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A NONSHRINKABLE DECOMPOSITION OF S^n INVOLVING A NULL SEQUENCE OF CELLULAR ARCS[1]

BY

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ABSTRACT. This paper presents a decomposition G of S^n ($n \geqslant 3$) into points and a null sequence of cellular arcs such that S^n/G is not a manifold; furthermore, the union of the nondegenerate elements from G lies in a 2-cell in S^n and the image in S^n/G of this union has 0-dimensional closure. Examples of nonshrinkable decompositions with a null sequence of cellular arcs have been constructed in the case $n = 3$ by D. S. Gillman and J. M. Martin and by R. H. Bing and M. Starbird. We construct another example in this dimension, for which all the arcs lie in the boundary of a crumpled cube C, and then produce higher dimensional examples by spinning C.

0. Introduction. At the foundation of manifold decomposition theory, marking its subtlety, rests a strikingly simple example due to R. H. Bing [$\mathbf{B_3}$] of a decomposition G of the 3-sphere S^3 into points and a null sequence (that is, only a finite number have diameter greater than any positive amount) of cellular sets for which the associated decomposition space S^3/G is *not* a topological manifold. No comparably simple example has been discovered in S^n when $n > 3$. Daverman [$\mathbf{D_1}$] has shown that examples do exist in S^n, $n \geqslant 5$, and D. G. Wright [**W**] has shown they exist in S^4 as well. However, their work establishes existence results, and their examples are implicit. Here we present specific examples.

Moreover, what we do is exhibit a decomposition G of S^n, $n \geqslant 3$, involving a null sequence of cellular arcs such that S^n/G is not a topological manifold. In the case $n = 3$ this was done in 1963 by D. S. Gillman and J. M. Martin [**GM**] and later redone by R. H. Bing and M. Starbird [**BS**] in 1977, the latter being the first to publish theirs. Our work provides still another example in S^3 (see §3), having the supplemental feature that all of its arcs are contained in a 1-dimensional compactum in the boundary of a wild 3-cell; this example then serves as the main ingredient in the higher dimensional ones, which are generated by spinning the crumpled cube complementary to the 3-cell.

As a natural outgrowth of the spinning process, all the arcs in the promised decomposition of S^n lie in a 1-dimensional compactum in some 2-cell, no matter

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what the value of n. This constitutes the most austere superstructure possible, because any decomposition of S^n, $n \geqslant 5$, into points and (a null sequence of) cellular sets all of which are contained in some arc or finite graph is shrinkable (see §5).

Coping with wildness is an absolute necessity, because Bing [$\mathbf{B_1}$] proved that any upper semicontinuous decomposition of S^n into points and countably many tame arcs is shrinkable. It should be no surprise, therefore, that the actual construction of the wild 3-cell in S^3 forms the heart of this paper. Since many of the techniques involved, which were developed originally by Bing [$\mathbf{B_2}$] and exploited heavily during the early 1960's, have fallen into disuse, we review them at some length in §2.

We express our thanks to Dušan Repovš, who provided us with an expanded set of notes which were based on a talk given by the first author and which included detailed drawings upon which the figures in this paper are modeled.

1. Preliminaries. The decompositions discussed in this paper are upper semicontinuous, abbreviated simply as usc; explicitly, a decomposition G of a space X is usc iff each $g \in G$ is compact and the natural map π: $X \to X/G$ is closed. For such decompositions G, we use H_G to denote the set of nondegenerate elements and N_G to denote the union of these nondegenerate elements. In case A is a compact subset of X, we also use the suggestive shorthand X/A for the decomposition space associated with the decomposition whose only nondegenerate element is A.

A decomposition G of a compact metric space X is *shrinkable* if, for $\varepsilon > 0$, there exists a homeomorphism h of X onto itself such that the diameter of $h(g)$ is less than ε for every $g \in G$ and such that the distance (in X/G) between $\pi(x)$ and $\pi h(x)$ is less than ε for every $x \in X$. A concept introduced and exercised extensively by Bing, its significance stems from the result that G is shrinkable iff π: $X \to X/G$ can be approximated by homeomorphisms. More importantly, when G is a usc decomposition of a closed n-manifold X into cellular sets and $n \neq 4$, G is shrinkable iff X/G is an n-manifold, necessarily then homeomorphic to X (see [**S** or **Ed**] in case $n \geqslant 5$ and [**Ar**] in case $n = 3$). According to R. D. Edwards' Cell-Like Approximation Theorem [**Ed**], when G is a cellular decomposition of an n-manifold X and $n \geqslant 5$, X/G is an n-manifold iff X/G is a finite dimensional space satisfying the Disjoint Disks Property (namely, any two maps of a 2-cell I^2 into X/G can be approximated, arbitrarily closely, by maps having disjoint images). This property, written as DDP, is closely connected to one employed here and designed for applications involving 3-dimensional objects.

A *crumpled n-cube* C is a space homeomorphic to the closure of one of the complementary domains determined by an $(n-1)$-sphere topologically (think: wildly) embedded in S^n. The *boundary of* C, written Bd C, is the subset corresponding to that $(n-1)$-sphere. When $n = 3$, we often refer simply to a crumpled cube.

A map f: $S^1 \to X$ is *essential* if f is not homotopic to a constant map. Somewhat similarly, we say that a map f of a disk with holes H (a compact, connected, planar 2-manifold) to a manifold with boundary M is *I-essential* (for *interior-essential*) if $f(\partial H) \subset \partial M$ and there is no map F: $H \to \partial M$ such that $F \mid \partial H = f \mid \partial H$.

Following a spreading practice, we say that a map f: $X \to Y$ is 1-1 *over a subset* Z of Y if $f \mid f^{-1}(Z)$ is 1-1.

