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FREE PRODUCTS OF TOPOLOGICAL GROUPS WITH CENTRAL AMALGAMATION. I

BY

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ABSTRACT. It is proved that the amalgamated free product of any two Hausdorff topological groups exists and is Hausdorff, providing the subgroup which is being amalgamated is closed and central.

1. Introduction. In 1950 M. I. Graev [3] proved that the free product of any two Hausdorff topological groups exists and is Hausdorff. His proof is quite technical, and although some special cases of Graev's result have been proved more easily, no one has produced an easier proof of his general result. In this paper we show that the amalgamated free product of any two Hausdorff topological groups exists and is Hausdorff, providing the subgroup which is being amalgamated is closed and central. This result, of course, includes Graev's and our method of proof is an extension of his. The two extra ingredients of our proof are a nice algebraic representation of free products of groups with a central amalgamated subgroup, and our earlier work on amalgamated direct products of topological groups [4].

The only other published work on amalgamated free products of topological groups of which we know is that by E. T. Ordman [7], who showed that the amalgamated free product of certain locally invariant Hausdorff topological groups exists and is Hausdorff providing the subgroup being amalgamated is closed.

In the sequel to this paper we study the topology of the amalgamated free product more carefully. In particular, when the groups being considered are k_ω-spaces, we are able to give a complete description of it. We also settle the question of when the amalgamated free product is locally compact.

2. Notation and preliminaries. The standard references for amalgamated free products of groups are B. H. Neumann [6] and Magnus, Karrass and Solitar [5].

For completeness we include some definitions here.

DEFINITION. Let A be a common subgroup of groups G and H. The group $G *_A H$ is said to be the *free product of G and H with amalgamated subgroup A* if

(i) G and H are subgroups of $G *_A H$,

(ii) $G \cup H$ generates $G *_A H$, and

(iii) every pair ϕ_1, ϕ_2 of homomorphisms of G and H, respectively, into any group D which agree on A, extend to a homomorphism Φ of $G *_A H$ into D.

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DEFINITION. Let A be a common subgroup of topological groups G and H. The topological group $G \amalg_A H$ is said to be the *free product of the topological groups G and H with amalgamated subgroup A* if

(i) G and H are topological subgroups of $G \amalg_A H$,

(ii) $G \cup H$ algebraically generates $G \amalg_A H$, and

(iii) every pair ϕ_1, ϕ_2 of continuous homomorphisms of G and H, respectively, into any topological group D, which agree on A, extend to a continuous homomorphism of $G \amalg_A H$ into D.

REMARKS. By not restricting ourselves to Hausdorff topological groups in the above definition it follows (by considering indiscrete groups D) that if $G \amalg_A H$ exists then its underlying group structure is $G *_A H$.

Standard categorical arguments show that $G \amalg_A H$ exists if and only if there exists a topological group E having G and H as topological subgroups, with $G \cap H = A$ of course. This, for example, says that if $A = \{e\}$, the identity element, then the free product $G \amalg H$ exists since we can put $E = G \times H$. When $A \neq \{e\}$ but is central in G and H we can put E equal to the topological direct product of G and H with A amalgamated.

DEFINITION. The group E is said to be the *direct product of its subgroups G and H with amalgamated subgroup A* if

(i) E is generated by $G \cup H$,

(ii) $G \cap H = A$, and

(iii) G is contained in the centralizer of H in E.

It is readily seen that the amalgamated direct product exists if and only if A is a central subgroup of G and H.

DEFINITION [4]. A topological group $G \times_A H$ is said to be *the (topological) direct product of its topological subgroups G and H with amalgamated subgroup A* if it has the properties:

(i) $G \times_A H$ is generated algebraically by $G \cup H$,

(ii) $G \cap H = A$,

(iii) G is contained in the centralizer of H in $G \times_A H$, and

(iv) if ϕ_1 and ϕ_2 are any continuous homomorphisms of G and H, respectively, into any topological group D such that ϕ_1 and ϕ_2 agree on A and $\phi_2(H)$ is contained in the centralizer in D of $\phi_1(G)$, then there exists a continuous homomorphism Φ: $G \times_A H \to D$ which extends ϕ_1 and ϕ_2.

In [4] it is shown that the underlying group of $G \times_A H$ is the amalgamated direct product of the underlying groups G and H. Moreover, $G \times_A H$ exists if and only if A is a central subgroup of G and H. The topological group $G \times_A H$ is topologically isomorphic to $(G \times H)/\{(a, a^{-1}): a \in A\}$.

So, returning to our earlier remarks, we see that $G \amalg_A H$ exists if A is central in G and H. Questions about the Hausdorffness of $G \amalg_A H$ are not so easily settled.

We shall show that $G \amalg_A H$ is Hausdorff by verifying that if A is central and closed in the Hausdorff topological groups G and H, then it is possible to put a Hausdorff group topology on $G *_A H$ which induces the given topologies on G and H. This topology will not, in general, be the topology of $G \amalg_A H$. However, it is easily seen that $G \amalg_A H$ must have the finest group topology on $G *_A H$ which induces the given topologies on G and H. Thus $G \amalg_A H$ will be Hausdorff (under the above-mentioned assumptions).

