TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 273, Number 1, September 1982

BRANCHING DEGREES ABOVE LOW DEGREES

BY

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ABSTRACT. In this paper, we investigate the location of the branching degrees within the recursively enumerable (r.e.) degrees. We show that there is a branching degree below any given nonzero r.e. degree and, using a new branching degree construction and a technique of Robinson, that there is a branching degree above any given low r.e. degree. Our results extend work of Shoenfield and Soare and Lachlan on the generalized nondiamond question and show that the branching degrees form an automorphism base for the r.e. degrees.

1. Introduction. In [**1**] we began a program of taking certain natural and important definable subclasses of the recursively enumerable (r.e.) degrees and studying their relation to the r.e. degrees as a whole. Our hope is that this program will tend to illuminate uniformities in the structure of the r.e. degrees, just as early work in the field (e.g. the Sacks' Splitting and Density Theorems) did, rather than demonstrate pathological aspects of the structure.

In [**1**], the particular class of r.e. degrees considered was the nonbranching degrees. We were able to obtain a strong uniformity result there, namely, that the nonbranching degrees are dense in the r.e. degrees. In the present paper we consider the branching degrees, the class complementary to the nonbranching degrees, and give certain uniformity results concerning this class of degrees. The techniques involved are more complicated than those required in [**1**]. Before describing our results, we give the definition and discuss previous results concerning branching degrees.

DEFINITION 1.1. An r.e. degree is *branching* if it is the infimum of two incomparable r.e. degrees.

As shown in [**1**], a result of Lachlan [**2**, Lemma 18] easily implies that in Definition 1.1 it does not matter if infima are taken with respect to all degrees or with respect to just the r.e. degrees.

The first result concerning branching degrees was the proof, obtained independently by Lachlan [**2**] and Yates [**14**], that **0** is a branching degree. (A pair of nonzero r.e. degrees which has infimum **0** is called a minimal pair.) This minimal pair result confirmed a conjecture of Sacks [**6**, p. 170, first ed.] and disproved a

Received by the editors July 9, 1981.

1980 *Mathematics Subject Classification*. Primary 03D25.

Key words and phrases. Recursively enumerable degree, branching degree, low degree, generalized nondiamond question, automorphism base.

[1]The results in this paper are part of the author's doctoral dissertation, written at the University of Chicago under Professor Robert I. Soare.They were presented on November 11, 1979 at the Conference on Mathematical Logic which was part of Logic Year 1979–80 sponsored by the Mathematics Department of the University of Connecticut.

0002-9947/82/0000-0105/$07.75

conjecture of Shoenfield [7] that the r.e. degrees satisfy a certain homogeneity property. Lachlan, in the same paper [**2**], showed the existence of a nonzero branching degree. Later results on embedding lattices into the r.e. degrees involve constructions of branching degrees. For example, Thomason's proof [**13**] that the countable atomless Boolean algebra can be embedded into the r.e. degrees is an extension of the techniques which Lachlan used to show the existence of a nonzero branching degree and this result shows that there are infinitely many branching degrees.

In §2 we show how Lachlan's construction of a nonzero branching degree can be used to show that below any given nonzero r.e. degree there is a nonzero branching degree (Theorem 2.1). Further work with branching degrees seems to require more complicated branching degree constructions. In §3 we introduce the new branching degree construction which we need for later work by giving a new proof of the answer to the generalized nondiamond question which was first obtained by Shoenfield and Soare [**9**] and Lachlan [**4**] (Theorem 3.1). The construction involves a tree of strategies as used by Lachlan in [**3**, **4**]. In §4 we illustrate a technique of Robinson for lifting a construction in the r.e. degrees above a given low r.e. degree by proving a special case of the Robinson Splitting Theorem [**5**, Corollary 9]. In §5 we combine techniques from §§3 and 4 to show that there is a branching degree above any given low r.e. degree. In fact, we show (Theorem 5.1) that the diamond lattice can be embedded into the r.e. degrees above any given low r.e. degree with top element preserved, thereby extending the result of Shoenfield and Soare and Lachlan mentioned above. Theorem 5.4 is a strengthened version of Theorem 5.1 which allows us to conclude that the branching degreees form an automorphism base for the r.e. degrees (Corollary 5.5) and that, given a nonzero, incomplete r.e. degree, there is a branching degree incomparable with the given r.e. degree (Corollary 5.6).

The question of density for the branching degrees is open as is the question of whether there is a branching degree above any given incomplete r.e. degree.[2]

We now fix some notations and conventions which are used throughout the paper. A *string* is an element of $2^{<\omega}$. If $\sigma \in 2^n$ we call n the *length* of σ ($\text{lh}\sigma = n$). We write $\subseteq$ for inclusion and $\subset$ for strict inclusion. A set is often identified with its characteristic function, so if A is a set, $A \restriction n$ is a string of length n and if σ is a string, it makes sense to write $\sigma \subseteq A$.

By degree we mean Turing degree. The least upper bound of two degrees **a** and **b** is called their join and is written $\mathbf{a} \cup \mathbf{b}$; the greatest lower bound of **a** and **b**, if it exists, is called their inf and is written $\mathbf{a} \cap \mathbf{b}$.

Let $\{W_e\}_{e\in\omega}$ be some standard enumeration of the r.e. sets and let $\{W_{e,s}\}_{e,s\in\omega}$ be a recursive collection of finite sets such that for all e and s, $W_{e,s} \subseteq W_{e,s+1}$, and for all e, $W_e = \bigcup_s W_{e,s}$. In a construction involving W_e, we assume that the elements of $W_{e,s} - W_{e,s-1}$ are enumerated into W_e at the beginning of stage s, before any action is taken for the construction. (Take $W_{e,-1}$ to be $\varnothing$.) If the construction involves a

[2] NOTE ADDED IN PROOF. Theodore Slaman has recently informed us that, using the "monstrous" injury technique, he has now shown that the branching degrees are dense in the r.e. degrees.

given r.e. set A, then we assume an effective stage-by-stage enumeration of A with new elements entering A only at the beginning of each stage and only finitely many elements entering at any stage. Which effective enumeration of A we use generally does not matter, but if we use in the construction of one-one effective enumeration $\{a_s\}_{s\in\omega}$ of A, then we assume that a_s is the unique element which enters A at stage s. Of course, if *we* are constructing the set A, then we may put new elements into A at any time during a stage.

By a *partial recursive functional* Ξ of n set variables, we mean an r.e. set of axioms with no redundancy. An *axiom* consists of n strings $\sigma_1,\ldots,\sigma_n$ of some common length k, an argument x, and a value y; we assume that $k > x$. If the axiom belongs to Ξ and for each i, $1 \leqslant i \leqslant n$, $\sigma_i \subseteq A_i$, then $\Xi(A_1,\ldots,A_n;\ x)$ converges to y. We use $\Xi(A_1,\ldots,A_n;\ x)\downarrow$ to mean that $\Xi(A_1,\ldots,A_n;\ x)$ converges to some value. The *length* of an axiom is the common length of the strings σ_i. By "no redundancy" we mean that for any sets $A_1,\ldots,A_n$ and number x, at most one axiom in Ξ applies to $A_1,\ldots,A_n$ and x. We always denote partial recursive functionals by capital Greek letters and their use functions by the corresponding small Greek letter. That is, if $\Xi(A_1,\ldots,A_n;\ x)\downarrow$, there is a unique axiom in Ξ applying to $A_1,\ldots,A_n$ and x and $\xi(A_1,\ldots,A_n;\ x)$ is the length of this axiom, which is $> x$. If $\Xi(A_1,\ldots,A_n;\ x)$ does not converge, then $\xi(A_1,\ldots,A_n;\ x) = 0$.

Let $\{\Psi_e\}_{e\in\omega}$ and $\{\Phi_e\}_{e\in\omega}$ be effective listings of all partial recursive functionals of one and two set variables respectively. Let $\{\Psi_{e,s}\}_{e,s\in\omega}$ be a recursive double array of finite sets of axioms such that for all e and s, each axiom in $\Psi_{e,s}$ has length $< s$, and $\Psi_{e,s} \subseteq \Psi_{e,s+1}$, and, for all e, $\Psi_e = \bigcup_s \Psi_{e,s}$. In any construction involving Ψ_e, we assume that the elements of $\Psi_{e,s} - \Psi_{e,s-1}$ are enumerated into Ψ_e at the beginning of stage s, before any action is taken. (Let $\Psi_{e,-1}$ be $\varnothing$.) We make similar conventions for $\{\Phi_e\}_{e\in\omega}$ and for the other sequences of functionals which we define later.

When we refer, during a construction, to sets and functionals which are being enumerated, we are actually referring to the finite sets and functionals consisting of those elements and axioms so far enumerated. For example, if at some point in a construction we say that $\Psi_e(A;\ x) = 0$, we mean that the functional consisting of those axioms so far enumerated into Ψ_e when applied to the finite set consisting of those numbers so far enumerated into A converges on x to 0. If we put $[s]$ after an expression involving sets and functionals being enumerated, then we are actually referring to as much of those sets and functionals as has been enumerated at the point in the construction just *before* stage s. Similarly, $[s, t]$ after an expression refers to the point in stage s just before substage t of stage s begins. After we have completed the description of a construction, when we refer to sets or functionals which were enumerated, we usually are referring to the final values. When there is a possibility for confusion, we add $[\omega]$ after an expression to emphasize that we are referring to the final values.

Suppose that Ξ is a partial recursive functional of n set variables and $A_1,\ldots,A_n$ are being constructed. If, at some point during the construction, $\Xi(A_1,\ldots,A_n;\ x)\downarrow$ and $A_1 \cup \cdots \cup A_n \upharpoonright u$ does not later change, where $u = \xi(A_1,\ldots,A_n;\ x)$, then the computation is "correct," i.e., $\Xi(A_1,\ldots,A_n;\ x)\downarrow[\omega]$ by the same computation.

We use $'$ for the usual jump operation on sets and degrees. A set A is *low* if $A' \equiv_T \varnothing'$, and a degree $\mathbf{a}$ is *low* if $\mathbf{a}' = \mathbf{0}'$.

We let $\langle\ \rangle: \omega^2 \to \omega$ be some fixed recursive bijection and write $\langle x, y\rangle$ for $\langle\ \rangle(x, y)$. If A and B are sets, we define $A \oplus B$ to be $\{2x: x \in A\} \cup \{2x+1: x \in B\}$, so $\deg(A) \cup \deg(B) = \deg(A \oplus B)$.

If σ and τ are strings, then we write $\sigma * \tau$ for σ concatenated with τ. If $i = 0$ or 1, we let i also denote the string of length 1 whose value at 0 is i. We linearly order the set of strings by saying that $\sigma \leqslant \tau$ if either $\sigma \subseteq \tau$ or else $\sigma(x) < \tau(x)$ where x is the least number such that $\sigma(x) \neq \tau(x)$.

We let $\mathcal{D}_x$ be the finite set with canonical index x. If m is defined to be the largest number satisfying some property and there are no such numbers, then we take m to be 0.

We denote the end of the proof of a theorem or corollary with the symbol ■. The symbol □ denotes the end of a construction or of the proof of a lemma to a theorem.

2. Nonzero branching degrees below nonzero r.e. degrees. We now give the Lachlan nonzero branching degree construction. In fact, we combine the construction with permitting to show that there is a nonzero branching degree below any given nonzero r.e. degree. Our presentation of the basic construction is similar to that of Soare **[12]**. We refer the reader to that paper for a clear discussion of the motivation for the method of proof. Although the addition of permitting to the basic nonzero branching degree construction of Lachlan is not difficult, the observation that it can be done appears to be new.

THEOREM 2.1. *If* $\mathbf{d}$ *is a nonzero r.e. degree, then there is a nonzero branching degree* $\mathbf{c}$ *with* $\mathbf{c} \leqslant \mathbf{d}$.

