q-EXTENSION OF THE p-ADIC GAMMA FUNCTION. II

BY

NEAL KOBLITZ

ABSTRACT. Taylor series and asymptotic expansions are developed for q-extensions of the p-adic psi (derivative of log-gamma) function “twisted” by roots of unity. Connections with p-adic L-functions and q-expansions of Eisenstein series are discussed. The p-adic series are compared with the analogous classical expansions.

Introduction. We shall study q-extensions of the ψ-function (derivative of log-gamma) and its “twists” (by roots of unity, Dirichlet characters, etc.) in the complex analytic, and especially the p-adic analytic cases. Using expressions for these functions as convolution transforms, we derive two types of expansions for them: Taylor expansions near x = 0 (or x = 1), and Stirling series for x large. For the usual type of ψ-function (which is the limit of the q-extension as q → 1 in both the classical and p-adic cases), the coefficients in the Taylor and Stirling series are essentially the values of the Riemann zeta-function (or Dirichlet L-functions) at positive integers (Taylor series) and at negative integers (Stirling series). For the q-extensions, these coefficients involve Eisenstein series, as well as values of zeta- or L-functions; in this context, the kth normalized Eisenstein series G_k, for variable k, play the role of q-extensions of ψ(k).

In the complex analytic case this occurrence of G_k as Taylor coefficient, generalizing ψ(k), is related in a simple way to the appearance of 1/2ψ(1 − k) as the constant term in G_k(q). Namely, a weight-k modular form f(z), q = e^{2πiz}, satisfies f(−1/z) = z^k f(z), so its behavior as q = e^{2πiz} → 1− (i.e., as z → i0+) is directly determined by its behavior as q → 0+. In the p-adic case, the connection between the Taylor coefficient, which is a function of q near 1, and the q-expansion of the corresponding p-adic modular form, which is a function of q near 0, is more indirect. The connection is by p-adic analytic continuation, a purely analytic procedure which does not, so far as we know, admit any interpretation in terms of moduli of elliptic curves.

The p-adic construction of the convolution transform requires us to twist the p-adic ψ_q by a number z outside the unit disc around 1, e.g., z ∈ \sqrt{1}, z ≠ 1, p | d. (The use of the letter z in this context should not be confused with its use in q = e^{2πiz}.) Such a twist, sometimes called “regularization”, is routinely needed in order to make a bounded p-adic measure.

Received by the editors June 12, 1981.

1980 Mathematics Subject Classification. Primary 12B40; Secondary 33A15, 12B30, 10D30.

Key words and phrases. Gamma function, psi function, p-adic functions, q-extension, Stirling series, Eisenstein series.

*Partially supported by NSF grant #MCS80-02271.

©1982 American Mathematical Society
0002-9947/81/0000-0625/00.00

111

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
In the complex analytic construction, one is not required to twist, and in fact, classical discussions of psi and q-psi functions generally treat the untwisted functions. However, the following classical examples illustrate that the twisted log-gamma and q-log-gamma functions have arisen naturally in the history of the subject.

**Example 1.** Let \( p > 2 \) be a prime, let \( S_n = \{ j = 1, 2, \ldots, pn - 1 \mid (\frac{j}{p}) = 1 \} \) be the set of numbers less than \( pn \) which are squares modulo \( p \) (here \((\frac{\cdot}{p})\) is the Legendre symbol), and let \( NS_n = \{ j = 1, 2, \ldots, pn - 1 \mid (\frac{j}{p}) = -1 \} \) be the set of nonsquares.

**Problem.** Does the ratio

\[
\prod_{j \in NS_n} j / \prod_{j \in S_n} j
\]

approach a finite nonzero limit as \( n \to \infty \), and, if so, what is it?

**Solution.** Using the formula

\[
\Gamma(x) = \lim_{n \to \infty} \frac{(n - 1)! n^x}{x(x + 1) \cdots (x + n - 1)},
\]

we see that \( n \) to the power \( 2^{\sum_{j=1}^{n-1} (\frac{j}{p})} \) times (0.1) approaches the limit

\[
\prod_{j \in S_n} \Gamma(j/p) / \prod_{j \in NS_n} \Gamma(j/p)
\]

as \( n \to \infty \). The exponent \( \sum_{j=1}^{n-1} (\frac{j}{p}) \) is zero if and only if \( p \equiv 1 \pmod{4} \). If \( p \equiv -1 \pmod{4} \) and \( p > 3 \), then this exponent is a negative integer equal to minus the class number of \( \mathbb{Q}(\sqrt{-p}) \), according to a well-known formula of Dirichlet. Thus, in the former case (0.1) approaches the limit (0.3), while in the latter case it diverges to \(+ \infty\).

Here the logarithm of (0.3) is the value at 0 of the log-gamma function twisted by the quadratic Dirichlet character \( \chi = (\frac{\cdot}{p}) \),

\[
(log \Gamma)_x(0) = \sum_{j=0}^{p-1} \chi(j) \log \Gamma((x + j)/p).
\]

The derivative of this function is typical of the twisted \( \psi \)-functions whose \( q \)-extensions we shall be studying.

**Example 2.** More generally, for any nontrivial Dirichlet character \( \chi \) modulo \( d \), the value \( (\log \Gamma)_x(0) \) is essentially the derivative at 0 of the corresponding Dirichlet \( L \)-series (see [23, p. 271]):

\[
L'(0, \chi) = B_{1, \chi} \log d + (\log \Gamma)_x(0),
\]

where \( B_{1, \chi} = \frac{1}{2} \chi(1) \) (\( B_{1, \chi} = 0 \) if and only if \( \chi(-1) = 1 \)).

**Example 3.** Among the simplest of the Rogers-Ramanujan identities are the following power series identities:

\[
\prod_{n=0}^{\infty} (1 - q^{5n+1})^{-1}(1 - q^{5n+4})^{-1} = \sum_{n=0}^{\infty} \frac{q^n}{(q)_n};
\]

\[
\prod_{n=0}^{\infty} (1 - q^{5n+2})^{-1}(1 - q^{5n+3})^{-1} = \sum_{n=0}^{\infty} \frac{q^{n+n^2}}{(q)_n},
\]

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
where

\[(q)_n = (q; q)_n \overset{\text{def}}{=} (1 - q)(1 - q^2) \cdots (1 - q^n)\]

(more generally,

\[(x; q)_n \overset{\text{def}}{=} (1 - x)(1 - xq) \cdots (1 - xq^{n-1})\].

Both sides of (0.4) converge if \(q\) is a real or complex parameter with \(|q| < 1\). It can also be shown that the ratio of the first product to the second in (0.4) is equal to the continued fraction

\[
1 + \frac{q}{1 + \frac{q^2}{1 + \frac{q^3}{1 + \cdots}}}
\]

The identities (0.4) are equivalent to a statement about partitions (see [1, Chapter 7] for a detailed discussion).

