \theta_2(f) \gamma(S_2) = \theta_2(f) \gamma(S_2 \cap S_3) \gamma(eSe) \). Also, \( \gamma(S) = \gamma(S_2) \gamma(eSe) = \theta_1(f) \gamma(S_2 \cap S_3) \gamma(eSe) \). It follows that \( \theta_1(f) = \theta_2(f) \) for all \( f \in E(\bar{T}) \). Let \( \mathfrak{U} = \{J \cap E(\bar{T}) \in \mathfrak{U}(S)\} \). Since \( E(\bar{T}) = E(\bar{T}) \) and since \( \theta_2(f) = \theta_2(f) \) for all \( f \in E(\bar{T}) \), we see that \( \mathfrak{U} = \{J \cap E(\bar{T}) : J \in \mathfrak{U}(S)\} \). If \( A_1, A_2 \in \mathfrak{U} \), then define \( A_1 > A_2 \) if there exist \( f_i \in A_i \), \( i = 1, 2 \), such that \( f_1 > f_2 \). Then by Lemma 1.7, \( \mathfrak{U}(S) = (\mathfrak{U}, >) \). In particular \( \beta(S) = \beta(S_2) \). Clearly \( \alpha(S) = |E(\bar{T})| = |E(\bar{T})| = \alpha(S_2) \).

**Theorem 2.9.** Let \( S \) be a connected monoid with group of units \( G \) and let \( T \) be a maximal torus of \( G \). Let \( S_1 \) be a closed connected submonoid of \( S \) such that \( N_G(T) \subseteq S_1 \). Then \( \mathfrak{U}(S) \cong \mathfrak{U}(S_1) \), \( \alpha(S) = \alpha(S_1) \) and \( \beta(S) = \beta(S_1) \).

**Proof.** We see that \( T \subseteq N_G(T) \subseteq S_1 \). So \( \bar{T} \subseteq S_1 \). Let \( \mathfrak{U} = \{J \cap E(\bar{T}) : J \in \mathfrak{U}(S)\} \). If \( A_1, A_2 \in \mathfrak{U} \), then define \( A_1 < A_2 \) if there exist \( e_i \in A_i \), \( i = 1, 2 \), such that \( e_1 < e_2 \). By Lemma 1.7, \( \mathfrak{U}(S) = (\mathfrak{U}, >) \). Since \( N_G(T) \subseteq S_1 \), we see by Lemma 1.7 that \( \mathfrak{U} = \{J \cap E(\bar{T}) : J \in \mathfrak{U}(S)\} \). Also, by Lemma 1.7, \( \mathfrak{U}(S_1) = (\mathfrak{U}, <) \). Hence \( \mathfrak{U}(S) \cong \mathfrak{U}(S_1) \).
**Theorem 2.10.** Let $S$ be a connected monoid and let $e$ be a minimal idempotent of $S$. Then there exists a closed connected submonoid $S_1$ of $S$ such that $e$ is the zero of $S_1$, $e \in C(S_1)^c$, $\alpha(S) = \alpha(S_1)$, $\beta(S) = \beta(S_1)$ and $\mathcal{U}(S) \cong \mathcal{U}(S_1)$. In particular, $\dim C(S_1) \geq 1$.

**Proof.** By Theorem 2.8, we can assume that $e = 0$ is the zero of $S$. Let $G$ denote the group of units of $S$ and let $T$ be a maximal torus of $G$. By [3, Chapter IX], $\Omega = N_G(T)/C_G(T)$ is a finite group. If $u \in N_G(T)$, then $a \to u^{-1}au$ is an automorphism of $T$. So $\Omega$ is a finite group of automorphisms of $T$. By Lemma 1.2, $N_G(T) = N_G(T)$ and $C_G(T) = C_G(T)$. So $\Omega$ is a finite group of automorphisms of $T$. By Lemma 1.13, there exists a subtorus $T_1$ of $T$ such that $0 \in T_1$ and $u^{-1}au = a$ for all $a \in T_1$ and $u \in N_G(T)$. Hence $N_G(T) \subseteq C_G(T_1)$. By [3, Theorem 22.3], $C_G(T_1)$ is a connected group. Let $S_1 = C_G(T_1)$. Then $N_G(T) \subseteq S_1$. By Theorem 2.9, $\mathcal{U}(S) \cong \mathcal{U}(S_1)$. Clearly $0 \in T_1 \subseteq C(S_1)^c$.

