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PRODUCTS OF $\kappa$-SPACES AND
SPACES OF COUNTABLE TIGHTNESS
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Abstract. In this paper, we obtain results of the following type: if $f: X \to Y$ is a closed map and $X$ is some “nice” space, and $Y^2$ is a $\kappa$-space or has countable tightness, then the boundary of the inverse image of each point of $Y$ is “small” in some sense, e.g., Lindelöf or $\omega_1$-compact. We then apply these results to more special cases. Most of these applications combine the “smallness” of the boundaries of the point-inverses obtained from the earlier results with “nice” properties of the domain to yield “nice” properties on the range.

Introduction. Recall the following theorem due to Morita and Hanai [14] and Stone [17].

Theorem. If $f: X \to Y$ is closed and $X$ is metrizable, then the following are equivalent.
(a) $Y$ is first countable;
(b) For each $y \in Y$, $\partial f^{-1}(y)$ is compact;
(c) $Y$ is metrizable.

The (c) $\Rightarrow$ (b) part is due to Vaïnštein [22]. But even the (a) $\Rightarrow$ (b) part holds under much more general conditions: Michael [7] showed (b) holds if $X$ is paracompact, and $Y$ is locally compact or first-countable.

Note that the assumptions on $Y$ in Michael’s theorem could not be weakened to “$Y$ is a $\kappa$-space” or “$Y$ has countable tightness”: the map identifying the limit points of a topological sum of $\kappa$ convergent sequences is a closed map from a metrizable space $X$ to a Fréchet space $Y$, and $|\partial f^{-1}(y)| = \kappa$ for some $y \in Y$. In this paper, we show that the situation is different if we require $Y^2$ to be a $\kappa$-space or have countable tightness. (Recall that the square of a $\kappa$-space or a space of countable tightness need not have the same property.) We will usually not be able to show that the boundaries of point-inverses are compact, but we will often (depending upon conditions imposed on $X$ or $Y$) be able to show that they are “small” in some sense, e.g., Lindelöf or $\omega_1$-compact. In the second section, we apply general results of this type to more special cases, often combining the “smallness” of the boundaries of point-inverses with “nice” properties of $X$ to obtain “nice” properties of $Y$.

We mention the following earlier result of the second author [21] which is related to this topic.

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Theorem. If \( f: X \to Y \) is closed and \( X \) is metrizable, then the following are equivalent.

(a) For each \( y \in Y \), \( \partial f^{-1}(y) \) is Lindelöf.

(b) \( Y \) has a point-countable k-network [see §2, Definition 2.1].

(c) \( Y \) has a \( \sigma \)-locally-countable k-network.

See [7] and [21] for other related results.

We will often make use of the following well-known property of closed maps (cf. [3, p. 52]): If \( f: X \to Y \) is closed, then for each \( y \in Y \) and open \( U \subseteq X \) such that \( f^{-1}(y) \subseteq U \), there is a neighborhood \( V \) of \( y \) such that \( f^{-1}(V) \subseteq U \).

1. General results. All our spaces are assumed to be regular and \( T_1 \). We consider cardinals to be initial ordinals. We now recall some basic definitions.

Definition 1.1. A space \( X \) has the weak topology with respect to a collection \( \mathcal{C} \) of sets if a subset \( A \) of \( X \) is closed (resp., open) in \( X \) if and only if \( A \cap C \) is closed (resp., open) in \( C \) for each \( C \in \mathcal{C} \).

Definition 1.2. A space \( X \) is a k-space (quasi-k-space) if \( X \) has the weak topology with respect to its compact (countably compact) subsets. \( X \) is sequential if \( X \) has the weak topology with respect to its compact metric subspaces (equivalently, with respect to its subspaces homeomorphic to \( \omega + 1 \), a sequence with its limit point). \( X \) has countable tightness (denoted by \( t(X) \leq \omega \)) if it has the weak topology with respect to its countable subsets.

We will be using the following elementary facts about these concepts.

(i) If \( X \) has the weak topology with respect to a collection \( \mathcal{C} \), and \( f: X \to Y \) is a quotient map, then \( Y \) has the weak topology with respect to \{ \( f(C) : C \in \mathcal{C} \) \}. Thus all properties named in Definition 1.2 are preserved by quotient maps.

(ii) If \( X \) satisfies any of the properties in Definition 2.2 locally, then the whole space has the property.

(iii) If \( X \) has a locally finite cover by a family \( \mathcal{C} \) of closed sets, then \( X \) has the weak topology with respect to \( \mathcal{C} \).

Definition 1.3. A space \( X \) is (strongly) collectionwise Hausdorff if whenever \{ \( x_\alpha : \alpha \in A \) \} is a closed discrete subset of \( X \), there exists a (discrete) disjoint collection \{ \( U_\alpha : \alpha \in A \) \} of open sets such that \( x_\alpha \) \( \in U_\alpha \) for each \( \alpha \in A \).

Note that every normal collectionwise Hausdorff space is strongly collectionwise Hausdorff.

Let \( c \) denote the cardinality of the continuum.

Theorem 1.4. Suppose \( f: X \to Y \) is closed, with \( X \) strongly collectionwise Hausdorff. Then the boundary, \( \partial f^{-1}(y) \), of \( f^{-1}(y) \) is \( c \)-compact for each \( y \in Y \) if either

(a) \( Y^2 \) is quasi-k and \( t(Y) \leq \omega \)

(b) \( t(Y^2) \leq \omega \).