3. The main result.

LEMMA 1. *Let G and H be groups with $G \cap H = A$ a central subgroup of both G and H. Let Γ be the canonical homomorphism of $G * H$ onto $G *_A H$. For $g \in G$ and $h \in H$, $\Gamma([g, h]) = e$ if and only if $g \in A$ or $h \in A$.*

PROOF. Now $\Gamma([g, h]) = g^{-1}h^{-1}gh$. So if $g \notin A$ and $h \notin A$ then $g^{-1}h^{-1}gh$ is in reduced form and so does not equal e. Thus $\Gamma([g, h]) = e$ implies $g \in A$ or $h \in A$.

Conversely, if $g \in A$ or $h \in A$, then by the centrality of A in G and H, $g^{-1}h^{-1}gh = e$ in $G *_A H$, as required.

LEMMA 2. *Let G, H, A and Γ be as in Lemma 1. Further, let $g_1, g_2 \in G$ and $h_1, h_2 \in H$ be such that $\Gamma([g_1, h_1]) \neq e$. Then $\Gamma([g_1, h_1]) = \Gamma([g_2, h_2])$ if and only if $g_2 \in g_1A$ and $h_2 \in h_1A$.*

PROOF. Now $\Gamma([g_1, h_1]) = \Gamma([g_2, h_2])$ implies $g_1^{-1}h_1^{-1}g_1h_1 = g_2^{-1}h_2^{-1}g_2h_2$ in the group $G *_A H$. By Lemma 1 none of g_1, g_2, h_1, h_2 is in A, since $\Gamma([g_1, h_1]) = \Gamma([g_2, h_2]) \neq e$. Thus $g_1^{-1}h_1^{-1}g_1h_1$ and $g_2^{-1}h_2^{-1}g_2h_2$ are in reduced form. Therefore (see, for example, Epstein [**1**]), $g_2 = a_1g_1a_2$ and $h_2 = a_3h_1a_4$, where each $a_i \in A$. So $g_2 = g_1(a_1a_2) \in g_1A$ and $h_2 = h_1(a_3a_4) \in h_1A$, as required.

The converse follows immediately from the centrality of A.

PROPOSITION 1. *Let G and H be groups with $G \cap H = A$ a central subgroup of both G and H. Let $\{s_i\colon i \in I\}$ be a complete set of coset representatives of A in G and $\{s_j\colon j \in J\}$ a complete set of coset representatives of A in H. Then the subgroup $K(G *_A H)$ of $G *_A H$ generated by $\{[g, h]\colon g \in G, h \in H\}$ is a free group with free basis $\{[s_i, s_j]\colon i \in I, j \in J\} \setminus \{e\}$.*

PROOF. Recall that if C and D are any groups, the cartesian subgroup $K(C * D)$ of the free product $C * D$ is the group generated by $\{[c, d]\colon c \in C, d \in D\}$ and is a free group with free basis $\{[c, d]\colon c \in C, d \in D\} \setminus \{e\}$. If we put $C = G/A$ and $D = H/A$ and let Φ be the canonical homomorphism of $G *_A H$ onto $G/A * H/A$ we see that $\Phi(K(G *_A H)) = K(G/A * H/A)$. Indeed by Lemmas 1 and 2, the set $\{[s_i, s_j]\colon i \in I, j \in J\} \setminus \{e\}$ is mapped by Φ bijectively onto a free basis for the free group $K(G/A * H/A)$. Thus $\{[s_i, s_j]\colon i \in I, j \in J\} \setminus \{e\}$ is a free basis for the group it generates. Hence $K(G *_A H)$ is a free group with the stated free basis.

LEMMA 3. *Let G, H, A and $K(G *_A H)$ be as in Proposition 1. Then every element u in $G *_A H$ can be written uniquely in the form $u = yk$, where $k \in K(G *_A H)$ and $y = gh$, $g \in G$, $h \in H$.*

PROOF. It is well known that each element in $G * H$ can be written (uniquely) in the form ghk, where $g \in G$, $h \in H$ and k is in the cartesian subgroup of $G * H$. Using the canonical homomorphism Γ of $G * H$ onto $G *_A H$ we see that every element $u \in G *_A H$ can be written in the required form.

Suppose $u = y_1k_1 = y_2k_2$, where $y_1 = g_1h_1$, $y_2 = g_2h_2$, $g_1 \in G$, $g_2 \in G$, $h_1 \in H$, $h_2 \in H$, $k_1 \in K(G *_A H)$ and $k_2 \in K(G *_A H)$. Consider the canonical homomorphism Λ of $G *_A H$ onto the amalgamated direct product $G \times_A H$. Then $\Lambda(u) = \Lambda(y_1k_1) = g_1h_1 = \Lambda(y_2k_2) = g_2h_2$. But $g_1h_1 = g_2h_2$ in $G \times_A H$ implies $g_1 = ag_2$ and $h_1 = a^{-1}h_2$ for some a in A, which in turn implies $y_1 = g_1h_1 = g_2h_2 = y_2$ in $G *_A H$. So $y_1k_1 = y_2k_2 = y_1k_2$. Thus $k_1 = k_2$, and the proof is complete.

REMARK. In Lemma 3, y can be written *uniquely* in the form $y = gh$, $g \in G$, $h \in H$, if and only if $A = \{e\}$.

THEOREM 1. *Let G and H be Hausdorff topological groups with $G \cap H = A$ a closed central subgroup of both G and H. Then the amalgamated free product $G *_A H$ admits a Hausdorff group topology which induces the given topologies on G and H.*

PROOF. By Lemma 3 and Proposition 1 each element u in $G *_A H$ can be written uniquely in the form $u = yk$, where $y = gh$, $g \in G$, $h \in H$, and k is in $K(G *_A H)$, the free group on the set $X = \{[s_i, s_j]: i \in I, j \in J\} \setminus \{e\}$.

