REPRODUCING KERNELS AND BILINEAR SUMS
FOR q-RACAH AND q-WILSON POLYNOMIALS

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ABSTRACT. A five-parameter family of reproducing kernels is constructed for q-Racah polynomials. Special cases for q-Hahn and little q-Jacobi polynomials are considered by selecting the parameters appropriately. Corresponding bilinear sums are also obtained for a whole range of q-orthogonal polynomials. As a special case, some product formulas are obtained for q-Racah and q-Wilson polynomials.

1. Introduction. In [9] we constructed a five-parameter family of reproducing kernels for Racah polynomials defined by a balanced \( _4F_3 \) series:

\[
W_n(x) = \binom{-n, n + \alpha + \beta + 1, -x, x + \gamma - N}{\alpha + 1, -N, \beta + \gamma + 1},
\]

where \(-N\) is the largest negative integer that appears in the denominator and \(x, n = 0, 1, \ldots, N\).

The purpose of this paper is to seek q-analogues of the main results of [9] and to discuss some interesting special cases. With the recent discovery of a q-analogue of (1.1) by Askey and Wilson [4] it is natural to expect that most, if not all, known results of Racah polynomials ought to have q-extensions. This remark seems especially true in view of the transformation properties of these basic polynomials which are defined by

\[
W_n(x; q) = W_n(x; a, b, c, N; q)
\]

and are called q-Racah polynomials, where \(x, n\) and \(N\) have the same meaning as in (1.1), and, \(a, b, c\) are complex parameters that are restricted only by the requirement that the denominator parameters \(aq\) and \(bcq\) do not lead to a zero factor before any in the numerator does. The \( _4F_3 \) series is balanced, which means that the product of
the denominator parameters is \( q \) times that of the numerator parameters, and is a special type of a basic hypergeometric series defined by

\[
\Phi_k^{a_1, a_2, \ldots, a_{k+1}}(b_1, b_2, \ldots, b_k; q, x) = \sum_{n=0}^{\infty} \frac{(a_1)_n(a_2)_n \cdots (a_{k+1})_n}{(q)_n(b_1)_n \cdots (b_k)_n} x^n,
\]

where

\[
(a)_n \equiv (a; q)_n = \begin{cases} 1, & \text{if } n = 0, \\ (1 - a)(1 - qa) \cdots (1 - aq^{n-1}), & n = 1, 2, \ldots. \end{cases}
\]

The transformation property we just alluded to is a \( q \)-analogue of Whipple's formula for the transformation of a balanced \( _4F_3 \) of type (1.1) and is given by

\[
\Phi_3^{q^{-n}, q^{-n+1}ab, q^{-x}, cq^{-N}}(aq, bcq, q^{-N}; q, q) = \frac{(bq)_n(ac^{-1}q)_n}{(aq)_n(bcq)_n} \Phi_3^{q^{-n}, q^{-n+1}ab, q^{-x}, cq^{-N}}(bq, ac^{-1}q, q^{-N}; q, q).
\]

Using (1.2), (1.5) and the well-known sum of a very well-poised \( \Phi_5 \) [11],

\[
\Phi_5\left[ a, \sqrt{a}, \sqrt{a}, \sqrt{a}, -q\sqrt{a}, b, c, q^{-n}, \frac{aq}{bc}, \frac{aq}{bc} \right] = \frac{(aq)_n(aq/bc)_n}{(aq/b)_n(aq/c)_n},
\]

one can show that the polynomials \( W_n(x; q) \) satisfy an orthogonality relation [4]

\[
\sum_{x=0}^{N} \rho(x) W_m(x; q) W_n(x; q) = h_n \delta_{m,n}
\]

where the weight function \( \rho(x) \) is defined by

\[
\rho(x) = \frac{(cq^{-N})_x(q^{-N})_x(-q^{-N})_x (aq)_x(bcq)_x(q^{-N})_x}{(cq^{-N})_x(q^{-N})_x(-q^{-N})_x (ca^{-1}q^{-N})_x(b^{-1}q^{-N})_x(cq)_x (abq)^{-x}}.
\]

and the normalization constant \( h_n \) by

\[
h_n = h_0 \frac{(q)_n(\sqrt{abq})_n(-\sqrt{abq})_n(bq)_n(ac^{-1}q)_n(abq^{N+2})_n(cq^{-N})_n}{(abq)_n(\sqrt{abq})_n(-\sqrt{abq})_n(aq)_n(bcq)_n(q^{-N})_n},
\]

with

\[
h_0 = \frac{(cq^{-N})_N(a^{-1}b^{-1}q^{-N-1})_N}{(ca^{-1}q^{-N})_N(b^{-1}q^{-N})_N} = \frac{(c^{-1})_N(abq^2)_N}{(cq^{-N})_N(abq^2)_N}.
\]