Proof. Suppose \( \partial f^{-1}(y) \) is not \( c \)-compact. Then there is a closed discrete subset \( D \subseteq \partial f^{-1}(y) \), with \( |D| = c \). For each \( d \in D \), let \( U_d' \) be an open set containing \( d \) such that \( \{ U_d' : d \in D \} \) is discrete.

Let \( d \in U_d \subseteq \overline{U_d} \subseteq U_d' \), where \( U_d \) is open. Note that \( \{ U_d : d \in D \} \) is also discrete.
For each $d \in D$, $y \in \overline{(U_d - f^{-1}(y))}$. Since $\tau(Y) \leq \omega$, there is a countable set 
\[ \{y_d, n : n \in \omega\} \subset f(U_d - f^{-1}(y)) \] such that $y \in \{y_d, n : n \in \omega\}$.

Let $X_{d,n} = f^{-1}(y_{d,n}) \cap \overline{U_d}$, and let $X_d = f^{-1}(y) \cap \overline{U_d}$.

If $O$ is open and contains $X_d$, then there is an open set $O'$ such that $O' \cap \overline{U_d} = \emptyset$, and $f^{-1}(y) \subset O \cup O'$. Let $W$ be the complement in $Y$ of $f(X - (O \cup O'))$. Then $y \in W$, so there is $n \in \omega$ such that $y_{d,n} \in W$, and hence $f^{-1}(y_{d,n}) \subset O \cup O'$. Thus $X_{d,n} \subset (O \cup O') \cap \overline{U_d} = O \cap \overline{U_d} \subset O$. Choose $x_{d,n} \in X_{d,n}$, and let $A_d = \{x_{d,n} : n \in \omega\}$. By the above argument, every open set containing $X_d$ contains infinitely many elements of $A_d$.

For $x \in A_d$, let $D_x = \{d' \in D :$ there exists $x' \in U_{d'}$, with $f(x) = f(x')\}$. Let $B_d = \{x \in A_d : D_x$ is uncountable\}.

**Claim 1.** $X_d \cap B_d = \emptyset$. To see this, let $B_d = \{x_0, x_1, \ldots\}$. Inductively choose a sequence $d_0, d_1, \ldots$ of distinct elements of $D$, and points $x'_n \in \overline{U_{d_n}}$ such that $f(x'_{d_n}) = f(x_{d_n})$. Then $\{x'_0, x'_1, \ldots\}$ is a closed subset of $X$, so $f(\{x'_0, x'_1, \ldots\}) = f(B_d)$ is closed. Thus $y \notin f(B_d) = f(B_d) = f(\overline{B_d})$, so $X_d \cap B_d = \emptyset$.

Let $C_d = A_d - B_d$. By Claim 1, $X_d \cap C_d \neq \emptyset$. Pick $d(0) \in D$. Let $D(d(0)) = \bigcup\{D_x : x \in C_d(0)\}$. Observe that $D(d(0))$ is countable. If $d(\beta)$ has been chosen for all $\beta < \alpha < c$, then $d(\alpha) \in D - \bigcup_{\beta < \alpha} D(d(\beta))$. Observe that if $x \in C_{d(\beta)}$ and $x' \in C_{d(\alpha)}$ with $\alpha \neq \beta$, then $f(x) \neq f(x')$.

Now let $\mathfrak{d} = \{E_\alpha : \alpha < c\}$ index all subsets of $\bigcup_{n \in \omega} C_d(n)$ such that $|E_\alpha \cap C_d(n)| = 1$ for each $\alpha < c$ and $n \in \omega$. Let $E_\alpha = \{e_{\alpha, n} : n \in \omega\}$ such that $e_{\alpha, n} \in C_d(n)$. Let $C_{d(\alpha)} = \{c_{\alpha, n} : n \in \omega\}$.

For $x \in X$, denote $f(x)$ by $x^*$. Let $H_\alpha = \{(e_{\alpha, n}, c_{\alpha, n}^* : n \in \omega\} \subset Y^2$, where $\lambda_\alpha$ is the $\alpha$th limit ordinal, and let $H = \bigcup_{\alpha < c} H_\alpha$.

**Claim 2.** $(y, y) \in H - H$. To see this, suppose $y \in O$, $O$ open in $Y$. We know $X_{d(\alpha)} \cap C_{d(\alpha)} \neq \emptyset$, so $y \in f(C_{d(\alpha)}) = f(\overline{C_{d(\alpha)}})$. For each $n \in \omega$, choose $x_n \in C_{d(n)}$ such that $x_n^* \in O$. Then $\{x_n : n \in \omega\} = E_\alpha$ for some $\alpha$, and $e_{\alpha, n} = x_n$ for each $n$. There is $n \in \omega$ such that $c_{\alpha, n}^* \in O$. Thus $(e_{\alpha, n}^*, c_{\alpha, n}^*) \in O^2 \cap H_\alpha$, which proves the claim.

The next claim completes the proof of part (a).