PROOF. Let D be an r.e. set in $\mathbf{d}$ and let $\{d_s\}_{s\in\omega}$ be an effective enumeration of D. Let $\{\langle\Theta_e^0, \Theta_e^1\rangle\}_{e\in\omega}$ be an effective enumeration of all pairs of partial recursive functionals of two set variables. We construct r.e. sets A_0, A_1, C with $C \leqslant_T D$ by permitting, $A_i \nleqslant_T C$, $i = 0$ or 1, and $\deg(C)$ the inf of $\deg(A_0 \oplus C)$ and $\deg(A_1 \oplus C)$. Then if $\mathbf{c} = \deg(C)$, $\mathbf{c}$ is as desired. We wish to meet, for all $e \in \omega$ and $i = 0$ or 1, the requirements

$$P_e^i: \Psi_e(C) \neq A_i, \tag{2.1}$$

$$P_e^C: \overline{C} \neq W_e, \tag{2.2}$$

$$N_e: \Theta_e^0(A_0, C) = \Theta_e^1(A_1, C) = f,\ f \text{ total} \to f \leqslant_T C. \tag{2.3}$$

At any point in the construction we let

$$l(e) = \max\{x: (\forall y < x)(\Theta_e^0(A_0, C; y) = \Theta_e^1(A_1, C; y))\}.$$

Just before stage s, we define, by induction on e, a restraint function $r(e)[s]$ and hence $R(e)[s] = \max\{r(e')[s]: e' < e\}$. Suppose that we have $r(e')[s]$ for all $e' < e$, so we have $R(e)[s]$. We call s *e-maximal* if

$$(\forall t < s)[R(e)[t] = R(e)[s] \to l(e)[t] < l(e)[s]]$$

and we let $r(e)[s]$ be the maximum of

(i) those $t < s$ such that $R(e)[t] < R(e)[s]$, and

(ii) if s is not e-maximal, those $t < s$ such that $R(e)[t] = R(e)[s]$ and t is e-maximal.

We will have for each e, $\liminf_s R(e)[s] < \infty$, but in general $\limsup_s R(e)[s] = \infty$. We will still be able to use permitting since numbers put into C are not subject to the $R(e)$ restraints.

Let $\{P_n\}_{n \in \omega}$ be some effective listing of the P_e^i's and P_e^C's. During the construction, the P_n's are assigned followers which may later be canceled. During stage s we say that $P_e^i = P_n$ *requires attention* if no follower of P_e^i is in A_i and either

(2.4) $$\text{for some follower } x \text{ of } P_e^i, \quad x \geqslant R(n)[s] \text{ and } \Psi_e(C; x) = 0,$$

or

(2.5) $$\text{for every follower } x \text{ of } P_e^i, \quad \Psi_e(C; x) = 0,$$

and we say that P_e^C *requires attention* if $C \cap W_e = \varnothing$ and either

(2.6) $$\text{for some follower } x \text{ of } P_e^C, \quad x \in W_e \text{ and } d_s < x,$$

or

(2.7) $$\text{for every follower } x \text{ of } P_e^C, \quad x \in W_e.$$

We now give the construction.

Stage s. Find the least n such that P_n requires attention. (This n exists since if P_e^i has no followers then P_e^i requires attention.) Cancel all followers of all $P_{n'}$ with $n' > n$. If P_n is P_e^i and there is an x satisfying (2.4), take the least such x and put it into A_i; otherwise, appoint s to be a follower of P_n. If P_n is P_e^C and (2.6) holds for some x, put the least such x into C; otherwise, appoint s to be a follower of P_n. We say that P_n *receives attention* at stage s. □

Note that by cancellation if, at some point in the construction, x is a follower of P_n and y is a follower of $P_{n'}$ with $x < y$, then $n \leqslant n'$.

LEMMA 1. $C \leqslant_T D$.

PROOF. By permitting, i.e., if $D \upharpoonright x[s] = D \upharpoonright x$, then $C \upharpoonright x[s] = C \upharpoonright x$. □

LEMMA 2. *For all* n, $\liminf_s R(n)[s] < \infty$.

PROOF. For all s, $R(0)[s] = 0$, so $\liminf_s R(0)[s] = 0$. Suppose that $\liminf_s R(n)[s] = w < \infty$. Let v be the greatest s such that $R(n)[s] < w$. Let $S = \{s: R(n)[s] = w\}$. Then S is infinite. If there are infinitely many $s \in S$ such that s is n-maximal, then for all such s which are $> v$,

$$R(n+1)[s] = \max\{R(n)[s], r(n)[s]\} = \max\{w, v\}.$$

If there are only finitely many $s \in S$ with s n-maximal, let s_0 be the greatest such s. Then for any $s \in S$ with $s > v, s_0$, $R(n+1)[s] = \max\{w, v, s_0\}$. Hence the result holds for $n + 1$. □

LEMMA 3. *For each* n, P_n *is met and receives attention only finitely often.*

PROOF. Suppose that the conclusion of the lemma holds for all $n' < n$. Let t_0 be the least stage t such that for no $s \geqslant t$ does a $P_{n'}$ with $n' < n$ receive attention. Then P_n has no followers just before stage t_0; a follower appointed to P_n at a stage $\geqslant t_0$ is never canceled; and if P_n requires attention at a stage $\geqslant t_0$, then it receives attention at that stage.

Suppose that P_n is P_e^i. If P_n receives attention at a stage $s \geqslant t_0$ at which (2.4) holds, then the follower of P_e^i put into A_i at stage s is never canceled, so P_e^i never later requires attention, so P_e^i receives attention only finitely often; also, a computation $\Psi_e(C; x) = 0$ exists at stage s and is correct because any followers appointed at stages $\geqslant s$ will be too large to destroy the computation, all followers of $P_{n'}$ with $n' > n$ are canceled at stage s, P_n does not put numbers into C, and no $P_{n'}$ with $n' < n$ receives attention at a stage $\geqslant s$. Thus in this case P_n is met since $A_i(x)[\omega] = 1 \neq 0 = \Psi_e(C; x)[\omega]$.

Suppose that P_e^i receives attention infinitely often. Then if P_e^i receives attention at a stage $s \geqslant t_0$, (2.4) fails, so P_e^i is appointed infinitely many followers. In particular, at some stage $x_0 \geqslant t_0$ with $x_0 \geqslant \liminf_s R(n)[s]$, P_e^i receives attention, so x_0 is assigned as a follower of P_e^i and x_0 is never canceled. Let $t_1 > x_0$ be a stage at which P_e^i receives attention. Then (2.5) holds at t_1, so $\Psi_e(C; x_0)[t_1 + 1] = 0$ and since $P_n = P_e^i$ receives attention at stage t_1, this computation is correct, by the argument of the preceding paragraph. Now take $t_2 > t_1$ such that $R(n)[t_2] \leqslant x_0$. Then at stage $t_2 > t_0$, no follower of P_e^i is in A_i (else that follower will never be canceled and P_e^i receives attention only finitely often) and (2.4) holds through x_0, so P_e^i requires attention. But then P_e^i receives attention at stage t_2 and (2.4) holds, a contradiction. Thus P_e^i receives attention finitely often.

Suppose that P_e^i fails. Then at no stage $s \geqslant t_0$ at which P_e^i receives attention does (2.4) hold, so for no $s \geqslant t_0$ is there a follower of P_e^i which is in A_i. Since P_e^i receives attention only finitely often, only finitely many followers of P_e^i are appointed at stages $\geqslant t_0$ and none of these followers is put into A_i, so for each such follower x, $0 = A_i(x)[\omega] = \Psi_e(C; x)[\omega]$. Take $s \geqslant t_0$ such that P_e^i does not receive attention at stage s and such that for all followers x of P_e^i appointed at or after stage t_0, $\Psi_e(C; x)[s] = 0$. Then P_e^i requires attention at stage s, so receives attention, a contradiction. Thus P_e^i is met.

Now suppose that P_n is P_e^C. If P_e^C ever receives attention at a stage s such that (2.6) holds at stage s, then $C \cap W_e \neq \varnothing$, so P_e^C is met and P_e^C never requires attention after stage s, so receives attention only finitely often.

Suppose that P_e^C receives attention infinitely often. Then each time P_e^C receives attention (2.6) fails, so infinitely many followers are appointed to P_e^C. For each follower x appointed to P_e^C after stage t_0, $x \in W_e$, else P_e^C does not receive attention after x is appointed to follow P_e^C (since x is never canceled). Thus, given y, we can effectively find a stage $s \geqslant t_0$ such that for some follower $x > y$ of P_e^C appointed after stage t_0 and before stage s, $x \in W_e[s]$. Now if, for $s' \geqslant s$, $d_{s'} < x$, then at stage s', $C \cap W_e[s'] = \varnothing$ (else P_e^C receives attention only finitely often) and (2.6) holds for x, so P_e^C requires attention at stage $s' \geqslant t_0$, so receives attention and (2.6) holds, a contradiction. Thus $D \upharpoonright x[s] = D \upharpoonright x$. But this procedure gives D recursive, a contradiction. Hence P_e^C receives attention only finitely often.

Now suppose that P_e^C fails. Then whenever P_e^C receives attention, (2.6) fails. Thus for every follower x appointed to P_e^C, $x \in \overline{C} = W_e$. Take $s \geqslant t_0$ so large that P_e^C does not receive attention at stage s and for every follower x ever appointed to P_e^C (there are only finitely many such x), $x \in W_e[s]$. Then $C \cap W_e[s] = \varnothing$ (since $\overline{C} = W_e$) so P_e^C requires attention at stage s, so P_e^C receives attention at stage s, a contradiction. Thus P_e^C is met. □

LEMMA 4. *For each e, N_e is met.*

PROOF. Suppose that $\Theta_e^0(A_0, C) = \Theta_e^1(A_1, C) = f$, f total. We show how to compute f recursively in C. Let $w = \liminf_s R(e)[s]$. Take t_0 such that

$$s \geqslant t_0 \rightarrow R(e)[s] \geqslant w, \tag{2.8}$$

and

$$s \geqslant t_0, n \leqslant e \rightarrow P_n \quad \text{does not receive attention at stage } s. \tag{2.9}$$

Let $S = \{s\colon R(e)[s] = w\}$, so S is infinite. We have $\lim_s l(e)[s] = \infty$, so there are infinitely many e-maximal stages in S. (For each n, let s_n be the least $s \in S$ such that $l(e)[s] \geqslant n$. Then s_n is e-maximal and as n runs over all integers, s_n takes on infinitely many values.) Given p, we compute $f(p)$ recursively in C as follows. Find a stage $s \geqslant t_0$ with $s \in S$, s is e-maximal, $l(e)[s] > p$, and, for some $i = 0$ or 1, $\Theta_e^i(A_i, C; p)[s+1]$ converges by a C-correct computation, i.e., C is correct through the use $\theta_e^i(A_i, C; p)[s+1]$. We claim that if $q = \Theta_e^i(A_i, C; p)[s+1]$, then $q = f(p)$. In fact we show that for all $t > s$, there is an $m < t$ and an $i = 0$ or 1 such that $m \in S$, m is e-maximal, $\Theta_e^i(A_i, C; p)[t] = q$ by a C-correct computation and $\theta_e^i(A_i, C; p)[t] \leqslant m$. For $t = s + 1$ we have the claim. Suppose that the claim holds for t. If the computation $\Theta_e^i(A_i, C; p)[t] = q$ is not injured at stage t, then the result holds for $t + 1$. Otherwise, since the computation is C-correct, a number $<$ $\theta_e^i(A_i, C; p)[t] \leqslant m$ enters A_i at stage t and destroys the C-correct computation. Since $t > s \geqslant t_0$, $R(e)[t] \geqslant w$. If $R(e)[t] > w$ or if $R(e)[t] = w$ and t is not e-maximal, then $r(e)[t] \geqslant m$, and, since $t \geqslant t_0$, no P_n with $n \leqslant e$ receives attention at stage t, so no number $< m$ could enter A_i at stage t. Thus $R(e)[t] = w$ (i.e., $t \in S$) and t is e-maximal. Hence $p < l(e)[s] < l(e)[t]$, so a computation $\Theta_e^{1-i}(A_{1-i}, C; p)[t]$ exists and equals $\Theta_e^i(A_i, C; p)[t] = q$. This computation is not injured at stage t, so exists at the end of stage t and we claim that it is C-correct. For the number x which enters A_i at stage t is a follower of some P_n. Any followers of $P_{n'}$ with $n' > n$ which exist at the beginning of stage t are canceled at stage t. Followers of P_n are not put into C. Any follower of a $P_{n'}$ with $n' < n$ which exists at the beginning of stage t is $< x$; but then such a follower cannot be later put into C since by induction hypothesis C is correct through x prior to stage t. Any followers appointed after stage t will be too large to injure the computation. Thus the $\Theta_e^{1-i}(A_{1-i}, C; p)[t+1]$ computation is C-correct, so the result holds for $t + 1$ with $1 - i$ in place of i and t the new e-maximal stage in S. This establishes the lemma. □

The theorem follows from the lemmas. Let $\mathbf{c} = \deg(C)$. Then $\mathbf{c} \leqslant \mathbf{d}$ by Lemma 1, $\mathbf{0} < \mathbf{c}$ by Lemma 3, $\mathbf{c}$ is the inf of $\deg(A_0 \oplus C)$, $\deg(A_1 \oplus C)$ by Lemma 4, and these latter two degrees are $> \mathbf{c}$ by Lemma 3. ■

3. A new branching degree construction. The delicate argument of Lemma 4 in the previous theorem does not combine well with the injuries which result from trying to code in a given incomplete r.e. degree into the branching degree **c**, even if the degree being coded in is low. We present a new branching degree construction which we will later use to show that there is a branching degree above any given low r.e. degree. In the Lachlan branching degree construction, the minimal pair type requirements N_e are met solely by negative restraint. (It is for this reason that they have traditionally been called N_e.) In our new construction, the N_e's are met solely by positive action.

