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PRODUCTS OF k-SPACES AND SPACES OF COUNTABLE TIGHTNESS

BY

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ABSTRACT. In this paper, we obtain results of the following type: if $f: X \to Y$ is a closed map and X is some "nice" space, and Y^2 is a k-space or has countable tightness, then the boundary of the inverse image of each point of Y is "small" in some sense, e.g., Lindelöf or ω_1-compact. We then apply these results to more special cases. Most of these applications combine the "smallness" of the boundaries of the point-inverses obtained from the earlier results with "nice" properties of the domain to yield "nice" properties on the range.

Introduction. Recall the following theorem due to Morita and Hanai [**14**] and Stone [**17**].

THEOREM. *If $f: X \to Y$ is closed and X is metrizable, then the following are equivalent.*

(a) *Y is first countable;*

(b) *For each $y \in Y$, $\partial f^{-1}(y)$ is compact;*

(c) *Y is metrizable.*

The (c) $\Rightarrow$ (b) part is due to Vaĭnšteĭn [**22**]. But even the (a) $\Rightarrow$ (b) part holds under much more general conditions: Michael [**7**] showed (b) holds if X is paracompact, and Y is locally compact or first-countable.

Note that the assumptions on Y in Michael's theorem could not be weakened to "Y is a k-space" or "Y has countable tightness": the map identifying the limit points of a topological sum of κ convergent sequences is a closed map from a metrizable space X to a Fréchet space Y, and $|\partial f^{-1}(y)| = \kappa$ for some $y \in Y$. In this paper, we show that the situation is different if we require Y^2 to be a k-space or have countable tightness. (Recall that the square of a k-space or a space of countable tightness need not have the same property.) We will usually not be able to show that the boundaries of point-inverses are compact, but we will often (depending upon conditions imposed on X or Y) be able to show that they are "small" in some sense, e.g., Lindelöf or ω_1-compact. In the second section, we apply general results of this type to more special cases, often combining the "smallness" of the boundaries of point-inverses with "nice" properties of X to obtain "nice" properties of Y.

We mention the following earlier result of the second author [**21**] which is related to this topic.

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THEOREM. *If f: $X \to Y$ is closed and X is metrizable, then the following are equivalent.*

(a) *For each $y \in Y$, $\partial f^{-1}(y)$ is Lindelöf.*

(b) *Y has a point-countable k-network* [*see* §2, *Definition* 2.1].

(c) *Y has a σ-locally-countable k-network.*

See [7] and [**21**] for other related results.

We will often make use of the following well-known property of closed maps (cf. [**3**, p. 52]): If f: $X \to Y$ is closed, then for each $y \in Y$ and open $U \subset X$ such that $f^{-1}(y) \subset U$, there is a neighborhood V of y such that $f^{-1}(V) \subset U$.

1. General results. All our spaces are assumed to be regular and T_1. We consider cardinals to be initial ordinals. We now recall some basic definitions.

DEFINITION 1.1. A space X has the *weak topology with respect to a collection $\mathcal{C}$ of sets* if a subset A of X is closed (resp., open) in X if and only if $A \cap C$ is closed (resp., open) in C for each $C \in \mathcal{C}$.

DEFINITION 1.2. A space X is a *k-space* (*quasi-k-space*) if X has the weak topology with respect to its compact (countably compact) subsets. X is *sequential* if X has the weak topology with respect to its compact metric subspaces (equivalently, with respect to its subspaces homeomorphic to $\omega + 1$, a sequence with its limit point). X has *countable tightness* (denoted by $t(X) \leqslant \omega$) if it has the weak topology with respect to its countable subsets.

We will be using the following elementary facts about these concepts.

(i) If X has the weak topology with respect to a collection $\mathcal{C}$, and f: $X \to Y$ is a quotient map, then Y has the weak topology with respect to $\{f(C)\colon C \in \mathcal{C}\}$. Thus all properties named in Definition 1.2 are preserved by quotient maps.

(ii) If X satisfies any of the properties in Definition 2.2 locally, then the whole space has the property.

(iii) If X has a locally finite cover by a family $\mathcal{C}$ of closed sets, then X has the weak topology with respect to $\mathcal{C}$.

DEFINITION 1.3. A space X is (*strongly*) *collectionwise Hausdorff* if whenever $\{x_\alpha\colon \alpha \in A\}$ is a closed discrete subset of X, there exists a (discrete) disjoint collection $\{U_\alpha\colon \alpha \in A\}$ of open sets such that $x_\alpha \in U_\alpha$ for each $\alpha \in A$.

Note that every normal collectionwise Hausdorff space is strongly collectionwise Hausdorff.

Let c denote the cardinality of the continuum.

THEOREM 1.4. *Suppose f: $X \to Y$ is closed, with X strongly collectionwise Hausdorff. Then the boundary, $\partial f^{-1}(y)$, of $f^{-1}(y)$ is c-compact for each $y \in Y$ if either*

(a) *Y^2 is quasi-k and $t(Y) \leqslant \omega$ or*

(b) *$t(Y^2) \leqslant \omega$.*

PROOF. Suppose $\partial f^{-1}(y)$ is not c-compact. Then there is a closed discrete subset $D \subset \partial f^{-1}(y)$, with $|D| = c$. For each $d \in D$, let U'_d be an open set containing d such that $\{U'_d\colon d \in D\}$ is discrete.