**Claim 3.** If $K \subset Y^2$ is countably compact, then $K \cap H$ is finite. To see this, suppose $\alpha_0, \alpha_1, \ldots$ are distinct ordinals such that for each $n \in \omega$, $K \cap H_{\alpha_n} \neq \emptyset$. Then we can find $(e_{\alpha_n, k_n}, c_{\alpha_n, k_n}^*) \in K \cap H_{\alpha_n}$. But $(c_{\alpha_n, k_n}^* : n \in \omega)$ is a closed discrete subset of $X$, since $c_{\alpha_n, k_n}^* \in \overline{U_{d(\alpha_n)}}$. Thus $\{(e_{\alpha_n, k_n}, c_{\alpha_n, k_n}^* : n \in \omega\}$ is an infinite closed discrete subset of $K$, contradicting that $K$ meets only finitely many $H_\alpha$'s. Now suppose that for fixed $\alpha$, $K \cap H_{\alpha}$ is infinite. Then for each $n \in \omega$, we can find $(e_{\alpha, k_n}, c_{\alpha, k_n}^*) \in K \cap H_{\alpha}$. But $(e_{\alpha, k_n} : n \in \omega)$ is an infinite closed discrete subset of $X$ and we get a contradiction as before. Thus each $K \cap H_{\alpha}$ is finite, and so $K \cap H$ is finite.

To complete the proof of part (b), we have the next claim.

**Claim 4.** No countable subset of $H$ contains $(y, y)$ in its closure. Suppose $C \subset H$, $|C| \leq \omega$. Then there exists a sequence $\alpha_0, \alpha_1, \ldots$ of distinct ordinals such that $C \subset \bigcup_{n \in \omega} H_{\alpha_n}$. For each $n$, let $U_n \subset U_d'$ be an open set in $X$ containing $X_{d(n)}$ such that $e_{\alpha_n, k} \notin U_n$ if $k \neq n$. Note $U_n \cap \overline{U_d} = \emptyset$ if $d \neq d(n)$. Let $V_n$ be an open set in $X$
containing $X_{d(\lambda_n)}$ such that $c_{\lambda_{a,n}} \notin V_n$ if $k \leq n$, and $V_n \cap U_d = \emptyset$ if $d \neq d(\lambda_n)$.

If $x \in f^{-1}(y) = \bigcup_{n \in \omega}(U_n \cup V_n)$, then there is an open set $O_x$ containing $x$ such that $O_x \cap U_d = \emptyset$ if $d \in \{d(n): n \in \omega\} \cup \{d(\lambda_n): n \in \omega\}$. Let

$$O = \bigcup \left\{ O_x: x \in f^{-1}(y) \cup \bigcup_{n \in \omega}(U_n \cup V_n) \right\} \cup \bigcup_{n \in \omega}(U_n \cup V_n)$$

There is an open set $W$ containing $y$ such that $f^{-1}(W) \subset O$. Suppose $W^2 \cap C \neq \emptyset$. Then there exists $m$ such that $W^2 \cap H_{a,m} \neq \emptyset$. Choose $n \in \omega$ such that $(e_{a,n}, c_{\lambda_{a,n}}) \in W^2 \cap H_{a,m}$. Then $(e_{a,n}, c_{\lambda_{a,n}}) \in O^2$. Recall $e_{a,n} \in C_d(n) \subset \overline{U_d}(n)$. Thus $e_{a,m} \in U_m$, and so $m > n$. Also recall $c_{\lambda_{m,n}} \in V_m$, which means $n > m$, a contradiction. This proves Claim 4. Hence $Y^2$ does not have countable tightness, a contradiction which proves the theorem. $\square$

Assuming the continuum hypothesis (CH), we have the following corollary.

**Corollary 1.5 (CH).** Suppose $f: X \to Y$ is closed, with $X$ paracompact. Then each $\partial f^{-1}(y)$ is Lindelöf if either $Y^2$ is a k-space with $t(Y) \leq \omega$, or $t(Y^2) \leq \omega$.

**Proof.** Immediate from Theorem 1.4 and the fact that $\omega_1$-compact paracompact spaces are Lindelöf [1].

**Remark.** By the proof below, if $Y^2$ is a k-space with $t(Y) \leq \omega$, then $t(Y^2) \leq \omega$. Thus the two conditions are not independent.

**Proof.** Since $Y^2$ is a k-space, it has the weak topology with respect to the collection of compact subsets of $Y^2$; that is $A \subset Y^2$ is closed whenever $A \subset C$ is closed in $C$ for every compact subset of $C$ of $Y^2$. Each compact subset $C$ of $Y^2$ is contained in $\pi(C)^2$, where $\pi$ is the projection from $Y^2$ onto $Y$. Then $Y^2$ has the weak topology with respect to $\{\pi(C)^2: C$ is compact in $Y^2\}$. Since each $\pi(C)$ is a compact space of countable tightness, by a result of V. I. Malyhin [5, Theorem 4], so is each $\pi(C)^2$. Then $t(Y^2) \leq \omega$. $\square$

We do not know if Corollary 1.5 is true without CH. The problem seems to hinge on strengthening the conclusion of Theorems 1.3 and 1.4 by replacing “c-compact” with “$\omega_1$-compact”. It turns out if we add the condition “$Y$ is sequential” to the hypotheses of these theorems, then we can do it.

**Theorem 1.6.** Suppose $f: X \to Y$ is closed with $X$ strongly collectionwise Hausdorff and $Y$ sequential. Then each $\partial f^{-1}(y)$ is $\omega_1$-compact if either $Y^2$ is a quasi-k-space or $t(Y^2) \leq \omega$.