The logarithm of the ratio of the two products on the left in (0.4) is a twisted version of the \(q\)-gamma function

\[
\Gamma_q(x) = \frac{(q; q)_\infty}{(q^x; q)_\infty} (1 - q)^{1-x} = \frac{(1 - q)(1 - q^2) \cdots}{(1 - q^x)(1 - q^{x+1}) \cdots} (1 - q)^{1-x}.
\]

Namely,

\[
\log \frac{\prod_{n=0}^{\infty} (1 - q^{5n+1})^{-1}(1 - q^{5n+4})^{-1}}{\prod_{n=0}^{\infty} (1 - q^{5n+2})^{-1}(1 - q^{5n+3})^{-1}} = 4 \sum_{j=1}^{4} \left( \frac{j}{5} \right) \log \Gamma_q \left( \frac{j}{5} \right).
\]

The right-hand side is the value at 0 of the \(q\)-log-gamma function twisted by the character \((\cdot)_5\), where one defines

\[
(0.6) \quad (\log \Gamma_q)_x(x) \overset{\text{def}}{=} \sum_{j=0}^{d-1} \chi(j) \log \Gamma_q((x + j)/d),
\]

for a character \(\chi\) modulo \(d\). It is the derivatives of functions like (0.6) which we shall be studying.

Although the classical and \(p\)-adic cases (the first and second sections of the present paper) are logically independent, there is a striking similarity between them. In fact, they are formally the same in the following sense. If the root of unity \(z\) used in the twist is replaced by \(z\) in the open unit disc about zero, then the same power series (in \(\mathbb{Z}[[z, q]]\)) occurs as Taylor coefficient classically and \(p\)-adically (except for "removal of the \(p\)-Euler factor" in the \(p\)-adic case). With \(z\) in this open disc the measures in the convolution transform are given by essentially the same formula in the classical and \(p\)-adic situations. Of course, the theory only has arithmetic or modular meaning when \(z\) is a root of unity; hence, the relation between the complex and \(p\)-adic functions is formal and indirect.
1. The classical case.

1. Taylor series. Let $\varphi(-x)$ be defined by convolution of a function $g(u)$ with a Stieltjes measure $df(u)$ on $R^+$:

$$\varphi(x) = \int_0^\infty g(u + x) \, df(u).$$

We shall suppose in what follows that $f$ and $g$ are such that the integrals below converge.

Expanding

$$g(u + x) = \sum_{j=0}^{\infty} \frac{g^{(j)}(u)}{j!} x^j$$

(or, alternately, noting that $\varphi^{(j)}(0) = \int_0^\infty g^{(j)}(u) \, df(u)$), we have for $x$ small:

$$\varphi(x) = \sum_{j=0}^{\infty} \frac{x^j}{j!} \int_0^\infty g^{(j)}(u) \, df(u).$$

The $\psi$-function, twisted $\psi$-functions, $q$-extension of the $\psi$-function, and $q$-extensions of twisted $\psi$-functions are special cases of (1.1) with rather simple choices of $f$ and $g$.

Examples. I. Let $f(u) = u - [u] - \frac{1}{2}$ be the first Bernoulli polynomial (made periodic with period 1), and let $g(u) = -\frac{1}{u}$. Start with the formula [23, p. 261]

$$\log(1 + x) = \int_1^\infty \frac{f(u)}{u + x} \, du.$$

Differentiating and replacing $x$ by $x + 1$, we obtain

$$\psi(1 + x) = \log(1 + x) - \frac{1}{2(1 + x)} + \int_1^\infty \frac{f(u)}{(u + x)^2} \, du.$$

Integrating by parts, we have

$$\psi(1 + x) = \log(1 + x) - \frac{1}{2(1 + x)} + \frac{f(u)}{u + x} \bigg|_{u=1}^{u=x} + \int_1^\infty \frac{df(u)}{u + x}.$$

To find the Taylor coefficients in the expansion of $\psi(1 + x)$, we have $\psi(1) = -\gamma$, and for $j \geq 1$,

$$\frac{1}{j!} \psi^{(j)}(1) = \frac{(-1)^{j-1}}{j} + \int_1^\infty (-1)^j \frac{df(u)}{u^{j+1}}.$$

$$= \frac{(-1)^{j-1}}{j} + (-1)^j \int_1^\infty \left( \frac{du}{u^{j+1}} - \frac{d[u]}{u^{j+1}} \right).$$

$$= (-1)^{j-1} \int_1^\infty \frac{d[u]}{u^{j+1}}.$$
But
\[ \int_{1^-}^{\infty} \frac{d[u]}{u^{j+1}} = \sum_{n=1}^{\infty} \frac{1}{n^{j+1}} = \zeta(j + 1). \]

This gives the expansion
\[ (1.6) \quad \psi(1 + x) = -\gamma + \sum_{j=2}^{\infty} (-1)^j \zeta(j) x^{j-1}. \]

Alternately, in the integral in (1.5) we could have expanded
\[ g(u + x) = -1/(u + x) = \sum_{j=1}^{\infty} (-1)^j x^{-j-1}/u^j \]
for \(|x| < 1 < u\) as in (1.2).

Our reason for belaboring this derivation of a well-known expansion (see, e.g., [23, p. 241]) is that it is the prototype for the examples that follow and for our discussion of the \(p\)-adic case in §2.

II. Let \(\rho: \mathbb{Z} \to \mathbb{C}\) be a periodic function of period \(d\) such that \(\sum_{a=0}^{d-1} \rho(a) = 0\). The key examples are: (1) \(\rho(j) = \chi(j)\), a nontrivial Dirichlet character modulo \(d\); and (2) \(\rho(j) = z^j\), for \(z \neq 1\) a \(d\)th root of unity. Let \(g(u) = -1/u\) be the same as in Example I, and set \(f_\rho(u) = \sum_{a=0}^{\infty} \rho(a)\).

We define the twisted \(\psi\)-function as follows:
\[ (1.7) \quad \psi_\rho(x) = \frac{d}{dx} \sum_{a=0}^{d-1} \rho(a) \log \Gamma \left( \frac{x + a}{d} \right). \]

Using the formula (0.2) for \(\Gamma(x)\), we easily see that
\[ (1.8) \quad \psi_\rho(x) = -\int_{0^-}^{\infty} \frac{df_\rho(u)}{u + x}. \]

If, for example, \(\rho = \chi\) is a nontrivial Dirichlet character, so that, in particular, \(\rho(0) = 0\), then the integral really goes from \(1^-\) to \(\infty\), and for \(|x| < 1 < u\) we have
\[ (1.9) \quad \psi_\rho(x) = \sum_{j=1}^{\infty} (-1)^j x^{-j-1} \int_{1^-}^{\infty} \frac{df_\chi(u)}{u^j} = \sum_{j=1}^{\infty} (-1)^j x^{-j-1} L(j, \chi). \]

since
\[ \int_{1^-}^{\infty} u^{-j} df_\chi(u) = \sum_{n=1}^{\infty} \chi(n)n^{-j} = L(j, \chi). \]

In the example \(\rho(j) = z^j\), we have \(df_z(u - 1) = z^{-1} df_z(u)\) for \(u \geq 1\) (we use the notation \(f_z = f_\rho\), \(\psi_z = \psi_\rho\) here), so that by (1.8),
\[ (1.10) \quad \psi_z(x + 1) = -\frac{1}{z} \int_{1^-}^{\infty} \frac{df_z(u)}{u + x} \]
\[ = \frac{1}{z} \sum_{j=1}^{\infty} (-1)^j x^{-j-1} \int_{1^-}^{\infty} \frac{df_z(u)}{u^j} \]
\[ = \frac{1}{z} \sum_{j=1}^{\infty} (-1)^j x^{-j-1} L(j, z), \]
where
\[ L(j, z) = \sum_{n=1}^{\infty} \frac{z^n}{n^j}. \]