**Problem 2.11.** Characterize the lattice $\mathcal{U}(S)$ where $S$ is a connected monoid. By Theorem 2.10, the problem reduces to the case when $S$ has zero 0 and $0 \in C(S)^c$.

See [9] for the following theorem.

**Theorem 2.12 [9].** Let $S$ be a connected monoid with group of units $G$ such that the maximal subgroup of the kernel of $S$ is solvable. Then the following conditions are equivalent:

1. $G$ is solvable.
2. $\alpha(S) = \beta(S)$.
3. $S$ is a semilattice of archimedean semigroups.
4. $J^2 \subseteq J$ for all $J \in \mathcal{U}(S)$.
5. For all $e, f \in E(S)$, any eigenvalue of $ef$ is 0 or 1.

**Theorem 2.13.** Let $S$ be a connected monoid with group of units $G$. Let $J \in \mathcal{U}(S)$. Then the following conditions are equivalent:

1. $J^2 \subseteq J$.
2. $\theta(J) = 1$.
3. $E(J) \subseteq B$ for some Borel subgroup $B$ of $G$.
4. $E(J) \subseteq B$ for every Borel subgroup $B$ of $G$.
5. For all $e, f \in E(J)$, any eigenvalue of $ef$ is 0 or 1.

**Proof.** (1) $\Rightarrow$ (2). Since $J^2 \subseteq J$, $J$ is completely simple. So for any $e, f \in E(J)$, $ef = fe$ implies $e = f$. It follows that $\theta(J) = 1$.

(2) $\Rightarrow$ (3). If $e \in E(J)$, then let $X_e = \{ f | f \in E(S), e \not{\in} f \}$, $Y_e = \{ f | f \in E(S), e \not{\in} f \}$. If $f \in X_e$, then $ef = f$ and $fe = e$. If $f \in Y_e$, then $ef = e$ and $fe = f$. Now fix $e \in E(J)$. Let $S_1$ be a closed connected submonoid of $S$ of smallest possible dimension such that $X_e \subseteq S_1$. Let $S_2 = \{ a | a \in S_1, ae = e \}$. Let $f \in X_e$. Then there exists, by [7, Lemma 1.1], a closed connected submonoid $S_3$ of $S_1$ such that $f \in S_3$ and $bf = fb = f$ for all $b \in S_3$. So $S_3 \subseteq S_2$ and $S_3 \subseteq S_2$. So $f \in S_2$. Hence $X_e \subseteq S_2$. By the minimality of $\dim S_1$, $S_1 = S_2$. Let $e_1 \in E(S_1)$. Then by [8, Theorem 1] there exists a closed connected submonoid $S'$ of $S_1$ such that $xe_1 = e_1xe_1$ for all $x \in S'$ and $e_1S_1 \subseteq S'$. Thus $X_e \subseteq S'$. By the minimality of $\dim S_1$, $S' = S_1$. So $xe_1 = e_1xe_1$.
for all \( e_1 \in E(S_1) \). Let \( e_1, e_2 \in E(S_1) \). Then \( e_1 e_2 e_1 e_2 = e_2 e_1 e_2 = e_1 e_2 \). Thus \( E(S_1)^2 \subseteq E(S_1) \). Also \( e_1 e = \{ e \} \). Thus the hypothesis of Theorem 2.12 is satisfied. Hence the group of units \( G_1 \) of \( S_1 \) is solvable. Now \( G_1 \subseteq B_1 \) for some Borel subgroup \( B_1 \) of \( G \). Then \( X_\varepsilon \subseteq \overline{B_1} \). Let \( f \in E(J) \cap \overline{B_1} \). Let \( T \) be a maximal torus of \( B_1 \) (and hence of \( G \)) such that \( e \in T \). By Theorem 2.1, \( x^{-1} f x \in \overline{T} \) for some \( x \in B_1 \). Then \( e, x^{-1} f x \in E(\overline{T}) \cap J \). Since \( \theta(J) = 1, e = x^{-1} f x \). So \( X_\varepsilon = x X_\varepsilon x^{-1} \subseteq x \overline{B_1} x^{-1} = \overline{B_1} \). Thus for all \( f \in \overline{B_1} \cap E(J), X_\varepsilon \subseteq \overline{B_1} \). Since \( y^{-1} E(J) y = E(J) \) for all \( y \in G \), and since all Borel subgroups of \( G \) are conjugate, we see that \( X_\varepsilon \subseteq \overline{B} \) for any Borel subgroup \( B \) of \( G \) and \( f \in \overline{B} \cap E(J) \). Similarly \( Y_\varepsilon \subseteq \overline{B} \) for any Borel subgroup \( B \) of \( G \) and \( f \in \overline{B} \cap E(J) \).