Let $d \in U_d \subset \overline{U}_d \subset U'_d$, where U_d is open. Note that $\{U_d\colon d \in D\}$ is also discrete.

For each $d \in D$, $y \in \overline{f(U_d - f^{-1}(y))}$. Since $t(Y) \leqslant \omega$, there is a countable set $\{y_{d,n}: n \in \omega\} \subset f(U_d - f^{-1}(y))$ such that $y \in \overline{\{y_{d,n}: n \in \omega\}}$.

Let $X_{d,n} = f^{-1}(y_{d,n}) \cap \overline{U}_d$, and let $X_d = f^{-1}(y) \cap \overline{U}_d$.

If O is open and contains X_d, then there is an open set O' such that $O' \cap \overline{U}_d = \varnothing$, and $f^{-1}(y) \subset O \cup O'$. Let W be the complement in Y of $f(X - (O \cup O'))$. Then $y \in W$, so there is $n \in \omega$ such that $y_{d,n} \in W$, and hence $f^{-1}(y_{d,n}) \subset O \cup O'$. Thus $X_{d,n} \subset (O \cup O') \cap \overline{U}_d = O \cap \overline{U}_d \subset O$. Choose $x_{d,n} \in X_{d,n}$, and let $A_d = \{x_{d,n}; n \in \omega\}$. By the above argument, every open set containing X_d contains infinitely many elements of A_d.

For $x \in A_d$, let $D_x = \{d' \in D$: there exists $x' \in \overline{U}_{d'}$ with $f(x) = f(x')\}$. Let $B_d = \{x \in A_d$: D_x is uncountable$\}$.

Claim 1. $X_d \cap \overline{B}_d = \varnothing$. To see this, let $B_d = \{x_0, x_1, \ldots\}$. Inductively choose a sequence $d_0, d_1, \ldots$ of distinct elements of D, and points $x'_n \in \overline{U}_{d_n}$ such that $f(x'_n) = f(x_n)$. Then $\{x'_0, x'_1, \ldots\}$ is a closed subset of X, so $f(\{x'_0, x'_1, \ldots\}) = f(B_d)$ is closed. Thus $y \notin f(B_d) = \overline{f(B_d)} = f(\overline{B}_d)$, so $X_d \cap \overline{B}_d = \varnothing$.

Let $C_d = A_d - B_d$. By Claim 1, $X_d \cap \overline{C}_d \neq \varnothing$. Pick $d(0) \in D$. Let $D(d(0)) = \bigcup \{D_x; x \in \overline{C}_{d(0)}\}$. Observe that $D(d(0))$ is countable. If $d(\beta)$ has been chosen for all $\beta < \alpha < c$, let $d(\alpha) \in D - \bigcup_{\beta<\alpha} D(d(\beta))$. Observe that if $x \in C_{d(\beta)}$ and $x' \in C_{d(\alpha)}$ with $\alpha \neq \beta$, then $f(x) \neq f(x')$.

Now let $\mathcal{E} = \{E_\alpha: \alpha < c\}$ index all subsets of $\bigcup_{n\in\omega} C_{d(n)}$ such that $|E_\alpha \cap C_{d(n)}| = 1$ for each $\alpha < c$ and $n \in \omega$. Let $E_\alpha = \{e_{\alpha n}; n \in \omega\}$ such that $e_{\alpha n} \in C_{d(n)}$. Let $C_{d(\alpha)} = \{c_{\alpha n}; n \in \omega\}$.

For $x \in X$, denote $f(x)$ by x^*. Let $H_\alpha = \{(e^*_{\alpha n}, c^*_{\lambda_\alpha n}): n \in \omega\} \subset Y^2$, where λ_α is the αth limit ordinal, and let $H = \bigcup_{\alpha<c} H_\alpha$.

Claim 2. $(y, y) \in \overline{H} - H$. To see this, suppose $y \in O$, O open in Y. We know $X_{d(\alpha)} \cap \overline{C}_{d(\alpha)} \neq \varnothing$, so $y \in f(\overline{C}_{d(\alpha)}) = \overline{f(C_{d(\alpha)})}$. For each $n \in \omega$, choose $x_n \in C_{d(n)}$ such that $x^*_n \in O$. Then $\{x_n: n \in \omega\} = E_\alpha$ for some α, and $e_{\alpha n} = x_n$ for each n. There is $n \in \omega$ such that $c^*_{\lambda_\alpha n} \in O$. Thus $(e^*_{\alpha n}, c^*_{\lambda_\alpha n}) \in O^2 \cap H_\alpha$, which proves the claim.

The next claim completes the proof of part (a).