**Proof.** Suppose $Y^2$ is a quasi-k-space. Since $Y$ is sequential, by [18, Theorem 2.2] $Y^2$ is sequential, hence $t(Y^2) \leq \omega$. Thus we can assume that $t(Y^2) \leq \omega$. Suppose $\partial f^{-1}(y)$ is not $\omega_1$-compact. Then there is a closed discrete set $D \subset \partial f^{-1}(y)$ with $|D| = \omega_1$. Let $\{U_d: d \in D\}$ be a discrete collection of open sets in $X$ with $d \in U_d$. Then $f(\overline{U}_d)$ is a closed subset of the sequential space $Y$, and is therefore sequential. Since $y$ is not isolated in $f(\overline{U}_d)$, there exists a sequence $y_{d,n} \to y$, with $y_{d,n} \in f(\overline{U}_d) - \{y\}$ for each $n \in \omega$. Choose $x_{d,n} \in \overline{U}_d \cap f^{-1}(y_{d,n})$. As in the proof of Theorem 1.3, we can construct $\{d(\alpha): \alpha < \omega_1\} \subset D$ and an infinite set $C_{d(\alpha)} \subset \{x_{d(\alpha),n}: n \in \omega\}$ such that $f$ is 1-1 on $\bigcup_{\alpha < \omega_1} C_{d(\alpha)}$. Let $X_{d(\alpha)} = \overline{U}_{d(\alpha)} \cap f^{-1}(y)$. Observe that every open set containing $X_{d(\alpha)}$ contains all but finitely many elements of $C_{d(\alpha)}$, and that
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$f^{-1}(y) \cup \left( \bigcup_{\alpha < \omega_1} C_d(\alpha) \right)$ is closed. Thus $Y$ contains a closed copy of the space obtained by identifying the limit points of $\omega_1$ convergent sequences. In [4], this space is denoted by $S_{\omega_1}$, and it is proved there that $S_{\omega_1}^2$ is not a $k$-space.

To complete the proof, it is sufficient to show that $S_{\omega_1}^2$ does not have countable tightness. For each $\alpha < \omega_1$, let $S_\alpha \subset S_{\omega_1}$ be the union of the first $\alpha$ sequences (with limit point). The closure of a countable subset of $S_{\omega_1}^2$ is contained in some $S_\alpha^2$. Thus if $S_{\omega_1}^2$ had countable tightness, then it would have the weak topology with respect to $\{S_\alpha^2; \alpha < \omega_1\}$. But each $S_\alpha^2$ is a $k$-space (cf. [9, (7.5)]), so then $S_{\omega_1}^2$ would be a $k$-space, contradiction. □

**Corollary 1.7.** Suppose $f: X \to Y$ is closed with $X$ paracompact and $Y$ sequential. Then each $\partial f^{-1}(y)$ is Lindelöf if either $Y^2$ is a $k$-space or $t(Y^2) < \omega_1$.

The following example shows that the assumption “$Y^2$ is a $k$-space” is not sufficient to obtain “$\partial f^{-1}(y)$ Lindelöf” in Corollary 1.5.

**Example 1.8.** There exists $f: X \to Y$ closed with $X$ locally compact and paracompact, such that $Y^2$ is a $k$-space, but $\partial f^{-1}(y)$ is not Lindelöf for some $y \in Y$.

**Proof.** For each $\alpha < \omega_1$, let $S(\alpha)$ be a copy of ordinal space $\omega_1 + 1$. Let $X$ be the free union of $\{S(\alpha); \alpha < \omega_1\}$. Let $Y$ be the space obtained from $X$ by identifying the point $\omega_1$ in each copy to a single point $\infty$. Let $f: X \to Y$ be the quotient map. Then $X$ is paracompact and locally compact, $f$ is closed, and $\partial f^{-1}(\infty)$ is not Lindelöf. $X$ is a $k$-space (being locally compact), hence so is $Y$.

It remains to prove that $Y^2$ is a $k$-space. First we introduce some notation. For each $\alpha, \beta \leq \omega_1$, let $\beta(\alpha)$ be the image under $f$ of the element of $S(\alpha)$ corresponding to the ordinal number $\beta$. If $\beta < \beta' \leq \omega_1$, let $[\beta(\alpha), \beta'(\alpha)] = (\gamma(\alpha); \beta \leq \gamma \leq \beta')$, and let $[\beta(\alpha), \infty] = [\beta(\alpha), \omega_1(\alpha)]$.

Suppose $A \subset Y^2$, with $A$ $k$-closed, but not closed. Since for each $\alpha, \beta \leq \omega_1$, $[O(\alpha), \beta(\alpha)] \times Y$ and $Y \times [O(\alpha), \beta(\alpha)]$ are clopen $k$-subspaces of $Y^2$, it must be true that $(\infty, \infty) \in \tilde{A} - A$.

Since $[f(S(O))]^2 \cap A$ is closed, there exists $\gamma_0 < \omega_1$ such that $[\gamma_0(O), \infty]^2 \cap A = \emptyset$. Now suppose $\gamma_0$ has been defined for all $\alpha < \beta$, where $\beta < \omega_1$, in such a way that the following property $P_\alpha$ holds.

$$P_\alpha: (\beta_1(\alpha_1), \beta_2(\alpha_2)) \in A \quad \text{and} \quad \alpha_1, \alpha_2 \leq \alpha \implies \beta_1 < \gamma_{\alpha_1} \lor \beta_2 < \gamma_{\alpha_2}.$$ It is easy to check that $P_0$ holds from the way $\gamma_0$ has been defined. We will show how to define $P_\beta$ in such a way that $P_\beta$ holds.