We proceed with the proof in 4 steps.

Step 1. We define a topology on $X \cup \{e\}$. For each continuous right invariant pseudometric ρ_G on G and each continuous right invariant pseudometric ρ_H on H we define a pseudometric ρ_X on $X \cup \{e\}$. We show that the family of all such ρ_X gives rise to a Hausdorff topology on $X \cup \{e\}$.

Step 2. Having given $X \cup \{e\}$ a (completely regular Hausdorff) topology we use Graev's now standard technique to topologize $K(G *_A H)$, the free group on X, such that the topology is a Hausdorff group topology and induces on $X \cup \{e\}$ the topology defined in Step 1.

Step 3. Let $P = \{gh: g \in G, h \in H\} \subseteq G *_A H$. Define an operation $\circ$ on P as follows: for $g_1, g_2 \in G$ and $h_1, h_2 \in H$,

$$(g_1h_1) \circ (g_2h_2) = (g_1g_2)(h_1h_2).$$

Then $(P, \circ)$ is a group and induces the given group structures on G and H.

Given ρ_G and ρ_H as in Step 1, define a pseudometric ρ_P on P as

$$\rho_P(g_1h_1, g_2h_2) = \inf_{a \in A} \{\rho_G(g_1a, g_2) + \rho_H(h_1a^{-1}, h_2)\}.$$

Clearly, there is a canonical isomorphism f of the amalgamated direct product $G \times_A H$ onto $(P, \circ)$. Indeed if $G \times_A H$ is given the amalgamated direct product topology as described in Khan and Morris **[4]**, then f is a homeomorphism and an isomorphism of $G \times_A H$ onto $(P, \circ)$, where $(P, \circ)$ is given the topology generated by all the ρ_P. Thus $(P, \circ)$ with the topology generated by all the ρ_P is a Hausdorff topological group which has G and H (with their given topologies) as topological subgroups.

Step 4. Finally, if ρ_G and ρ_H are any continuous right invariant pseudometrics on G and H, respectively, ρ_X is as in Step 1 and ρ_X also denotes its extension to $K(G *_A H)$ as in Step 2, and ρ_P is as in Step 3 then define a pseudometric ρ on $G *_A H$ by

$$\rho(y_1k_1, y_2k_2) = \rho_P(y_1, y_2) + \rho_X(k_1, k_2)$$

where $y_1, y_2 \in P$ and $k_1, k_2 \in K(G *_A H)$.

We shall show that the family of all such pseudometrics ρ defines a group topology τ on $G *_A H$. Then $(G *_A H, \tau)$ will be homeomorphic (but not isomorphic) to $(G \times_A H) \times K(G *_A H)$, where $G \times_A H$ has the amalgamated direct product topology and $K(G *_A H)$ has the topology of Graev mentioned in Step 2. Thus τ is a Hausdorff topology and so we shall have the required result.

Now let us turn to the details of the proof.

Step 1. Let $g_1, g_2 \in \{s_i: i \in I\}$ and $h_1, h_2 \in \{s_j: j \in J\}$. Define

$$\begin{aligned} \rho_X([g_1, h_1], [g_2, h_2]) = \inf_{a_1, a_2 \in A} \big(\min\{&\min[\rho_G(g_1, a_1); \rho_H(h_1, a_1)] \\ &+ \min[\rho_G(g_2, a_2); \rho_H(h_2, a_2)]; \\ &\rho_G(g_1 a_1, g_2) + \rho_H(h_1 a_2, h_2)\}\big). \end{aligned} \tag{1}$$

If $[g_1, h_1] \neq [g_2, h_2]$ in $G *_A H$, we shall show that there is a ρ_X such that $\rho_X([g_1, h_1], [g_2, h_2]) > 0$. From this it follows that the topology defined by the family of all ρ_X is Hausdorff.

Without loss of generality, assume $[g_1, h_1] \neq e$. By Lemma 1 this implies $g_1 \notin A$ and $h_1 \notin A$. As $[g_1, h_1] \neq [g_2, h_2]$ Lemma 2 then implies that $g_2 \notin g_1 A$ or $h_2 \notin h_1 A$. Without loss of generality, assume $h_2 \notin h_1 A$. Since A is closed in G and H, there are continuous right invariant pseudometrics ρ_G and ρ_H on G and H, respectively, such that for all $a \in A$ we have $\rho_G(g_1, a) > 1$, $\rho_H(h_1, a) > 1$ and $\rho_H(h_1 a, h_2) > 1$. Clearly then the corresponding ρ_X satisfies

$$\rho_X([g_1, h_1], [g_2, h_2]) > 1.$$

Hence the topology determined by all the ρ_X is Hausdorff.