Claim 3. If $K \subset Y^2$ is countably compact, then $K \cap H$ is finite. To see this, suppose $\alpha_0, \alpha_1 \ldots$ are distinct ordinals such that for each $n \in \omega$, $K \cap H_{\alpha_n} \neq \varnothing$. Then we can find $(e^*_{\alpha_n k_n}, c^*_{\lambda_{\alpha_n} k_n}) \in K \cap H_{\alpha_n}$. But $\{c_{\lambda_{\alpha_n} k_n}: n \in \omega\}$ is a closed discrete subset of X, since $c_{\lambda_{\alpha_n} k_n} \in \overline{U}_{d(\lambda_{\alpha_n})}$. Thus $\{(e^*_{\alpha_n k_n}, c^*_{\lambda_{\alpha_n} k_n}); n \in \omega\}$ is an infinite closed discrete subset of K, contradiction. Thus K meets only finitely many H_α's. Now suppose that for fixed α, $K \cap H_\alpha$ is infinite. Then for each $n \in \omega$, we can find $(e^*_{\alpha k_n}, c^*_{\lambda_\alpha k_n}) \in K \cap H_\alpha$. But $\{e_{\alpha k_n}: n \in \omega\}$ is an infinite closed discrete subset of X and we get a contradiction as before. Thus each $K \cap H_\alpha$ is finite, and so $K \cap H$ is finite.

To complete the proof of part (b), we have the next claim.

Claim 4. No countable subset of H contains (y, y) in its closure. Suppose $C \subset H$, $|C| \leqslant \omega$. Then there exists a sequence $\alpha_0, \alpha_1, \ldots$ of distinct ordinals such that $C \subset \bigcup_{n\in\omega} H_{\alpha_n}$. For each n, let $U_n \subset U'_d$ be an open set in X containing $X_{d(n)}$ such that $e_{\alpha_k n} \notin U_n$ if $k \leqslant n$. Note $U_n \cap \overline{U}_d = \varnothing$ if $d \neq d(n)$. Let V_n be an open set in X

containing $X_{d(\lambda_{\alpha_n})}$ such that $c_{\lambda_{\alpha_n k}} \notin V_n$ if $k \leq n$, and $V_n \cap \overline{U}_d = \varnothing$ if $d \neq d(\lambda_{\alpha_n})$.

If $x \in f^{-1}(y) - \bigcup_{n\in\omega}(U_n \cup V_n)$, then there is an open set O_x containing x such that $O_x \cap \overline{U}_d = \varnothing$ if $d \in \{d(n): n \in \omega\} \cup \{d(\lambda_{\alpha_n}): n \in \omega\}$. Let

$$O = \bigcup \left\{O_x : x \in f^{-1}(y) - \bigcup_{n\in\omega}(U_n \cup V_n)\right\} \cup \left(\bigcup_{n\in\omega}(U_n \cup V_n)\right)$$

There is an open set W containing y such that $f^{-1}(W) \subset O$. Suppose $W^2 \cap C \neq \varnothing$. Then there exists m such that $W^2 \cap H_{\alpha_m} \neq \varnothing$. Choose $n \in \omega$ such that $(e^*_{\alpha_m n}, c^*_{\lambda_{\alpha_m n}}) \in W^2 \cap H_{\alpha_m}$. Then $(e_{\alpha_m n}, c_{\lambda_{\alpha_m n}}) \in O^2$. Recall $e_{\alpha_m} \in C_{d(n)} \subset \overline{U}_{d(n)}$. Thus $e_{\alpha_m n} \in U_n$, and so $m > n$. Also recall $c_{\lambda_{\alpha_m n}} \in C_{d(\lambda_{\alpha_m})} \subset \overline{U}_{d(\lambda_{\alpha_m})}$. Thus $c_{\lambda_{\alpha_m n}} \in V_m$, which means $n > m$, a contradiction. This proves Claim 4. Hence Y^2 does not have countable tightness, a contradiction which proves the theorem. □

Assuming the continuum hypothesis (CH), we have the following corollary.

COROLLARY 1.5 (CH). *Suppose f: $X \to Y$ is closed, with X paracompact. Then each $\partial f^{-1}(y)$ is Lindelöf if either Y^2 is a k-space with $t(Y) \leq \omega$, or $t(Y^2) \leq \omega$.*

PROOF. Immediate from Theorem 1.4 and the fact that ω_1-compact paracompact spaces are Lindelöf [**1**].

REMARK. By the proof below, if Y^2 is a k-space with $t(Y) \leq \omega$, then $t(Y^2) \leq \omega$. Thus the two conditions are not independent.

PROOF. Since Y^2 is a k-space, it has the weak topology with respect to the collection of compact subsets of Y^2; that is $A \subset Y^2$ is closed whenever $A \subset C$ is closed in C for every compact subset of C of Y^2. Each compact subset C of Y^2 is contained in $\pi(C)^2$, where π is the projection from Y^2 onto Y. Then Y^2 has the weak topology with respect to $\{\pi(C)^2;\ C$ is compact in $Y^2\}$. Since each $\pi(C)$ is a compact space of countable tightness, by a result of V. I. Malyhin [**5**, Theorem 4], so is each $\pi(C)^2$. Then $t(Y^2) \leq \omega$. □

We do not know if Corollary 1.5 is true without CH. The problem seems to hinge on strengthening the conclusion of Theorems 1.3 and 1.4 by replacing "c-compact" with "ω_1-compact". It turns out if we add the condition "Y is sequential" to the hypotheses of these theorems, then we can do it.