In [**2**], Lachlan, in what has become known as the nondiamond theorem, showed that if two incomparable r.e. degrees have join $\mathbf{0}'$, then they do not have inf $\mathbf{0}$. Lachlan asked if two incomparable r.e. degrees which join to $\mathbf{0}'$ can have any degree as their inf. This question, known as the generalized nondiamond question, was answered affirmatively by Shoenfield and Soare [**9**] and, independently, Lachlan proved a result (the Lachlan Splitting Theorem [**4**]) which as a corollary also gives an affirmative answer to his earlier question. We introduce our new branching degree construction by using it to give a new proof of this answer to Lachlan's question.

We wish to construct r.e. sets A_0, A_1 and C to meet requirements P_e^i and N_e, for all e and for $i = 0$ or 1, given by (2.1) and (2.3). Let $\{R_n\}_{n\in\omega}$ be some effective ordering of the P_e^i's. We meet the N_e's as follows. When $\Theta_e^0(x) = \Theta_e^1(x)$ (we drop set arguments during this discussion), we put down a marker λ_x^e on some large number not yet in C and set Λ_x^e to be $\Theta_e^0(x)$. Then if later in the construction neither $\Theta_e^0(x)$ nor $\Theta_e^1(x)$ gives the answer Λ_x^e, we need to change our mind on Λ_x^e, so we put λ_x^e into C which frees us to later define a new value for λ_x^e and set Λ_x^e equal to the new common value $\Theta_e^0(x) = \Theta_e^1(x)$. If the $\Theta_e^0(x)$, $\Theta_e^1(x)$ values change again, we repeat the process. The N_e's will be met as long as they face a finite amount of restraint.

To meet P_e^i we try a Friedberg-Muchnik type argument, i.e., $R_n = P_e^i$ has a diagonalization witness x_n which is kept out of A_i until $\Psi_e(x_n) = 0$. When this happens, we want to put x_n into A_i and hold C on the use of the computation. We can let R_n cancel $N_{e'}$ if $e' \geq n$, i.e., remove $\lambda_x^{e'}$ markers which threaten to later go into C to destroy the computation, but R_n still has to deal with the $\lambda_x^{e'}$ markers with $e' < n$ which might harm the computation. The essential idea is that, although R_n cannot keep such a $\lambda_x^{e'}$ from entering C if it later wants to, R_n can do something to keep $\lambda_x^{e'}$ from ever wanting to go into C. The fact that $\lambda_x^{e'}$ has a value means that at least one of $\Theta_{e'}^0(x)$,$\Theta_{e'}^1(x)$ has a value. If R_n can restrain $A_0 \cup C$ or $A_1 \cup C$ (whichever is appropriate) on the use, then the computation $\Theta_{e'}^0(x)$ or $\Theta_{e'}^1(x)$ will not be destroyed and $\lambda_x^{e'}$ will never want to enter C. The problem is that we are only guaranteed convergence of one of $\Theta_{e'}^0(x)$, $\Theta_{e'}^1(x)$ and this convergence may be destroyed when R_n puts x_n into A_i to diagonalize, so R_n cannot keep $\lambda_x^{e'}$ from wanting to go into C. The solution is for R_n to make a guess about each $N_{e'}$ with $e' < n$. The guess is whether or not the apparent length of agreement between $\Theta_{e'}^0$ and $\Theta_{e'}^1$ will be unbounded. If not, then, if we require that new values of $\lambda_x^{e'}$ be put down only at "e'-maximal" stages, i.e., at stages at which there is a longer length of agreement between $\Theta_{e'}^0$ and $\Theta_{e'}^1$ than ever before, then R_n can simply ignore $\lambda_x^{e'}$

markers since after some stage such numbers stop going into C. If, on the other hand, there will be infinitely many e'-maximal stages, then R_n can wait for an e'-maximal stage before acting. At such a stage, both $\Theta^0_{e'}(x)$ and $\Theta^1_{e'}(x)$ converge for all x for which $\lambda^{e'}_x$ is defined. Thus R_n can put on restraint to protect the one of these computations which its attack will not destroy and hence keep $N_{e'}$ from wanting to put $\lambda^{e'}_x$ into C.

Thus R_n has 2^n strategies, one for each string of length n. If δ_n is the highest priority strategy of length n which looks correct infinitely often, then $\{\delta_n\}_{n\in\omega}$ is a path through $2^{<\omega}$, the δ_n strategy for R_n wins R_n, and δ_n acts only finitely often. The overall requirement R_n, through its 2^n strategies, may act infinitely often, but the strategies for R_n which act infinitely often will be of lower priority than the correct path through the tree of strategies, so will not interfere with strategy $\delta_{n'}$, even when n' is $> n$. This tree of strategies technique was begun by Lachlan in his "monster" paper [**3**]. Our tree of strategies is similar to that in [**4**].

At stage s we define inductively β_s in 2^s such that for each $n \leqslant s$, $\beta_s \restriction n$ is the strategy for R_n which looks correct at stage s. For $e < s$, we set $\beta_s(e) = 0$ iff s is an e-maximal stage, i.e., iff for all $t < s$, if $\beta_t \restriction e = \beta_s \restriction e$, then $l(e)[t] < l(e)[s]$ where $l(e)$ is as defined in the proof of Theorem 2.1. We want that if $\beta_s(e) = 0$ and λ^e_x is defined at stage s, then $x < l(e)[s]$, so both $\Theta^0_e(x)$ and $\Theta^1_e(x)$ are defined. But for some $t < s$ with $\beta_t \restriction e \neq \beta_s \restriction e$ we could have had $l(e)[t]$ very large so that λ^e_x could become defined for some $x > l(e)[s]$. To get around this problem, we let N_e build 2^e reductions Λ^α, one for each string α of length $e+1$ which ends in a 0; Λ^α is played at stage s only if $\alpha \subseteq \beta_s$. Then if s is e-maximal and $\lambda^{\beta_s \restriction e+1}_x$ is defined, we must have $x < l(e)[s]$ as we wanted. At stage s we may cancel any values of λ^α_x with $\beta_s \restriction e+1 < \alpha$, so these reductions will not bother us.

We now turn to the detailed construction.

THEOREM 3.1. *There are r.e. degrees* $\mathbf{c}, \mathbf{a}_0$ *and* $\mathbf{a}_1$ *such that* $\mathbf{a}_0$ *and* $\mathbf{a}_1$ *are incomparable*, $\mathbf{c}$ *is the* inf *of* $\mathbf{a}_0$ *and* $\mathbf{a}_1$, *and* $\mathbf{0}'$ *is the join of* $\mathbf{a}_0$ *and* $\mathbf{a}_1$.

PROOF. We construct r.e. sets C, A_0 and A_1 and set $\mathbf{c} = \deg(C)$, $\mathbf{a}_i = \deg(A_i \oplus C)$, $i = 0$ or 1. We wish to meet for all $e \in \omega$ and $i = 0$ or 1 requirements N_e and P^i_e given by (2.3) and (2.1). Let $\{R_n\}_{n\in\omega}$ be some effective ordering of the P^i_e's. For each string α of length n, there is a strategy for R_n. We will identify α with this strategy, so we will say "α requires (receives) attention" instead of "strategy α of R_n requires (receives) attention." If $\mathrm{lh}\alpha = n$ and R_n is P^i_e, we write A_α for A_i, Ψ_α for Ψ_e, and ψ_α for ψ_e. Strategy α for R_n tries to win R_n by diagonalization on a number x_α. The value of x_α may be canceled and redefined throughout the construction. For each α of length $e+1$ with $\alpha(e) = 0$, there are markers λ^α_x and values Λ^α_x, for each x, which may be assigned and canceled throughout the construction.

At any point in the construction we say that α *requires attention* if either

(3.1) $$x_\alpha \text{ is undefined,}$$

or

(3.2) $$x_\alpha \text{ is defined, } x_\alpha \notin A_\alpha, \text{ and } \Psi_\alpha(C; x_\alpha) = 0.$$

If, at some point in the construction, we have x_α defined with $x_\alpha \in A_\alpha$ and $\Psi_\alpha(C; x_\alpha)$ does not converge to 0, then x_α is canceled.

At any point in the construction we let, as in the proof of Theorem 2.1,

$$l(e) = \max\{x\colon (\forall y < x)[\Theta_e^0(A_0, C; y) = \Theta_e^1(A_1, C; y)]\}$$

and (say we are in stage s) we define $\beta \in 2^\omega$ by induction on e by

$$\beta(e) = 0 \leftrightarrow (\forall t)[(e \leqslant t < s) \wedge \beta_t \restriction e = \beta \restriction e \to l(e)[t, 2] < l(e)],$$

where β_t is an element of 2^t which is defined at stage t.

Let K be a complete r.e. set and let $\{k_s\}_{s \in \omega}$ be a one-one recursive enumeration of K. We now give the construction.

Stage s. Substage 1. For each α and x such that λ_x^α is defined, if neither $\Theta^0_{\mathrm{lh}\alpha - 1}(A_0, C; x)$ nor $\Theta^1_{\mathrm{lh}\alpha - 1}(A_1, C; x)$ converges to Λ_x^α, then enumerate λ_x^α into C and cancel λ_x^α and Λ_x^α. Repeat this process until for every λ_x^α which is defined, at least one of $\Theta^0_{\mathrm{lh}\alpha - 1}(A_0, C; x)$, $\Theta^1_{\mathrm{lh}\alpha - 1}(A_1, C; x)$ converges to Λ_x^α.

Let β_s be the current value of β, restricted to s.

Substage 2. See if for any $n \leqslant s$, $\beta_s \restriction n$ requires attention. If not, then substage 2 is over. If so, take n minimal and set $\alpha = \beta_s \restriction n$. Then α receives attention at stage s. If x_α is undefined, then let x_α be $2s + 1$ and substage 2 is over. Otherwise, enumerate x_α into A_α, enumerate a restraint of priority α equal to s, and cancel all $\lambda_x^{\alpha'}$, $\Lambda_x^{\alpha'}$, and $x_{\alpha'}$ with $\alpha < \alpha'$.

Substage 3. For each $e < s$ such that $\beta_s(e) = 0$, find the least $p < l(e)[s, 2]$, if any, such that $\Theta_e^0(A_0, C; p) = \Theta_e^1(A_1, C; p) = q$, say, and $\lambda_p^{\beta_s \restriction e+1}$ is undefined and then define $\lambda_p^{\beta_s \restriction e+1}$ to be a number larger than any used in the construction so far and define $\Lambda_p^{\beta_s \restriction e+1} = q$.

Substage 4. Find the least α, if any, such that $2 \cdot k_s$ is $<$ a restraint of priority α. If this α exists, put $2 \cdot k_s$ into A_α. Otherwise, put $2 \cdot k_s$ into A_0. □

LEMMA 1. $K \leqslant_T A_0 \oplus A_1$.