Now fix a Borel subgroup \( B \) of \( G \). Then by Theorem 2.1, there exists \( e_1 \in E(J) \cap \overline{B} \). Let \( e_2 \in E(J) \). Then by Lemma 1.12, there exist \( e_3, e_4 \in E(J) \) such that 

\[
e_1 \mathbb{R} e_3 \mathbb{R} e_4 \mathbb{R} e_2.\]

Then \( e_3 \in X_{e_1} \subseteq \overline{B} \). Hence \( e_4 \in Y_{e_3} \subseteq \overline{B} \). So \( e_2 \in X_{e_4} \subseteq \overline{B} \). Hence \( E(J) \subseteq \overline{B} \).

(3) \( \Rightarrow \) (4). This follows since \( X^{-1} E(J) x = E(J) \) for all \( x \in G \).

(4) \( \Rightarrow \) (5) follows from Theorem 2.12.

(5) \( \Rightarrow \) (1) follows from Lemma 1.11.

**Corollary 2.14.** Let \( S \) be a connected monoid with group of units \( G \) and kernel \( M \). Then \( E(M) \subseteq \overline{B} \) for every Borel subgroup \( B \) of \( G \).

**Theorem 2.15.** Let \( S \) be a connected monoid with group of units \( G \). Let \( e \) be a minimal idempotent of \( S \) and let \( G_0 = \{ a \mid a \in G, ae = ea = e \} \). Then the following conditions are equivalent:

1. \( \alpha(S) = \beta(S) \).
2. \( S \) is a semilattice of archimedean semigroups.
3. \( G_0 \) is solvable.
4. \( E(S) \subseteq \overline{B} \) for some (every) Borel subgroup \( B \) of \( G \).
5. For all \( e, f \in E(S) \), any eigenvalue of \( ef \) is 0 or 1.

**Proof.** That (1) \( \Rightarrow \) (4) follows from Corollary 2.7 and Theorem 2.13. That (4) \( \Rightarrow \) (5) follows from Theorem 2.12. That (5) \( \Rightarrow \) (2) \( \Rightarrow \) (1) follows from Corollary 2.7, Theorem 2.13 and [9, Proposition 21]. That (1) \( \Leftrightarrow \) (3) follows from Theorems 2.8 and 2.12.

The following is immediate from Theorem 2.1 and Corollary 2.7.