THEOREM 1.6. *Suppose f: $X \to Y$ is closed with X strongly collectionwise Hausdorff and Y sequential. Then each $\partial f^{-1}(y)$ is ω_1-compact if either Y^2 is a quasi-k-space or $t(Y^2) \leq \omega$.*

PROOF. Suppose Y^2 is a quasi-k-space. Since Y is sequential, by [**18**, Theorem 2.2] Y^2 is sequential, hence $t(Y^2) \leq \omega$. Thus we can assume that $t(Y^2) \leq \omega$. Suppose $\partial f^{-1}(y)$ is not ω_1-compact. Then there is a closed discrete set $D \subset \partial f^{-1}(y)$ with $|D| = \omega_1$. Let $\{U_d;\ d \in D\}$ be a discrete collection of open sets in X with $d \in U_d$. Then $f(\overline{U}_d)$ is a closed subset of the sequential space Y, and is therefore sequential. Since y is not isolated in $f(\overline{U}_d)$, there exists a sequence $y_{d,n} \to y$, with $y_{d,n} \in f(\overline{U}_d) - \{y\}$ for each $n \in \omega$. Choose $x_{d,n} \in \overline{U}_d \cap f^{-1}(y_{d,n})$. As in the proof of Theorem 1.3, we can construct $\{d(\alpha);\ \alpha < \omega_1\} \subset D$ and an infinite set $C_{d(\alpha)} \subset \{x_{d(\alpha),n};\ n \in \omega\}$ such that f is 1-1 on $\bigcup_{\alpha<\omega_1} C_{d(\alpha)}$. Let $X_{d(\alpha)} = \overline{U}_{d(\alpha)} \cap f^{-1}(y)$. Observe that every open set containing $X_{d(\alpha)}$ contains all but finitely many elements fo $C_{d(\alpha)}$, and that

$f^{-1}(y) \cup (\bigcup_{\alpha<\omega_1} C_{d(\alpha)})$ is closed. Thus Y contains a closed copy of the space obtained by identifying the limit points of ω_1 convergent sequences. In [**4**], this space is denoted by S_{ω_1}, and it is proved there that $S^2_{\omega_1}$ is not a k-space.

To complete the proof, it is sufficient to show that $S^2_{\omega_1}$ does not have countable tightness. For each $\alpha < \omega_1$, let $S_\alpha \subset S_{\omega_1}$ be the union of the first α sequences (with limit point). The closure of a countable subset of $S^2_{\omega_1}$ is contained in some S^2_α. Thus if $S^2_{\omega_1}$ had countable tightness, then it would have the weak topology with respect to $\{S^2_\alpha;\ \alpha < \omega_1\}$. But each S^2_α is a k-space (cf. [**9**, (7.5)]), so then $S^2_{\omega_1}$ would be a k-space, contradiction. □

COROLLARY 1.7. *Suppose f: $X \to Y$ is closed with X paracompact and Y sequential. Then each $\partial f^{-1}(y)$ is Lindelöf if either Y^2 is a k-space or $t(Y^2) \leqslant \omega$.*

The following example shows that the assumption "Y^2 is a k-space" is not sufficient to obtain "$\partial f^{-1}(y)$ Lindelöf" in Corollary 1.5.

EXAMPLE 1.8. There exists f: $X \to Y$ closed with X locally compact and paracompact, such that Y^2 is a k-space, but $\partial f^{-1}(y)$ is not Lindelöf for some $y \in Y$.

PROOF. For each $\alpha < \omega_1$, let $S(\alpha)$ be a copy of ordinal space $\omega_1 + 1$. Let X be the free union of $\{S(\alpha)\colon \alpha < \omega_1\}$. Let Y be the space obtained from X by identifying the point ω_1 in each copy to a single point ∞. Let f: $X \to Y$ be the quotient map. Then X is paracompact and locally compact, f is closed, and $\partial f^{-1}(\infty)$ is not Lindelöf. X is a k-space (being locally compact), hence so is Y.

It remains to prove that Y^2 is a k-space. First we introduce some notation. For each $\alpha, \beta \leqslant \omega_1$, let $\beta(\alpha)$ be the image under f of the element of $S(\alpha)$ corresponding to the ordinal number β. If $\beta < \beta' \leqslant \omega_1$, let $[\beta(\alpha), \beta'(\alpha)] = \{\gamma(\alpha)\colon \beta \leqslant \gamma \leqslant \beta'\}$, and let $[\beta(\alpha), \infty] = [\beta(\alpha), \omega_1(\alpha)]$.