2. Wild 3-cells constructed by stringing eyebolts. To construct the desired crumpled 3-cube we intend to exploit those techniques for stringing eyebolts together first introduced by Bing [$\mathbf{B_2}$] and later modified by D. S. Gillman [**G**] and W. R. Alford [**Al**]. Among these methods Alford's is the closest to ours. To their earlier discussion we shall add further terminology and then shall review the basic properties of this eyebolt technology.

An *eyebolt* is just a (3-dimensional) solid torus. In our work an eyebolt will always be tamely embedded in S^3 and will always be attached to some tame 3-cell B in S^3, in such a way that their common intersection is a 2-cell in the boundary of each. Intuitively one should imagine an eyebolt T to be a thin regular neighborhood (in $S^3 - \operatorname{Int} B$) of the wedge of an arc and a circle, attached to B at the other ($\neq$ wedge point) endpoint of the arc, so as to divide the eyebolt into two parts—its *stem* S (the regular neighborhood of the arc) and its *handle* H (the regular neighborhood of the circle). The handle part must be unknotted; we require that there be a *membrane* (a tame 2-cell) μ in $S^3 - B$ such that $\mu \cap T = \partial\mu$ is a nonseparating curve in $\partial T - S = \partial H - S$.

Let $D_1,\ldots,D_q$ denote a chain of 2-cells in ∂B, with $D_i \cap D_{i+1}$ an arc in the boundary of each ($i = 1,\ldots,q-1$). In practice, $\bigcup_i D_i$ will be a long, narrow strip, "partitioned" via the D_i's into small consecutive sections having disjoint interiors (relative to ∂B). A *string of eyebolts* (*associated with such a chain*, a phrase usually suppressed) is a chain of eyebolts $T_1,\ldots,T_q$ attached to ∂B such that $T_i \cap \partial B = D_i$, $T_i \cap T_{i+1}$ is a disk in the boundary of each (implying that $\bigcup T_i$ contains a neighborhood of $\operatorname{Int}(\bigcup D_i)$ in $S^3 - \operatorname{Int} B$), and the interior of the membrane μ_i for T_i intersects only T_{i+1}, with $\mu_i \cap T_{i+1}$ being a 2-cell in the stem of T_{i+1} separating the handle from the base $\partial B \cap T_{i+1}$ ($i = 1,\ldots,q-1$). Generally these stems are allowed to wind around other stems and could even be knotted (although none we construct ever is), as shown in Figure 1.

A *dangling string* of eyebolts is a string $T_1,\ldots,T_q$ such that the membrane μ_q for the last eyebolt T_q meets no other eyebolt, exactly as in Figure 1; a *tied down string* of eyebolts is a dangling string $T_1,\ldots,T_q$ together with another eyebolt T, disjoint from the others, such that T_q and T link in the strong geometric sense that $\mu_q \cap T$ and $\mu \cap T_q$ (where μ denotes the membrane for T) are nonseparating disks in the handles of T and T_q, respectively, and such that μ meets no eyebolt from the string other than T_q. One should be prepared to allow T to loom up from some extraneous region, not connected to the strip $\bigcup D_i$ on which the string is based. See Figure 2.

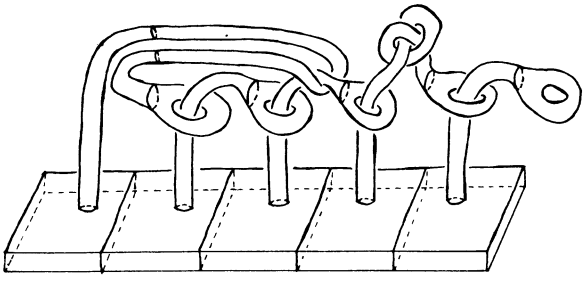

FIGURE 1

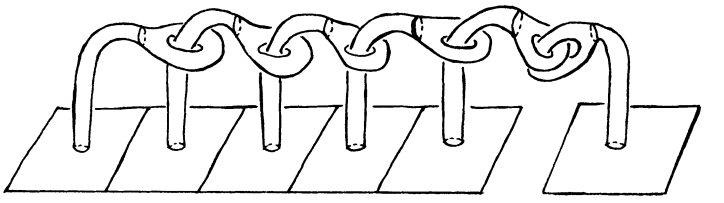

FIGURE 2

With such strings, dangling or tied down, the linking pattern creates a flow from one end of the strip $\bigcup D_i$ to the other. That unidirectional flow restricts us too much, for we must have the freedom to reverse directions. As a result, we refer to a collection of eyebolts $T_1, \ldots, T_m$ associated with a chain of disks $D_1, \ldots, D_{2q}$ on ∂B, as before, where $q < m \leqslant 2q$, as a *folded dangling string* provided: (1) $T_i \cap \partial B = D_{2i-1}$ $(1 \leqslant i \leqslant q)$ and $T_i \cap \partial B = D_{4q-2i+2}$ $(q < i \leqslant m)$; (2) T_i meets T_j precisely when the disks on which they are based intersect and then $(i \neq j)$ $T_i \cap T_j$ is a disk in the boundary of each; (3) the interior of the membrane μ_i for T_i $(1 \leqslant i < m)$ meets only the eyebolt T_{i+1} and $\mu_i \cap T_{i+1}$ is a disk in the stem of T_{i+1} separating the handle from $T_{i+1} \cap \partial B$; and (4) the interior of the membrane μ_m misses all the eyebolts. See Figure 3. Similarly, we refer to a collection of eyebolts $T_1, \ldots, T_m$, T as a *folded tied down string* if $T_1, \ldots, T_m$ is a folded dangling string and T is an eyebolt, disjoint from the others, based on the same 3-cell B, such that T links T_m, exactly as in the definition of tied down string, and the membrane for T meets no eyebolts besides T_m.