Note that (1.8), combined with the rule \( df_z(u - 1) = z^{-1} df_z(u) \), gives the functional relation
\[ Z \psi_z(x + 1) - \psi_z(x) = 1/x, \]
which generalizes the relation \( \psi(x + 1) - \psi(x) = 1/x \) for the usual \( \psi \)-function. Combining (1.11) with the above Taylor series for \( \psi_z(x + 1) \) gives
\[ \psi_z(x) = -\frac{1}{x} + \sum_{j=1}^{\infty} (-1)^j x^{j-1} L(j, z). \]

III. Let \( 0 < |q| < 1 \), and take \( f(u) = [u], g(u) = (\log q)(q^u/(1 - q^u)) \). The \( q \)-gamma function defined by (0.5) has the properties:
1. \( \Gamma_q(x + 1) = ((1 - q^x)/(1 - q))\Gamma_q(x) \),
2. \( \Gamma_q(x) \to \Gamma(x) \) as \( q \to 1^- \).

The logarithmic derivative of \( \Gamma_q(x) \) is, by (0.5),
\[ \psi_q(x) = -\log(1 - q) + \log q \sum_{j=0}^{\infty} \frac{q^{x+j}}{1 - q^{x+j}}. \]

Hence,
\[ \psi_q(1 + x) = -\log(1 - q) + \log q \sum_{j=1}^{\infty} \frac{q^{x+j}}{1 - q^{x+j}} \]
\[ = -\log(1 - q) + \int_{1^-}^{\infty} g(u + x) df(u). \]

Note that for \( q < 1 \) the integral always converges, whereas in the limit when \( q \to 1^- \) and \( g(u) \) becomes \(-1/u\) (Example I) one must modify \( f(u) \) to make the integral converge.

Now define
\[ P_k(q) = \left( q \frac{d}{dq} \right)^{k-1} \frac{q}{1 - q} = \sum_{n=1}^{\infty} n^{k-1} q^n, \]
so that \( (1 - q)^k P_k(q) \in \mathbb{Z}[q] \). Since \( g(u) = (\log q)(q^u/(1 - q^u)) \), we have \( g^{(k-1)}(u) = (\log q)^k P_k(q) u^{(k-1)} \), so that for \( k \geq 2 \),
\[ \psi_q^{(k-1)}(1) = \frac{(\log q)^k}{(k-1)!} \int_{1^-}^{\infty} P_k(q) u^{(k-1)} df(u); \]
this integral is equal to \( \sum_{j=1}^{\infty} P_k(q^j) = \sum_{m=1}^{\infty} \sigma_{k-1}(m) q^m \), where
\[ \sigma_{k-1}(m) = \sum_{d|m} d^{k-1}. \]

It is common to set \( \psi_q(1) = -\gamma_q \) and call \( \gamma_q \) the \( q \)-Euler constant. Thus
\[ \gamma_q = \log(1 - q) - \log q \sum_{j=1}^{\infty} \frac{q^j}{1 - q^j}. \]
Now for \( k \geq 4 \) an even integer, the weight-\( k \) Eisenstein series

\[
E_k(z) = \sum_{m,n \in \mathbb{Z}} \frac{1}{(mz + n)^k}
\]

has the following expression in terms of \( q = e^{2\pi iz} \): \(^1\)

\[
E_k(z) = \frac{2(2\pi i)^k}{(k - 1)!} G_k(q),
\]

where

\[
G_k(q) = \frac{1}{2} \xi(1 - k) + \sum_{m=1}^{\infty} \sigma_{k-1}(m) q^m = -\frac{B_k}{2k} + \sum_{m=1}^{\infty} \sigma_{k-1}(m) q^m,
\]

in which \( B_k \) is the \( k \)th Bernoulli number:

\[
\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}.
\]

The definition (1.18) makes sense for any \( k \neq 0 \) (since \( \xi(1 - k) \) and \( \sigma_{k-1}(m) = \sum_{d \mid m} d^{k-1} \) make sense), but the relation to modular forms for \( SL(2, \mathbb{Z}) \) only holds for \( k = 4, 6, 8, \ldots \).

Returning to \( \psi_q \), we thus have

\[
\psi_q(1 + x) = -\gamma_q + \sum_{k=1}^{\infty} \frac{x^{k-1}}{(k - 1)!} (\log q)^k \left( G_k(q) + \frac{B_k}{2k} \right).
\]

Using (1.19) with \( t \) replaced by \( x \log q \), we can rewrite this in the form

\[
\psi_q(1 + x) = -\gamma_q + \frac{1}{4} \log q - \frac{1}{2x} + \frac{1}{2} \log q - \frac{1}{q^2 - 1} + \sum_{k=1}^{\infty} \frac{(\log q)^k}{(k - 1)!} G_k(q) x^{k-1}.
\]

Comparing with the Taylor expansion (1.6) for the usual \( \psi \)-function, we see that

\[
\frac{(-\log q)^k}{(k - 1)!} G_k(q) \quad \text{corresponds to } \xi(k);
\]

in particular,

\[
\lim_{q \to 1^-} \frac{(-\log q)^k}{(k - 1)!} G_k(q) = \xi(k).
\]

(Thus one might want to take \((-\log q)^k G_k(q)/T(s)\) as a \( q \)-extension of \( \xi(s) \).)

**Remarks.** 1. For even \( k \geq 4 \), the relationship (1.22) follows from the fact that \( G_k(0) = \frac{1}{2} \xi(1 - k) \) (see (1.18)). Namely, for \( q = e^{2\pi iz} \), we have

\[
\frac{(-\log q)^k}{(k - 1)!} G_k(q) = \left( -\frac{2\pi iz}{(k - 1)!} \right) G_k(q) = \frac{z^k}{2} E_k(z) = \frac{1}{2} E_k \left( -\frac{1}{z} \right)
\]

by the modular property of \( E_k \). But as \(-1/z \to i\infty\) along the positive imaginary axis (i.e., as \( q \to 1^- \)), this gives

\[
\frac{1}{2} E_k(i\infty) = \frac{1}{2} \sum_{n \in \mathbb{Z}, n \neq 0} \frac{1}{n^k} = \xi(k).
\]
2. If one takes \((d^2/dx^2)\) log of both sides of the relation \(\Gamma(x)\Gamma(1-x) = \pi/\sin \pi x\), one obtains

\[
\psi'(x) + \psi'(1-x) = \pi^2/\sin^2 \pi x.
\]

The right side has the partial fraction decomposition

\[
\frac{\pi^2}{\sin^2 \pi x} = (2\pi i)^2 \frac{e^{2\pi ix}}{(1 - e^{2\pi ix})^2} = \sum_{n \in \mathbb{Z}} \frac{1}{(x + n)^2},
\]

while \(\psi'(x)\) is equal to \(\sum_{n=0}^{\infty} 1/(x + n)^2\). Thus, (1.23) expresses the fact that the sum for \(\psi'(x)\) is “half” of the sum for \(\pi^2 \csc^2 \pi x\).