Suppose $A \subset Y^2$, with A k-closed, but not closed. Since for each $\alpha, \beta < \omega_1$, $[O(\alpha), \beta(\alpha)] \times Y$ and $Y \times [O(\alpha), \beta(\alpha)]$ are clopen k-subspaces of Y^2, it must be true that $(\infty, \infty) \in \bar{A} - A$.

Since $[f(S(O))]^2 \cap A$ is closed, there exists $\gamma_0 < \omega_1$ such that $[\gamma_0(O), \infty]^2 \cap A = \varnothing$. Now suppose γ_α has been defined for all $\alpha < \beta$, where $\beta < \omega_1$, in such a way that the following property P_α holds.

$$P_\alpha\colon (\beta_1(\alpha_1), \beta_2(\alpha_2)) \in A \quad \text{and} \quad \alpha_1, \alpha_2 \leqslant \alpha \text{ implies } \beta_1 < \gamma_{\alpha_1} \text{ or } \beta_2 < \gamma_{\alpha_2}.$$

It is easy to check that P_0 holds from the way γ_0 has been defined. We will show how to define γ_β in such a way that P_β holds.

For each $\alpha \leqslant \beta$, $f(S(\alpha) \times S(\beta)) \cap A$ and $f(S(\beta) \times S(\alpha)) \cap A$ are closed, so there exists $\delta_{\alpha,\beta} < \omega_1$ such that

(i) $\gamma_\alpha \leqslant \delta_{\alpha,\beta}$;

(ii) $([\delta_{\alpha,\beta}(\alpha), \infty] \times [\delta_{\alpha,\beta}(\beta), \infty]) \cap A = \varnothing$; and

(iii) $([\delta_{\alpha,\beta}(\beta), \infty] \times [\delta_{\alpha,\beta}(\alpha), \infty]) \cap A = \varnothing$.

For each $\alpha < \beta$, and each $\beta' \in [\gamma_\alpha, \delta_{\alpha,\beta})$, we have by P_α that $(\beta'(\alpha), \infty) \notin A$ and $(\infty, \beta'(\alpha)) \notin A$ and $(\infty, \beta'(\alpha)) \notin A$. Thus there exists $\delta^\alpha_{\beta',\beta} < \omega_1$ such that

(a) $([\delta^\alpha_{\beta',\beta}(\beta), \infty] \times \{\beta'(\alpha)\}) \cap A = \varnothing$ and

(b) $(\{\beta'(\alpha)\} \times [\delta^\alpha_{\beta',\beta}(\beta), \infty]) \cap A = \varnothing$.

Let $\gamma_\beta = \sup\{\delta_{\alpha,\beta}: \alpha \leqslant \beta\} + \sup\{\delta^\alpha_{\beta',\beta}: \alpha < \beta, \gamma_\alpha \leqslant \beta' < \delta_{\alpha,\beta}\}$. Then $\gamma_\beta < \omega_1$. To show that P_β holds, we can suppose $(\beta_1(\alpha), \beta_2(\beta)) \in A$, with $\alpha \leqslant \beta$. If $\alpha = \beta$, then either $\beta_1 < \delta_{\beta,\beta} \leqslant \gamma_\beta$ or $\beta_2 < \delta_{\beta,\beta} \leqslant \gamma_\beta$, so P_β holds. If $\alpha < \beta$, we can suppose $\beta_2 \geqslant \gamma_\beta$. Then it must be true that $\beta_1 < \delta_{\alpha,\beta}$ (by (ii) above). If $\gamma_\alpha \leqslant \beta_1 < \delta_{\alpha,\beta}$, then since $\gamma_\beta \geqslant \delta^\alpha_{\beta_1,\beta}$, we have $(\beta_1(\alpha), \beta_2(\beta)) \in (\{\beta_1(\alpha)\} \times [\delta^\alpha_{\beta_1,\beta}(\beta), \infty]) \cap A$, a contradiction. Thus $\beta_1 < \gamma_\alpha$, so P_β holds.

Thus we can define $\{\gamma_\alpha: \alpha < \omega_1\}$ in such a way that P_α holds for each $\alpha < \omega_1$. Let $U = \{\beta(\alpha): \beta > \gamma_\alpha, \alpha < \omega_1\}$. Then U is an open set in Y containing ∞. Since $(\infty, \infty) \in \bar{A}$, there exists $(\beta_1(\alpha_1)\beta_2(\alpha_2)) \in U^2 \cap A$. Since $P_{\alpha_1+\alpha_2}$ holds, either $\beta_1 < \gamma_{\alpha_1}$ or $\beta_2 < \gamma_{\alpha_2}$. But then either $\beta_1(\alpha_1) \notin U$ or $\beta_2(\alpha_2) \notin U$, contradiction. Thus Y^2 is a k-space.

2. Applications. As applications of results in §1, we shall consider the products of k-spaces and spaces of countable tightness in more special cases.