Let $W = S^3 - B$ and let $T_1, \ldots, T_m$, T denote a tied down string, folded or not. Two contrasting aspects of this eyebolt technology are the facts that any loop in

$$W - \bigcup_i (\mu_i \cup T_i)$$

is contractible in $W - \bigcup_i T_i$ and that any loop in $W - \bigcup_i(\mu_i \cup T_i \cup T)$ contractible in $W - \bigcup_i(T_i \cup T)$ must also be contractible in $W - \bigcup_i(\mu_i \cup T_i \cup T)$. The next proposition sets forth a technical refinement to the first of these; a refinement to the second can be found at the end of this section.

PROPOSITION 2.1. *Let $T_1, \ldots, T_m$ denote a dangling string (possibly folded) of eyebolts based on a 3-cell B, with membranes $\mu_1, \ldots, \mu_m$; let $W = S^3 - B$; let U denote a neighborhood of $\bigcup(\mu_i \cup T_i)$; and let $f\colon I^2 \to W$ be a map. Then there exists a map $F\colon I^2 \to W - \bigcup T_i$ agreeing with f outside $f^{-1}(U)$.*

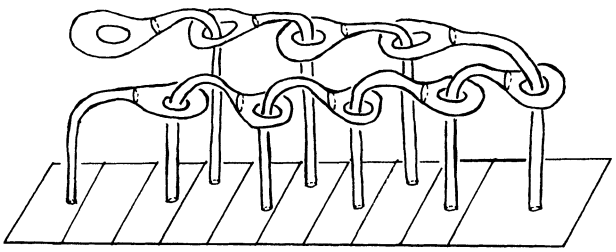

FIGURE 3

The crux of the proof is the observation that, since the string is dangling, the looping of handles around stems can be undone by a homeomorphism supported in U. See [**G**, Theorem 3].

Briefly, here are the essential steps involved in producing a wild embedding of a 3-cell by attaching eyebolts. The process requires an infinite number of stages, starting with an initial 3-cell B and some eyebolts, and continuing generally with a kth-stage 3-cell B_k and some attached eyebolts $T_1, \ldots, T_q$, each of diameter less than $1/k$, somehow organized into a finite collection of pairwise disjoint (here: dangling) strings. To iterate, one pushes each T_i into $(\partial B_k \cap T_i) \cup \operatorname{Int} T_i$ and onto an eyebolt T_i', removes a thin slice P_i from the handle of T_i' so that $\operatorname{Cl}(T_i' - P_i)$ is a 3-cell Q_i, and produces the next stage 3-cell $B_{k+1} = B_k \cup (\bigcup_i Q_i)$. There is a straightforward homeomorphism $B_k \to B_{k+1}$, affecting no points of the domain except those near $\bigcup \operatorname{Int} D_i$ and moving no point more than $1/k$. Then one can attach new eyebolts to B_{k+1} in $\bigcup T_i$, about which more will be written in the ensuing paragraphs. For now, we simply assert that useful new eyebolts can be affixed, each having diameter less than $1/(k+1)$; indeed, the usual methods, just like those we shall employ, readily permit this provided the slices P_i are all thin. The limit of these B_i's then will be a 3-cell B, for there exists a Cauchy sequence of homeomorphisms $e_k\colon B^3 \to B_k$ converging to a map f of B^3 onto B, and by tuning those controls mentioned above one can compel f to be an embedding.

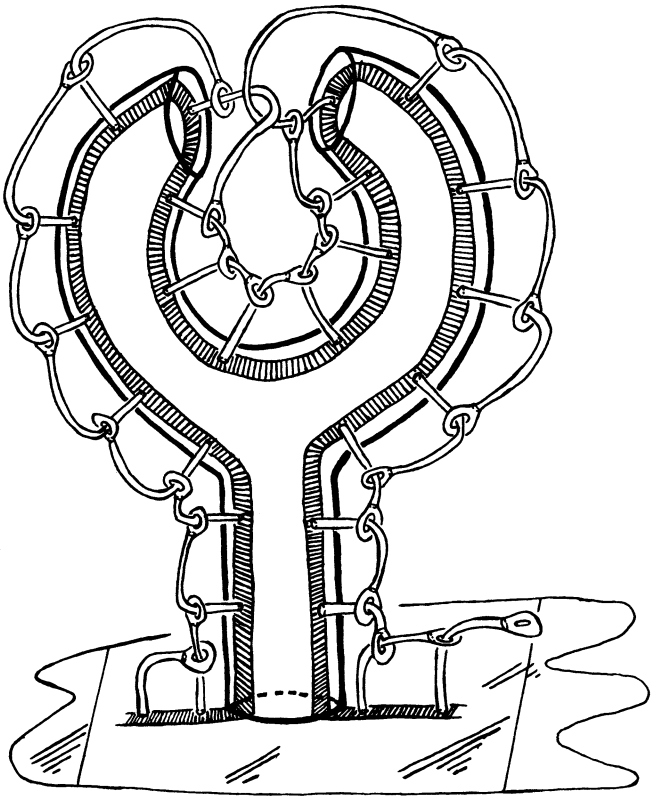

FIGURE 4

Now let $T_1, \ldots, T_m$ denote one of those dangling strings S of eyebolts, possibly folded, attached to B_k. The way the next stage eyebolts will thread through S is similar to the way Alford threads eyebolts in [**Al**]. Basically this can be illustrated in the T_i's one by one. Little eyebolts are constructed, flowing through the eyebolt T attached to B_k in $D_i = \partial B_k \cap T$ from $D_i \cap D_{i-1}$ towards $D_i \cap D_{i+1}$, as shown in Figure 4; the number of them employed is not significant, but the looping of eyes around stems and the intertwining of stems in the removed slice $\mathrm{Cl}(T' - P)$ is. All of the new eyebolts and their membranes associated with T are to be contained in T except the last (as long as D_i is not the final one in the strip $\bigcup D_i$), which is contained in the union of T and the eyebolt T_j ($T_j = T_{i+1}$ if S is not a folded string attached to D_{i+1}), and that last eyebolt stretches a short distance to loop the stem of the first little eyebolt in T_j.