**Theorem 2.16.** Let \( S \) be a connected monoid such that \( \alpha(S) = \beta(S) \). Then \( \mathcal{U}(S) \) is a relatively complemented lattice (in fact the lattices of all faces of a polytope in some \( \mathbb{R}^n \)).

**Conjecture 2.17.** By Theorems 2.10, 2.12, 2.15, 2.16, the following conjectures are equivalent:

1. If \( S \) is a connected monoid, then \( \mathcal{U}(S) \) is relatively complemented if and only if \( \alpha(S) = \beta(S) \).
2. If \( S \) is a connected monoid with zero and group of units \( G \), then \( G \) is solvable if and only if \( \mathcal{U}(S) \) is relatively complemented.
3. If \( S \) is a connected monoid with zero 0 and group of units \( G \), if \( 0 \in C(S)^c \), and if \( \mathcal{U}(S) \) is relatively complemented, then \( G \) is solvable.
We now prove Conjecture 2.17 in a special case.

**Theorem 2.18.** Let $S$ be a connected monoid such that $\mathcal{U}(S)$ is relatively complemented. If $\alpha(S) \leq 13$ or $\beta(S) \leq 7$, then $S$ is a semilattice of archimedean semigroups.

**Proof.** By Theorem 2.15, it suffices to show that $\alpha(S) = \beta(S)$. Assume that this is not true and find a counterexample $S$ with $\dim S$ minimal. Let $G$ denote the group of units of $S$. Let $e \in E(S)$, $e \neq 1$. Then $\mathcal{U}(eSe)$ is relatively complemented by Corollary 1.9. By Lemma 1.8, $\beta(eSe) \leq \beta(S)$. By Lemma 1.4, $\alpha(eSe) \leq \alpha(S)$. By the minimality of $\dim S$, $eSe$ is a semilattice of archimedean semigroups. Since $\alpha(S) \neq \beta(S)$, we see by Theorem 2.6 that $\theta(J_0) > 1$ for some $J_0 \in \mathcal{U}(S)$. Choose $J_0$ maximal in $\mathcal{U}(S)$. Let $J' \in \mathcal{U}(S)$, $J' > J_0$. Then $\theta(J') = 1$. Let $T$ be a maximal torus of $G$. Then $J' \cap E(\overline{T}) = \{\xi\}$ and there exist $e_0, e'_0 \in J_0 \cap E(\overline{T})$ such that $e_0 \neq e'_0$. By Lemma 1.7, $\xi > e_0, \xi > e'_0$. If $\xi \neq 1$, we get a contradiction to Lemma 1.10. So $\xi = 1$ and $J' = G$. Thus $G$ covers $J_0$.