PROOF. We have $x \in K \leftrightarrow 2x \in A_0 \cup A_1$. □

For each $s \geqslant n$, $\beta_s \restriction n$ is a string of length n. Since there are only 2^n strings of length n, we can define δ_n to be the least string of length n which is equal to $\beta_s \restriction n$ for infinitely many $s \geqslant n$. Furthermore, for all n,

$$(3.3) \qquad \delta_n \subseteq \delta_{n+1}.$$

To see (3.3), let $\alpha = \delta_{n+1} \restriction n$. If $s \geqslant n + 1$ and $\beta_s \restriction n + 1 = \delta_{n+1}$, then $\beta_s \restriction n = \delta_{n+1} \restriction n = \alpha$. Since there are infinitely many $s \geqslant n + 1$ with $\beta_s \restriction n + 1 = \delta_{n+1}$, $\delta_n \leqslant \alpha$. Conversely, if $s \geqslant n + 1$ and $\beta_s \restriction n = \delta_n$, then for some $i = 0$ or 1, $\beta_s \restriction n + 1 = \delta_n * i$. Hence there is a fixed $i_0 = 0$ or 1 such that for infinitely many $s \geqslant n + 1$, $\beta_s \restriction n + 1 = \delta_n * i_0$. Thus $\alpha * \delta_{n+1}(n) = \delta_{n+1} \leqslant \delta_n * i_0$. It follows that $\alpha \leqslant \delta_n$, so $\alpha = \delta_n$, as desired.

LEMMA 2. *For each n, R_n is met and δ_n receives attention only finitely often.*

PROOF. Assume that the result holds for all $n' < n$. Suppose that at stage s some α with $\alpha < \delta_n$, $\alpha \subseteq \delta_n$ receives attention. Then, by (3.3), $\alpha = \delta_{n'}$ with $n' = \mathrm{lh}\alpha < n$, so by induction hypothesis there are only finitely many such s. Also, suppose that $s \geqslant n$ and at stage s some α with $\alpha < \delta_n$, $\alpha \not\subseteq \delta_n$ receives attention. Then, by construction, $\alpha \subseteq \beta_s$, so $\beta_s \upharpoonright n < \delta_n$; but by definition of δ_n there are only finitely many such s. Thus, only finitely often does a strategy of higher priority than δ_n receive attention, so there is only finitely much restraint of priority $< \delta_n$ put on during the construction. Let q be the largest such restraint.

Now also, if $s \geqslant n$ and at stage s a value is assigned to λ^α_m for some m and α with $\alpha < \delta_n$, $\alpha \not\subseteq \delta_n$, then $\alpha \subseteq \beta_s$, so $\beta_s \upharpoonright n < \delta_n$. Hence only finitely many numbers are ever assigned to be λ^α_m for some m and α with $\alpha < \delta_n$, $\alpha \not\subseteq \delta_n$.

Thus we may take $s_0 \geqslant n$ such that

(3.4) $\qquad \alpha < \delta_n, s \geqslant s_0 \to \alpha$ does not receive attention at stage s,

(3.5) $\qquad \alpha < \delta_n$, $\alpha \not\subseteq \delta_n \to$ if x is ever assigned to be λ^α_m for some m, and $x \in C$, then $x \in C[s_0]$,

and

(3.6) $$K \upharpoonright q[s_0] = K \upharpoonright q.$$

Let $A_{\delta_n} = A_i$. Then it follows from (3.6) that at no stage $s \geqslant s_0$ does $2 \cdot k_s$ enter A_{1-i} if $2 \cdot k_s$ is $<$ a restraint of priority δ_n which exists at substage 4 of stage s.

Suppose that at some stage $s \geqslant s_0$, δ_n receives attention and x_{δ_n} is put into $A_{\delta_n} = A_i$. Then $\Psi_{\delta_n}(C; x_{\delta_n})[s, 2] = 0$. We claim that this computation is correct. In fact we show that $A_{1-i} \cup C \upharpoonright s[s, 2] = A_{1-i} \cup C \upharpoonright s$ which suffices to show the claim. Suppose that w is the first number $< s$ which enters $A_{1-i} \cup C$ after substage 1 of stage s, say w enters $A_{1-i} \cup C$ at stage $t \geqslant s \geqslant s_0$. Then w is not $2 \cdot k_t$ since at substage 2 of stage s a restraint of priority δ_n equal to s is enumerated. Suppose that $w = x_\alpha$ for some α. When x_{δ_n} is put into A_i at stage s, we cancel x_γ for $\gamma > \delta_n$ and any later values of x_γ will be $> s$, so we cannot have $\alpha > \delta_n$. We rule out $\alpha = \delta_n$ since no values of x_{δ_n} are put into A_{1-i} and $\alpha < \delta_n$ is ruled out by (3.4) since x_α put into A_{1-i} at stage s' implies α receives attention at stage s'. Thus $w = x_\alpha$ is impossible. The only remaining possibility is $w = \lambda^\alpha_m$ for some α and m. When x_{δ_n} is put into A_i at stage s, we cancel λ^α_m for all $\alpha > \delta_n$ and any later values of λ^α_m will be $> s$, so $\alpha > \delta_n$ is impossible. We cannot have $\alpha < \delta_n$, $\alpha \not\subseteq \delta_n$ by (3.5), so we must have $\alpha \subseteq \delta_n$. Since $w < s$, λ^α_m must have been assigned to w before stage s, say at stage $v < s$ where we must have $\alpha \subseteq \beta_v$(so $\mathrm{lh}\alpha \leqslant v$) and $m < l(\mathrm{lh}\alpha - 1)[v, 2]$. Now $\alpha(\mathrm{lh}\alpha - 1) = 0$ and $\alpha \subseteq \delta_n \subseteq \beta_s$, so $\beta_s(\mathrm{lh}\alpha - 1) = \alpha(\mathrm{lh}\alpha - 1) = 0$, i.e., $\beta(\mathrm{lh}\alpha - 1)[s, 2] = 0$. We have $\mathrm{lh}\alpha - 1 \leqslant v < s$ and $\beta_v \upharpoonright \mathrm{lh}\alpha - 1 = \alpha \upharpoonright \mathrm{lh}\alpha - 1 = \beta_s \upharpoonright \mathrm{lh}\alpha - 1 = \beta \upharpoonright \mathrm{lh}\alpha - 1[s, 2]$, so by definition of $\beta(\mathrm{lh}\alpha - 1) = 0$ we must have $m < l(\mathrm{lh}\alpha - 1)[v, 2] < l(\mathrm{lh}\alpha - 1)[s, 2]$. Thus, just prior to substage 2 of stage s, both $\Theta^0_{\mathrm{lh}\alpha - 1}(A_0, C; m)$ and $\Theta^1_{\mathrm{lh}\alpha - 1}(A_1, C; m)$ converge to some number, say p. We must have $p = \Lambda^\alpha_m[s, 2]$, else at substage 1 of stage s, λ^α_m would be canceled. But when $w = \lambda^\alpha_m$ is put into C at stage t, $\Theta^{1-i}_{\mathrm{lh}\alpha - 1}(A_{1-i}, C; m)$ does not converge to p, so at some point after substage 1 of stage s and before w is put into C a number $< s$ has entered $A_{1-i} \cup C$ to destroy the computation $\Theta^{1-i}_{\mathrm{lh}\alpha - 1}(A_{1-i}, C; m)[s, 2]$. This contradicts choice of w. Thus $w = \lambda^\alpha_m$ is ruled out and the claim is proved. But then, since $s \geqslant s_0$ and $\Psi_{\delta_n}(C;\, x_{\delta_n})[s,2] = 0$ by a correct computation, x_{δ_n} is not canceled at a stage $\geqslant s$, so δ_n never requires attention at a stage $> s$, so δ_n receives attention only finitely often.

If the case of the preceding paragraph fails to hold, so that if δ_n receives attention at a stage $s \geqslant s_0$ then x_{δ_n} is not put into A_i at stage s, then we again have that δ_n receives attention only finitely often. For suppose that at some stage $t \geqslant s_0$, δ_n receives attention. Then x_{δ_n} is assigned a value at stage t and this value is not in A_i when x_{δ_n} is assigned to it, so, by our assumption, this value will never be put into A_i. But then, by (3.4), x_{δ_n} is never canceled from the value assigned to it at stage t, so no new value will ever be assigned to x_{δ_n}. Thus again δ_n receives attention only finitely often.

Finally, to see that R_n is met, since δ_n receives attention finitely often, x_{δ_n} is either eventually permanently defined or eventually permanently undefined. Suppose that the latter case holds. Then for some $s \geqslant s_0$ with $\delta_n \subseteq \beta_s$, x_{δ_n} is permanently undefined by stage s. But then δ_n requires attention, via (3.1), throughout stage s, so, by (3.4), δ_n receives attention at stage s and x_{δ_n} is assigned a value, a contradiction. Thus x_{δ_n} is eventually permanently defined. If the final value $x_{\delta_n}[\omega]$ is in A_i, then we cannot have $\Psi_{\delta_n}(C;\, x_{\delta_n}[\omega]) = 1$, else x_{δ_n} would be canceled and this value would not be the final value of x_{δ_n}, so R_n is won on $x_{\delta_n}[\omega]$. If $x_{\delta_n}[\omega]$ is not in A_i, then we cannot have $\Psi_{\delta_n}(C;\, x_{\delta_n}[\omega]) = 0$ else at some large enough stage $s \geqslant s_0$ with $\delta_n \subseteq \beta_s$, δ_n would require attention through (3.2) and $x_{\delta_n}[\omega]$ would be put into A_i, a contradiction, so again R_n is won on $x_{\delta_n}[\omega]$. □

LEMMA 3. *For all e, N_e is met.*

PROOF. Suppose that $\Theta_e^0(A_0, C) = \Theta_e^1(A_1, C) = f$, f total. Then we claim that $\delta_{e+1}(e) = 0$. To see this, let $S = \{s \geqslant e\colon\ \beta_s \restriction e = \delta_e\}$. Then S is infinite and $\lim_{s \in S} l(e)[s,2] = \infty$. Given n, let s_n be the least $s \in S$ with $l(e)[s,2] \geqslant n$. Then if $e \leqslant t < s_n$ and $\beta_t \restriction e = \beta \restriction e[s_n,2] = \beta_{s_n} \restriction e = \delta_e$, then $t \in S$, so by definition of s_n, $l(e)[t,2] < n \leqslant l(e)[s_n,2]$, and hence, by definition of β, $\beta(e)[s_n,2] = 0$. If $s_n > e$, then $\beta_{s_n}(e) = \beta(e)[s_n,2] = 0$, so $\beta_{s_n} \restriction e+1 = \delta_e * 0$. As n runs over all integers, s_n takes on infinitely many values, so by (3.3), $\delta_{e+1} = \delta_e * 0$ and $\delta_{e+1}(e) = 0$.

As in the proof of Lemma 2, there is a stage s_0 such that if $s \geqslant s_0$ and $\alpha < \delta_{e+1}$, then α does not receive attention at stage s. Thus if a value of $\lambda_m^{\delta_{e+1}}$ is canceled at a stage $\geqslant s_0$, then $\lambda_m^{\delta_{e+1}}$ is put into C (for the first time) at that stage. Thus if each $\lambda_m^{\delta_{e+1}}$ comes to a final value, these final values can be found recursively in C and then the corresponding final values of $\Lambda_m^{\delta_{e+1}}$ can be determined. Hence it only remains to show that for each m, $\lambda_m^{\delta_{e+1}}$ reaches a final value and that the corresponding final value of $\Lambda_m^{\delta_{e+1}}$ is $f(m)$. Suppose that this holds for all $m' < m$. Then take $s_1 \geqslant s_0$ such that

(3.7) $\quad m' < m \to \lambda_{m'}^{\delta_{e+1}}$ has obtained its final value prior to stage s_1,

and

(3.8) $\quad l(e)[s_1] > m$ and all the computations for Θ_e^0, Θ_e^1 on numbers $\leqslant m$ existing just before stage s_1 are correct.