The situation with the \(q\)-extension \(\psi_q(x)\) is similar. For simplicity, we take one more derivative, and we use (1.13) and (1.24) to write

\[
\psi''_q(x) = \frac{d}{dx} \left( \log q \right)^2 \sum_{j=0}^{\infty} \frac{q^{x+j}}{(1 - q^{x+j})^2}
\]

\[
= (2\pi iz)^2 \sum_{j=0}^{\infty} \frac{d}{dx} \frac{e^{2\pi iz(x+j)}}{(1 - e^{2\pi iz(x+j)})^2}
\]

\[
= z^2 \sum_{j=0}^{\infty} \frac{d}{dx} \sum_{n \in \mathbb{Z}} \frac{1}{(z(x+j) + n)^2}
\]

\[
= -2 \sum_{j>0, n \in \mathbb{Z}} \frac{1}{(x + j + n(-1/z))^3};
\]

\[
\psi''_q(1-x) = -2 \sum_{j<0, n \in \mathbb{Z}} \frac{1}{(x + j + n(-1/z))^3}.
\]

So \(\psi''_q\) is “half” of the sum for \(\varphi'(x, -1/z)\) (where \(\varphi'\) denotes the derivative of the Weierstrass \(\varphi\)-function); that is, subtracting (1.26) from (1.25) gives

\[
\frac{d^3}{dx^3} \log \Gamma_q(x) \Gamma_q(1-x) = -2 \sum_{j,n \in \mathbb{Z}} \frac{1}{(x + j + n(-1/z))^3} = \varphi'(x, -1/z).
\]

Thus, the expansion (1.20) for \(\psi_q(x + 1)\) is closely related to the expansion

\[
\varphi \left( x, -\frac{1}{z} \right) = \frac{1}{x^2} + \sum_{k=4}^{\infty} E_k \left( -\frac{1}{z} \right) (k-1) x^{k-2}
\]

\[
= \frac{1}{x^2} + \sum_{k \text{ even}} \frac{2(\log q)^k}{(k-1)!} G_k(q) (k-1) x^{k-2}.
\]

Namely, if one adds (1.20) plus \((\log q) (q^*/(1-q^*))\) (which will give \(\psi_q(x)\) on the left) to (1.20), with \(x\) replaced by \(-x\), then the odd terms cancel in the sum on the right, and the derivative of the resulting expression is (up to a constant not containing \(x\)) equal to (1.27).

IV. Let \(\rho\) and \(f_\rho\) be as in Example II, and let \(g(u) = (\log q)(q^u/(1-q^u))\) as in Example III. We define the \(q\)-extension of the twisted \(\psi\)-function (1.7) as follows:

\[
\psi_{q,\rho}(x) = \frac{1}{d} \sum_{a=0}^{d-1} \rho(a) \psi_q \left( \frac{x + a}{d} \right).
\]
By (1.13),

\[(1.29) \quad \psi_{q,p}(x) = \log q \sum_{j=0}^{\infty} \frac{\rho(j)q^{x+j}}{1-q^{x+j}} = \int_{0}^{\infty} g(u + x) \, df_p(u). \]

If \( \rho = \chi \) is a nontrivial Dirichlet character, then for \( |x| < 1 \leq u \),

\[ \psi_{q,\chi}(x) = \sum_{k=1}^{\infty} x^{k-1} \frac{(\log q)^k}{(k-1)!} \int_{1}^{\infty} P_k(q^u) \, df_\chi(u), \]

where \( P_k \) is defined by (1.15). The integral here is equal to \( \sum_{j=1}^{\infty} \rho(j)P_k(q^j) = \sum_{m=1}^{\infty} \sigma_{k-1,\chi}(m)q^m \), with

\[ \sigma_{k-1,\chi}(m) = \sum_{\bar{m}} \chi \left( \frac{m}{d} \right) d^{k-1}. \]

If \( \rho(j) = z^j \), where \( z \neq 1 \) is a \( d \)th root of unity, then

\[ \psi_{q,z}(x + 1) = \frac{1}{z} \int_{1}^{\infty} g(u + x) \, df_z(u) \]

\[ = \frac{1}{z} \sum_{k=1}^{\infty} x^{k-1} \frac{(\log q)^k}{(k-1)!} \int_{1}^{\infty} P_k(q^u) \, df_z(u), \]

in which the integral is equal to

\[(1.30) \quad \sum_{m=1}^{\infty} \sigma_{k-1,z}(m)q^m, \quad \text{with} \quad \sigma_{k-1,z}(m) = \sum_{\bar{m}} z^{m/d} d^{k-1}. \]

In analogy to (1.11), \( \psi_{q,z} \) satisfies the relationship \( z\psi_{q,z}(x + 1) - \psi_{q,z}(x) = -(\log q)(q^x/(1 - q^x)) \), which combines with the above Taylor series for \( \psi_{q,z}(x + 1) \) to give

\[(1.31) \quad \psi_{q,z}(x) = \log q \frac{q^x}{1 - q^x} + \sum_{k=1}^{\infty} x^{k-1} \frac{(\log q)^k}{(k-1)!} \int_{1}^{\infty} P_k(q^u) \, df_z(u). \]

2. Stirling series. As before, let \( \varphi(x) \) be of the form \( \int_{0}^{\infty} g(u + x) \, df(u) \). Heuristically, we would like to consider \( x \) to be large, \( u \) to be small compared to \( x \); then expand \( g(u + x) \) with \( x \) (rather than \( u \)) as the center: \( g(u + x) = \sum (g^{(j)}(x)/j!)u^j \); and finally write

\[(1.32) \quad \varphi(x) \sim \sum_{j=0}^{\infty} \frac{g^{(j)}(x)}{j!} \int_{0}^{\infty} u^j \, df(u). \]

Unfortunately, for positive \( j \), the integrals \( \int_{0}^{\infty} u^j \, df(u) \) diverge in the examples I–IV above, although heuristically we might take, for example, \( \int_{0}^{\infty} u^j \, d[u] = \sum n^j = \zeta(-j) \). Because of the divergence, we have to proceed somewhat differently to obtain the asymptotic series in the classical case; but we shall see that in the \( p \)-adic case there will be no convergence problem, and the \( p \)-adic version of (1.32) will be literally correct.

We now make the additional assumption that \( f(u) \) is periodic of period \( d \) and

\[(1.33) \quad \int_{0}^{d} f(u) \, du = 0. \]
We write $f$ as a Fourier series

$$f(u) = \sum_{0 \neq n \in \mathbb{Z}} a_n e^{2\pi inu/d}.$$  

Let $f^{(-1)}$ denote the integral of $f$ normalized so that $\int_0^d f^{(-1)}(u)\,du = 0$, and define $f^{(-j)}$ inductively as $(f^{(-j+1)})(-1)$, $j = 2, 3, \ldots$. Then

$$f^{(-j)}(u) = \left(\frac{d}{2\pi i}\right)^j \sum_{0 \neq n \in \mathbb{Z}} \frac{a_n}{n^j} e^{2\pi inu/d}.$$  

In particular, $f^{(-j)}(0)$ is the value of the corresponding Dirichlet series:

$$(1.34) \quad f^{(-j)}(0) = \left(\frac{d}{2\pi i}\right)^j \sum_{0 \neq n \in \mathbb{Z}} \frac{a_n}{n^j}.$$  

**Examples revisited.** I. If $f(u) = u - \lfloor u \rfloor - \frac{1}{2} = B_1(u - \lfloor u \rfloor)$, then $f^{(-j)}(u) = (1/(j + 1)!B_{j+1}(u - \lfloor u \rfloor)$ are the successive Bernoulli polynomials, and

$$(1.35) \quad f^{(-j)}(0) = \frac{1}{(j + 1)!} B_{j+1} = -\frac{1}{j!} \zeta(-j).$$

(Alternately, we have $f(u) = -\frac{1}{2\pi i} \sum_{n \neq 0} e^{2\pi inu}$, and so by (1.34):

$$f^{(-j)}(0) = -\frac{1}{(2\pi i)^{j+1}} \sum_{0 \neq n \in \mathbb{Z}} \frac{1}{n^{j+1}} = -\frac{1}{j!} \zeta(j)$$

by the functional equation for $\zeta(s)$.)