Let $X = \{ e \mid e \in E(\overline{T}), 1 \text{ covers } e \}$, $Y = \{ f \mid f \in E(\overline{T}), e \text{ covers } f \text{ for some } e \in X \}$. Then clearly $X \cap Y = \emptyset$. By Proposition 1.14, $J_0 \cap E(\overline{T}) = J_0 \cap E$. If $e, e' \in X, e \neq e'$, then by Lemma 1.15, there exist distinct $e_1, \ldots, e_{k+1} \in X, f_1, \ldots, f_k \in Y$ such that $e = e_1, e' = e_{k+1}, e_i > f_i, e_{i+1} > f_i, i = 1, \ldots, k$. Let $\lambda(e, e')$ denote the smallest such possible $k$. Let $\Omega = \{(e, e') \mid e, e' \in X, e \neq e', \lambda(e, e') \neq \emptyset \}$. Choose $(e, e') \in \Omega$ with $\lambda(e, e') = k$ minimal. Then there exist distinct $e_1, \ldots, e_{k+1} \in X, f_1, \ldots, f_k \in Y$ such that $e = e_1, e' = e_{k+1}, e_i > f_i, e_{i+1} > f_i, i = 1, \ldots, k$. Let $e_i \in J_i \in \mathcal{U}(S), f_i \in J_i^* \in \mathcal{U}(S), i = 1, \ldots, k$. Then with $J_{k+1} = J_1$ we have $J_i \supset J_i^*, J_{i+1} \supset J_i, i = 1, \ldots, k$. Let $\Xi = \{ J \mid J \in \mathcal{U}(S), G \text{ covers } J \}$, $\mathcal{Y} = \{ J^* \mid J^* \in \mathcal{U}(S) \}$, there exists $J \in \Xi$ such that $J$ covers $J^*$. Then by Proposition 1.14, $J_1, \ldots, J_k \in \Xi$ and $J_1^*, \ldots, J_k^* \in \mathcal{Y}$. We claim that $J_1, \ldots, J_k$ are distinct. For suppose that for some $\mu, \nu \in Z^+$, with $1 \leq \mu < \nu < k$, $J_\mu = J_\nu$. Then $e_\mu^\lambda e_\nu$ and therefore $(e_\mu, e_\nu) \in \Omega$. Clearly $\lambda(e_\mu, e_\nu) < k$. This contradiction shows that $J_1, \ldots, J_k$ are distinct. Next we show that $J_1^*, \ldots, J_k^*$ are distinct. For suppose $J_\mu^* = J_\nu^*$ for some $\mu, \nu \in Z^+$ with $1 \leq \mu < \nu < k$. First assume $\nu > \mu + 1$. Then $J_\mu > J_\mu^*, J_{\mu+1} > J_\mu^*, J_\nu > J_\mu^*$, contradicting Proposition 1.14. Hence $\nu = \mu + 1$. But then $e_{\mu+1} > f_{\mu}, e_{\mu+1} > f_{\mu+1}, f_{\mu}^\lambda f_{\mu+1}$, contradicting Lemma 1.10. Hence $J_1^*, \ldots, J_k^*$ are distinct. Next assume $k = 1$. Then $e_1^\lambda e_2, e_1 > f_1, e_2 > f_1$. This contradicts Proposition 1.14. Thus $k > 1$. Now assume $k = 2$. Then $J_\mu > J_\nu^*$ for $\mu = 1, 2$, $\nu = 1, 2$. Hence $J_1^* = J_1 \wedge J_2 = J_2^*$, a contradiction. Thus $k \geq 3$. Since $J_1, \ldots, J_k, J_1^*, \ldots, J_k^*, J_1^* \wedge J_k^*$ are distinct elements of $\mathcal{U}(S)$, $\beta(S) \geq 2k + 2 > 8$. Now let $e_i^\prime = e_i \in J_i$. Then $e_i^\prime \neq e_i$. By Proposition 1.14, $J_i \cap E(\overline{T}) \subseteq X, J_i^* \cap E(\overline{T}) \subseteq Y_i \subseteq X, i = 1, \ldots, k$. By Lemma 1.5, there exists $f_i^\prime \in J_i^* \cap E(\overline{T})$ such that $e_i^\prime > f_i^\prime$. If $f_1 = f_1^\prime$, then $e_1^\prime > f_1, e_i > f_1, e_i^\prime > e_i, e_i \neq e_i^\prime$. This contradicts Proposition 1.14(7). So $f_1 \neq f_1^\prime$. Since $J_2 > J_1^*$, there exists by Lemma 1.5, $e_2^\prime \in J_2 \cap E(\overline{T})$ such that $e_2^\prime > f_2^\prime$. Suppose $e_2^\prime = e_2$. Then $e_2 > f_1, e_2 > f_1^\prime, e_2^\prime > f_1^\prime$, contradicting Lemma 1.10. Hence $e_2 \neq e_2^\prime$. Continuing this process, we can find $e_i^\prime \in J_i \cap E(\overline{T}), f_i^\prime \in J_i^* \cap E(\overline{T}), i = 1, \ldots, k$, such that for any $i = 1, \ldots, k, e_i \neq e_i^\prime$ and $f_i \neq f_i^\prime$. Hence 1, $e_1, \ldots, e_k, e_1^\prime, \ldots, e_k^\prime, f_1, \ldots, J_k, f_1^\prime, \ldots, J_k^\prime$ are distinct elements of $E(\overline{T})$. So $\alpha(S) \geq 4k + 2 > 14$. Thus $\alpha(S) \geq 14$ and $\beta(S) \geq 8$, contradicting the hypothesis of the theorem. This proves the theorem.
By Theorems 2.12, 2.16, 2.18 we have