DEFINITION 2.1 [**8, 16**]. A collection $\mathcal{P}$ of (not necessarily open) subsets of a space X is a *k-network* for X if, whenever $C \subset U$ with C compact and U open, then $C \subset \bigcup \mathcal{F} \subset U$ for some finite subcollection $\mathcal{F}$ of $\mathcal{P}$. An *$\aleph$-space* is a space with a σ-locally finite k-network, and an *$\aleph_0$-space* is a space with a countable k-network.

Note that metrizable spaces are $\aleph$-spaces, and separable metrizable spaces are $\aleph_0$-spaces.

We say that X is a *locally $\aleph_0$-space* if each point of X has a neighborhood which is an $\aleph_0$-space.

THEOREM 2.2 (CH) *Let $f: X \to Y$ be a closed map. Let X be a paracompact, locally $\aleph_0$-space. Then the following are equivalent.*

(a) $t(Y^2) \leqslant \omega$;

(b) *each $\partial f^{-1}(y)$ is Lindelöf*;

(c) *Y is a locally $\aleph_0$-space*; *and*

(d) *Y is locally separable.*

Furthermore, if Y is sequential, then the CH *assumption can be omitted.*

PROOF. (a) $\Rightarrow$ (b): This is Corollary 1.5.

(b) $\Rightarrow$ (c): Since each subset of a locally $\aleph_0$-space is locally $\aleph_0$, as in the proof of [**7**, Corollary 1.2], we can assume that each $f^{-1}(y)$ is Lindelöf. Thus, f is a closed map with each $f^{-1}(y)$ Lindelöf. Then, for each $y \in Y$, there is a closed neighborhood W of y in Y, and open subsets V_i of X which are $\aleph_0$-spaces such that $f^{-1}(W) \subset \bigcup_{i=1}^\infty V_i$. Since $\bigcup_{i=1}^\infty V_i$ is an $\aleph_0$-space, so is $f^{-1}(W)$. Since the closed image of an $\aleph_0$-space is also $\aleph_0$ by [**8**, G], W is an $\aleph_0$-space. This implies (c).

(c) $\Rightarrow$ (a) and (c) $\Rightarrow$ (d): By [**8**, F], Y^2 is a locally $\aleph_0$-space. Then, by [**8**, D, E] Y^2 is locally a hereditarily separable space. Hence $t(Y^2) \leqslant \omega$.

(d) $\Rightarrow$ (b): This follows from [**21**, Proposition 1], because Y is paracompact, hence is locally Lindelöf by (d).

From Theorem 2.2 and some results in [**21**], we have

COROLLARY 2.3. *Let $f: X \to Y$ be a closed map with X locally separable metric. Then the following are equivalent.*

(a) $t(Y^2) \leqslant \omega$;
(b) *each* $\partial f^{-1}(y)$ *is Lindelöf*;
(c) *Y is locally separable*;
(d) *Y is locally Lindelöf*;
(e) *Y is an $\aleph$-space.*

DEFINITION 2.4. A decreasing sequence (A_n) in a space X is a *k-sequence* [**10**], if it is an outer network at a compact subset K of X; that is, $K = \bigcap_{n=1}^{\infty} A_n$ and every neighborhood of K contains some A_n. By regularity, if a compact subset K has an outer network, it has one in which each A_n is closed in X.

Let Y be a space. Then Y *satisfies condition* $K(\aleph_0)$ [**20**] if, for any k-sequence (A_n) in Y, some A_n is countably compact.

It is shown that [**20**, Proposition 2.4] a space Y satisfies $K(\aleph_0)$ if and only if each closed subset of Y which is a paracompact M-space is locally compact.

DEFINITION 2.5 [**10**, Lemma 3.E.2]. A space Y is a *bi-k-space* if, whenever a filter base $\mathfrak{F}$ accumulates at y in Y, then there exists a k-sequence (A_n) in Y such that $y \in \overline{F \cap A_n}$ for all $n \in N$ and all $F \in \mathfrak{F}$.

It is shown that [**10**, Theorem 3.E.3] Y is a bi-k-space if and only if Y is a bi-quotient image of a paracompact M-space X. Then by a result of H. Wicke [**23**], spaces of pointwise countable type [**2**] are bi-k.

DEFINITION 2.6. A space X is a k_ω-space [**9**] (K. Morita [**13**] called it a *space of class* $\mathfrak{S}'$), if it has the weak topology with respect to a countable covering of compact subsets of X.

For a space Y we shall say that Y is a *locally k_ω-space*, if each point of Y has a neighborhood whose closure is a k_ω-space.

It is implicit in a result of J. Milnor [**12**, Lemma 2.1] that the product of two k_ω-spaces is k_ω. This fact implies the following lemma.

LEMMA 2.7. *Let Y be a locally k_ω-space. Then Y^2 is a locally k_ω-space, hence, a k-space.*

LEMMA 2.8. *Let f: $X \to Y$ be a closed map with each $\partial f^{-1}(y)$ Lindelöf. If X is bi-k and Y satisfies $K(\aleph_0)$, then Y is a locally k_ω-space.*

PROOF. Since each closed subset of X is a bi-k, as in the proof of [**7**, Corollary 1.2], we can assume that each $f^{-1}(y)$ is Lindelöf. Let $y \in Y$. Then we will prove that each point of $f^{-1}(y)$ has a neighborhood contained in the inverse image of some compact subset of Y. To see this, suppose not. Then there is a point a_0 of $f^{-1}(y)$ such that for every neighborhood V of a_0 and for every compact subset K of Y, $V \not\subseteq f^{-1}(K)$.