Differentiating (1.4) and successively integrating by parts, we have

$$\psi(x) = \log x - \frac{1}{2x} + \int_0^x \frac{f(u)}{(u + x)^2} du

= \log x - \frac{1}{2x} - \frac{f^{(-1)}(0)}{x^2} + 2\int_0^x \frac{f^{(-1)}(u)}{(u + x)^3} du

= \cdots = \log x - \frac{1}{2x} - \frac{f^{(-1)}(0)}{x^2} - \frac{2f^{(-2)}(0)}{x^3}

- \cdots - \frac{j! f^{(-j)}(0)}{x^{j+1}} + (j + 1)! \int_0^x \frac{f^{(-j)}(u)}{(u + x)^{j+2}} du.$$  

Making obvious estimates for the integral and using (1.35), we see that

$$(1.36) \quad \log x + \sum_{j=0}^{\infty} \frac{\zeta(-j)}{(-x)^{j+1}}$$

is an asymptotic series for $\psi(x)$.

II. Let $\rho$ be as before, but now modify $f_\rho$ by setting $f_\rho(u) = B_1 u + \sum_{a=0}^{d-1} a \rho(a)$, where $B_1 \rho = (1/d)\sum_{a=0}^{d-1} a \rho(a)$; in that way we have (1.33). As in Example I, we first
rewrite \( \int g(u + x)\,df(u) \) as \( \int g'(u + x)f(u)\,du \) and then continue integrating by parts \( j \) more times:

\[
\psi_p(x) = -\int_0^\infty df_p(u) = -\frac{B_{1,p}}{x} - \int_0^\infty \frac{f_p(u)}{(u + x)^2}\,du
\]

\[
= \frac{B_{1,p}}{x} + \frac{f_p^{(1)}(0)}{x^2} + \ldots + \frac{j!f_p^{(j)}(0)}{x^{j+1}} - (j + 1) \int_0^\infty \frac{f_p^{(j+1)}(u)}{(u + x)^{j+2}}\,du.
\]

We find, as in Example I, that \( \psi_p(x) \) has asymptotic series

\[
\psi_p(x) \sim \sum_{j=0}^\infty \frac{L(-j, \rho)}{(-x)^{j+1}}.
\]

III. Let \( f(u) = u - \lfloor u \rfloor - \frac{1}{2} \) be as in Example I, and let

\[
g(u) = \frac{\log q}{1 - q^u},
\]

where \( q \) is a parameter, \( |q| < 1 \). By (1.13) we have

\[
\psi_q(x) = -\log(1 - q) + \int_0^\infty g(u + x)\,d\lfloor u \rfloor
\]

\[
= \log \frac{1 - q^x}{1 - q} + \frac{1}{2} \int_0^\infty g(u + x)\,df(u) + \int_0^\infty g(u + x)\,du
\]

\[
= \log \frac{1 - q^x}{1 - q} + \frac{1}{2} \int_0^\infty g(u + x)\,f(u)\,du.
\]

Proceeding just as in Example I, we obtain

\[
\psi_q(x) \sim \log \frac{1 - q^x}{1 - q} + \sum_{j=0}^\infty \frac{\zeta(-j)}{j!} g^{(j)}(x),
\]

where

\[
g^{(j)}(x) = (\log q)^{j+1} \left( t \frac{d}{dt} \right)^j \frac{t}{1 - t} \bigg|_{t=q^x} = (\log q)^{j+1} P_{j+1}(q^x).
\]

This is D. Moak’s [20] asymptotic series for \( \psi_q \). (Note that our use of the notation \( P_j \) is slightly different from Moak’s.)

IV. Let \( f(u) = f_p(u) \) be as in Example II, and let \( g(u) = (\log q)(q^u/(1 - q^u)) \).

Following the procedure in Example III, we find the asymptotic series,

\[
\psi_{q,\rho}(x) \sim \sum_{j=0}^\infty \frac{L(-j, \rho)}{j!} g^{(j)}(x).
\]

Note that in Examples III and IV, as \( q \to 1^- \) we have

\[
\frac{1}{j!} g^{(j)}(x) \to \frac{1}{j!} \frac{d^j}{dx^j} \left( \frac{-1}{x} \right) = \frac{1}{(-x)^{j+1}},
\]

so we obtain the asymptotic series in Examples I and II, as expected.
2. The $p$-adic case. Let $\mathbb{Q}_p$ denote the field of $p$-adic numbers, $\mathbb{Z}_p$ denote the ring of $p$-adic integers, $\mathbb{Z}_p^* = \mathbb{Z}_p - p\mathbb{Z}_p$, $a + (p^N) = \{ x \in \mathbb{Z}_p \mid x \equiv a \pmod{p^N} \}$, and $\mathbb{C}_p$ denote the completion of the algebraic closure of $\mathbb{Q}_p$. The $p$-adic absolute value $\mid \mid_p$ on $\mathbb{C}_p$ is normalized so that $\mid p \mid_p = 1/p$.

The compact-open subsets of $\mathbb{Z}_p$ are finite disjoint unions of sets of the form $a + (p^N)$. A $\mathbb{C}_p$-valued measure $\mu$ on $\mathbb{Z}_p$ is a bounded, finitely additive map from compact-open subsets to $\mathbb{C}_p$. The integral of a continuous function $g: \mathbb{Z}_p \to \mathbb{C}_p$ with respect to $\mu$, defined as the usual limit of Riemann sums, always exists. These facts are easy to prove; for details, see, for example, [12, Chapter 2].