For each $\alpha \leq \beta$, $f(S(\alpha) \times S(\alpha)) \cap A$ and $f(S(\beta) \times S(\alpha)) \cap A$ are closed, so there exists $\delta_{\alpha, \beta} < \omega_1$ such that

(i) $\gamma_\alpha \leq \delta_{\alpha, \beta}$;

(ii) $[\delta_{\alpha, \beta}(\alpha), \infty] \times [\delta_{\alpha, \beta}(\beta), \infty]) \cap A = \emptyset$; and

(iii) $[\delta_{\alpha, \beta}(\beta), \infty] \times [\delta_{\alpha, \beta}(\alpha), \infty]) \cap A = \emptyset$.

For each $\alpha < \beta$, and each $\beta' \in [\gamma_\alpha, \delta_{\alpha, \beta})$, we have by $P_\alpha$ that $(\beta'(\alpha), \infty) \notin A$ and $(\infty, \beta'(\alpha)) \notin A$. Thus there exists $\delta_{\alpha, \beta}^a < \omega_1$ such that

(a) $[\delta_{\alpha, \beta}^a(\beta), \infty] \times (\beta'(\alpha))] \cap A = \emptyset$ and

(b) $[(\beta'(\alpha)) \times [\delta_{\alpha, \beta}^a(\beta), \infty]) \cap A = \emptyset$. 
Let \( \gamma_\beta = \sup \{ \delta_{a, \beta} : a < \beta \} + \sup \{ \delta_{\beta, a} : \alpha < \beta, \gamma_a < \delta_{\beta, a} \} \). Then \( \gamma_\beta < \omega_1 \). To show that \( P_\beta \) holds, we can suppose \( (\beta_1(\alpha), \beta_2(\beta)) \in A \), with \( \alpha < \beta \). If \( \alpha = \beta \), then either \( \beta_1 < \delta_{\beta, \beta} \leq \gamma_\beta \) or \( \beta_2 < \delta_{\beta, \beta} \leq \gamma_\beta \), so \( P_\beta \) holds. If \( \alpha < \beta \), we can suppose \( \beta_2 > \gamma_\beta \). Then it must be true that \( \beta_1 < \delta_{\alpha, \beta} \) (by (ii) above). If \( \gamma_\alpha < \beta_1 < \delta_{\alpha, \beta} \), then since \( \gamma_\beta > \delta_{\beta, \beta} \), we have \( (\beta_1(\alpha), \beta_2(\beta)) \in (\{ \beta_1(\alpha) \} \times [\delta_{\beta, \beta}(\beta), \infty) \cap A) \), a contradiction. Thus \( \beta_1 < \gamma_\alpha \), so \( P_\beta \) holds.

Thus we can define \( \{ \gamma_\alpha : \alpha < \omega_1 \} \) in such a way that \( P_\alpha \) holds for each \( \alpha < \omega_1 \). Let \( U = \{ \beta(\alpha) : \beta > \gamma_\alpha, \alpha < \omega_1 \} \). Then \( U \) is an open set in \( Y \) containing \( \infty \). Since \( (\infty, \infty) \in A \), there exists \( (\beta_1(\alpha_1), \beta_2(\alpha_2)) \in U^2 \cap A \). Since \( P_{\alpha_1 + \alpha_2} \) holds, either \( \beta_1 < \gamma_{\alpha_1} \) or \( \beta_2 < \gamma_{\alpha_2} \). But then either \( \beta_1(\alpha_1) \notin U \) or \( \beta_2(\alpha_2) \notin U \), contradiction. Thus \( Y^2 \) is a \( k \)-space.

2. Applications. As applications of results in §1, we shall consider the products of \( k \)-spaces and spaces of countable tightness in more special cases.

**Definition 2.1** [8, 16]. A collection \( \mathcal{G} \) of (not necessarily open) subsets of a space \( X \) is a \( k \)-network for \( X \) if, whenever \( C \subseteq U \) with \( C \) compact and \( U \) open, then \( C \subseteq U \cap C \subseteq U \cap C \) for some finite subcollection \( \mathcal{G} \) of \( \mathcal{G} \). An \( \mathcal{N} \)-space is a space with a \( \sigma \)-locally finite \( k \)-network, and an \( \mathcal{X}_0 \)-space is a space with a countable \( k \)-network.

Note that metrizable spaces are \( \mathcal{N} \)-spaces, and separable metrizable spaces are \( \mathcal{N}_0 \)-spaces.

We say that \( X \) is a locally \( \mathcal{N}_0 \)-space if each point of \( X \) has a neighborhood which is an \( \mathcal{N}_0 \)-space.

**Theorem 2.2** (CH) Let \( f : X \to Y \) be a closed map. Let \( X \) be a paracompact, locally \( \mathcal{N}_0 \)-space. Then the following are equivalent.

(a) \( t(Y^2) \leq \omega \);
(b) each \( df^{-1}(y) \) is Lindelöf;
(c) \( Y \) is a locally \( \mathcal{N}_0 \)-space; and
(d) \( Y \) is locally separable.