Before proceeding to Step 2, note how formula (1) simplifies when $[g_2, h_2] = e$;

$$\rho_X([g_1, h_1], e) = \inf_{a \in A} \big(\min[\rho_G(g_1, a); \rho_H(h_1, a)]\big). \tag{2}$$

Step 2. Given a family of pseudometrics ρ_X which defines the topology of $X \cup \{e\}$, Graev's method of topologizing the free group on X is as follows: Each ρ_X determines a pseudometric, also called ρ_X, on the free group on X. Let a and b be any elements of the free group. We can write

$$\left.\begin{aligned} a &= r_1 r_2 \cdots r_m, \\ b &= w_1 w_2 \cdots w_m \end{aligned}\right\} \quad \text{same length not necessarily reduced words,}$$

where $r_i \in X \cup X^{-1} \cup \{e\}$ and $w_i \in X \cup X^{-1} \cup \{e\}$. We define $\rho_X(a, b)$ to be the infimum, over all equal-length representations of a and b, of $\sum_{i=1}^m \rho_X(r_i, w_i)$, where $\rho_X(x_1, x_2^{-1}) = \rho_X(x_1^{-1}, x_2) = \rho(x_1, e) + \rho(x_2, e)$ and $\rho_X(x_1^{-1}, x_2^{-1}) = \rho_X(x_1, x_2)$, for all x_1 and x_2 in X. Graev **[2]** shows that this infimum is achieved. Noting that the given family of ρ_X defined a Hausdorff topology on $X \cup \{e\}$, it then follows that the family of ρ_X just defined induces a Hausdorff topology on the free group. That the topology so defined is a group topology follows from the fact that each ρ_X is two-sided invariant; that is, $\rho_X(a, b) = \rho_X(at, bt) = \rho_X(ta, tb)$, for all a, b and t in the free group $K(G *_A H)$.

Step 3. As mentioned earlier $G \times_A H$ is topologically isomorphic to

$$(G \times H) / \{(a, a^{-1}): a \in A\}.$$

Thus for each pair of continuous right invariant pseudometrics ρ_G on G and ρ_H on H, there is a pseudometric ρ_q on $G \times_A H$ given by

$$\rho_q(g_1h_1, g_2h_2) = \inf\{\rho_G(x_1, x_2) + \rho_H(y_1, y_2) \colon \psi(x_1, y_1) = g_1h_1 \text{ and } \psi(x_2, y_2) = g_2h_2\}$$

where ψ is the canonical homomorphism of $G \times H$ onto $G \times_A H$.

Indeed the family of all such ρ_q define the required quotient topology on $G \times_A H$.

Now $\psi(x_1, y_1) = g_1h_1$ if and only if $x_1 = g_1a_1$ and $y_1 = h_1a_1^{-1}$, for some $a_1 \in A$. Also $\psi(x_2, y_2) = g_2h_2$ if and only if $x_2 = g_2a_2$ and $y_2 = h_2a_2^{-1}$, for some $a_2 \in A$. Thus

$$\begin{aligned}\rho_q(g_1h_1, g_2h_2) &= \inf\{\rho_G(g_1a_1, g_2a_2) + \rho_H(h_1a_1^{-1}, h_2a_2^{-1}) \colon a_1, a_2 \in A\}\\ &= \inf\left\{\rho_G(g_1a_1a_2^{-1}, g_2) + \rho_H\left(h_1(a_1a_2^{-1})^{-1}, h_2\right) \colon a_1, a_2 \in A\right\}\\ &= \inf\{\rho_G(g_1a, g_2) + \rho_H(h_1a^{-1}, h_2) \colon a \in A\}\\ &= \rho_P(g_1h_1, g_2h_2).\end{aligned}$$

Thus $G \times_A H$ with the topology defined by the family of all pseudometrics ρ_P is indeed the quotient topology from $G \times H$; that is, f is a homeomorphism of $G \times_A H$ onto $(P, \circ)$.

Step 4. As indicated earlier this final step is to show that the family of all ρ defines a group topology on $G *_A H$.

We need a preliminary lemma.

LEMMA 4. *Let k_1 and k_2 be in $K(G *_A H)$ and let g be in G. Then*

$$\rho_X(g^{-1}k_1g, g^{-1}k_2g) \leqslant 2\rho_X(k_1, k_2). \tag{3}$$

PROOF OF LEMMA 4. First note that $g^{-1}k_1g$ and $g^{-1}k_2g$ are indeed in $K(G *_A H)$, so the left-hand side of the inequality makes sense! As indicated in Step 2, there exist representations $x_{11}^{j_1}x_{12}^{j_2}\cdots x_{1l}^{j_l}$ and $x_{21}^{n_1}x_{22}^{n_2}\cdots x_{2l}^{n_l}$ of k_1 and k_2, respectively, such that

$$\rho_X(k_1, k_2) = \sum_{i=1}^{l} \rho_X\left(x_{1i}^{j_i}, x_{2i}^{n_i}\right) \tag{4}$$

where $j_i = +1$ or -1, $n_i = +1$ or -1 and x_{1i}, x_{2i} are in $X \cup \{e\}$, for $i = 1, \ldots, l$.