If $\lambda_m^{\delta_{e+1}}$ is defined just prior to stage s_1 and $\Lambda_m^{\delta_{e+1}}[s_1] = f(m)$, then $\lambda_m^{\delta_{e+1}}$ will never be canceled (by (3.8) and the fact that $s_1 \geqslant s_0$), as desired. If $\lambda_m^{\delta_{e+1}}[s_1]$ is defined and $\Lambda_m^{\delta_{e+1}}[s_1] \neq f(m)$, then $\lambda_m^{\delta_{e+1}}$ is canceled at stage s_1. If $\lambda_m^{\delta_{e+1}}$ is ever assigned a value at a stage $\geqslant s_1$, then at this stage, by (3.8), $\Lambda_m^{\delta_{e+1}}$ is set equal to $f(m)$ and is never canceled from this value. Hence we will be done if we can show that if $\lambda_m^{\delta_{e+1}}$ is undefined at some point during a stage $s \geqslant s_1$, then it will later be defined. Let $s' > s, e + 1$ be such that $\beta_{s'} \upharpoonright e + 1 = \delta_{e+1}$. If $\lambda_m^{\delta_{e+1}}$ is not yet defined, then we have $\beta_{s'}(e) = 0$, $m < l(e)[s', 2]$ (by (3.8)) and m is the least m' such that $\lambda_{m'}^{\delta_{e+1}} = \lambda_{m'}^{\beta_{s'} \upharpoonright e+1}$ is undefined (by (3.7)). Thus $\lambda_m^{\delta_{e+1}}$ is defined at stage s', completing the proof. □

Now by Lemma 1, $\mathbf{a}_0 \cup \mathbf{a}_1 = \mathbf{0}'$; by Lemma 2, $\mathbf{a}_0$ and $\mathbf{a}_1$ are $> \mathbf{c}$; and by Lemma 3, $\mathbf{a}_0$ and $\mathbf{a}_1$ inf to $\mathbf{c}$, as desired. ■

4. The Robinson technique. We would like to carry out the construction of Theorem 3.1 above an arbitrary low r.e. degree. In **[5]**, Robinson introduced a technique for lifting a construction in the r.e. degrees above a given low r.e. degree. We discuss the technique and then illustrate its use by proving a special case of the Robinson Splitting Theorem.

Say D is a low r.e. set. Roughly speaking, Robinson discovered that there is an oracle procedure for answering questions of the form "is $\mathcal{D}_x \subseteq \overline{D}$?" The oracle procedure may give a false positive answer, but never a false negative answer. Under certain conditions, the number of false positive answers is manageable. A typical situation is where we have a strategy for requirement R_n which from time to time sees an apparent computation $\Phi_e(A, D; x)$, where A is a set under construction, and would like to make an attack if the computation is correct. The strategy can take steps to ensure that the computation is A-correct, but needs the oracle procedure to know whether $\mathcal{D}_x \subseteq \overline{D}$ where $\mathcal{D}_x$ consists of those numbers $<$ the use in the computation which have not yet appeared in D. If the oracle procedure gives a positive answer, an attack is made. The positive answer may turn out to be false, or a higher priority strategy may force a number into A destroying the computation even though the oracle procedure's positive answer was correct. If we ensure that the latter case can happen only finitely often, then the former case can happen only finitely often, and R_n is met. The result which gives us this oracle procedure is contained in the following theorem.

THEOREM 4.1. *If D is an r.e. set, then D is low iff there is a recursive function f such that for all j,*

$$W_j \cap \{x : \mathcal{D}_x \subseteq \overline{D}\} = W_{f(j)} \cap \{x : \mathcal{D}_x \subseteq \overline{D}\}, \tag{4.1}$$

and

$$W_j \cap \{x : \mathcal{D}_x \subseteq \overline{D}\} \text{ finite} \rightarrow W_{f(j)} \text{ finite}. \tag{4.2}$$

PROOF. This result is similar to Theorem 2.7 of **[11]**. It can be obtained by combining these two results from **[11]**: the characterization of low r.e. sets given just prior to Theorem 2.7 and Theorem 2.6. ■

The theorem is used to give the oracle procedure in the following way. If R_n wants to know whether or not $\mathfrak{D}_x \subseteq \overline{D}$, then x is enumerated into a set W_{j_n} (the recursion theorem allows us to assume that for each n we know the r.e. index for the set of characteristic indices about which R_n asks) and we look ahead in the enumeration of D to see if a number in $\mathfrak{D}_x$ later appears in D while simultaneously looking for x in $W_{f(j_n)}$. By (4.1) one of these two searches must succeed. If the latter one succeeds first, R_n makes an attack on the assumption that the computation is correct. Equation (4.2) implies that if R_n asks about only finitely many $\mathfrak{D}_x$'s with $\mathfrak{D}_x \subseteq \overline{D}$, then the oracle procedure can give only finitely many different false positive answers.

We illustrate this technique by proving a result which is a weak form of our main theorem, Theorem 5.1, and is also a special case of the Robinson Splitting Theorem [**5**, Corollary 9], one of the original results obtained with the Robinson technique. In this proof we show explicitly how the recursion theorem is used. In later proofs we will use the recursion theorem implicitly and not give details.

THEOREM 4.2. *Let* $\mathbf{d}$ *be a low r.e. degree. Then there are incomparable r.e. degrees* $\mathbf{a}_0$ *and* $\mathbf{a}_1$, *both* $\geqslant \mathbf{d}$, *such that* $\mathbf{a}_0 \cup \mathbf{a}_1 = \mathbf{0}'$.

PROOF. Let D be an r.e. set in $\mathbf{d}$. We construct r.e sets A_0 and A_1 and set $\mathbf{a}_i = \deg(A_i \oplus D)$. For each $e \in \omega$ and $i = 0$ or 1 we have the requirement

$$R_e^i: \Phi_e(A_{1-i}, D) \neq A_i. \tag{4.3}$$

Let $\{R_n\}_{n\in\omega}$ be an effective listing of all the R_e^i's. We attempt to meet R_n by diagonalization on a number x_n. The value assigned to x_n may be defined, canceled, and redefined throughout the construction.

During stage s we say that $R_n = R_e^i$ *requires attention* if either

$$x_n \text{ is undefined,} \tag{4.4}$$

or

$$x_n \text{ is defined, } x_n \notin A_i, \text{ and } \Phi_e(A_{1-i}, D; x_n) = 0. \tag{4.5}$$

If at some point in the construction $x_n \in A_i$ and $\Phi_e(A_{1-i}, D; x_n)$ does not converge to 0, then x_n is canceled.

Since D is low, there is a recursive function f satisfying (4.1) and (4.2). We give a construction which depends on a parameter r. In the rth construction, we construct a set of Q_r which consists of those numbers $\langle n, x\rangle$ where R_n asks about $\mathfrak{D}_x$. In the rth construction, we proceed as if $Q_r = W_r$. By the recursion theorem, for some construction, say the r_0th, this assumption will be true, and it is the r_0th construction which works. Let $\{h_n\}_{n\in\omega}$ be a uniformly recursive sequence of recursive functions such that for all x, $W_{h_n(x)} = \{y: \langle n, y\rangle \in W_x\}$.

Let K be an r.e. set of degree $\mathbf{0}'$ and let $\{k_s\}_{s\in\omega}$ be a one-one recursive enumeration of K. We now give the construction.

Stage s. Substage 1. For each n, $n \leqslant s$, such that R_n requires attention, starting with the least such and continuing in increasing order until either some R_n receives attention or else no more n's are left, proceed as follows. Let R_n be R_e^i. If (4.4) holds for n, then appoint $2s+1$ to be x_n and R_n receives attention. Otherwise, (4.5) holds. Let $\mathfrak{D}_x = \overline{D} \upharpoonright \varphi_e(A_{1-i}, D; x_n)$ and put $\langle n, x\rangle$ into Q_r. Begin looking ahead in the enumeration of D for an element of $\mathfrak{D}_x$ to appear in D and at the same time search for x in $W_{fh_n(r)}$. (If neither search terminates, the construction peters out here.) If the latter search terminates first, enumerate x_n into A_i, enumerate a restraint of priority n equal to s and cancel all $x_{n'}$ with $n' > n$; R_n receives attention. If the former search terminates first, R_n does not receive attention; go on to the next n, if any.

Substage 2. Find the least n such that $2 \cdot k_s$ is $<$ a restraint of priority n. If n exists and $R_n = R^i_e$, put $2 \cdot k_s$ into A_i. Otherwise, put $2 \cdot k_s$ into A_0. □

The instructions for this construction depend uniformly on r. Hence there is a recursive function g such that $Q_r = W_{g(r)}$ for all r. By the recursion theorem, there is an r_0 with $W_{g(r_0)} = W_{r_0}$, so $Q_{r_0} = W_{r_0}$. In the r_0th construction, if $\langle n, x\rangle$ is put into $Q_{r_0} = W_{r_0}$ (so $x \in W_{h_n(r_0)}$), then either $\mathfrak{D}_x \not\subseteq \overline{D}$ or else

$$x \in W_{h_n(r_0)} \cap \{x\colon \mathfrak{D}_x \subseteq \overline{D}\} = W_{fh_n(r_0)} \cap \{x\colon \mathfrak{D}_x \subseteq \overline{D}\},$$

so $x \in W_{fh_n(r_0)}$. Thus the r_0th construction never peters out. From now on we refer only to the r_0th construction.

LEMMA 1. $K \leqslant_T A_0 \oplus A_1$.

PROOF. We have $x \in K \leftrightarrow 2x \in A_0 \cup A_1$. □

LEMMA 2. *For all n, R_n receives attention only finitely often and is met.*

PROOF. Suppose that the result holds for all $n' < n$. Then there is only finitely much restraint of priority $< n$ put on during the construction. Let q be the largest such restraint. Let s_0 be such that

(4.6) $$s \geqslant s_0, \quad n' < n \rightarrow R_{n'} \text{ does not receive attention at stage } s,$$

and

(4.7) $$K \upharpoonright q[s_0] = K \upharpoonright q.$$

Let R_n be R^i_e. If x_n is put into A_i at a stage $s \geqslant s_0$, then all values of $x_{n'}$ with $n' > n$ are canceled and a restraint of priority n equal to s is enumerated, so by (4.6) and (4.7), $A_{1-i} \upharpoonright s[s] = A_{1-i} \upharpoonright s$.

Now suppose that at some stage $s \geqslant s_0$ a number $\langle n, x\rangle$ is put into $Q_{r_0} = W_{r_0}$ and $\mathfrak{D}_x \subseteq \overline{D}$. Then x_n is put into A_i at stage s and the computation $\Phi_e(A_{1-i}, D; x_n) = 0$ which exists at stage s is correct. Hence, by (4.6), x_n is never later canceled and R_n never later requires attention, so no further numbers $\langle n, x'\rangle$ are put into W_{r_0}. Thus $W_{h_n(r_0)} \cap \{x\colon \mathfrak{D}_x \subseteq \overline{D}\}$ is finite, so, by (4.2), $W_{fh_n(r_0)}$ is finite.

Suppose that x_n is put into A_i at stages s_1 and s_2 with $s_2 > s_1 \geqslant s_0$. Then, for $j = 1$ or 2, at stage s_j, $\langle n, y_j\rangle$ is put into W_{r_0} where $\mathfrak{D}_{y_j} \subseteq \overline{D}[s_j, 1]$ and $y_j \in W_{fh_n(r_0)}$. When x_n is put into A_i at stage s_1, there is a computation $\Phi_e(A_{1-i}, D; x_n)[s_1, 1] = 0$. As long as this computation remains, x_n is not canceled and R_n does not receive attention. Since the computation is A_{1-i}-correct, before or at stage s_2 a number in $\mathfrak{D}_{y_1}$ must appear in D to destroy the computation. Since $\mathfrak{D}_{y_2} \subseteq \overline{D}[s_2, 1]$, $y_2 \neq y_1$. Since $W_{fh_n(r_0)}$ is finite, it follows that for some $t_0 \geqslant s_0$, x_n is not put into A_i at any stage $t \geqslant t_0$. Now if R_n receives attention at a stage $t \geqslant t_0$, (4.4) must hold, so x_n is appointed a new value. This new value is not in A_i and will never be put in, so by (4.6) x_n will never be canceled from this value, so R_n never later receives attention. Hence R_n receives attention only finitely often.