We modify this Alford construction in two major ways. The simpler to describe is that we thread two strings through T, basing the second on another strip in $T \cap \partial B_{k+1}$ "parallel" to and disjoint from the shaded strip shown in Figure 4, permitting neither linking of handles nor intertwining of stems between the eyebolts on the two separate strips. The other change is that we fold the string of next stage eyebolts on one of these strips (extending all along S). Thus, in each eyebolt T we must intersperse new disks between those of the shaded strip and must carve out new eyebolts flowing in the opposite direction from $D_i \cap D_{i+1}$ towards $D_i \cap D_{i-1}$; for these there need be no intertwining of stems in the removed slice and there should be no linking of eyebolts with those flowing in the forward direction and based on the same strip, although there will be some linking with those based on the parallel strip. That linking, a minor additional variation to the modifications just described, will be spelled out in the construction of the example itself.

The proposition below provides a slight hint about the flexibility of the wildness produced in this way.

PROPOSITION 2.2. *Let T be an eyebolt attached to the 3-cell B_k, let N_k be a neighborhood of B_k containing the disks which T has in common with adjacent eyebolts, and let f_e: $M_e \to T$ be maps of 2-manifolds M_e such that $f_e(\partial M_e) \subset \partial T - N_k$ ($e = 1, 2$). Then there exist maps F_e: $M_e \to T - B_{k+1}$ such that $F_e \mid \partial M_e = f_e \mid \partial M_e$ ($e = 1, 2$) and no eyebolt attached to B_{k+1} (as above) meets both $F_1(M_1)$ and $F_2(M_2)$.*

To prove this, one simply adjusts the maps f_e so that the only next stage eyebolts intersected by their images are based near opposite ends of the slice removed from T.

Pinning down the significant global features of our linking pattern is aided by still another definition. Given a string S of eyebolts $T_1, \ldots, T_m$ based on the 3-cell B_k ($k = 1, 2, \ldots$), we say that a tied down string $T_1^*, \ldots, T_q^*$, T^* in $\bigcup T_i$ and based on B_{k+1} *entangles* S if T^* misses the slices removed from $T_1, \ldots, T_m$ and, for each I-essential map f; $H \to (\bigcup T_i) - B_{k+1}$ such that f is transverse with respect to $\bigcup(\partial T^* \cup \partial T_j^*)$, there exists a disk with holes component H^* of $f^{-1}(\bigcup(T^* \cup T_j^*))$ such that $f \mid H^*$: $H^* \to (\bigcup(T^* \cup T_j^*)) - B_{k+1}$ is I-essential.

PROPOSITION 2.3. *Let S denote a string of eyebolts $T_1, \ldots, T_m$ based on the 3-cell B_k; let $T_1^*, \ldots, T_q^*$ denote a string (possibly folded) in $\bigcup T_i$, based on B_{k+1}, and carved out by the modified Alford method outlined above; and let T^* denote another eyebolt in $\bigcup T_i$ and based on B_{k+1} such that $T_1^*, \ldots, T_q^*$, T^* is a tied down string and T^* misses the slices removed from $\bigcup T_i$. Then $T_1^*, \ldots, T_q^*$, T^* entangles S.*

The argument proceeds like the one given in §3 of [$\mathbf{B_2}$]. Accommodation must be made, however, for the required change from consideration of a map on a disk, as in [$\mathbf{B_2}$], to an I-essential map f of a disk with holes. Let $X = T^* \cup (\bigcup_j T_j^*)$. For $i = 1, \ldots, m$ let μ_i^* denote a membrane for T^* and N_i a regular neighborhood of μ_i^* in $S^3 - \operatorname{Int}(B_{k+1} \cup T_i^*)$ such that these N_i's are pairwise disjoint and only N_m meets T^*. Then by repeated application of [$\mathbf{B_2}$, Theorem 9] one proves that $f^{-1}(X \cup (\bigcup N_j))$ has a component on which f is I-essential (here one uses that part of the hypothesis guaranteeing that T^* avoids the locale where the intertwined stems hook up each eyebolt attached to B_k) and by repeated applications of [$\mathbf{B_2}$, Theorem 11] one proves that, for $p = 2, 3, \ldots, m+1$, $f^{-1}(X \cup (\bigcup_{j=p}^m N_j))$ has such a component as well. Typically the essence of these individual steps (say the second kind, for $p = 2$) is the proof showing each simple closed curve J in $f^{-1}(\mu_2^* - T_3^*)$ is mapped inessentially into $\mu_2^* - T_3^*$, which itself depends on the observation that each loop in $f(\partial H)$ is contractible in the complement of $B_{k+1} \cup T_3^* \cup \mu_3^* \cup T_4^*$. Then instead of the mapwise disk-trading methods of [$\mathbf{B_2}$], one adjusts f near J, piping it off $\mu_2^* \cup T_2^* \cup B_{k+1}$, working near $N_2 \cup T_2^*$ and introducing no new intersections with T_2^* (nor any other cells or eyebolts of interest).

3. A null sequence of cellular arcs in S^3. As a variation to the Disjoint Disks Property, in the spirit of [**Ea, D$_2$, CD**], we shall say that a crumpled n-cube C satisfies the *Boundary Mismatch Property*, abbreviated as BMP, if any two maps f_1, f_2: $I^2 \to C$ can be approximated arbitrarily closely by maps F_1, F_2: $I^2 \to C$ such that $F_1(I^2) \cap F_2(I^2) \cap \operatorname{Bd} C = \varnothing$. It is easy to show that, whenever C is at least 5-dimensional, C satisfies the BMP iff it satisfies the DDP. Most often, however, we will be dealing with a 3-dimensional crumpled cube C, in which case C satisfies the BMP iff $C \cup_{\mathrm{Id}} C$ (the space obtained from two copies of C by identifying corresponding points from their boundaries) is topologically S^3 [**Ea**, Theorem 3]. More generally, given a usc decomposition G of a crumpled n-cube C with decomposition map π: $C \to C/G$, we shall say that C/G satisfies the BMP if any two maps f_1, f_2: $I^2 \to C/G$ can be approximated by maps F_1, F_2: $I^2 \to C/G$ such that $F_1(I^2) \cap F_2(I^2) \cap \pi(\operatorname{Bd} C) = \varnothing$.