Let $z \in \mathbb{C}_p$ be outside the open unit disc about $1$: $\mid z - 1 \mid_p \geq 1$. Define

$$\mu_z(a + (p^N)) = z^a / (1 - z^{p^N}),$$

where $a$ is the least nonnegative integer representative: $0 \leq a < p^N$. Then it is easy to prove [14, Chapter 2] that:

1. $\mu_z$ extends to a measure on $\mathbb{Z}_p$, with $\mid \mu_z(U) \mid_p \leq 1$ for all compact-open $U$.
2. If $z$ is a root of unity, $p(j) = z^j$ (see Examples II and IV in §1), and $L(s, z) = L(s, \rho) = \sum_{n=1}^{\infty} z^n / n^s$ (continued analytically onto the complex $s$-plane), then the number $L(-k, z) \in \mathbb{Q}(z)$ is given by the $p$-adic formula

$$L(-k, z) = \int_{\mathbb{Z}_p^*} x^k \, d\mu_z(x).$$

3. If $\omega: \mathbb{Z}_p^* \to 1^{1/(p-1)}$ is the Teichmüller character (considered to be either $\mathbb{C}$-valued or $\mathbb{C}_p$-valued; we suppose we have fixed imbeddings of the algebraic numbers in both $\mathbb{C}$ and $\mathbb{C}_p$), $\langle x \rangle = x / \omega(x)$ for $x \in \mathbb{Z}_p^*$, and

$$L^*(s, \omega^k, z) = \sum_{n \equiv 1, p \mid n} \omega^k(n) \frac{z^n}{n^s},$$

then the $p$-adic function

$$L_p(s, \omega^j, z) = \int_{\mathbb{Z}_p^*} \langle x \rangle^{-s} \omega^j - 1(x) \, d\mu_z(x),$$

$\mid z - 1 \mid_p \geq 1, s \in \mathbb{Z}_p$, when $z$ is a root of unity interpolates the algebraic values

$$L^*(-k, \omega^{-k-1}, z) = \int_{\mathbb{Z}_p^*} x^k \omega^{-k-1}(x) \, d\mu_z(x).$$

4. If a function $\varphi: \mathbb{C}_p \to \mathbb{C}_p$ is defined by $\varphi(x) = \int_{\mathbb{Z}_p^*} g(u + x) \, d\mu_z(u)$ (where $g: \mathbb{C}_p \to \mathbb{C}_p$ is continuous; in examples, $\varphi$ and $g$ might only be defined on a subset of $\mathbb{C}_p$), then $\varphi$ satisfies

$$z\varphi(x + 1) - \varphi(x) = -g(x).$$

5. If $\varphi^*(x)$ is defined by $\varphi^*(x) = \int_{\mathbb{Z}_p^*} g(u + x) \, d\mu_z(u)$, then

$$z^p\varphi^*(x + p) - \varphi^*(x) = -\sum_{a=1}^{p-1} z^a g(x + a).$$
A basic example of the construction of \( \varphi \) and \( \varphi^* \) occurs when \( g(u) = -1/u \). For any \( z \in C_p \) with \( |z - 1|_p \geq 1 \), we define (see [13 or 14])

\[
(2.5) \quad \psi_{p,z}(x) = -\int_{\mathbb{Z}_p} \frac{1}{u + x} \, d\mu_z(u), \quad x \in C_p - \mathbb{Z}_p;
\]

\[
(2.6) \quad \psi_{p,z}^*(x) = -\int_{\mathbb{Z}_p^*} \frac{1}{u + x} \, d\mu_z(u), \quad x \in C_p - \mathbb{Z}_p^*.
\]

By (2.3), \( \psi_{p,z} \) satisfies the relationship

\[
(2.7) \quad z \psi_{p,z}(x + 1) - \psi_{p,z}(x) = 1/x
\]

(compare with (1.11)). It is also easy to check that \( \psi_{p,z} \) satisfies the “Gauss multiplication formula”

\[
(2.8) \quad \psi_{p,z}(x) = \frac{1}{m} \sum_{h=0}^{m-1} z^h \psi_{p,z}(x + h/m),
\]

for any positive integer \( m \), and the “Euler parity relation”

\[
(2.9) \quad \psi_{p,z}(x) = z^{-1} \psi_{p,z}(-1 - x).
\]

The function \( \psi_{p,z}^*(x) \) relates to \( \psi_{p,z}(x) \) as follows (“removing the \( p \)-Euler factor”):

\[
\psi_{p,z}^*(x) = \psi_{p,z}(x) - \frac{1}{p} \psi_{p,z}(\frac{x}{p}).
\]

**Remark.** Compare with Example II of §1 when \( \rho(z) = z^j \). In the classical case the measure \( d\mu_z(u) \) gives the integer \( u = a \) the point mass \( z^a \). In the \( p \)-adic case, if we take \( |z|_p < 1 \), then as \( N \to \infty \), the measure of the interval \( a + (p^N) \) around \( a \) has measure \( \mu_z(a + (p^N)) = z^a/(1 - z^p) \), which approaches \( z^a \); that is, formally the measure \( \mu_z(u) \) also gives \( u = a \) point mass \( z^a \). Thus, for \( |z|_p < 1 \) the \( p \)-adic construction is formally the same as the classical one: \( \psi_z(x) \) and \( \psi_{p,z}(x) \) are given by the same series \(-\sum_{a=0}^{\infty} z^a/(x + a)\). However, the functions \( \psi_z \) and \( \psi_{p,z} \) are of arithmetic interest (relate to Dirichlet L-series or modular forms, see below) only when \( z \) is a root of unity. In other words, we must extend analytically beyond the disc where \( \psi_z \) and \( \psi_{p,z} \) are formally the same in order to reach values of \( z \) for which the functions are of number-theoretic interest. Note that \( \psi_{p,z} \) and \( \psi_z \) satisfy the same relations (2.7) and (2.8).

Now let \( q \in C_p \) be a parameter with \( |q - 1|_p < 1 \). We want the function \( q^u \) to make sense for certain \( u \in C_p \). If \( u \in \mathbb{Z}_p \), then \( q^u \) converges for any \( |q - 1|_p < 1 \). More generally, \( q^u \) can be defined as \( \exp(u \log_p q) \) for any \( u \in C_p \) with \( |u|_p < r_q \), where \( r_q = |\log_p q|_p^{1/p^{1/(p-1)}} \). We shall usually assume that \( |q - 1|_p < p^{-1/(p-1)} \), in which case

\[
r_q = \frac{1}{|q - 1|_p p^{1/(p-1)}} > 1.
\]

Let

\[
g(u) = \log_p q^{\frac{q^u}{1-q^u}} = -\frac{d}{du} \log_p \frac{1-q^u}{1-q}.
\]
Note that \( \lim_{q \to 1} g(u) = -1/u \). Also define
\[
g^*(u) = g(u) - g(pu) = \log_p q \left( \frac{q^u}{1 - q^u} - \frac{q^{pu}}{1 - q^{pu}} \right).
\]