The formula for the total weight \( h_0 \) follows from (1.6) and the use of the identity [11, p. 241]

\[
(aq^{-n})_n = (-a)^n q^{-n(n+1)/2}(q/a)_n.
\]
The duality of $W_n(x; q)$ in $x$ and $n$ is obvious in definition (1.2) and one can exploit that to write the dual orthogonality relation

$$\sum_{n=0}^{N} (h_n h_n^{-1}) W_n(x; q) W_n(y; q) = \frac{h_n \delta_{x,y}}{\rho(x)}.$$  

A limiting case of the $q$-Racah polynomials, discovered by Hahn in 1949 [7], corresponds to setting $c = 0$ in (1.2):

$$Q_n(x; q) = \Phi_2 \left[ q^{-n}, q^{n+1}ab, q^{-x}; aq, q^{-N}; q, q \right].$$

There are transformation formulas for $Q_0(x; q)$. But the most important one can be worked out from (1.5) by taking the limit $c \to 0$. Thus one has

$$Q_n(x; q) = (-1)^n \left( \frac{bq}{aq} \right)_n \frac{a^n q^{n(n+1)/2}}{2} \Phi_2 \left[ q^{-n}, q^{n+1}ab, q^{x-N}; bq, q^{-N}; q, q^{-x}/a \right].$$

Andrews and Askey studied these polynomials and found a weight function for them a few years ago [3]. Another limiting case that will be of interest to us is known in the current literature as little $\psi$-Jacobi polynomials and is obtained from (1.13) by taking the limits $x, N \to \infty$ such that $N - x = r$ where $r$ takes on nonnegative integer values. These polynomials were also studied by Hahn [6], as well as Andrews and Askey [2, 3], and are defined by

$$p_n(x; \alpha, \beta | q) = \Phi_1 \left[ q^{-n}, q^{n+1} \alpha \beta; aq, qx \right]$$

where $x = q^r, r = 0, 1, \ldots$

In [1] Al-Salam and Ismail found a reproducing kernel for $p_n(x; \alpha, \beta | q)$ and a corresponding bilinear formula which will be seen in this paper to follow as a very special case of a rather general family of reproducing kernels for the $q$-Racah polynomials $W_n(x; q)$. We first take five arbitrary parameters $a_1, a_2, b_1, b_2, b_3$ and define two different $q$-Racah polynomials:

$$W^{(1)}_n(x; q) = \Phi_3 \left[ q^{-n}, q^{n+1}a_1 a_2 b_2 b_3, q^{-x}, b_1 b_2^{-1} cq^{x-N}; a_1 b_1 q, q^{-N}, a_2 b_3 cq \right]$$

and

$$W^{(2)}_n(x; q) = \Phi_3 \left[ q^{-n}, q^{n+1}a_1 a_2 b_2 b_3, q^{-x}, cq^{x-N}; a_1 b_2 q, q^{-N}, a_2 b_3 cq \right].$$

Obviously $W^{(1)}_n(x; q) = W^{(2)}_n(x; q)$ when $b_1 = b_2$.

We then show that they are related by a connection relation

$$\sum_{j=0}^{N} K_N(x, y) W^{(2)}_n(y; q) = \lambda_n W^{(1)}_n(x; q).$$
where the “eigenvalue” \( \lambda_n \) is given by a balanced 4\( \phi_3 \):

\[
\lambda_n = 4\phi_3 \left[ q^{-n}, \quad q^{n+1}a_1a_2b_2b_3, \quad b_2, \quad b_1^{-1}b_2b_3 \right] \left[ a_1b_2q, \quad b_2b_3, \quad a_2b_1^{-1}b_2b_3q \right]_q, \quad q, q
\]

and the kernel \( K_N(x, y) \) has the double sum form:

\[
K_N(x, y) = A_N(x, y) \sum_{i=0}^{x \wedge y} \left( \frac{b_1b_2cq^{-N}}{(b_1b_2cq^{-N})(b_1b_2cq^{-N})} \right) \left( \frac{a_1q}{a_1q} \right) \left( \frac{a_1q}{a_1q} \right) \left( \frac{a_1q}{a_1q} \right)
\]

\[
(1.19)
\]

\[
(1.20)
\]

where

\[
A_N(x, y) = \frac{(q)_N(a_2b_2cq)_N(a_1b_2c^{-1}q)_N}{(b_2b_3)_N(b_2c)_N(b_2c^{-1})_N} \left( \frac{(b_1)_x(b_2c)_x(b_2)_y(b_2c)_y}{(a_1b_1q)_x(a_2b_2cq)_x(q)_y(q)_y} \right) \frac{(b_2c^{-1})_{N-x}(b_2b_3b_1^{-1})_{N-y}(b_3c^{-1})_{N-y}}{(a_1b_2c^{-1}q)_{N-x}(a_2b_2b_3b_1^{-1}q)_{N-x}(q)_{N-x}(q)_{N-y}(c^{-1})_{N-y}} \frac{1 - cq^{2y-N}}{1 - cq^{2y-N}}
\]

and \( x \wedge y = \min(x, y) \).