In this section we fabricate our own nonshrinkable cellular decomposition G of S^3 into points and a null sequence of arcs. It differs from those built previously by Gillman and Martin [**GM**] and by Bing and Starbird [**BS**] in that its nondegenerate elements all lie in the boundary of a common 3-cell. Specifically, our plan is to construct a 3-cell B in S^3 whose closed complement is a crumpled cube satisfying the following:

THEOREM 3.1. *There exists a crumpled cube C in S^3 such that:*

(1) *C satisfies the BMP;*

(2) Bd C *is locally flat modulo a compact* 1-*dimensional set* Y, *the nondegenerate components of which form a null sequence of cellular arcs*;

(3) *for each component* A *of* Y, C/A *satisfies the BMP*;

(4) *for the decomposition* G *of* C *into points and the components of* Y, C/G *fails to satisfy the BMP*.

Before unrolling the blueprint for this construction, one may find it helpful to have a crude conception of what Y looks like. The first three stages in a description of Y as a nested intersection of 2-manifolds in E^2 are depicted in Figure 5. The ultimate intersection Y can be conceived as lying in the product of the standard middle-thirds Cantor set Z with $I = [0, 1]$, so that the nondegenerate components of Y lie in $\{z\} \times I$, for some $z \in Z$ of the form $z = m/3^k$, where $m \geqslant 0$ is an even integer.

PROOF OF THEOREM 3.1. Consider a tame 3-cell B_1 in S^3. Attach an eyebolt T_1 along a long and thin disk D_1 in ∂B_1. For iterative purposes T_1 should be regarded as a dangling string on B_1.

Remove a small slice from the eye of T_1 to form a new 3-cell B_2, with $B_1 \subset B_2 \subset B_1 \cup T_1$, as described in §2, and thread strings of eyebolts through T_1 using the

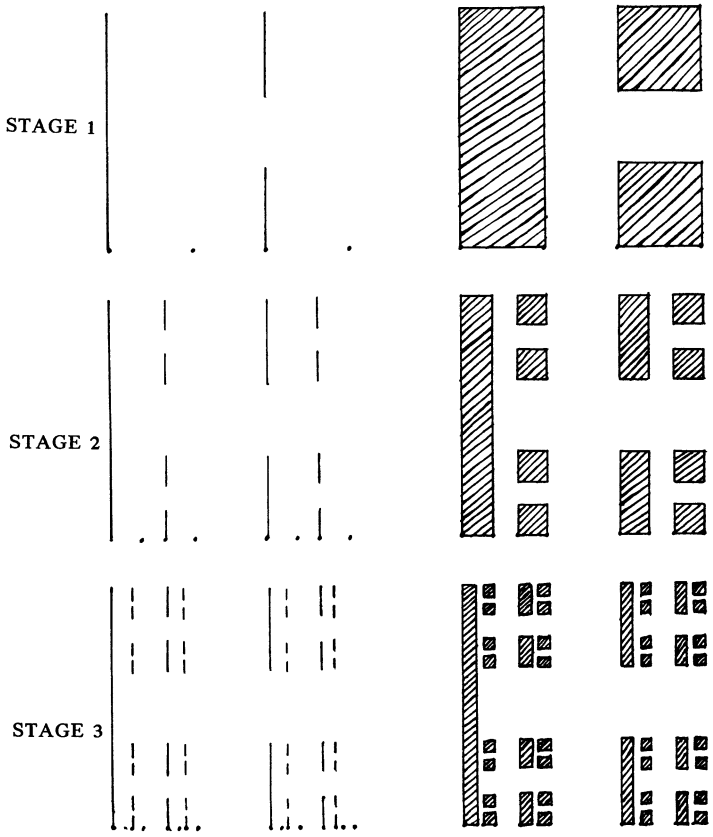

FIGURE 5

pattern illustrated in Figure 6. In particular, these strings should consist of a single folded dangling string $S_{F(1)}$ and another finite collection of pairwise disjoint, dangling strings $S_1, \ldots, S_{s(1)}$ such that (†) their union, were it not for the small gaps between adjacent strings, contains a tied down string entangling T_1 and (††) each $S_{F(1)} \cup S_i$ $(i = 1, \ldots, s(1))$ contains a tied down string entangling T_1 as well.

Inductively we presume that we have formed a tame 3-cell B_k and to it have attached eyebolts, which are organized as a finite collection of (pairwise disjoint) dangling strings, some folded but others not. As before, we remove slices from each eyebolt to form a new tame 3-cell B_{k+1} and attach some new and very small eyebolts, this time threading new strings through each string from the previous stage according to the pattern illustrated in Figure 6. In particular, for each string S associated with B_k, which itself is based on some long and thin disk D, at the next stage we carve out some strings of eyebolts, forming a single folded dangling string $S^*_{F(k)}$ and another finite collection of small, mutually exclusive, dangling strings $S^*_1, \ldots, S^*_{s(k)}$ such that (†) the union of the S^*_i $(i = 1, \ldots, s(k))$, except for some gaps, contains a tied down string entangling S and (††) for each $i = 1, \ldots, s(k)$, $S^*_{F(k)} \cup S^*_i$ also contains a tied down string entangling S. In order to maintain the process outlined in §2, we do this by interspersing small gaps near the disks of intersection from adjacent eybolts of S, thereby breaking up the nonfolded part, $\bigcup S^*_i$, into a union of small dangling (individually) strings. We simply add an extra eyebolt at the very end to tie down this new string of strings. The primary extra wrinkle, unmentioned in §2, is to have the handle of the last eyebolt in each dangling string S^*_i link a nearby handle of $S^*_{F(k)}$ arising after the fold, so that this last handle ties down part of $S^*_{F(k)}$ in a manner that entangles S.