Let $\mathfrak{F} = \{X - f^{-1}(K);\ K \text{ is compact in } Y\}$. Then $\mathfrak{F}$ is a filter base accumulating at the point a_0. Since X is bi-k, there exists a k-sequence (A_n) in X such that $a_0 \in \overline{F \cap A_n}$ for all $n \in \omega$ and all $F \in \mathfrak{F}$. Obviously, $(f(A_n))$ is a k-sequence in Y. Thus, by condition $K(\aleph_0)$, some $f(A_{n_0})$ is compact. Let $K_0 = f(A_{n_0})$. Then, $a_0 \in \overline{(X - f^{-1}(K_0)) \cap A_{n_0}} \subset \overline{(X - f^{-1}(K_0)) \cap f^{-1}(K_0)} = \varnothing$. This is a contradiction.

Thus, each point x of $f^{-1}(y)$ has a neighborhood V_x which is contained in the inverse image of some compact subset of Y. Since $f^{-1}(y)$ is Lindelöf, $\{V_x: x \in f^{-1}(y)\}$ contains a countable subcover $\{V_n\}_{n\in\omega}$ of $f^{-1}(y)$. For each n, let K_n be a compact subset of Y such that $V_n \subset f^{-1}(K_n)$. Since f is closed and Y is regular there exists a neighborhood W of Y such that $f^{-1}(\overline{W}) \subset \bigcup_{n\in\omega} V_n$. Let $F = f^{-1}(\overline{W})$ and $\mathcal{V} = \{F \cap V_i: i \in \omega\}$. Then, since $\mathcal{V}$ is an open covering of F, F has the weak topology with respect to $\mathcal{V}$. Since $F \cap V_i \subseteq F \cap f^{-1}(K_i)$ for each $i \in \omega$, F has the weak topology with respect to $\{F \cap f^{-1}(K_i); i \in \omega\}$. Since $f \mid F$ is closed, hence quotient, $f(F) = \overline{W}$ has the weak topology with respect to $\{\overline{W} \cap K_i: i \in \omega\}$. Thus $\overline{W}$ is a k_ω-space, and so Y is a locally k_ω-space. □

LEMMA 2.9. *Let f: $X \to Y$ be a closed map with X normal and $t(Y) \leqslant \omega$. If Y^2 is a k-space, then either Y satisfies condition $K(\aleph_0)$ or each $\partial f^{-1}(y)$ is countably compact.*

PROOF. According to [**20**, Theorem 4.2], if the product of two spaces is quasi-k, and one factor is not an inner-one A-space in the sense of E. Michael, R. C. Olson and F. Siwiec [**11**], then the other factor satisfies $K(\alpha)$, where α is its tightness. Y satisfies condition $K(\aleph_0)$, or Y is an inner-one A-space. If Y is inner-one A, by [**10**, Theorem 9.9] each $\partial f^{-1}(y)$ is countably compact.

LEMMA 2.10 [**10**]. *Bi-k-spaces are preserved by perfect images and countable products.*

By invoking Corollary 1.5, and Lemmas 2.7, 2.8, 2.9 and 2.10, we obtain the following theorem.

THEOREM 2.11 (CH). *Let f: $X \to Y$ be a closed map with X paracompact bi-k. If $t(Y) \leqslant \omega$, then the following are equivalent. When Y is sequential, the* CH *assumption can be omitted.*

(a) *Y^2 is a k-space.*

(b) *Y is locally k_ω, or each $\partial f^{-1}(y)$ is compact.*

(c) *Y is locally k_ω, or bi-k.*

COROLLARY 2.12. *Let f: $X \to Y$ be a closed map with X or Y sequential. Let X be a paracompact space of pointwise countable type. Then Y^2 is a sequential space (equivalently, a k-space by* [**18**, *Theorem* 2.2]) *if and only if Y is locally k_ω or bi-k.*

Before proceeding with the next lemma, we remind the reader that the perfect image of an $\aleph$-space is an $\aleph$-space, but the closed image of a locally compact metric space need not be $\aleph$-space (cf. [**21**, Theorem 7]).

LEMMA 2.13. *Let f: $X \to Y$ be a closed map with each $\partial f^{-1}(y)$ Lindelöf. If X is an $\aleph$-space, and Y satisfies condition $K(\aleph_0)$, then Y is also an $\aleph$-space.*

PROOF. Let $\mathcal{P} = \bigcup_{i=1}^{\infty} \mathcal{P}_i$ be a σ-locally finite k-network for X satisfying the following conditions: Each element of P is closed, $\mathcal{P}_i \subseteq \mathcal{P}_{i+1}$ and $\mathcal{P}_i$ is closed with respect to finite intersections. Let K be an arbitrary compact subset of Y. Since each subset of an $\aleph$-space is an $\aleph$-space, as in the proof of [**7**, Corollary 1.2], we can assume that each $f^{-1}(y)$ is Lindelöf and that there exists a compact subset C of X with $f(C) = K$.