Note that the \( _{10}\phi_9 \) series in (1.20) is terminating and very well-poised, and so is the second series implied in it. However, transformations of these series are, in general, not possible unless they have the additional property of being balanced, that is, the product of the denominator parameters is \( q \) times that of the numerator parameters. For the \( _{10}\phi_9 \) shown explicitly in (1.20) this would require \( a_2q = b_1/b_3 \), while for the second series one would need \( a_1q = b_3/b_1 \).

One may also note the kernel \( K_N(x, y) \) is positive under the conditions:

\[
0 < a_1q, a_2q, b_1, b_2, b_3, b_2b_3/b_1 < 1, \quad ca_i^{-1}b_j^{-1}q^{-N} < 1
\]

or,

\[
a_1q^{1+N}, a_2q^{1+N}, b_1q^N, b_2q^N, b_3q^N, a_2b_2b_3b_1^{-1}q^{N+1} > 1, \quad ca_i,b_jq < 1
\]

where \( 0 < q < 1 \) and \( i = 1, 2; j = 1, 2, 3 \).
Using (1.5) and (1.19) one finds that

\[
\lambda_n = \frac{(a_1^{-1}q^{-n})_n(b_2^{-1}q^{1-n})_n(a_1b_2b_3q^n)_n}{(a_1b_2q)_n(b_2b_3)_n}
\frac{\phi_3\left[q^{-n}, b_2, a_2q, a_1^{-1}b_1^{-1}q^{-n}; q, q\right]}{a_2b_1^{-1}b_2b_3q, a_1^{-1}q^{-n}, b_3^{-1}q^{1-n}; q, q}
\]

(1.24)

\[
= \frac{(q)_n(a_1^{-1}b_1^{-1}q^{-n})_n}{(a_1b_2q)_n(b_2b_3)_n}
\frac{(a_2b_3)q^n}{(a_2b_3)q^n(b_3)k_{n-k}(a_1b_2b_3q)_n}
\sum_{k=0}^{n} \frac{(q^{-n})_k(b_2)_k(a_2q)_k(a_1^{-1}q^{k-n})_n-k(b_3^{-1}q^{1+k-n})_n-k}{(q)_n(q)_k(a_1^{-1}b_1^{-1}q^{k-n})_n-k(a_2b_3)q^k}
\]

by (1.11).

It is interesting to note that \( \lambda_n \) is independent of \( c \) and \( N \) and is positive under conditions (1.22) and (1.23).

In the next section we prove the connection relation (1.18). In §3 we discuss some special cases and in §4 we obtain some bilinear formulas. In §5 we deduce some Bateman-type and Watson-type product formulas for \( q \)-Racah and \( q \)-Wilson polynomials in general, and, for the continuous \( q \)-Jacobi polynomials, in particular.

2. Construction of the kernel \( K_n(x, y) \). Applying (1.5) to the balanced series in (1.17) we obtain

\[
W_n^{(2)}(y; q) = \frac{(a_2b_3q)_n(a_1a_2b_2b_3q^{N+2})_n}{(a_1b_2q)_n(q^{-N})_n}(a_2b_3q^{N+1})^nW_n'(y; q),
\]

where

\[
W_n'(y; q) = \sum_{r=0}^{n} \frac{(q^{-n})_r(q^{-N}a_1a_2b_2b_3)_r(a_2b_3c)^{r+1},(a_2b_3q^{N+y})_r}{(q)_r(a_2b_3q)_r(a_2b_3c)_r(a_1a_2b_2b_3q^{N+2})_r}q^r.
\]

Using the \( q \)-analogue of the Pfaff-Saalschutz theorem [11, equation (IV, 4), p. 247], namely,

\[
\phi_2\left[q^{-k}, a, b; c, abc^{-1}q^{-k}; q, q\right] = \frac{(c/a)_k(c/b)_k}{(q)_k(c/abc)_k},
\]

we obtain

\[
(a_2b_3c^{1+y})_r(a_2b_3q^{N+y})_r = (a_2b_3c^{1+y})_r
\]