Keeping in mind how Y is supposed to appear, we do all this so that the union D^* of disks on which the folded string is based stretches about as long as D but is much thinner, the union of the disks on which any other string is based has small diameter, and the union of all the latter disks is essentially, except for small gaps, another long and thin disk in D parallel to but disjoint from D^*.

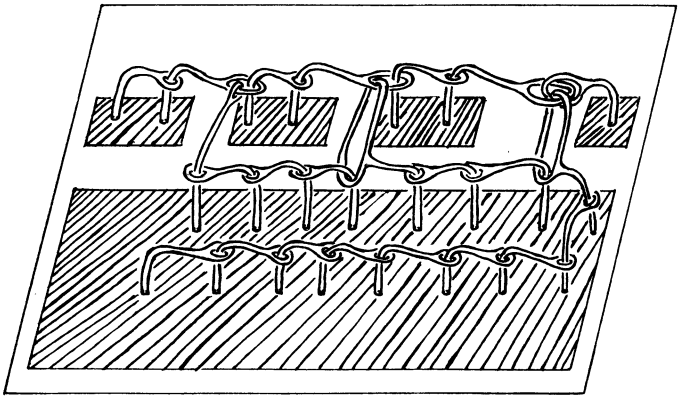

FIGURE 6

As mentioned in §2, the classical methods of [**B**$_2$, **G**, **Al**] permit controls on these eyebolt construction techniques guaranteeing that the limit of these B_k's is a 3-cell B. The desired crumpled cube C is defined as $C = S^3 - \operatorname{Int} B$. The rest of the argument is devoted to establishing that C satisfies the conclusions of Theorem 3.1.

A property shared among all those crumpled 3-cubes C^* obtained by threading strings of eyebolts is that $C^* - K$ is 1-ULC for each Cantor set K in $\operatorname{Bd} C^*$. Consequently, according to [**Ea**, Theorem 8], C^* is universal (see [**Ea**] for the definition), and $C^* \cup_{\mathrm{Id}} C^* = S^3$, which implies that our crumpled cube C satisfies the BMP. One can also devise a more direct argument for this, based on repeated application of Proposition 2.2.

Next, consider a nondegenerate component A in the set Y of points at which $\operatorname{Bd} C$ is wildly embedded. Lying in the homeomorphic images of long but successively thinner disks D_k in ∂B_k, A must be an arc, and it must be cellular for the same reasons that the arc of wild points on Alford's wild 2-sphere [**Al**] is cellular. Explicitly, given a neighborhood U of A, one can produce an index k and a string of eyebolts $T_1, \ldots, T_q$ attached to B_k so that $A \subset \bigcup T_i \subset U$. At the next stage, attached to B_{k+1} is a dangling string of eyebolts $T_1^*, \ldots, T_t^*$ such that $A \subset \bigcup T_j^* \subset \bigcup T_i$. Since the string $T_1^*, \ldots, T_t^*$ is dangling, S^3 admits a homeomorphism h fixed on B_{k+1} as well as on each $\partial\mu_j$ and outside $\bigcup T_i$ adjusting the interiors of membranes μ_j^* (for T_j^*) so that $h(\operatorname{Int} \mu_j^*)$ meets none of the eyebolts T_m^* $(j, m = 1, \ldots, t)$, which homeomorphism arises exactly as in [**G**, Theorem 3]. An appropriate 3-cell then would be a regular neighborhood of $\bigcup (T_i^* \cup h(\mu_i^*))$ in U. Moreover, there is a detail in the construction (that each string S of eyebolts at one stage contains only one long string at the next, each of the remaining next stage strings being small relative to the diameter of S) that forces the nondegenerate components of Y to be a null sequence.

To prove conclusion (3), consider a map $f\colon I^2 \to C$. From the existence of the homeomorphism h mentioned in the preceding paragraph it follows that there exists a map

$$f'\colon I^2 \to \left(C \cup \left[\left(S^3 - B_{k+1}\right) \cap \left(\bigcup T_i\right)\right]\right) - \bigcup T_j^*$$

such that f' agrees with f outside $f^{-1}(U)$. Based upon a more or less natural retraction of $C \cup [(S^3 - B_{k+1}) \cap (\bigcup T_i)]$ to C, which sends the complement of $\bigcup T_j^*$ into $C - A$, it also follows that there exists a map $F\colon I^2 \to C - A$ agreeing with f outside $f^{-1}(U)$. This means that the complement in C/A of the point image p_A of A is 1-ULC, or that any two maps $f_1, f_2\colon I^2 \to C/A$ can be approximated by maps F_1, F_2 whose images avoid p_A. Such maps lift automatically to maps into C. As a result, C/A satisfies the BMP because C does.

Finally, let $f_1, f_2\colon I^2 \to C$ so that $f_e(I^2) \subset T_1$, the eyebolt attached to B_1, and that $f_e(\partial I^2)$ is essential on $\partial T_1 - B_1$ $(e = 1, 2)$. The maps $\pi f_1, \pi f_2\colon I^2 \to C/G$ reveal that C/G fails to satisfy the BMP. Supposing otherwise, we could produce very close approximations to the πf_e and then approximately lift them [**AP**, Lemma 4.2 or **L**, Lemma 2.3] to maps $F_e\colon I^2 \to C$ $(e = 1, 2)$ such that no component of Y meets both $F_1(I^2)$ and $F_2(I^2)$. Given maps $F_e\colon I^2 \to C$ such that $F_e \mid \partial I^2$ is homotopic in