If x_n is undefined just before some stage $s \geqslant s_0$, n, then R_n requires attention, so receives it and x_n is assigned a value. Thus x_n has a final value. If the final value of x_n is in A_i, then we cannot have $\Phi_e(A_{1-i}, D; x_n[\omega]) = 1$, else x_n would be canceled and $x_n[\omega]$ would not be the final value, so R_n is won on $x_n[\omega]$. If $x_n[\omega] \notin A_i$, then we cannot have $\Phi_e(A_{1-i}, D; x_n[\omega]) = 0$ else at some large enough stage $s \geqslant s_0$, R_n would require attention via (4.5) and the number $\langle n, x \rangle$ put into W_{r_0} would have $\mathfrak{D}_x \subseteq \bar{D}$, since s is so large that the true computation holds at stage s, so $x_n[\omega]$ is put into A_i, a contradiction. Thus again R_n is won on $x_n[\omega]$. □

The theorem follows from the lemmas. Let $\mathbf{a}_i = \deg(A_i \oplus D)$ for $i = 0$ or 1. Then $\mathbf{d} \leqslant \mathbf{a}_0, \mathbf{a}_1$. By Lemma 1, $\mathbf{0}' \leqslant \mathbf{a}_0 \cup \mathbf{a}_1$ and by Lemma 2, $\mathbf{a}_0$ and $\mathbf{a}_1$ are incomparable. ■

5. Branching degrees above low degrees. In this section we use the technique of Robinson described in §4 to show that the construction of Theorem 3.1 can be carried out above a given low r.e. degree. We thereby simultaneously show that there is a branching degree above any given low r.e. degree and strengthen the positive answer to the generalized nondiamond question given by Shoenfield and Soare, and Lachlan.

Let $\mathbf{d}$ be a low r.e. degree and D be an r.e. set in $\mathbf{d}$. We wish to show that there are r.e. degrees $\mathbf{c}, \mathbf{a}_0$, and $\mathbf{a}_1$ with $\mathbf{d} \leqslant \mathbf{c}$, $\mathbf{a}_0$ and $\mathbf{a}_1$ incomparable, $\mathbf{c} = \mathbf{a}_0 \cap \mathbf{a}_1$ and $\mathbf{0}' = \mathbf{a}_0 \cup \mathbf{a}_1$. We will construct r.e. sets C, A_0 and A_1 to meet, for all $e \in \omega$ and $i = 0$ or 1, requirements

(5.1) $$N_e\colon \Theta_e^0(A_0, C, D) = \Theta_e^1(A_1, C, D) = f,\ f \text{ total} \rightarrow f \leqslant_T C,$$

and

(5.2) $$R_e^i\colon \Phi_e(C, D) \neq A_i,$$

where $\langle \Theta_e^0, \Theta_e^1 \rangle$ is an effective enumeration of all pairs of partial recursive functionals of three set variables. We will then set

$$\mathbf{c} = \deg(C \oplus D) \quad \text{and} \quad \mathbf{a}_i = \deg(A_i \oplus C \oplus D).$$

Let $\{R_n\}_{n\in\omega}$ be an effective listing of the R_e^i's.

We meet the N_e requirements as in the proof of Theorem 3.1, i.e., N_e works on 2^e reductions Λ^α, one for each string α with $\mathrm{lh}\alpha = e + 1$ and $\alpha(e) = 0$. Also as in the proof of Theorem 3.1 each R_n has 2^n strategies, one for each string α with $\mathrm{lh}\alpha = n$. A strategy α attempts to win $R_{\mathrm{lh}\alpha}$ by diagonalizing on a witness x_α. In the proof of Theorem 4.2, when R_n sees a computation on its witness x_n against which it wishes to diagonalize, it uses its oracle procedure to ask whether D is correct through the use in the computation. If the oracle procedure gives a positive answer an attack is made; if the answer is negative, the computation can be ignored because it is not correct. In the construction under discussion we define a set $\hat{D}$ which is a speeded-up enumeration of D. If strategy α sees at stage s a computation $\Phi_e(C, \hat{D}; x_\alpha) = 0$

against which it wishes to diagonalize (say $R_{\text{lh}\alpha}$ is R_e^i), it is not sufficient to ask the oracle procedure if $\hat{D}$ is correct through the use in this computation. For there may be a marker λ_m^γ, with $\gamma \subseteq \alpha$, whose value is less than the use $\varphi_e(C, \hat{D}; x_\alpha)$, but with $\theta_{\text{lh}\gamma-1}^{1-i}(A_{1-i}, C, \hat{D}; m)$ very large. Then, even though $\hat{D}$ is correct on the use $\varphi_e(C, \hat{D}; x_\alpha)$, at some later stage a number $< \theta_{\text{lh}\gamma-1}^{1-i}(A_{1-i}, C, \hat{D}; m)$ but $\geqslant \varphi_e(C, \hat{D}; x_\alpha)$ may enter $\hat{D}$, thereby destroying the computation $\Theta_{\text{lh}\gamma-1}^{1-i}(A_{1-i}, C, \hat{D}; m)$. Then strategy γ may have to put λ_m^γ into C and destroy the computation against which α is trying to diagonalize. Thus when α wishes to diagonalize at stage s, it asks the oracle procedure whether $\hat{D}$ is correct through s. If the oracle procedure gives a positive answer, then an attack is made, while if the answer is negative, then a number $< s$ has entered $\hat{D}$ and we repeat the process, using whatever strategies now look correct.

We now give the complete proof.

THEOREM 5.1. *If* $\mathbf{d}$ *is a low r.e. degree, then there are r.e. degrees* $\mathbf{c}, \mathbf{a}_0$, *and* $\mathbf{a}_1$ *such that* $\mathbf{d} \leqslant \mathbf{c}$, $\mathbf{a}_0$ *and* $\mathbf{a}_1$ *are incomparable*, $\mathbf{a}_0 \cap \mathbf{a}_1 = \mathbf{c}$, *and* $\mathbf{0}' = \mathbf{a}_0 \cup \mathbf{a}_1$.

PROOF. Let D be an r.e. set in $\mathbf{d}$. We construct r.e. sets C, A_0, and A_1. For each $e \in \omega$ and $i = 0$ or 1 we wish to meet N_e and R_e^i as given by (5.1) and (5.2). Let $\{R_n\}_{n\in\omega}$ be an effective listing of the R_e^i's. If $R_{\text{lh}\alpha}$ is R_e^i, we write Φ_α for Φ_e, φ_α for φ_e and A_α for A_i. We have x_α, Λ_m^α, λ_m^α as discussed previously.

As usual, we assume that we are given some standard effective stage-by-stage enumeration of D. In the course of the construction we define a set $\hat{D}$. The numbers put into $\hat{D}$ are exactly those numbers which are in D, put in in the same order as they appear in the standard enumeration of D. Substage 4 of the construction ensures that "in the limit," i.e., at the end of the construction, $\hat{D}$ equals D, but, in general, at a given point in the construction, the numbers which we have put into $\hat{D}$ (by looking ahead in the enumeration of D) will not appear in D until a much later stage. Thus $\hat{D}$ is the result of a "speeded-up" enumeration of D.

In our construction, substage 1 of stage s will be divided into a varying number of subsubstages, each of which has two parts a and b. We write substage $1.t$.a or $1.t$.b for parts a or b of subsubstage t of substage 1. We also let u_s be the last subsubstage of substage 1 of stage s.

During stage s of the construction we define

$$l(e) = \max\{x\colon (\forall y < x)[\Theta_e^0(A_0, C, \hat{D}; y) = \Theta_e^1(A_1, C, \hat{D}; y)]\},$$

and we define inductively $\beta \in 2^\omega$ by

$$\beta(e) = 0 \leftrightarrow (\forall t)[(e \leqslant t < s \wedge \beta_t \restriction e = \beta \restriction e) \to l(e)[t, 1.u_t.\text{b}] < l(e)],$$

where β_t is a string of length t defined at stage t.

At any point in the construction we say that α *requires attention for* $R_{\text{lh}\alpha}$ if either

(5.3) $$x_\alpha \text{ is undefined,}$$

or

(5.4) $$x_\alpha \text{ is defined, } x_\alpha \notin A_\alpha \text{ and } \Phi_\alpha(C, \hat{D}; x_\alpha) = 0.$$

If at some point in the construction we have x_α defined with $x_\alpha \in A_\alpha$ and $\Phi_\alpha(C, \hat{D}; x_\alpha)$ does not converge to 0, we cancel x_α.

Since D is low, there is a recursive function f satisfying (4.1) and (4.2). From time to time during the construction a strategy α for $R_{\text{lh}\alpha}$ wishes to know if $\mathfrak{D}_x \subseteq \overline{D}$, so enumerates x into a set. We assume, by the recursion theorem, that for each α we know the index j_α such that W_{j_α} equals the set of canonical indices about which α asks.

Let K be an r.e. set of degree $\mathbf{0}'$ and let $\{k_s\}_{s\in\omega}$ be a one-one recursive enumeration of K.

We now present the construction.

Stage s. Substage 1. Set $t = 0$.

Substage 1.t.a. For each α and n such that λ_n^α is defined, if neither $\Theta^0_{\text{lh}\alpha-1}(A_0, C, \hat{D}; n)$ nor $\Theta^1_{\text{lh}\alpha-1}(A_1, C, \hat{D}; n)$ converges to Λ_n^α, then enumerate λ_n^α into C and cancel λ_n^α and Λ_n^α. Repeat this process until for every λ_n^α which is defined at least one of $\Theta^0_{\text{lh}\alpha-1}(A_0, C, \hat{D}; n)$ and $\Theta^1_{\text{lh}\alpha-1}(A_1, C, \hat{D}; n)$ converges to Λ_n^α.

Substage 1.t.b. Let $\beta_{s,t}$ be the current value of β restricted to s. See if, for any $n \leqslant s$, $\beta_{s,t} \upharpoonright n$ requires attention for R_n. If not, then substage 1 is over. If so, let n be the least such and set $\alpha = \beta_{s,t} \upharpoonright n$. If x_α is undefined, let x_α be $2s + 1$; then α receives attention at stage s for R_n and substage 1 is over. Otherwise, let $\mathfrak{D}_x = \overline{\hat{D}} \upharpoonright s$, enumerate x into W_{j_α}, and then simultaneously enumerate new elements from D into $\hat{D}$ and search for x in $W_{f(j_\alpha)}$. Either an element of $\mathfrak{D}_x$ will appear in $\hat{D}$ or else $\mathfrak{D}_x \subseteq \overline{D}$, so x will appear in $W_{f(j_\alpha)}$. Stop the searches when one of these two events occurs. If the former occurs first, increase t by 1 and begin subsubstage t. Otherwise, enumerate x_α into A_α, cancel all $\lambda_m^{\alpha'}$, $\Lambda_m^{\alpha'}$ and $x_{\alpha'}$ with $\alpha < \alpha'$, and enumerate a restraint of priority α equal to s; α receives attention at stage s and substage 1 is over.

Let u_s be the final subsubstage of substage 1, and let $\beta_s = \beta_{s,u_s}$.

Substage 2. For each $e < s$ such that $\beta_s(e) = 0$, find the least $p < l(e)[s, 1.u_s.\text{b}]$, if any, such that $\Theta^0_e(A_0, C, \hat{D}; p) = \Theta^1_e(A_1, C, \hat{D}; p) = q$, say, and $\lambda_p^{\beta_s\upharpoonright e+1}$ is undefined and then define $\lambda_p^{\beta_s\upharpoonright e+1}$ to be a number larger than any used in the construction so far and define $\Lambda_p^{\beta_s\upharpoonright e+1} = q$.

Substage 3. Find the least α, if any, such that $2 \cdot k_s$ is $<$ a restraint of priority α. If this α exists, put $2 \cdot k_s$ into A_α. Otherwise, put $2 \cdot k_s$ into A_0.

Substage 4. Enumerate a new element from D into $\hat{D}$. □

Note that if t is not the final subsubstage of substage 1 of stage s, then during substage 1.t.b of stage s a number $< s$ is enumerated into $\hat{D}$. Thus for each s, there are only finitely many subsubstages in substage 1 of stage s.

LEMMA 1. $K \leqslant_T A_0 \oplus A_1$.

PROOF. We have $x \in K \leftrightarrow 2x \in A_0 \cup A_1$. □

For each n, let δ_n be the least string of length n which is equal to $\beta_s \upharpoonright n$ for infinitely many $s \geqslant n$. Then for all n, $\delta_n \subseteq \delta_{n+1}$, as in the proof of Theorem 3.1.