**Theorem 1.** Let \( |z - 1|_p > 1, |q - 1|_p < 1 \). The functions
\[
\psi_{p,q,z}(x) = \int_{\mathbb{Z}_p^*} g(u + x) \, d\mu_z(u)
\]
\[
= \log_p q \int_{\mathbb{Z}_p^*} \frac{q^{u+x}}{1 - q^{u+x}} \, d\mu_z(u), \quad |x|_p < r_q, x \notin \mathbb{Z}_p;
\]
\[
\psi^*_{p,q,z}(x) = \int_{\mathbb{Z}_p^*} g^*(u + x) \, d\mu_z(u)
\]
\[
= \log_p q \int_{\mathbb{Z}_p^*} \frac{q^{u+x}}{1 - q^{u+x}} \, d\mu_z(u), \quad |x|_p < r_q, x \notin \mathbb{Z}_p^*;
\]
\[
\psi^{**}_{p,q,z}(x) = \int_{\mathbb{Z}_p^*} g^{**}(u + x) \, d\mu_z(u)
\]
\[
= \log_p q \int_{\mathbb{Z}_p^*} \left( \frac{q^{u+x}}{1 - q^{u+x}} - \frac{q^{pu+x}}{1 - q^{pu+x}} \right) \, d\mu_z(u), \quad |x|_p < r_q, x \notin \mathbb{Z}_p^*,
\]
satisfy the following relations:
\[
\text{(2.10)} \quad \psi_{p,q,z}(x + 1) - \psi_{p,q,z}(x) = -\log_p q \frac{q^x}{1 - q^x} = \frac{d}{dx} \log_p \frac{1 - q^x}{1 - q};
\]
\[
\text{(2.11)} \quad \psi_{p,q,z}(x) = \frac{1}{m} \sum_{h=0}^{m-1} z^h \psi_{p,q,z^m}(\frac{x + h}{m}), \quad \text{for } m = 1, 2, 3, \ldots;
\]
\[
\text{(2.12)} \quad \psi_{p,q,z}(x) = z^{-1} \psi_{p,q^{-1},z^{-1}}(1 - x);
\]
\[
\text{(2.13)} \quad \psi_{p,q,z}(x) = \psi_{p,q,z}(x) + \log_p q \frac{1}{1 - z};
\]
\[
\text{(2.14)} \quad \psi^{*}_{p,q,z}(x) = \psi_{p,q,z}(x) - \frac{1}{p} \psi_{p,q,z^p}(x/p);
\]
\[
\text{(2.15)} \quad z^p \psi^{*}_{p,q,z}(x + p) - \psi^{*}_{p,q,z}(x) = -\sum_{a=1}^{p-1} z^a \log_p q \frac{q^{x+a}}{1 - q^{x+a}};
\]
\[
\text{(2.16)} \quad \psi^{**}_{p,q,z}(x) = \psi^{*}_{p,q,z}(x) - \frac{1}{p} \psi^{*}_{p,q,z^p}(x).
\]

**Proof.** (2.10) is an immediate consequence of (2.3). To prove the “Gauss multiplication formula” (2.11), we write the right side as
\[
\lim_{N \to \infty} \frac{1}{m} \sum_{h=0}^{m-1} z^h \log_p q \sum_{j=0}^{p^{N-1}} \frac{q^{m(j+(x+h)/m)}}{1 - q^{m(j+(x+h)/m)}} \frac{z^{mj}}{1 - z^{mp^N}}
\]
\[
= \lim_{N \to \infty} \log_p q \sum_{h,j} \frac{q^{mj+h+x}}{1 - q^{mj+h+x}} \frac{z^{mj+h}}{1 - z^{mp^N}}.
\]
If we write \( mj + h = p^N k + l, k = 0, 1, \ldots, m - 1, l = 0, 1, \ldots, p^N - 1 \); and if we use the fact that \( q^{p^N k + l} \to q^l \) as \( N \to \infty \), we find that the last limit is equal to

\[
\lim_{N \to \infty} \log_p q \sum_{l=0}^{p^N-1} \frac{q^{l+x}}{1 - q^{l+x}} \sum_{k=0}^{m-1} z^{p^N k} = \lim_{N \to \infty} \log_p q \sum_{l=0}^{p^N-1} \frac{q^{l+x}}{1 - q^{l+x}} \frac{z^l}{1 - z^{p^N}}
\]

\[
= \log_p q \int_{I_p} \frac{q^{u+x}}{1 - q^{u+x}} \, d\mu_z(u),
\]

as desired. The proofs of (2.12)-(2.16) are equally straightforward. Q.E.D.

Remark. The “multiplication formula” (2.11) and the parity relation (2.12) can be combined into the following more general identity:

\[
\psi_{p,q,z}(x) = \frac{1}{m!} \sum z^k \psi_{p,q,m}(x + h),
\]

where the summation is over \( \min(m, 0) < h < \max(m, 0) \) and \( m \) is any positive or negative integer.

Returning to the general situation, \( \varphi(x) = \int_{I_p} g(u + x) \, d\mu(u) \), we note that the following two expansions are possible:

for \( x \) small, \( g(u + x) = \sum_{j=0}^{\infty} \frac{x^j}{j!} g^{(j)}(u) \)

(2.17)

\[
\Rightarrow \varphi(x) = \sum_{j=0}^{\infty} \frac{x^j}{j!} \int_{I_p} g^{(j)}(u) \, d\mu(u);
\]

for \( x \) large, \( g(u + x) = \sum_{j=0}^{\infty} \frac{u^j}{j!} g^{(j)}(x) \)

(2.18)

\[
\Rightarrow \varphi(x) = \sum_{j=0}^{\infty} \frac{g^{(j)}(x)}{j!} \int_{I_p} u^j \, d\mu(u).
\]

The second of these expansions holds whenever the infinite sum converges; since the integral is bounded, we have convergence if \( \| g^{(j)}(x)/j! \|_p \to 0 \). The first expansion holds for \( |x|_p < p^{-1/(p-1)} \) provided that the \( g^{(j)}(u) \) are defined, continuous, and bounded (uniformly in \( j \)) on the support of \( \mu \). We can apply the second expansion for \( \mu = \mu_z \) and \( \mu = \mu_z |_{I_p} \) and for \( g(u) = -1/u \) and \( g(u) = (\log_p q)(q^u/(1 - q^u)) \). We can apply the first expansion for \( \mu = \mu_z |_{I_p} \) and for \( g(u) = -1/u \), \( g(u) = (\log_p q)(q^u/(1 - q^u)) \) and

\[
g^*(u) = \log_p q \left( \frac{q^u}{1 - q^u} - \frac{q^{pu}}{1 - q^{pu}} \right).
\]

Theorem 2. Let \( \psi_{p,z}(x), \psi_{p,q,z}(x), \psi_{p,q,z}(x), \psi_{p,q,z}(x), \psi_{p,q,z}(x) \) be the functions defined in (2.5)-(2.6) and in Theorem 1. In (2) below we suppose that \( |q - 1|_p < p^{-1/(p-1)} \), and in (4) we suppose that \( |q - 1|_p < p^{-2/(p-1)} \) (i.e., \( r_q > p^{1/(p-1)} \)). Then

(1) for \( |x|_p < 1 \),

\[
\psi_{p,z}(x) = \sum_{j=1}^{\infty} (-1)^j x^{j-1} L_p(j, \omega^{-1}, z)
\]