$C - Y$ to $f_e \mid \partial I^2$ $(e = 1, 2)$, we can append the homotopy to ∂I^2 to presume $F_e \mid \partial I^2 = f_e \mid \partial I^2$ and will reach a contradiction by proving that some component A of Y must touch both $F_1(I^2)$ and $F_2(I^2)$. For $k = 2, 3, \ldots$ let M_k denote the (tame) 2-manifold bounding the 3-cell B_k plus the eyebolts attached to it. Without loss of generality, we may assume that $F_e(I^2)$ meets M_k transversely (as mapped in), so that $F_e^{-1}(M_k)$ is a 1-manifold. Applying the ensuing Proposition 3.2 recursively for $k = 1, 2, \ldots$ and $e = 1, 2$, we can produce a string of eyebolts $T_{k1}, T_{k2}, \ldots, T_{kq}$ attached to B_k (one of those complete strings identified when B_k was formed) and a disk with holes J_{ek} in I^2 such that $J_{e(k+1)} \subset J_{ek}$, and F_e maps J_{ek} I-essentially into $(\bigcup_j T_{kj}) - B_k$. Thus, both $F_1(I^2)$ and $F_2(I^2)$ meet the component $\bigcap_k(\bigcup_j T_{kj})$ of Y.

The analysis in the next proposition reveals the crucial aspects to the linking pattern employed through this construction, as illustrated in Figure 6.

PROPOSITION 3.2. *Suppose $S = T_1, \ldots, T_q$ is one of the strings of eyebolts attached to the 3-cell B_k $(k = 1, 2, \ldots)$ and f_e is an I-essential map of a disk with holes J_e $(e = 1, 2)$ into $(\bigcup T_i) - B_k$ such that f_e maps J_e transversely with respect to M_{k+1}. Then there exists a string $S^* = \{T_1^*, \ldots, T_t^*\}$ attached to B_{k+1}, with $\bigcup_j T_j^* \subset \bigcup_i T_i$, and there exists a disk with holes J_e^* in J_e such that $f_e \mid J_e^*$ is an I-essential map of J_e^* into $(\bigcup T_j^*) - B_{k+1}$ $(e = 1, 2)$.*

PROOF. We shall presume that for each string $S^* = T_1^*, \ldots, T_t^*$ at the $(k+1)$-stage, each component of $f_e^{-1}(\bigcup T_j^*)$ is mapped by f_e in I-essential fashion into $(\bigcup T_j^*) - B_{k+1}$; on those components K that are not, f_e is simply changed on K first to send it into $\partial(\bigcup T_j^*) - B_{k+1}$ and then modified near K to push the resultant image entirely off $\bigcup T_j^*$. If there exist K_1 and K_2 under both f_1^{-1} and f_2^{-1}, respectively, of those eyebolts in the folded string S_F^* inside S, the proposition clearly holds. In case there exists no such component under f_1^{-1}, then by condition (††) in the construction and by Proposition 2.3 each string S^* other than S_F^* gives rise to a component under f_1^{-1} and by condition (†) and Proposition 2.3 some string S^*, other than S_F^*, does the same under f_2^{-1}, completing the proof in that case and the symmetric one. In the remaining case, in which there is no component associated with S_F^* under either f_1^{-1} or f_2^{-1}, by condition (†) as above each of the strings $S^* \neq S_F^*$ gives rise to an I-essential component under both f_1^{-1} and f_2^{-1}.

4. A null sequence of cellular arcs in S^n. Let k denote a nonnegative integer. By the *k-spin* $\mathrm{Sp}^k(C^*)$ of a crumpled n-cube C^* we mean the decomposition space $(C^* \times S^k)/D$ associated with the usc decomposition D of $C^* \times S^k$ into points and the spheres $c^* \times S^k$, $c^* \in \mathrm{Bd}\, C^*$. For expanded discussions of spinning the reader may wish to consult [**C**, Appendix III and $\mathbf{D_2}$, §11].

Throughout this section n will denote an integer larger than 3. The main result about spinning needed here is the following.

PROPOSITION 4.1. *Let C denote the crumpled cube of Theorem 3.1. Then $\mathrm{Sp}^{n-3}(C)$ is topologically S^n.*

PROOF. See conclusion (1) of Theorem 3.1 and Theorem 8.1 of [**CD**], and recall that C satisfies the BMP iff $C \cup_{\mathrm{Id}} C = S^3$.

As a result, we can define a map i: $C \to S^n$ by choosing an arbitrary point $p \in S^k$ and setting $i(c)$ equal to the image of $c \times p$ in $(C \times S^k)/D = S^n$, $c \in C$, as well as a map ψ: $S^n \to C$ as the one induced by the projection $C \times S^k \to C$. The basic properties are set forth in the next elementary result.

PROPOSITION 4.2. *The composition ψi equals the identity map on C, and ψ is* 1-1 *over* Bd C.

For our purposes an important consequence of Proposition 4.2 is that S^n contains a natural copy of a 2-sphere somehow associated with C, namely, the image of the embedding e of Bd C in S^n where $e = i \mid \mathrm{Bd}\, C$. In a reasonable sense G then induces a decomposition $\tilde{G}$ of S^n into points and the arcs $e(g)$, $g \in H_G$.

PROPOSITION 4.3. *The decomposition $\tilde{G}$ is usc and $H_{\tilde{G}}$ consists of a null sequence of cellular arcs.*

PROOF. Obviously $H_{\tilde{G}}$ consists of a null sequence of arcs. Thus, $\tilde{G}$ is a cell-like usc decomposition.

The nontrivial aspect of this proposition is the claim that $\tilde{G}$ is cellular. Fix $A \in H_G$. It suffices to show that $e(A)$ is cellular. From conclusion (2) of Theorem 3.1, the cellularity of A, it follows that $C_A = C/A$ is a crumpled cube, and from conclusion (3) it follows that $C_A \cup_{\mathrm{Id}} C_A = S^3$. By [**CD**, Theorem 8.1], $\mathrm{Sp}^{n-3}(C_A)$ is homeomorphic to S^n. It should be transparent how $\mathrm{Sp}^{n-3}(C_A)$ is naturally equivalent to $\mathrm{Sp}^{n-3}(C)/e(A) = S^n/e(A)$. Consequently, the quotient map $S^n \to S^n/e(A) = S^n$ of S^n to itself having $e(A)$ as its only inverse set reveals the cellularity of $e(A)$ [**Br**, Theorem 3].