Noting that if $g \in G$, $g_1 \in G$ and $h_1 \in H$ then $g^{-1}[g_1, h_1]g = [g_1g, h_1][g, h_1]^{-1}$, we see that in conjugating k_1 and k_2 by g, the element $x_{1i} = [g_1, h_1]$, say, from k_1 gets transformed into the element $c_1 = [g_1g, h_1][g, h_1]^{-1}$ and the element $x_{2i} = [g_2, h_2]$, say, from k_2 gets transformed into the element $c_2 = [g_2g, h_2][g, h_2]^{-1}$. Therefore to prove (3) it suffices to show that

$$\rho_X\left(c_1^{j_i}, c_2^{n_i}\right) \leqslant 2\rho_X\left(x_{1i}^{j_i}, x_{2i}^{n_i}\right). \tag{5}$$

Using the two-sided invariance of ρ_X on $K(G *_A H)$ we have

$$
\begin{aligned}
\rho_X(c_1,c_2) &\leqslant \rho_X([g_1g, h_1],[g_2g, h_2]) + \rho_X([g, h_1],[g, h_2]) \\
&\leqslant \inf_{a_1,a_2\in A}(\rho_H(h_1, a_1) + \rho_H(h_2, a_2)) \\
&\quad + \inf_{a_1,a_2\in A}(\rho_H(h_1, a_1) + \rho_H(h_2, a_2)) \\
&= 2\inf_{a_1,a_2\in A}(\rho_H(h_1,a_1) + \rho_H(h_2, a_2)).
\end{aligned} \tag{6}
$$

In due course we shall return to inequality (6) but first we prove some similar inequalities.

Once again we write

$$
\begin{aligned}
\rho_X(c_1, c_2) &\leqslant \rho_X([g_1g, h_1],[g_2g, h_2]) + \rho_X([g, h_1],[g, h_2]) \\
&\leqslant \inf_{a_1,a_2\in A}(\rho_G(g_1ga_1, g_2g) + \rho_H(h_1a_2, h_2)) \\
&\quad + \inf_{a_1,a_2\in A}(\rho_G(ga_1, g) + \rho_H(h_1a_2, h_2)) \\
&\leqslant 2\inf_{a_1,a_2\in A}(\rho_G(g_1a_1, g_2) + \rho_H(h_1a_2, h_2))
\end{aligned} \tag{7}
$$

by the centrality of A in G, the right invariance of ρ_G, and the fact that $\inf_{a_1\in A}(\rho_G(ga_1, g)) = 0$.

Further, using the representations

$$
\begin{aligned}
c_1 &= [g_1g, h_1][g, h_1]^{-1}[g, h_2][g, h_2]^{-1}, \\
c_2 &= [g, h_1][g, h_1]^{-1}[g_2g, h_2][g, h_2]^{-1}
\end{aligned}
$$

we obtain

$$
\begin{aligned}
\rho_X(c_1, c_2) &\leqslant \rho_X([g_1g, h_1],[g, h_1]) + \rho_X([g, h_2],[g_2g, h_2]) \\
&\leqslant \inf_{a_1,a_2\in A}(\rho_G(g_1ga_1, g) + \rho_H(h_1a_2, h_1)) \\
&\quad + \inf_{a_1,a_2\in A}(\rho_G(ga_1, g_2g) + \rho_H(h_2a_2, h_2)) \\
&= \inf_{a_1\in A}(\rho_G(g_1a_1, e)) + \inf_{a_1\in A}(\rho_G(a_1, g_2)) \\
&\leqslant 2\inf_{a_1,a_2\in A}(\rho_G(g_1, a_1) + \rho_G(g_2, a_2)).
\end{aligned} \tag{8}
$$

Finally we use the representations

$$
\begin{aligned}
c_1 &= [g_1g, h_1][g, h_1]^{-1}.e.e, \\
c_2 &= [g, h_1][g, h_1]^{-1}[g_2g, h_2][g, h_2]^{-1}
\end{aligned}
$$

to obtain

$$(9)\qquad \begin{aligned}\rho_X(c_1, c_2) &\leqslant \rho_X([g_1g, h_1],[g, h_1]) + \rho_X(e,[g_2g, h_2]) \\ &\quad +\rho_X\left(e,[g, h_2]^{-1}\right) \\ &\leqslant \inf_{a_1,a_2\in A} (\rho_G(g_1ga_1, g) + \rho_H(h_1a_2, h_1)) \\ &\quad + \inf_{a\in A} (\rho_H(h_2, a)) + \inf_{a\in A} (\rho_H(h_2, a)) \\ &\leqslant 2 \inf_{a_1,a_2\in A} (\rho_G(g_1, a_1) + \rho_H(h_2, a_2)).\end{aligned}$$

In similar fashion we obtain

$$(10)\qquad \rho_X(c_1, c_2) \leqslant 2 \inf_{a_1,a_2\in A} (\rho_G(g_2, a_2) + \rho_H(h_1, a_1)).$$

Examining (1) and (6)–(10) we obtain

$$\rho_X(c_1, c_2) \leqslant 2\rho_X(x_{1i}, x_{2i}).$$

From this we get

$$\rho_X\left(c_1^{-1}, c_2^{-1}\right) = \rho_X(c_1, c_2) \leqslant 2\rho_X(x_{1i}, x_{2i}) = 2\rho_X\left(x_{1i}^{-1}, x_{2i}^{-1}\right)$$

and

$$\begin{aligned}\rho_X\left(c_1^{-1}, c_2\right) = \rho_X\left(c_1, c_2^{-1}\right) &\leqslant \rho_X(c_1, e) + \rho_X(c_2, e) \\ &\leqslant 2\rho_X(x_{1i}, e) + 2\rho_X(x_{2i}, e) \\ &= 2\rho_X\left(x_{1i}, x_{2i}^{-1}\right) = 2\rho_X\left(x_{1i}^{-1}, x_{2i}\right).\end{aligned}$$

Therefore we have in general that

$$\rho_X\left(c_1^{j_i}, c_2^{n_i}\right) \leqslant 2\rho_X\left(x_{1i}^{j_i}, x_{2i}^{n_i}\right)$$

as required. So the inequality (5) is true. This completes the proof of Lemma 4.