(compare with (1.12)).
(2) for $|x|_p < p^{-1/(p-1)}$,
\[\psi_{p,q,z}(x) = \sum_{j=1}^{\infty} \frac{\log_p q}{(j-1)!} j \int_{\mathbb{Z}_p} P_j(q^u) \, d\mu_z(u),\]
\[\psi_{p,q,z}(x) = \sum_{j=1}^{\infty} \frac{\log_p q}{(j-1)!} j \int_{\mathbb{Z}_p} P_j^*(q^u) \, d\mu_z(u),\]
where $P_j$ was defined in (1.15) and
\[P_j^*(q) = P_j(q) - p^{j-1} P_j(q^p) = \left( q \frac{d}{dq} \right)^{j-1} \left( \frac{q}{1-q} - \frac{q^p}{1-q^p} \right)\]
(compare with (1.31));
(3) for $|x|_p > 1$,
\[\psi_{p,z}(x) = \sum_{j=0}^{\infty} \frac{L(-j, z)}{(-x)^{j+1}}, \quad \psi_{p,z}(x) = \sum_{j=0}^{\infty} \frac{L_p(-j, \omega^{j+1}, z)}{(-x)^{j+1}}\]
(compare with (1.37)). Here in the first sum $L(-j, z) = \int_{\mathbb{Z}_p} u^j \, d\mu_z(u)$, which coincides with the classical value $L(-j, z)$ when $z$ is a root of unity;
(4) for $p^{1/(p-1)} < |x|_p < r_q$,
\[\psi_{p,q,z}(x) = \sum_{j=0}^{\infty} \frac{L(-j, z)}{j!} (\log_p q)^j P_{j+1}(q^x),\]
\[\psi_{p,z}(x) = \sum_{j=0}^{\infty} \frac{L_p(-j, \omega^{j+1}, z)}{j!} (\log_p q)^j P_{j+1}(q^x),\]
\[\psi_{p,q,z}(x) = \sum_{j=0}^{\infty} \frac{L_p(-j, \omega^{j+1}, z)}{j!} (\log_p q)^j P_{j+1}(q^x)\]
(compare with (1.39)).

The proof of Theorem 2 follows immediately from (2.17), (2.18), (2.1), (2.2) and the following observations concerning convergence: in (1), (3) and (4), $L_p(j, \omega^{j+1}, z)$, $L(-j, z)$, $L_p(-j, \omega^{j+1}, z)$ are bounded with respect to $j$; in (2), $(\log_p q)^j P_j(q^x)$ and $(\log_p q)^j P_j^*(q^x)$ are bounded over $u \in \mathbb{Z}_p$ uniformly with respect to $j$; and in (4), we have
\[| (\log_p q)^j P_{j+1}(q^x) |_p \leq \left| \frac{\log_p q}{1-q^x} \right|_p = \frac{1}{x} |^{j+1}\]
and the same relation for $| (\log_p q)^j P_{j+1}(q^x) |_p$.

Remark. The integrals $a_j = a_{j,q,z}$ in the coefficients of the Taylor series for $\psi_{p,q,z}(x)$ (part (2) of Theorem 2) are related to $p$-adic Eisenstein series, but more indirectly than in the classical case. To see this connection, first suppose that
If we also suppose that $|q|_p < 1$, then we have

$$P_j^*(q) = \sum_{n=1, p|m} n^{-1}q^n$$

and

$$\sum_{m=1, p|m} P_j^*(q^m)z^m = \sum_{m=1, p|m} q^m \sum_{dd' = m} d^{-1}z^{dd'}.$$

Now set $z$ equal to a nontrivial $D$th root of unity, and define $f$ on $\mathbb{Z}/D\mathbb{Z} \times \mathbb{Z}/D\mathbb{Z}$ by $f(u, v) = z^{uv}$. Then

$$a_{j, q, z} + (-1)^j a_{j, q, z}^{-1} = \sum_{m=1, p|m} q^m \sum_{dd' = m} d^{-1}(f(d, d') + (-1)^j f(-d, -d'))$$

is the series Katz denotes $2\Phi_j^*_{-1,0, f}$ (see (6.4.1) of [8]; note that the definitions and results generalize trivially to $\mathbb{Z}_p\{z\}$-valued functions $f$). It is a $p$-adic modular form for the congruence-subgroup $\Gamma(D)$ (see [8]).

In other words, we can consider the power series

$$a_j = \sum_{m=1, p|m} q^m \sum_{dd' = m} d^{-1}z^{dd'} \in \mathbb{Z}[z][[q]] \subset \mathbb{Z}[[z, q]]$$

as a $p$-adic analytic function in the two variables $z, q$ for $|z|_p \leq 1, |q|_p < 1$. When $z^D = 1$ we essentially have a $p$-adic Eisenstein series. On the other hand, for any fixed $z$ with $|z|_p < 1$, the same series $a_j$ has a unique analytic continuation

$$a_j = \sum_{m=1, p|m} P_j^*(q^m)z^m$$

to the region $\{q \in \mathbb{C}_p \mid |q|_p \leq 1, q^n \neq 1 \text{ for all } n, p^2 \nmid n\}$. This $p$-adic analytic function is given on the region $0 < |q - 1|_p < p^{-1/(p-1)}$ by the integral

$$\int_{\mathbb{Z}_p} P_j^*(q^u) d\mu_z(u).$$

Finally, for any fixed $q$ with $0 < |q - 1|_p < p^{-1/(p-1)}$, this integral gives a unique analytic continuation (in $z$) to the region $|z - 1|_p \gg 1$. In particular, for $z$ a $D$th root of unity we obtain essentially the $j$th Taylor expansion coefficient of the twisted psi-function $\psi_{p, q, z}^*$. This is a $p$-adic analogy to the occurrence discussed in §1 of classical Eisenstein series in the expansion of $\psi_q(x + 1)$ and $\psi_q(x + 1)$.

Finally, we discuss the relationship between $\psi_{p, q, z}$ and the $q$-extensions defined in [15] for J. Diamond’s $p$-adic log-gamma function [5]. We temporarily use $\tilde{\psi}$ to denote the functions defined in [15] by means of limits of the form

$$\lim_{N \to \infty} \frac{1}{dp^N} \sum_{0 \leq j < dp^N} g(x + j).$$
In what follows $\Sigma'$ means that indices divisible by $p$ are omitted. Thus for $z^d = 1$ (with $z \neq 1$, $p \nmid d$) and for $|x|_p < r_q$, $x \in \mathbb{Z}_p$,
\[
\Psi_{p,q,z}(x) = \lim_{N \to \infty} \frac{1}{dp^N} \sum_{0 \leq j < dp^N} z^j \log_p \frac{1 - q^{x+j}}{1 - q} ;
\]
\[
\Psi_{p,q,z}^*(x) = \lim_{N \to \infty} \frac{1}{dp^N} \sum_{0 \leq j < dp^N} z^j \log_p \frac{1 - q^{x+j}}{1 - q} .
\]

**Theorem 3.** (1) $\Psi_{p,q,z}(x) = \Psi_{p,q,z}^*(x)$ for $|x|_p < r_q$, $x \in \mathbb{Z}_p$.  
(2) $\Psi_{p,q,z}^*(x) = \Psi_{p,q,z}^*(x)$ for $|x|_p < r_q$, $x \in \mathbb{Z}_p^*$.  

The proof is similar to the proof that $G_{p,z}(x) = G_{p,z}^*(x)$ in [14, p. 51]. Namely, one first shows that (2.11) also holds for $\tilde{\Psi}$. Then we can use (2.11) with $m = p^n$ for both $\psi$ and $\tilde{\psi}$ to see that it suffices to prove that
\[
\tilde{\Psi}_{p,q,z}^*(x) = \frac{1}{p} \Psi_{p,q,z}^*(x/p),
\]
i.e., to prove that $\tilde{\Psi}_{p,q,z}(x) = \Psi_{p,q,z}(x)$ for $|x|_p$ large. But this is true because both sides have the same Stirling series. The proof that $\tilde{\Psi} = \Psi^*$ is similar.

**REFERENCES**


School of Mathematics, Institute for Advanced Study, Princeton, New Jersey 08540

Department of Mathematics, University of Washington, Seattle, Washington 98195 (Current address)