LEMMA 2. *For all* n, R_n *is met and* δ_n *receives attention only finitely often.*

Proof. Suppose that the result holds for $n' < n$. Let q be the largest restraint of priority $< \delta_n$ put on during the construction. Take $s_0 \geqslant n$ so large that (3.4), (3.5), and (3.6) are satisfied. Let A_{δ_n} be A_i. Then, by (3.6), at no stage $s \geqslant s_0$ does $2 \cdot k_s$ enter A_{1-i} if $2 \cdot k_s$ is less than a restraint of priority δ_n which exists at substage 3 of stage s. Also

(5.5) if $s \geqslant s_0$ and $\delta_n \subseteq \beta_s$, then at substage $1.u_s$.b either δ_n does not require attention for R_n or else δ_n receives attention for R_n.

To see (5.5), note that at substage $1.u_s$.b either no initial segment of $\beta_{s,u_s} = \beta_s$ requires attention, so δ_n does not require attention, or else some initial segment α of β_s receives attention. In the latter case, if $\alpha \subset \delta_n$, then $\alpha = \delta_{n'}$, $n' = \text{lh}\alpha < n$, and this contradicts (3.4). If $\alpha = \delta_n$, then (5.5) holds. If $\delta_n \subset \alpha$, then we must have δ_n does not require attention at substage $1.u_s$.b (else α would not receive attention), so again (5.5) holds.

Now suppose that at stage $s \geqslant s_0$, x_{δ_n} is put into $A_{\delta_n} = A_i$ at substage $1.u_s$.b. Then we claim that the first change in $A_{1-i} \cup C \cup \hat{D}$ below s at or after substage $1.u_s$.b of stage s is a change in $\hat{D}$. The proof is similar to the proof of the corresponding claim in Lemma 2 of Theorem 3.1. Let w be the first number $< s$ to enter $A_{1-i} \cup C \cup \hat{D}$ at or after substage $1.u_s$.b of stage s, say w enters at stage $t \geqslant s$. We can rule out $w = 2 \cdot k_t$ since a restraint of priority δ_n equal to s is enumerated at substage $1.u_s$.b of stage s. By cancellation and (3.4), $w = x_\alpha$ for some α is ruled out. If $w = \lambda^\alpha_m$, then by (3.5) and cancellation we have $\alpha \subseteq \delta_n \subseteq \beta_{s,u_s}$, so $\beta_{s,u_s}(\text{lh}\alpha - 1) = 0$. Since w equalled λ^α_m before stage s, $m < l(\text{lh}\alpha - 1)[s, 1.u_s.\text{b}]$, so both $\Theta^0_{\text{lh}\alpha-1}(A_0, C, \hat{D}; m)[s, 1.u_s.\text{b}]$ and $\Theta^1_{\text{lh}\alpha-1}(A_1, C, \hat{D}; m)[s, 1.u_s.\text{b}]$ converge to a common value, z say, and $\Lambda^\alpha_m[s, 1.u_s.\text{b}] = z$, so before w is put into C at stage t, a number $< \theta^{1-i}_{\text{lh}\alpha-1}(A_{1-i}, C, \hat{D}; m)[s, 1.u_s.\text{b}] < s$ must enter $A_{1-i} \cup C \cup \hat{D}$. But then w was not the first number $< s$ to enter $A_{1-i} \cup C \cup \hat{D}$ at or after substage $1.u_s$.b of stage s. Thus w must enter $\hat{D}$.

Next, suppose that at stage $s \geqslant s_0$ a number x is put into $W_{j_{\delta_n}}$ and $\mathfrak{D}_x \subseteq \bar{D}$. Then x_{δ_n} is put into A_i at stage s and by the preceding paragraph and the fact that $\mathfrak{D}_x \subseteq \bar{D}$, the computation $\Phi_{\delta_n}(C, \hat{D}; x_{\delta_n})[s, 1.u_s.\text{b}] = 0$ is correct. Hence x_{δ_n} is never canceled and δ_n never later requires attention, so no further numbers are put into $W_{j_{\delta_n}}$. Thus $W_{j_{\delta_n}} \cap \{x: \mathfrak{D}_x \subseteq \bar{D}\}$ is finite, so by (4.2), $W_{f(j_{\delta_n})}$ is finite.

Suppose that x_{δ_n} is put into A_i at stages s_1 and s_2 with $s_2 > s_1 \geqslant s_0$. Then for $r = 1$ or 2, at stage s_r, y_r is put into $W_{j_{\delta_n}}$ where $\mathfrak{D}_{y_r} \subseteq \hat{D}[s_r, 1.u_{s_r}.\text{b}]$ and $y_r \in W_{f(j_{\delta_n})}$. After x_{δ_n} is put into A_{δ_n} at stage s_1, as long as no element of $\mathfrak{D}_{y_1}$ appears in $\hat{D}$, δ_n does not require attention for R_n, so $\mathfrak{D}_{y_1} \not\subseteq \hat{D}[s_2, 1.u_{s_2}.\text{b}]$ and $y_1 \neq y_2$. Since $W_{f(j_{\delta_n})}$ is finite, it follows that for some $t_0 \geqslant s_0$, x_{δ_n} is not put into A_i at a stage $\geqslant t_0$. Hence if δ_n receives attention for R_n at a stage $\geqslant t_0$, a value is appointed to x_{δ_n} at that stage. This value is not in A_i and will never be put in, so by (3.4), x_{δ_n} is never canceled from this value, so δ_n never later receives attention. Thus δ_n receives attention only finitely often.

To see that R_n is met, note that if x_{δ_n} were eventually permanently undefined, then δ_n would require attention through (5.3) cofinitely through the construction. Since there are infinitely many s with $\beta_s \restriction n = \delta_n$, this would contradict (5.5), so x_{δ_n} has a final value. If this final value is in A_i, then we cannot have $\Phi_{\delta_n}(C, D; x_{\delta_n}[\omega]) = 1$, else x_{δ_n} would be canceled from its final value, so R_n is won on $x_{\delta_n}[\omega]$. If this final value is not in A_i, then we cannot have $\Phi_{\delta_n}(C, D; x_{\delta_n}[\omega]) = 0$, else at some large enough stage s with $\delta_n \subseteq \beta_s$, δ_n would require attention throughout stage s by (5.4), again contradicting (5.5). Thus R_n is won on $x_{\delta_n}[\omega]$. □

LEMMA 3. *For all* e, N_e *is met.*

PROOF. The proof is virtually the same as the proof of Lemma 3 of Theorem 3.1. □

The theorem follows from the lemmas, for if

$$\mathbf{c} = \deg(C \oplus D) \quad \text{and} \quad \mathbf{a}_i = \deg(A_i \oplus C \oplus D)$$

for $i = 0$ or 1, then by Lemma 1, $\mathbf{0}' = \mathbf{a}_0 \cup \mathbf{a}_1$, by Lemma 2, $\mathbf{a}_0$ and $\mathbf{a}_1$ are $> \mathbf{c}$, and by Lemma 3, $\mathbf{a}_0 \cap \mathbf{a}_1 = \mathbf{c}$. ■

The "diamond lattice" is the four-element lattice with two incomparable elements. As immediate corollaries of Theorem 5.1 we have

COROLLARY 5.2. *If* $\mathbf{d}$ *is a low r.e. degree, then there is a branching degree* $\mathbf{c}$ *with* $\mathbf{d} \leqslant \mathbf{c}$. ■

COROLLARY 5.3. *The diamond lattice can be embedded into the r.e. degrees above any given low r.e. degree, with greatest element preserved.* ■

Corollary 5.3 extends the result of Shoenfield and Soare, and Lachlan, giving a positive answer to the generalized nondiamond question.

We would now like to strengthen Theorem 5.1 by adding avoiding a cone, i.e., in the notation of that theorem, if $\mathbf{b}$ is a degree with $\mathbf{b} \nleqslant \mathbf{d}$, then we would like to add to the conclusion that $\mathbf{b} \nleqslant \mathbf{c}$. With this strengthened theorem we can deduce that the branching degrees form an automorphism base for the r.e. degrees and that, given an incomplete, nonzero r.e. degree, there is a branching degree incomparable with the given degree. If $\mathbf{b} \nleqslant \mathbf{0}'$, then $\mathbf{b} \nleqslant \mathbf{c}$ is automatic, so we may take $\mathbf{b} \leqslant \mathbf{0}'$. Let $B \in \mathbf{b}$. Then by Shoenfield's Limit Lemma [**8**, p. 29], there is a recursive sequence of finite sets $\{B_s\}_{s \in \omega}$ which converges to B. When we refer to B during stage s, we mean B_s. The usual avoiding the cone technique is that of Sacks negative restraint. For each e we have the requirement

$$S_e\colon \Phi_e(C, D) \neq B. \tag{5.6}$$

For the Sacks negative restraint technique to work, B must not be recursive in the injury set to S_e. The usual way to ensure this would be for the set put into C for sake of any given N_e to be recursive in D. However, C itself can be recursive in the set put into C for sake of a single N_e and in general $D <_T C$. Hence the Sacks strategy cannot be used without modification; nevertheless, the basic idea behind our strategy is the same as that of the Sacks strategy.

Each S_e has 2^e strategies, one for each string α of length e. Strategy α is played when α looks like the correct guess; if $\text{lh}\alpha = e$ and y is $<$ the length of agreement between B and $\Phi_e(C, \hat{D})$ at stage s, then strategy α wants to verify the computation $\Phi_e(C, \hat{D}; y)$ by asking the oracle procedure if $\hat{D}$ is correct through s. If a positive answer is given, strategy α, through cancellation and restraint ensures that the apparent computation $\Phi_e(C, \hat{D}; y)$ is correct unless the oracle's answer is false. If S_e fails, then for every y a $\hat{D}$-correct computation $\Phi_e(C, \hat{D}; y)$ becomes verified by δ_e and then $B \leqslant_T D$, a contradiction. Here are the details.

THEOREM 5.4. *If* $\mathbf{d}$ *is a low r.e. degree and* $\mathbf{b} \nleqslant \mathbf{d}$, *then there are r.e. degrees* $\mathbf{c}, \mathbf{a}_0$, *and* $\mathbf{a}_1$ *with* $\mathbf{d} \leqslant \mathbf{c}$, $\mathbf{b} \nleqslant \mathbf{c}$, $\mathbf{a}_0$ *and* $\mathbf{a}_1$ *incomparable*, $\mathbf{a}_0 \cap \mathbf{a}_1 = \mathbf{c}$, *and* $\mathbf{a}_0 \cup \mathbf{a}_1 = \mathbf{0}'$.

PROOF. We indicate only the modifications which must be made to the proof of Theorem 5.1. Let $B \in \mathbf{b}$. We may assume $\mathbf{b} \leqslant \mathbf{0}'$ (else $\mathbf{b} \nleqslant \mathbf{c}$ is automatic) so there is a recursive sequence $\{B_s\}_{s \in \omega}$ of finite sets converging to B. When we refer to B during stage s, we mean B_s.

We have for each e requirement S_e given by (5.6). Each string α of length e besides being a strategy for R_e is also a strategy for S_e. We say that α requires (receives) attention for R_e (S_e) if strategy α for R_e (S_e) requires (receives) attention. If α requires (receives) attention for either R_e or S_e, then we say that α requires (receives) attention. When α receives attention for S_e, it does so through a certain number y. At this time $\langle \alpha, y \rangle$ becomes verified and a number is specified such that if $C \cup \hat{D}$ later changes below this number, then $\langle \alpha, y \rangle$ becomes unverified; $\langle \alpha, y \rangle$ remains verified until such a change occurs.

At any point during the construction we let

$$l^B(e) = \max\{x: (\forall y < x)[\Phi_e(C, \hat{D}; y) = B(y)]\}$$

and we say that α *requires attention for* $S_{\text{lh}\alpha}$ if there is a $y < l^B(\text{lh}\alpha)$ with $\langle \alpha, y \rangle$ unverified.

From time to time during the construction a strategy α for $S_{\text{lh}\alpha}$ wants to know, for sake of some y, if $\mathfrak{D}_x \subseteq \overline{D}$, so enumerates x into a set. We assume, by the recursion theorem, that for each α and y we know the index $v_{\langle \alpha, y \rangle}$ such that $W_{v_{\langle \alpha, y \rangle}}$ equals the set of canonical indices about which strategy α for $S_{\text{lh}\alpha}$ asks for sake of y.

The construction is identical to that of Theorem 5.1, except for substage 1.t.b which we give here.