**Theorem 2.19.** Let $S$ be a connected monoid with zero and group of units $G$. Assume that either $\alpha(S) \leq 13$ or $\beta(S) \leq 7$. Then $G$ is solvable if and only if $\mathcal{U}(S)$ is relatively complemented.

**Remark 2.20.** The proof of Theorem 2.18 suggests that a possible counterexample to Conjecture 2.17 has the following lattice structures:

$$\mathcal{U}(S) = \{G, J_1, J_2, J_3, J_1^*, J_2^*, J_3^*, 0\}$$

with the structure illustrated in Figure 1.

![Figure 1](image1.png)

$E(\overline{T}) = \{1, e_1, e_1', e_2, e_2', e_3, e_3', f_1, f_1', f_2, f_2', f_3, f_3', 0\}, e_i, e_i' \in J_i, i = 1, 2, 3, f_i, f_i' \in J_i^*, i = 1, 2, 3$ with the structure illustrated in Figure 2.

![Figure 2](image2.png)

**Theorem 2.21.** Let $S$ be a connected monoid with group of units $G$ and let $T$ be a maximal torus of $G$. Let $\mathcal{V} = \{J \mid J \in \mathcal{U}(S), \theta(J) = 1\}, \Gamma = \bigcup \{J \cap E(\overline{T}) \mid J \in \mathcal{V}\}$. Then $\Gamma$ is a subsemilattice of $(E(\overline{T}), \cdot)$, $\Gamma$ is the maximal semilattice image of $S$, and $\Gamma \models (\mathcal{V}, \land)$.

**Proof.** Let $S_\alpha (\alpha \in \Omega)$ denote the maximal semilattice decomposition of $S$. By Tamura [10, 11], each $S_\alpha$ is an $\mathcal{S}$-indecomposable semigroup. By [4], $\mathcal{U}(S) = \bigcup_{\alpha \in \Omega} \mathcal{U}(S_\alpha)$. If $\alpha \in \Omega$, then let $M_\alpha$ denote the kernel of $S_\alpha$. By Theorem 2.13 and Lemma 1.21, $\mathcal{V} = \{M_\alpha \mid \alpha \in \Omega\}$. If $\alpha \in \Omega$, then let $e_\alpha \in E(\overline{T}) \cap M_\alpha$. Let $\alpha, \beta \in \Omega$, $S_\alpha S_\beta \subset S_\gamma$. Then $e_\alpha e_\beta \in S_\gamma$. Let $J$ denote the $\beta$-class of $e_\alpha e_\beta$. Then $J \in \mathcal{U}(S_\alpha), M_\alpha \geq J, M_\beta \geq J$. Let $J' \in \mathcal{U}(S)$ such that $M_\alpha \geq J'$ and $M_\beta \geq J'$. Let $h \in J' \cap E(\overline{T})$. Then by Lemma 1.5, $e_\alpha \geq h, e_\beta \geq h$. So $e_\alpha e_\beta \geq h$. Thus $J \geq J'$. Hence $J = M_\gamma$. By Theorem 2.6(4), $\theta(J) = 1$. Hence $J = M_\gamma$ and $e_\gamma = e_\alpha e_\beta$. This proves the theorem.
3. Central idempotents. In this section, we study special conditions on idempotents.