Furthermore, if \( Y \) is sequential, then the CH assumption can be omitted.

**Proof.** (a) \( \Rightarrow \) (b): This is Corollary 1.5.

(b) \( \Rightarrow \) (c): Since each subset of a locally \( \mathcal{N}_0 \)-space is locally \( \mathcal{N}_0 \), as in the proof of [7, Corollary 1.2], we can assume that each \( f^{-1}(y) \) is Lindelöf. Thus, \( f \) is a closed map with each \( f^{-1}(y) \) Lindelöf. Then, for each \( y \in Y \), there is a closed neighborhood \( W \) of \( y \) in \( Y \), and open subsets \( V_i \) of \( X \) which are \( \mathcal{N}_0 \)-spaces such that \( f^{-1}(W) \subseteq \bigcup_{i=1}^{\infty} V_i \). Since \( \bigcup_{i=1}^{\infty} V_i \) is an \( \mathcal{N}_0 \)-space, so is \( f^{-1}(W) \). Since the closed image of an \( \mathcal{N}_0 \)-space is also \( \mathcal{N}_0 \) by [8, G], \( W \) is an \( \mathcal{N}_0 \)-space. This implies (c).

(c) \( \Rightarrow \) (a) and (c) \( \Rightarrow \) (d): By [8, F], \( Y^2 \) is a locally \( \mathcal{N}_0 \)-space. Then, by [8, D, E] \( Y^2 \) is locally a hereditarily separable space. Hence \( t(Y^2) \leq \omega \).

(d) \( \Rightarrow \) (b): This follows from [21, Proposition 1], because \( Y \) is paracompact, hence is locally Lindelöf by (d).

From Theorem 2.2 and some results in [21], we have

**Corollary 2.3.** Let \( f : X \to Y \) be a closed map with \( X \) locally separable metric. Then the following are equivalent.
(a) \( t(Y^2) \leq \omega \);
(b) each \( \partial f^{-1}(y) \) is Lindelöf;
(c) \( Y \) is locally separable;
(d) \( Y \) is locally Lindelöf;
(e) \( Y \) is an \( \aleph \)-space.

**Definition 2.4.** A decreasing sequence \((A_n)\) in a space \( X \) is a \( k \)-sequence \([10]\), if it is an outer network at a compact subset \( K \) of \( X \); that is, \( K = \bigcap_{n=1}^{\infty} A_n \) and every neighborhood of \( K \) contains some \( A_n \). By regularity, if a compact subset \( K \) has an outer network, it has one in which each \( A_n \) is closed in \( X \).

Let \( Y \) be a space. Then \( Y \) satisfies condition \( K(N_0) [20] \) if, for any \( k \)-sequence \((A_n)\) in \( Y \), some \( A_n \) is countably compact.

It is shown that \([20, \text{Proposition 2.4}]\) a space \( Y \) satisfies \( K(N_0) \) if and only if each closed subset of \( Y \) which is a paracompact \( M \)-space is locally compact.

**Definition 2.5** \([10, \text{Lemma 3.E.2}]\). A space \( F \) is a bi-\( k \)-space if, whenever a filter base \( \mathcal{F} \) accumulates at \( y \) in \( Y \), then there exists a \( k \)-sequence \((A_n)\) in \( Y \) such that \( y \in F \cap A_n \) for all \( n \in \mathbb{N} \) and all \( F \in \mathcal{F} \).

It is shown that \([10, \text{Theorem 3.E.3}]\) \( Y \) is a bi-\( k \)-space if and only if \( Y \) is a bi-quotient image of a paracompact \( M \)-space \( X \). Then by a result of H. Wicke \([23]\), spaces of pointwise countable type \([2]\) are bi-\( k \).

**Definition 2.6.** A space \( X \) is a \( k_\omega \)-space \([9]\) (K. Morita \([13]\) called it a space of class \( \mathcal{C}_\Gamma' \)), if it has the weak topology with respect to a countable covering of compact subsets of \( X \).

For a space \( Y \) we shall say that \( Y \) is a locally \( k_\omega \)-space, if each point of \( Y \) has a neighborhood whose closure is a \( k_\omega \)-space.

It is implicit in a result of J. Milnor \([12, \text{Lemma 2.1}]\) that the product of two \( k_\omega \)-spaces is \( k_\omega \). This fact implies the following lemma.

**Lemma 2.7.** Let \( Y \) be a locally \( k_\omega \)-space. Then \( Y^2 \) is a locally \( k_\omega \)-space, hence, a \( k \)-space.

**Lemma 2.8.** Let \( f: X \to Y \) be a closed map with each \( \partial f^{-1}(y) \) Lindelöf. If \( X \) is bi-\( k \) and \( Y \) satisfies \( K(N_0) \), then \( Y \) is a locally \( k_\omega \)-space.

**Proof.** Since each closed subset of \( X \) is a bi-\( k \), as in the proof of \([7, \text{Corollary 1.2}]\), we can assume that each \( f^{-1}(y) \) is Lindelöf. Let \( y \in Y \). Then we will prove that each point of \( f^{-1}(y) \) has a neighborhood contained in the inverse image of some compact subset of \( Y \). To see this, suppose not. Then there is a point \( a_0 \) of \( f^{-1}(y) \) such that for every neighborhood \( V \) of \( a_0 \) and for every compact subset \( K \) of \( Y \), \( V \nsubseteq f^{-1}(K) \).