Similarly it follows that if k_1 and k_2 are in $K(G *_A H)$ and h is in H, then

$$(11)\qquad \rho_X\left(h^{-1}k_1h, h^{-1}k_2h\right) \leqslant 2\rho_X(k_1, k_2).$$

We now turn to proving that the family of all ρ, defined earlier, gives rise to a group topology on $G *_A H$. To do this it suffices to show that:

Given ρ, $p_1 = g_1h_1$ in P, $p_2 = g_2h_2$ in P, k_1 and k_2 in $K(G *_A H)$, and $\varepsilon > 0$ there exists a $\delta > 0$ and a continuous pseudometric ρ^1 on $G *_A H$ such that whenever $p_3 = g_3h_3$ in P, $p_4 = g_4h_4$ in P, k_2 and k_4 are in $K(G *_A H)$ then

$$(12)\qquad \begin{cases}\rho^1(p_1k_1, p_3k_3) < \delta \text{ and } \rho^1(p_2k_2, p_4k_4) < \delta \\ \qquad\qquad \text{implies} \\ \rho\left(p_1k_1k_2^{-1}p_2^{-1}, p_3k_3k_4^{-1}p_4^{-1}\right) < \varepsilon M\end{cases}$$

where M is a real number dependent only on k_1 and k_2.

Let ρ arise from pseudometrics ρ_G and ρ_H on G and H, respectively. Let ρ_P be the pseudometric on P corresponding to ρ_G and ρ_H. As $(P, \circ)$ with the topology generated by all such ρ_P is a topological group we have the following.

There exists a pseudometric ρ_P^2 on P and a $\delta_1 > 0$ such that

$$(13) \quad \begin{cases} \rho_P^2(p_1, p_3) < \delta_1 \text{ and } \rho_P^2(p_2, p_4) < \delta_1 \\ \qquad \text{implies} \\ \rho_P(p_1 \circ p_2^{-1}, p_3 \circ p_4^{-1}) < \varepsilon \\ \qquad \text{that is} \\ \rho_P(g_1 g_2^{-1} h_1 h_2^{-1}, g_3 g_4^{-1} h_3 h_4^{-1}) < \varepsilon. \end{cases}$$

Note that, in particular,

$$(14) \quad \begin{cases} \rho_P(g_1 g_2^{-1} h_1 h_2^{-1}, g_3 g_4^{-1} h_3 h_4^{-1}) < \varepsilon \\ \qquad \text{implies} \\ \inf_{a \in A} \left(\rho_H(h_1 h_2^{-1} a, h_3 h_4^{-1})\right) < \delta. \end{cases}$$

Let ρ_G^2 and ρ_H^2 denote the restriction of ρ_P^2 to G and H, respectively.
Also there exists a pseudometric ρ_P^3 on P and a $\delta_2 > 0$ with the property that

$$(15) \quad \begin{cases} \rho_P^3(p_1, p_3) < \delta_2 \text{ implies } \rho_P(p_1^{-1}, p_3^{-1}) < \varepsilon \\ \qquad \text{and} \\ \rho_P^3(p_2, p_4) < \delta_2 \text{ implies } \rho_P(p_2^{-1}, p_4^{-1}) < \varepsilon. \end{cases}$$

Note that, in particular,

$$(16) \quad \begin{cases} \rho_P(p_2^{-1}, p_4^{-1}) < \varepsilon \text{ implies } \inf_{a \in A} \left(\rho_G(g_2^{-1} a, g_4^{-1})\right) < \varepsilon \\ \qquad \text{and} \\ \inf_{a \in A} \left(\rho_H(h_2^{-1} a, h_4^{-1})\right) < \varepsilon. \end{cases}$$

Now $k_1 k_2^{-1} = r_1^{d_1} r_2^{d_2} \cdots r_N^{d_N}$, where each $r_i = x_i^{-1} y_i^{-1} x_i y_i$, $x_i \in G$, $y_i \in H$ and $d_i = +1$ or -1 for $i = 1, \ldots, N$. There exists a ρ_P^4 on P and a $\delta_3 > 0$ with the property

$$(17) \quad \begin{cases} \rho_P^4(p_2, p_4) < \delta_3 \text{ implies } \rho_P(x_i p_2^{-1}, x_i p_4^{-1}) < \varepsilon \\ \qquad \text{and} \\ \rho_P(y_i p_2^{-1}, y_i p_4^{-1}) < \varepsilon, \; i = 1, \ldots, N. \end{cases}$$

Note that, in particular,

$$(18) \quad \begin{cases} \rho_P(x_i p_2^{-1}, x_i p_4^{-1}) < \varepsilon \\ \qquad \text{implies} \\ \inf_{a \in A} \left(\rho_G(x_i g_2^{-1} a, x_i g_4^{-1})\right) < \varepsilon \end{cases}$$

and also

$$(19) \quad \begin{cases} \rho_P(y_i p_2^{-1}, y_i p_4^{-1}) < \varepsilon \\ \qquad \text{implies} \\ \inf_{a \in A} \left(\rho_H(y_i h_2^{-1} a, y_i h_4^{-1})\right) < \varepsilon. \end{cases}$$