**Definition.** Let $S$ be a connected monoid. Then $S$ is $E$-central if $E(S) \subseteq C(S)$. $S$ is $E$-finite if $E(S)$ is finite. $S$ is $E$-commutative if idempotents of $S$ commute with each other.

The following theorem follows from [5, Theorem 2.8].

**Theorem 3.1** [5]. Let $S$ be a connected monoid with group of units $G$ and let $T$ be a maximal torus of $G$. Then the following conditions are equivalent:

1. $S$ is $E$-central.
2. $S$ is $E$-finite.
3. $S$ is $E$-commutative.
4. $E(T) \subseteq C(S)$.
5. $E(S) = E(T)$.
6. $S$ is a semilattice of nil extensions of groups.

If $A, B$ are matrices, then let $A \oplus B$ denote $[\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}]$. If $\mathcal{A}, \mathcal{B}$ are sets of matrices, then let $\mathcal{A} \oplus \mathcal{B} = \{A \oplus B \mid A \in \mathcal{A}, B \in \mathcal{B}\}$. If $a_1, \ldots, a_t \in K$, then let $\mathcal{C}(a_1, \ldots, a_t)$ denote the set of all $t \times t$ upper triangular matrices over $K$ having $(a_1, \ldots, a_t)$ as the diagonal.

**Example 3.2.** Let $r_1, \ldots, r_t \in \mathbb{Z}^+$. Choose $e_{ij} \in \mathbb{Z}^+$, $1 \leq i \leq t$, $1 \leq j \leq r_i$. Let $w_1, \ldots, w_m$ be words in the commuting variables $X_1, \ldots, X_t$, involving each variable at least once. If $a_1, \ldots, a_t \in K$, then let $\mathcal{C}(a_1, \ldots, a_t) = \mathcal{C}(a_1^{(1)}, a_1^{(2)}, \ldots) \oplus \mathcal{C}(a_2^{(1)}, \ldots) \oplus \cdots \oplus \mathcal{C}(w_i(a_1, \ldots, a_t), \ldots, w_m(a_1, \ldots, a_t))$. Let $S = \bigcup_{a_1, \ldots, a_t \in K} \mathcal{C}(a_1, \ldots, a_t)$. Then $S$ is a connected monoid with zero and $|E(S)| = |E(S)| = 2^t$ so $S$ is $E$-central. Is $S = \bar{S}$?

**Conjecture 3.3.** Let $S$ be an $E$-central connected monoid with zero. Then $S$ is isomorphic to a closed connected submonoid of a finite direct product of semigroups of the type given in Example 3.2.

**Theorem 3.4.** Let $S$ be a connected monoid with group of units $G$ and let $T$ be a maximal torus of $G$. Let $\Lambda$ be a maximal chain in $E(T)$. If $\Lambda \subseteq C(S)$, then $S$ is a semilattice of archimedean semigroups.

**Proof.** Let $\{1 > e_1 > \cdots > e_k\} = \Lambda$. Since $\Lambda \subseteq C(S)$, $\theta(e_i) = 1$, $i = 1, \ldots, k$. It follows by [9, Theorem 18], that $\gamma(S) = \gamma(e_k(S))$. Let $G_i = \{a \mid a \in G, ae_k = e_k\}$. It follows from Theorem 2.8 that $\gamma(G_i) = 1$. Hence [3, Theorem 25.2C], $G_i^c$ is solvable. By Theorem 2.15, $S$ is a semilattice of archimedean semigroups.

**Example 3.5.** We give an example to show that the hypothesis of Theorem 3.4 does not imply that $S$ is $E$-central. Let
\[
S = \left\{ \begin{bmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & 0 & ab \end{bmatrix} \mid a, b, c \in K \right\},
\]
\[
\bar{T} = \left\{ \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & ab \end{bmatrix} \mid a, b \in K \right\}.
\]
Then
\[
\Lambda = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}, \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}, \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]
is a maximal chain in \(E(T)\) and \(\Lambda \subseteq C(S)\). However
\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix} \notin C(S).
\]
Hence \(S\) is not \(E\)-central.