Substage 1.t.b. Let $\beta_{s,t}$ be the current value of β restricted to s. See if for any $n \leqslant s$, $\beta_{s,t} \upharpoonright n$ requires attention. If not, then substage 1 is over. If so, let n be the least such and set $\alpha = \beta_{s,t} \upharpoonright n$. If α requires attention for R_n, then proceed as before. Otherwise α requires attention for S_n. Let y be the least number $< l^B(n)$ such that $\langle \alpha, y \rangle$ is unverified, and let $\mathfrak{D}_x = \hat{D} \upharpoonright s$. Enumerate x into $W_{v_{\langle \alpha, y \rangle}}$ and then simultaneously enumerate new elements from D into $\hat{D}$ and search for x in $W_{f(v_{\langle \alpha, y \rangle})}$. Either an element of $\mathfrak{D}_x$ will appear in $\hat{D}$ or else x will appear in $W_{f(v_{\langle \alpha, y \rangle})}$. Stop the searches when one of these events occurs. If the former happens first, increase t by 1 and begin subsubstage t. Otherwise declare $\langle \alpha, y \rangle$ to be verified until $C \cup \hat{D}$ changes below s, cancel all $\lambda_m^{\alpha'}$, $\Lambda_m^{\alpha'}$ and $x_{\alpha'}$ with $\alpha < \alpha'$, and enumerate a restraint of priority α equal to s; α receives attention for S_n at stage s and substage 1 is over. □

Again, if t is not the final subsubstage of substage 1 of stage s, then a number $< s$ is enumerated into $\hat{D}$ during substage $1.t.$b of stage s, so for all s there are only finitely many subsubstages in substage 1 of stage s.

LEMMA 1. $K \leqslant_T A_0 \oplus A_1$. □

Define δ_n as before. Then for all n, $\delta_n \subseteq \delta_{n+1}$.

LEMMA 2. *For each n,*
(i) *R_n is met and δ_n receives attention for R_n only finitely often, and*
(ii) *S_n is met and δ_n receives attention for S_n only finitely often.*

PROOF. Suppose that the result holds for all $n' < n$. The proof of (i) is as before. Assuming that (i) holds for n, we may take $s_0 \geqslant n$ such that (3.4), (3.5), and (3.6) are satisfied and, in addition,

(5.7) $\qquad s \geqslant s_0 \to \delta_n$ does not receive attention for R_n at stage s.

Corresponding to (5.5) we now have

(5.8) $\qquad$ if $s \geqslant s_0$ and $\delta_n \subseteq \beta_s$, then at substage $1.u_s.$b either δ_n does not require attention for S_n or else δ_n receives attention for S_n.

Let $A_{\delta_n} = A_i$. Suppose that at stage $s \geqslant s_0$, δ_n receives attention for S_n through y. Then we claim that the first change in $A_{1-i} \cup C \cup \hat{D}$ below s at or after substage $1.u_s.$b of stage s is a change in $\hat{D}$. The proof of this claim is as before.

Next suppose that for some y a number x with $\mathfrak{D}_x \subseteq \bar{D}$ is put into $W_{v\langle\delta_n, y\rangle}$ at stage $s \geqslant s_0$. Then $\langle\delta_n, y\rangle$ is declared verified at stage s and $\hat{D} \upharpoonright s[s, 1.u_s.\text{b}] = \hat{D} \upharpoonright s$, so by the previous claim $C \cup \hat{D} \upharpoonright s[s, 1.u_s.\text{b}] = C \cup \hat{D} \upharpoonright s$. Hence $\langle\delta_n, y\rangle$ remains verified throughout the rest of the construction and no further numbers are put into $W_{v\langle\delta_n, y\rangle}$. It follows that

$$W_{v\langle\delta_n, y\rangle} \cap \{x : \mathfrak{D}_x \subseteq \bar{D}\}$$

is finite, so $W_{f(v\langle\delta_n, y\rangle)}$ is finite for each y.

Suppose that δ_n receives attention for S_n through y at stages s_1 and s_2 with $s_2 > s_1 \geqslant s_0$. Then for $r = 1$ or 2, at stage s_r, x_r is put into $W_{v\langle\delta_n, y\rangle}$ where $\mathfrak{D}_{x_r} \subseteq \hat{D}[s_r, 1.u_{s_r}.\text{b}]$ and $x_r \in W_{f(v\langle\delta_n, y\rangle)}$. As long as no element of $\mathfrak{D}_{x_1}$ appears in $\hat{D}$, $\langle\delta_n, y\rangle$ remains verified, so $\mathfrak{D}_{x_1} \not\subseteq \hat{D}[s_2, 1.u_{s_2}.\text{b}]$ and $x_1 \neq x_2$. Since, for all y, $W_{f(v\langle\delta_n, y\rangle)}$ is finite, for all y, δ_n receives attention for S_n through y only finitely often. Thus for each y, $\langle\delta_n, y\rangle$ is either eventually permanently verified or eventually permanently unverified.

Now suppose that S_n fails, so $\Phi_n(C, D) = B$. Then we claim that for all y, $\langle\delta_n, y\rangle$ is eventually permanently verified. Suppose not. Take y_0 least such that $\langle\delta_n, y_0\rangle$ is eventually permanently unverified. Let $s_1 \geqslant s_0$ be such that

(5.9) $\qquad \langle\delta_n, y_0\rangle$ is permanently unverified by the beginning of stage s_1,

(5.10) $\qquad y < y_0 \to \langle\delta_n, y\rangle$ is permanently verified by the beginning of stage s_1,

and

(5.11) $\qquad y_0 < l^B(n)$ throughout all stages $\geqslant s_1$.

Let $s \geqslant s_1$ be such that $\beta_s \upharpoonright n = \delta_n$. Then by (5.9) and (5.11), δ_n requires attention for S_n throughout stage s, so by (5.8) δ_n receives attention for S_n at stage s. By (5.9)–(5.11), δ_n receives attention for S_n through y_0 at stage s, contradicting (5.9). This proves the claim.

Now let y be such that δ_n does not receive attention for S_n through y prior to stage s_0 (this is true for almost all y). Then by the previous claim, at some stage $s \geqslant s_0$, δ_n receives attention for S_n through y at stage s and $\langle \delta_n, y \rangle$ remains verified throughout the rest of the construction. Thus, given a D oracle and such a y, we may successfully search for a stage $s \geqslant s_0$ such that δ_n receives attention for S_n through y at stage s and $\hat{D} \upharpoonright s[s, 1.u_s.\mathrm{b}] = \hat{D} \upharpoonright s$. By an earlier claim we in fact have $C \cup \hat{D} \upharpoonright s[s, 1.u_s.\mathrm{b}] = C \cup \hat{D} \upharpoonright s$ so since $\Phi_n(C, \hat{D}; y)\downarrow[s, 1.u_s.\mathrm{b}]$,

$$B(y) = \Phi_n(C, \hat{D}; y) = \Phi_n(C, \hat{D}; y)[s, 1.u_s.\mathrm{b}],$$

where s was found effectively in D. Thus $B \leqslant_T D$, contradicting $\mathbf{b} \nleqslant \mathbf{d}$. Hence S_n is met.

It remains to show that δ_n receives attention for S_n only finitely often. Since S_n holds, let $p = \mu p'[\neg(\Phi_n(C, D; p') = B(p'))]$. Then $\langle \delta_n, p \rangle$ is either eventually permanently verified or eventually permanently unverified. If $\langle \delta_n, p \rangle$ is eventually permanently verified, then $\Phi_n(C, D; p)[\omega]\downarrow \neq B(p)$, so throughout all sufficiently large stages $l^B(n) = p$. Thus for all sufficiently large s, if δ_n receives attention for S_n through y at stage s, then $y < p$. But we already know that for any fixed y, δ_n receives attention for S_n through y only finitely often, so in this case δ_n receives attention for S_n only finitely often. If $\langle \delta_n, p \rangle$ is eventually permanently unverified, then once $\langle \delta_n, p \rangle$ is permanently unverified, δ_n cannot receive attention for S_n through any $y > p$. Hence again δ_n receives attention for S_n only finitely often and the lemma is established. □

LEMMA 3. *For all e, N_e is met.*

PROOF. The proof is exactly the same as that of Lemma 3 of Theorem 5.1. □

The theorem follows from the lemmas. Let

$$\mathbf{c} = \deg(C \oplus D) \quad \text{and} \quad \mathbf{a}_i = \deg(A_i \oplus C \oplus D)$$

for $i = 0$ and 1. Then by Lemma 1, $\mathbf{0}' = \mathbf{a}_0 \cup \mathbf{a}_1$, by Lemma 2, $\mathbf{a}_i > \mathbf{c}$ for $i = 0$ and 1 and $\mathbf{b} \nleqslant \mathbf{c}$, and by Lemma 3, $\mathbf{c} = \mathbf{a}_0 \cap \mathbf{a}_1$. ■

An *automorphism* of the r.e. degrees is a map Θ taking r.e. degrees to r.e. degrees such that Θ is a bijection and for all r.e. degrees $\mathbf{a}$ and $\mathbf{b}$, $\mathbf{a} \leqslant \mathbf{b} \leftrightarrow \Theta(\mathbf{a}) \leqslant \Theta(\mathbf{b})$. A set S of r.e. degrees is called an *automorphism base for the r.e. degrees* if any two automorphisms of the r.e. degrees which agree on S are necessarily the same automorphism.

COROLLARY 5.5. *The branching degrees are an automorphism base for the r.e. degrees.*

PROOF. It is easily seen that a set S of r.e. degrees is an automorphism base for the r.e. degrees iff the only automorphism of the r.e. degrees which fixes each element of S is the identity map. By the Sacks Splitting Theorem, as given in [**10**], the low r.e.

degrees generate the r.e. degrees under join, so form an automorphism base for the r.e. degrees.

Suppose that Θ is an automorphism of the r.e. degrees which fixes each branching degree. We want to show that Θ is the identity. It suffices to show that Θ fixes each low r.e. degree. Suppose for a contradiction that for some low r.e. degree $\mathbf{d}$, $\Theta(\mathbf{d}) \neq \mathbf{d}$. If $\Theta(\mathbf{d}) \nleqslant \mathbf{d}$, let $\bar{\Theta} = \Theta$ and $\bar{\mathbf{d}} = \mathbf{d}$. If $\Theta(\mathbf{d}) \leqslant \mathbf{d}$, then in fact $\Theta(\mathbf{d}) < \mathbf{d}$; take $\bar{\Theta} = \Theta^{-1}$ and $\bar{\mathbf{d}} = \Theta(\mathbf{d})$. Then in either case, $\bar{\Theta}$ is an automorphism of the r.e. degrees which fixes each branching degree and $\bar{\mathbf{d}}$ is a low r.e. degree such that $\bar{\Theta}(\bar{\mathbf{d}}) \nleqslant \bar{\mathbf{d}}$. By Theorem 5.4, there is a branching degree $\mathbf{c}$ with $\bar{\mathbf{d}} \leqslant \mathbf{c}$ and $\bar{\Theta}(\bar{\mathbf{d}}) \nleqslant \mathbf{c}$. Then $\bar{\Theta}(\bar{\mathbf{d}}) \leqslant \bar{\Theta}(\mathbf{c}) = \mathbf{c}$, giving the desired contradiction. ■

COROLLARY 5.6. *For each r.e. degree* $\mathbf{b}$ *with* $\mathbf{0} < \mathbf{b} < \mathbf{0}'$, *there is a branching degree* $\mathbf{c}$ *with* $\mathbf{b} \mid \mathbf{c}$ (*i.e.*, $\mathbf{b} \nleqslant \mathbf{c}$ *and* $\mathbf{c} \nleqslant \mathbf{b}$).

PROOF. Since $\mathbf{0} < \mathbf{b} < \mathbf{0}'$, there is a low r.e. degree $\mathbf{d}$ with $\mathbf{b} \mid \mathbf{d}$. (This fact follows, nonuniformly in an r.e. index for a set in $\mathbf{b}$, from the version of the Sacks Splitting Theorem given in [**10**, Theorem 1.2 and Remark 4.5].) By Theorem 5.4, there is a branching degree $\mathbf{c}$ with $\mathbf{d} \leqslant \mathbf{c}$ and $\mathbf{b} \nleqslant \mathbf{c}$. We must also have $\mathbf{c} \nleqslant \mathbf{b}$, else $\mathbf{d} \leqslant \mathbf{c} \leqslant \mathbf{b}$, so $\mathbf{b} \mid \mathbf{c}$, as desired. ■

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