Let $\rho_G^1 = \rho_G + \rho_G^2 + \rho_G^3 + \rho_G^4$ and $\rho_H^1 = \rho_H + \rho_H^2 + \rho_H^3 + \rho_H^4$, where ρ_G^i and ρ_H^i denote the restriction of ρ_P^i to G and H, respectively, for $i = 3, 4$. Let ρ_X^1 be the two-sided invariant pseudometric on $K(G *_A H)$ corresponding to ρ_G^1 and ρ_H^1. Being two-sided invariant, ρ_X^1 gives rise to a group topology on $K(G *_A H)$. Therefore there exists a $\delta_4 > 0$ such that

$$\text{(20)} \qquad \begin{cases} \rho_X^1(k_1, k_3) < \delta_4 \text{ and } \rho_X^1(k_2, k_4) < \delta_4 \\ \qquad\qquad \text{imply} \\ \rho_X^1\left(k_1 k_2^{-1}, k_3 k_4^{-1}\right) < \varepsilon. \end{cases}$$

Put $\delta = \min\{\delta_1, \delta_2, \delta_3, \delta_4, \varepsilon\}$. Noting that $\rho_G^1 \geqslant \rho_G$ and $\rho_H^1 \geqslant \rho_H$, we see that $\rho_X^1 \geqslant \rho_X$. Thus by (20)

$$\text{(21)} \qquad \begin{cases} \rho_X^1(k_1, k_3) < \delta \text{ and } \rho_X^1(k_2, k_4) < \delta \\ \qquad\qquad \text{imply} \\ \rho_X\left(k_1 k_2^{-1}, k_3 k_4^{-1}\right) < \varepsilon. \end{cases}$$

Let ρ_P^1 be the pseudometric on P corresponding to ρ_G^1 and ρ_H^1. Let ρ^1 be the pseudometric on $G *_A H$ corresponding to ρ_P^1 and ρ_X^1. We are required to show that, for this choice of ρ^1 and δ, (12) is true.

Noting that

$$\begin{aligned} p_1 k_1 k_2^{-1} p_2^{-1} &= g_1 h_1 k_1 k_2^{-1} h_2^{-1} g_2^{-1} \\ &= \left(g_1 g_2^{-1}\right)\left(h_1 h_2^{-1}\right)\left(\left[g_2^{-1}, h_1 h_2^{-1}\right]^{-1}\right)\left(g_2 h_2 k_1 k_2^{-1} h_2^{-1} g_2^{-1}\right) \end{aligned}$$

and the analogous representation of $p_3 k_3 k_4^{-1} p_4^{-1}$ we see that

$$\text{(22)} \qquad \begin{aligned} \rho\left(p_1 k_1 k_2^{-1} p_2^{-1}, p_3 k_3 k_4^{-1} p_4^{-1}\right) \leqslant\ & \rho_P\left(g_1 g_2^{-1} h_1 h_2^{-1}, g_3 g_4^{-1} h_3 h_4^{-1}\right) \\ & + \rho_X\left(\left[g_2^{-1}, h_1 h_2^{-1}\right]^{-1}, \left[g_4^{-1}, h_3 h_4^{-1}\right]^{-1}\right) \\ & + \rho_X\left(g_2 h_2 k_1 k_2^{-1} h_2^{-1} g_2^{-1}, g_4 h_4 k_3 k_4^{-1} h_4^{-1} g_4^{-1}\right). \end{aligned}$$

Now

$$\text{(23)} \qquad \begin{aligned} & \rho_X\left(\left[g_2^{-1}, h_1 h_2^{-1}\right]^{-1}, \left[g_4^{-1}, h_3 h_4^{-1}\right]^{-1}\right) \\ & \quad \leqslant \inf_{a_1, a_2 \in A} \left(\rho_G\left(g_2^{-1} a_1, g_4^{-1}\right) + \rho_H\left(h_1 h_2^{-1} a_2, h_3 h_4^{-1}\right)\right) < 2\varepsilon. \end{aligned}$$

The last inequality follows from (13)–(16) and the observations

(i) $\rho^1(p_1 k_1, p_3 k_3) < \delta$ implies $\rho_P^1(p_1, p_3) < \delta$ which in turn implies $\rho_P^2(p_1, p_3) < \delta \leqslant \delta_1$;

(ii) similarly $\rho^1(p_2 k_2, p_4 k_4) < \delta$ implies $\rho_P^2(p_2, p_4) < \delta_2$.

We now examine the last term in (22).

$$
\begin{aligned}
&\rho_X\left(g_2h_2k_1k_2^{-1}h_2^{-1}g_2^{-1},\ g_4h_4k_3k_4^{-1}h_4^{-1}g_4^{-1}\right)\\
&\qquad \leqslant \rho_X\left(g_2h_2k_1k_2^{-1}h_2^{-1}g_2^{-1},\ g_4h_4k_1k_2^{-1}h_4^{-1}g_4^{-1}\right)\\
&\qquad\quad +\rho_X\left(g_4h_4k_1k_2^{-1}h_4^{-1}g_4^{-1},\ g_4h_4k_3k_4^{-1}h_4^{-1}g_4^{-1}\right)\\
&\qquad \leqslant \rho_X\left(g_2h_2k_1k_2^{-1}h_2^{-1}g_2^{-1},\ g_4h_4k_1k_2^{-1}h_4^{-1}g_4^{-1}\right)\\
&\qquad\quad +4\rho_X\left(k_1k_2^{-1},\ k_3k_4^{-1}\right)\quad \text{(by (3) and (11))}\\
&\qquad \leqslant \rho_X\left(g_2h_2k_1k_2^{-1}h_2^{-1}g_2^{-1},\ g_4h_4k_1k_2^{-1}h_4^{-1}g_4^{-1}\right)+4\varepsilon\quad \text{(by (21))}.
\end{aligned}\tag{24}
$$