**Theorem 3.6.** Let \(S\) be a connected monoid with group of units \(G, \alpha(S) = n\). Then
(1) If \(S_1\) is a connected \(E\)-central submonoid of \(S\), then \(|E(S_1)| \leq n\). If \(|E(S_1)| = n\), then \(S_1 \subseteq C_G(E(T))\) for some maximal torus of \(T\) of \(G\).

(2) If \(T\) is a maximal torus of \(G\), then \(C_G(E(T))\) is a maximal connected \(E\)-central submonoid of \(S\).

**Proof.** (1) Let \(G_1\) denote the group of units of \(S_1\), \(T_1\) a maximal torus of \(G_1\). Then \(T_1 \subseteq T\) for some maximal torus \(T\) of \(G\). Then \(E(S_1) = E(T_1) \subseteq C(S_1) \cap E(T)\). So \(|E(S_1)| \leq n\). If \(|E(S_1)| = n\), then \(E(T_1) = E(T)\) and \(E(T) \subseteq C(S_1)\). Then \(G_1 \subseteq C_G(E(T))\). So \(S_1 \subseteq C_G(E(T))\).

(2) If \(e \in E(T)\), then \(\overline{T} \subseteq C_G(e)\). Repeated application of Theorem 2.3 yields that \(C_G(E(T))\) is a connected group. Hence \(C_G(E(T))\) is a connected \(E\)-central submonoid of \(S\). That it is maximal follows from (1) since \(T \subseteq C_G(E(T))\).

**Example 3.7.** Let
\[
S = \left\{ \begin{bmatrix} a & c \\ 0 & b \end{bmatrix} | a, b, c \in K \right\}
\]
and let
\[
S_0 = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} | a, b \in K \right\},
\]
\[
S_j = \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} | a, b \in K \right\} \text{ for } j \in \mathbb{Z}^+.
\]
Then \(S_i\) is a maximal connected \(E\)-submonoid of \(S\) for all \(i \geq 0\). Let \(G_i\) denote the group of units of \(S_i, i = 0, 1, \ldots\). Then \(G_0\) is abelian, \(G_1\) is not abelian but nilpotent, \(G_i, (i \geq 2)\) is not even nilpotent. Moreover \(S_i\) is not conjugate to \(S_j\) for \(i \neq j\).

**Problem 3.8.** Let \(S\) be a connected monoid. Study the maximal \(E\)-central submonoids of \(S\).

**Theorem 3.9.** Let \(S\) be a connected monoid such that for all \(f \in E(S)\), the set \(\{e | e \in E(S), e \geq f\}\) is finite. Then \(S\) is a semilattice of archimedean semigroups.

**Proof.** Let \(e\) be a minimal idempotent of \(S\). Let \(S_1 = \{a | a \in S, ae = ea = e\}\). Then \(S_1^e\) is \(E\)-finite. By Theorem 3.1, \(\alpha(S_1^e) = \beta(S_1^e)\). By Theorem 2.8, \(\alpha(S) = \beta(S)\). By Theorem 2.15, \(S\) is a semilattice of archimedean semigroups.
As an immediate consequence of Theorem 3.9 and [6, Theorem 3.7] we have,

**Theorem 3.10.** Let $S$ be a connected regular monoid. Then the following conditions are equivalent:

1. For all $f \in E(S)$, the set $(e \mid e \in E(S), e \triangleright f)$ is finite.
2. $S$ is a Clifford semigroup.

**Remark 3.11.** Theorem 3.10 is not true for connected regular semigroups (see [6, Example 4.1]).

**Note added in the proof.** Theorem 2.18 is valid for any algebraic monoid with a dense group of units. Conjecture 2.17 has recently been proved by the author. See J. Algebra 73 (1981), 601–612.

**References**

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