Let \( \mathcal{F} = \{ X - f^{-1}(K); K \text{ is compact in } Y \} \). Then \( \mathcal{F} \) is a filter base accumulating at the point \( a_0 \). Since \( X \) is bi-\( k \), there exists a \( k \)-sequence \((A_n)\) in \( X \) such that \( a_0 \in F \cap A_n \) for all \( n \in \omega \) and all \( F \in \mathcal{F} \). Obviously, \( (f(A_n)) \) is a \( k \)-sequence in \( Y \). Thus, by condition \( K(N_0), \) some \( f(A_n) \) is compact. Let \( K_0 = f(A_{n_0}) \). Then, \( a_0 \in (X - f^{-1}(K_0)) \cap A_{n_0} \subset (X - f^{-1}(K_0)) \cap f^{-1}(K_0) = \emptyset \). This is a contradiction.
Thus, each point $x$ of $f^{-1}(y)$ has a neighborhood $V_x$ which is contained in the inverse image of some compact subset of $Y$. Since $f^{-1}(y)$ is Lindelöf, $\{V_x : x \in f^{-1}(y)\}$ contains a countable subcover $\{V_n\}_{n \in \omega}$ of $f^{-1}(y)$. For each $n$, let $K_n$ be a compact subset of $Y$ such that $V_n \subset f^{-1}(K_n)$. Since $f$ is closed and $Y$ is regular there exists a neighborhood $W$ of $y$ such that $f^{-1}(W) \subset \bigcup_{n \in \omega} V_n$. Let $F = f^{-1}(W)$ and $\mathcal{V} = \{F \cap V_i : i \in \omega\}$. Then, since $\mathcal{V}$ is an open covering of $F$, $F$ has the weak topology with respect to $\mathcal{V}$. Since $F \cap V_i \subset F \cap f^{-1}(K_i)$ for each $i \in \omega$, $F$ has the weak topology with respect to $\{F \cap f^{-1}(K_i) : i \in \omega\}$. Since $f|_F$ is closed, hence quotient, $f(F) = W$ has the weak topology with respect to $\{W \cap K_i : i \in \omega\}$. Thus $W$ is a $k_\omega$-space, and so $Y$ is a locally $k_\omega$-space.

**Lemma 2.9.** Let $f : X \to Y$ be a closed map with $X$ normal and $t(Y) = \omega$. If $Y^2$ is a $k$-space, then either $Y$ satisfies condition $K(\mathcal{N}_\omega)$ or each $\partial f^{-1}(y)$ is countably compact.

**Proof.** According to [20, Theorem 4.2], if the product of two spaces is quasi-$k$, and one factor is not an inner-one $A$-space in the sense of E. Michael, R. C. Olson and F. Siwiec [11], then the other factor satisfies $K(\alpha)$, where $\alpha$ is its tightness. $Y$ satisfies condition $K(\mathcal{N}_\omega)$, or $Y$ is an inner-one $A$-space. If $Y$ is inner-one $A$, by [10, Theorem 9.9] each $\partial f^{-1}(y)$ is countably compact.

**Lemma 2.10 [10].** Bi-$k$-spaces are preserved by perfect images and countable products.

By invoking Corollary 1.5, and Lemmas 2.7, 2.8, 2.9 and 2.10, we obtain the following theorem.

**Theorem 2.11 (CH).** Let $f : X \to Y$ be a closed map with $X$ paracompact $bi-k$. If $t(Y) = \omega$, then the following are equivalent. When $Y$ is sequential, the CH assumption can be omitted.

(a) $Y^2$ is a $k$-space.
(b) $Y$ is locally $k_\omega$, or each $\partial f^{-1}(y)$ is compact.
(c) $Y$ is locally $k_\omega$, or $bi-k$.

**Corollary 2.12.** Let $f : X \to Y$ be a closed map with $X$ or $Y$ sequential. Let $X$ be a paracompact space of pointwise countable type. Then $Y^2$ is a sequential space (equivalently, a $k$-space by [18, Theorem 2.2]) if and only if $Y$ is locally $k_\omega$ or $bi-k$.

Before proceeding with the next lemma, we remind the reader that the perfect image of an $\mathcal{N}$-space is an $\mathcal{N}$-space, but the closed image of a locally compact metric space need not be $\mathcal{N}$-space (cf. [21, Theorem 7]).

**Lemma 2.13.** Let $f : X \to Y$ be a closed map with each $\partial f^{-1}(y)$ Lindelöf. If $X$ is an $\mathcal{N}$-space, and $Y$ satisfies condition $K(\mathcal{N}_\omega)$, then $Y$ is also an $\mathcal{N}$-space.

**Proof.** Let $\mathcal{P} = \bigcup_{i=1}^\infty \mathcal{P}_i$ be a $\sigma$-locally finite $k$-network for $X$ satisfying the following conditions: Each element of $P$ is closed, $\mathcal{P}_i \subseteq \mathcal{P}_{i+1}$ and $\mathcal{P}_i$ is closed with respect to finite intersections. Let $K$ be an arbitrary compact subset of $X$. Since each subset of an $\mathcal{N}$-space is an $\mathcal{N}$-space, as in the proof of [7, Corollary 1.2], we can assume that each $f^{-1}(y)$ is Lindelöf and that there exists a compact subset $C$ of $X$ with $f(C) = K$. 