PROPOSITION 4.4. *The decomposition space $S^n/\tilde{G}$ does not satisfy the DDP.*

PROOF. Let π: $C \to C/G$ and $\tilde{\pi}$: $S^n \to S^n/\tilde{G}$ denote the decomposition maps. Define maps $\tilde{i}$: $C/G \to S^n/\tilde{G}$ as $\tilde{i} = \tilde{\pi} i \pi^{-1}$ and $\tilde{\psi}$: $S^n/\tilde{G} \to C/G$ as $\tilde{\psi} = \pi \psi \tilde{\pi}^{-1}$. Certainly $\tilde{\psi}\tilde{i} =$ identity on C/G and $\tilde{\psi}$ is 1-1 over $\pi(\mathrm{Bd}\, C)$.

Suppose to the contrary that $S^n/\tilde{G}$ did satisfy the DDP. Given maps β_1, β_2: $I^2 \to C/G$, one could then find maps f_1, f_2: $I^2 \to S^n/\tilde{G}$ so close to $\tilde{i}\beta_1$ and $\tilde{i}\beta_2$, respectively, that $\tilde{\psi} f_e$ is close to β_e $(e = 1, 2)$ and such that $f_1(I^2) \cap f_2(I^2) = \varnothing$. The fact that $\tilde{\psi}$ is 1-1 over $\pi(\mathrm{Bd}\, C)$ would then imply that

$$\tilde{\psi} f_1(I^2) \cap \tilde{\psi} f_2(I^2) \cap \pi(\mathrm{Bd}\, C) = \varnothing .$$

In effect, this would show that C/G satisfies the BMP, a contradiction to conclusion (4) of Theorem 3.1.

The next result summarizes the work of this section.

THEOREM 4.5. *The decomposition $\tilde{G}$ is a nonshrinkable usc decomposition of S^n $(n \geqslant 4)$ whose nondegenerate elements consist of a null sequence of cellular arcs which all lie in some* 1-*dimensional compactum in a* 2-*cell in S^n.*

Details of the proof in the case $n = 4$ involve a variation to Proposition 4.4 showing that if $\tilde{G}$ were shrinkable, then maps f_1, f_2 could be obtained such that $f_1(I^2) \cap f_2(I^2) \cap \tilde{\psi}^{-1}(\pi(\mathrm{Bd}\, C)) = \varnothing$.

5. Cellular decompositions supported on an arc. In this section we demonstrate that confinement of N_G to a 1-dimensional compactum in a 2-cell (at least for $n \geqslant 5$) is the simplest possibility. The following represents the primary result. Mike Starbird has pointed out that the Bing-Starbird example [**BS**] can be described in a way that shows the result to be false when $n = 3$.

THEOREM 5.1. *Let G be a cellular usc decomposition of an n-manifold M, $n \geqslant 5$, for which there exists an arc A such that $N_G \subset A \subset M$. Then G is shrinkable.*

PROOF. Obviously G must be a countable decomposition (i.e., H_G is a countable set), which forces M/G to be finite dimensional. According to Edwards Cell-like Approximation Theorem [**Ed**], it suffices to show that M/G has the DDP.

Towards that end consider maps $f_1, f_2\colon I^2 \to M/G$. These can be approximately lifted to maps $F_1, F_2\colon I^2 \to M$, with πF_i close to f_i [**AP**, Lemma 4.2; **L**, Lemma 2.3], where the F_i can be regarded as disjoint locally flat embeddings. Moreover, each F_i can be further approximated so as to miss the endpoints of all the elements in H_G. As a result, each $g \in H_G$ has a G-saturated neighborhood W_g such that whenever $g' \in G$ satisfies $g' \subset A \cap W_g$ and $g' \cap F_i(I^2) \neq \varnothing$ (i either 1 or 2), then $g' = g$. This implies that only a finite collection $g(1), \ldots, g(s) \in H_G$ intersect both $F_1(I^2)$ and $F_2(I^2)$.

(REMARK. The same conclusion can be easily reached in this case using the fact that H_G forms a null sequence; however, the argument above lends itself to subsequent generalizations.)

The proof will be completed by showing how to adjust F_1 to F_1' near any one of these $g(i)$'s, say $g(1)$, so that $F_1'(I^2) \cap g(1) = \varnothing$ and no new elements of G in A (other than $g(2), \ldots, g(s)$) meet both $F_1'(I^2)$ and $F_2(I^2)$. In order to accomplish this, one can use the cellularity of $g(1)$ to produce an n-cell C such that $g(1) \subset \mathrm{Int}\, C \subset C \subset W_{g(1)}$ and that the diameter of $\pi(C)$ is small; since $C - g(1)$ is simply connected, F_1 can be redefined, changing it only at points of $F_1^{-1}(C)$, to yield a map F_1' such that $F_1'(F_1^{-1}(C)) \subset C - g(1)$.

Exactly the same argument establishes the next result. With minor additions and modifications, the argument also establishes Theorem 5.3 below.

THEOREM 5.2. *Let G be a cellular usc decomposition of an n-manifold M, $n \geqslant 5$, into points and r-cells, $r \leqslant n - 2$, having locally flat boundaries such that there exists an r-manifold R topologically embedded in M as a closed subset and $N_G \subset R$. Then G is shrinkable.*

THEOREM 5.3. *Let G be a cellular usc decomposition of an n-manifold M, $n \geqslant 5$, for which there exists a 1-complex Γ topologically embedded in M as a closed subset such that $N_G \subset \Gamma$. Then G is shrinkable.*

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