Recall that $k_1k_2^{-1} = r_1^{d_1}r_2^{d_2}\cdots r_N^{d_N}$, where $r_i = x_i^{-1}y_i^{-1}x_iy_i$, $x_i \in G$, $y_i \in H$, $d_i = +1$ or -1, $i = 1,\ldots,N$. Noting that

$$g_2h_2x_i^{-1}y_i^{-1}x_iy_ih_2^{-1}g_2^{-1} = \left[g_2^{-1}, h_2^{-1}\right]\left[x_ig_2^{-1}, h_2^{-1}\right]^{-1}\left[x_ig_2^{-1}, y_ih_2^{-1}\right]\left[g_2^{-1}, y_ih_2^{-1}\right]^{-1}$$

and the analogous representation of $g_4h_4x_i^{-1}y_i^{-1}x_iy_ih_4^{-1}g_4^{-1}$ we see that for $i = 1,\ldots,N$

$$
\begin{aligned}
&\rho_X\left(g_2h_2x_i^{-1}y_i^{-1}x_iy_ih_2^{-1}g_2^{-1},\ g_4h_4x_i^{-1}y_i^{-1}x_iy_ih_4^{-1}g_4^{-1}\right)\\
&\quad\leqslant \rho_X\left(\left[g_2^{-1}, h_2^{-1}\right],\left[g_4^{-1}, h_4^{-1}\right]\right)+\rho_X\left(\left[x_ig_2^{-1}, h_2^{-1}\right],\left[x_ig_4^{-1}, h_4^{-1}\right]\right)\\
&\qquad+\rho_X\left(\left[x_ig_2^{-1}, y_ih_2^{-1}\right],\left[x_ig_4^{-1}, y_ih_4^{-1}\right]\right)\\
&\qquad+\rho_X\left(\left[g_2^{-1}, y_ih_2^{-1}\right],\left[g_4^{-1}, y_ih_4^{-1}\right]\right)\\
&\quad\leqslant \inf_{a_1,a_2\in A}\left(\rho_G\left(g_2^{-1}a_1, g_4^{-1}\right)+\rho_H\left(h_2^{-1}a_2, h_4^{-1}\right)\right)\\
&\qquad+\inf_{a_1,a_2\in A}\left(\rho_G\left(x_ig_2^{-1}a_1, x_ig_4^{-1}\right)+\rho_H\left(h_2^{-1}a_2, h_4^{-1}\right)\right)\\
&\qquad+\inf_{a_1,a_2\in A}\left(\rho_G\left(x_ig_2^{-1}a_1, x_ig_4^{-1}\right)+\rho_H\left(y_ih_2^{-1}a_2, y_ih_4^{-1}\right)\right)\\
&\qquad+\inf_{a_1,a_2\in A}\left(\rho_G\left(g_2^{-1}a_1, g_4^{-1}\right)+\rho_H\left(y_ih_2^{-1}a_2, y_ih_4^{-1}\right)\right)\\
&\quad< 8\varepsilon\quad \text{by (15)–(19)}.
\end{aligned}
$$

Similarly

$$\rho_X\left(g_2h_2y_i^{-1}x_i^{-1}y_ix_ih_2^{-1}g_2^{-1},\ g_4h_4y_i^{-1}x_i^{-1}y_ix_ih_4^{-1}g_4^{-1}\right) < 8\varepsilon.$$

From the last two inequalities we get

$$\rho_X\left(g_2h_2k_1k_2^{-1}h_2^{-1}g_2^{-1},\ g_4h_4k_1k_2^{-1}h_4^{-1}g_4^{-1}\right) < 8N\varepsilon. \tag{25}$$

So from (24) and (25) we have

$$\rho_X\left(g_2h_2k_1k_2^{-1}h_2^{-1}g_2^{-1},\ g_4h_4k_3k_4^{-1}h_4^{-1}g_4^{-1}\right) < (8N+4)\varepsilon. \tag{26}$$

Thus from (22), (13), (23) and (26) we obtain

$$\rho\left(p_1k_1k_2^{-1}p_2^{-1},\ p_3k_3k_4^{-1}p_4^{-1}\right) < (8N+7)\varepsilon.$$

The above inequality shows that (12) is satisfied and hence the family of all ρ does give a group topology on $G *_A H$. This completes the proof of the theorem.

As indicated in §2 we can immediately deduce the following result from Theorem 1.

THEOREM 2. *Let G and H be Hausdorff topological groups with $G \cap H = A$ a closed central subgroup of both G and H. Then $G \amalg_A H$ exists and is Hausdorff.*

REMARKS. It should be noted that the *proof* of Theorem 1 gives much more than the rather bland statement of Theorem 2. We know that $G \amalg_A H$ has the finest group topology on $G *_A H$ which induces the given topologies on G and H. The topology constructed in the proof of Theorem 1 provides a lower bound, then, for the topology of $G \amalg_A H$. For example, while we cannot describe the topology of $G \amalg_A H$ for general topological groups G and H, we can show that if G and H are Lie groups then the topology of Theorem 1 is such that it has no small subgroups. Therefore the finer topology of $G \amalg_A H$ must also be such that it has no small subgroups. This example and others are included in the sequel.

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