Let $\mathcal{P}' = \{ P \in \mathcal{P} : \mathcal{P} \cap C \neq \emptyset \}$, and let $\mathcal{C}$ be the collection of finite unions of elements of $\mathcal{P}'$ which contain the compact subset $C$. Then $\mathcal{C}$ is a nonempty, countable collection in $X$.

Let $\mathcal{C} = \{ P_i : i \in \omega \}$ and $C_n = \bigcap_{i=1}^{n} P_i$ for each $n$. Then $(C_n)$ is a $k$-sequence for $C$. Since $(f(C_n))$ is a $k$-sequence for $K$, by $K(\mathcal{N})$ there exists a compact subset $f(C_n)$ of $Y$. On the other hand, by the conditions of the collection $\mathcal{P}$, each $C_n$ can be expressed as a union of finitely many elements of $\mathcal{P}$. So, the compact subset $f(C_n)$ containing $K$ can be expressed as a union of finitely many elements of $f(\mathcal{P})$. Let $\mathcal{K}_i = \{ f(P) : P \in \mathcal{P} \text{ and } f(P) \text{ is compact in } Y \}$, and let $\mathcal{K}_i^*$ be the union of all elements of $\mathcal{K}_i$. Then, since $f(\mathcal{P}) \subseteq f(\mathcal{P}_{i+1})$, by the above, each compact subset of $Y$ is contained in some $\mathcal{K}_i^*$.

We will now prove that $Y$ is an $\mathcal{N}$-space. Each $\mathcal{K}_i$ is a hereditarily closure-preserving collection of compact subsets of $Y$, that is, whenever a subset $K'$ of $K$ is chosen for each $K \in \mathcal{K}_i$, the collection $\{ K' : K \in \mathcal{K}_i \}$ is closure-preserving. This is because $\mathcal{K}_i$ is the image of a locally finite, hence hereditarily closure-preserving, collection under a closed map. Then by a result of Michael [6, Theorem 1], each $\mathcal{K}_i^*$ is paracompact. Next, to see each $\mathcal{K}_i^*$ is locally $\mathcal{N}_0$, let $\mathcal{H}_i = \{ P \in \mathcal{P} : f(P) \in \mathcal{K}_i \}$ and let $\mathcal{K}_i^* = \bigcup \mathcal{H}_i$. Then $\mathcal{K}_i^*$ has the weak topology with respect to the locally finite closed collection $\mathcal{H}_i$. Also, $f(\mathcal{K}_i^*)$ is closed, hence quotient. Thus $\mathcal{K}_i^* = f(\mathcal{K}_i^*)$ has the weak topology with respect to $\mathcal{K}_i$. Since $f$ is closed and each $f^{-1}(y)$ is Lindelöf, $\mathcal{K}_i$ is locally countable. Hence each $\mathcal{K}_i^*$ is a locally $k_\omega$-space. Since each compact subset of $X$ is an $\mathcal{N}_0$-space, by [8, G] each compact subset of $Y$ is also $\mathcal{N}_0$ because it is the image of a compact subset of $X$. Then each $\mathcal{K}_i^*$ is a locally $\mathcal{N}_0$-space, since each point has a neighborhood which has the weak topology with respect to a countable collection of compact $\mathcal{N}_0$-spaces (see [8]). So, each $\mathcal{K}_i^*$ is a paracompact, locally $\mathcal{N}_0$-space. It follows that each $\mathcal{K}_i^*$ is also an $\mathcal{N}$-space. As is seen, each compact subset of $Y$ is contained in some $\mathcal{K}_i^*$. Since each $\mathcal{K}_i^*$ is an $\mathcal{N}$-space, it follows that $Y$ is also an $\mathcal{N}$-space. This completes the proof of the lemma.

**Lemma 2.14 [19, Theorem 3.1].** Let $Y$ be a $k$- and $\mathcal{N}$-space. Then $Y^2$ is a $k$- and $\mathcal{N}$-space if and only if $Y$ is metrizable, or $Y$ has the weak topology with respect to a countable covering of closed and locally compact subsets of $Y$.

Let a $k$-space $Y$ be the closed image of an $\mathcal{N}$-space. Since each closed subset of an $\mathcal{N}$-space is easily seen to be a $G_\delta$-set, each point of $Y$ is a $G_\delta$-set. Thus by [10, Theorem 7.3], $Y$ is sequential. Therefore, by Corollary 1.7, and Lemmas 2.13 and 2.14, we have

**Theorem 2.15.** Let $f : X \to Y$ be a closed map with $X$ a paracompact $\mathcal{N}$-space. Then $Y^2$ is a $k$-space if and only if $Y$ is metrizable, or $Y$ is an $\mathcal{N}$-space having the weak topology with respect to a countable covering of closed and locally compact subsets of $Y$.

**Remark.** Let $X$ be an $\mathcal{N}$-space each of whose countable (resp. uncountable) subset has an accumulation point. Then $X$ is an $\mathcal{N}_0$-space, and so $X$ is compact (resp. Lindelöf). Thus, by Theorem 1.6, we have the following.

If an $\mathcal{N}$-space $X$ is more generally strongly collectionwise Hausdorff, then the statement of Theorem 2.15 is also valid.
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