Let $\mathcal{P}' = \{P \in \mathcal{P}: \mathcal{P} \cap C \neq \varnothing\}$, and let $\mathcal{C}$ be the collection of finite unions of elements of $\mathcal{P}'$ which contain the compact subset C. Then $\mathcal{C}$ is a nonempty, countable collection in X.

Let $\mathcal{C} = \{P_i: i \in \omega\}$ and $C_n = \bigcap_{i=1}^{n} P_i$ for each n. Then (C_n) is a k-sequence for C. Since $(f(C_n))$ is a k-sequence for K, by $K(\aleph_0)$ there exists a compact subset $f(C_{n_0})$ of Y. On the other hand, by the conditions of the collection $\mathcal{P}$, each C_n can be expressed as a union of finitely many elements of $\mathcal{P}$. So, the compact subset $f(C_{n_0})$ containing K can be expressed as a union of finitely many elements of $f(\mathcal{P})$. Let $\mathcal{K}_i = \{f(P): P \in \mathcal{P}$ and $f(P)$ is compact in $Y\}$, and let $\mathcal{K}_i^*$ be the union of all elements of $\mathcal{K}_i$. Then, since $f(\mathcal{P}_i) \subset f(\mathcal{P}_{i+1})$, by the above, each compact subset of Y is contained in some $\mathcal{K}_i^*$.

We will now prove that Y is an $\aleph$-space. Each $\mathcal{K}_i$ is a hereditarily closure-preserving collection of compact subsets of Y, that is, whenever a subset K' of K is chosen for each $K \in \mathcal{K}_i$, the collection $\{K': K \in \mathcal{K}_i\}$ is closure-preserving. This is because $\mathcal{K}_i$ is the image of a locally finite, hence hereditarily closure-preserving, collection under a closed map. Then by a result of Michael [**6**, Theorem 1], each $\mathcal{K}_i^*$ is paracompact. Next, to see each $\mathcal{K}_i^*$ is locally $\aleph_0$, let $\mathcal{N}_i = \{P \in \mathcal{P}_i: f(P) \in \mathcal{K}_i\}$ and let $\mathcal{N}_i^* = \bigcup \mathcal{N}_i$. Then $\mathcal{N}_i^*$ has the weak topology with respect to the locally finite closed collection $\mathcal{N}_i$. Also, $f \mid \mathcal{N}_i^*$ is closed, hence quotient. Thus $\mathcal{K}_i^* = f(\mathcal{N}_i^*)$ has the weak topology with respect to $\mathcal{K}_i$. Since f is closed and each $f^{-1}(y)$ is Lindelöf, $\mathcal{K}_i$ is locally countable. Hence each $\mathcal{K}_i^*$ is a locally k_ω-space. Since each compact subset of X is an $\aleph_0$-space, by [**8**, G] each compact subset of Y is also $\aleph_0$ because it is the image of a compact subset of X. Then each $\mathcal{K}_i^*$ is a locally $\aleph_0$-space, since each point has a neighborhood which has the weak topology with respect to a countable collection of compact $\aleph_0$-spaces (see [**8**]). So, each $\mathcal{K}_i^*$ is a paracompact, locally $\aleph_0$-space. It follows that each $\mathcal{K}_i^*$ is also an $\aleph$-space. As is seen, each compact subset of Y is contained in some $\mathcal{K}_i^*$. Since each $\mathcal{K}_i^*$ is an $\aleph$-space, it follows that Y is also an $\aleph$-space. This completes the proof of the lemma.

LEMMA 2.14 [**19**, THEOREM 3.1]. *Let Y be a k- and $\aleph$-space. Then Y^2 is a k- and $\aleph$-space if and only if Y is metrizable, or Y has the weak topology with respect to a countable covering of closed and locally compact subsets of Y.*

Let a k-space Y be the closed image of an $\aleph$-space. Since each closed subset of an $\aleph$-space is easily seen to be a G_δ-set, each point of Y is a G_δ-set. Thus by [**10**, Theorem 7.3], Y is sequential. Therefore, by Corollary 1.7, and Lemmas 2.13 and 2.14, we have

THEOREM 2.15. *Let $f: X \to Y$ be a closed map with X a paracompact $\aleph$-space. Then Y^2 is a k-space if and only if Y is metrizable, or Y is an $\aleph$-space having the weak topology with respect to a countable covering of closed and locally compact subsets of Y.*

REMARK. Let X be an $\aleph$-space each of whose countable (resp. uncountable) subset has an accumulation point. Then X is an $\aleph_0$-space, and so X is compact (resp. Lindelöf). Thus, by Theorem 1.6, we have the following.

If an $\aleph$-space X is more generally strongly collectionwise Hausdorff, then the statement of Theorem 2.15 is also valid.

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