(2.4)

\[
\sum_{k=0}^{r} \frac{(b_3q^{1-y})_k(b_3cq^{N+y})_k(a_2b_3c^{1+y})_k}{(a_2b_3c^{1+y})_k}(a_2b_3c^{1+y})_r-k(a_2b_3c^{1+y})^{k},
\]
where \( j \) is a nonnegative integer such that \( y \leq j \leq N \) and \([\frac{s}{k}]\) is the \( q \)-binomial coefficient defined by \([\frac{s}{k}] = \frac{(q)_s}{(q)_k(q)^{s-k}}\). In deriving the r.h.s. of (2.4) from (2.3) we need to make use of the following identities [11, p. 241].

\[
(a)_{n-N} = \frac{(a)_{N}(q/a)_{n}}{(a^{-1}q_{1-N})_{n}q^{-n}}. \\
(aq^{-n})_{N} = \frac{(a)_{N}(q/a)_{n}}{(a^{-1}q_{1-N})_{n}q^{-n}}. 
\]

Let us now compute the sum

\[
L_1(i, j) = \sum_{y=i}^{j} b_2^{-y} \frac{1 - cq^2y^{-N}}{1 - cq^{2i-N}} \left( \frac{(cq^{-i-N})_{y}(b_3cq^i-N)_{y}}{(b_2q_{y-i})_{\infty}(b_3q_{y-1})_{\infty}} \right) W_{i}(y; q). 
\]

One may feel somewhat mystified by the appearance of the coefficients on the right-hand side, but this is the obvious \( q \)-analogue of a similar object we considered in [9] and is motivated simply by the requirement that these coefficients together with the \( y \)-dependent terms in (2.4) will be summable by a very well-poised \( \phi_3 \) sum (1.6).

Through a somewhat tedious but straightforward calculation one can, in fact, show that the use of (2.4), the identities (2.5), (2.6), and the summation formula (1.6) yield the following result:

\[
L_1(i, j) = C(i, j) \sum_{r=0}^{n} \frac{q^{-n}}{(q)_{r}(a_2b_3q)_{r}(a_2b_3cq)_{r}(a_2b_2c)_{r}} \left[ q^{-r}, \ b_3, \ b_3c^{-i-N}, \ b_2b_3q^{-i} \right] \left[ a_2b_3q^{i-N-r}, \ b_2b_3, \ a_2b_3c^{-i+j}, \ q, q \right] 
\]

where

\[
C(i, j) = \frac{(q)_{\infty}^2}{(b_2)_{\infty}(b_3)_{\infty}(b_2b_3)_{\infty}(b_2b_3)_{\infty}} \left( \frac{(b_2)_{\infty}(b_3)_{\infty}(b_2b_3)_{\infty}(b_2b_3)_{\infty}}{(q)_{\infty}(a_2b_3q)_{\infty}(a_2b_3cq)_{\infty}(a_2b_2c)_{\infty}} \right) \left( 1 - cq^{-N} \right). 
\]

One fortunate, though expected, feature of (2.8) is that the \( \phi_3 \) series on the r.h.s. is balanced, hence we may apply the transformation (1.5) on it as often as necessary. The transformation we require specifically is the following:

\[
s_{\phi_3} \left[ q^{-r}, \ b_3, \ b_3c^{-i-N}, \ b_2b_3q^{-i} \right] \left[ a_2b_3q^{i-N-r}, \ b_2b_3, \ a_2b_3c^{-i+j}, \ q, q \right] 
\]

\[
= \frac{(b_2)_{r}(a_2b_3c^{-i+j})_{r}}{b_3} b_3^{-r} \frac{(q)_{r}(a_2b_3c^{-i+j})_{r}}{(b_2b_3)_{r}(a_2b_3c^{-i+j})_{r}} \left[ q^{-r}, \ b_3, \ a_2b_3c^{-i-N-r}, \ a_2b_3b_2b_3c^{-i-j-N-r} \right] \left[ a_2b_3q^{i-N-r}, \ b_2b_3, \ a_2b_3c^{-i-j+N}, \ a_2b_3q^{-i-N-r} \right]. 
\]
It may be mentioned that the coefficient of $\phi_3$ on the r.h.s. does not follow directly from (1.5), but only after using (1.11). Now, by (2.5),

\[(b_2)_r(a_2 b_3 cq^{1+j})_r(a_2 q^{N-j+1})_r \]
\[= (b_2)_{r-k}(a_2 b_3 cq^{1+j})_{r-k}(a_2 q^{N-j+1})_{r-k}(-a_2^2 b_2 b_3 cq^{N+2})^k q^{3rk-3k(k+1)/2}.\]

Using this in (2.10) and then in (2.8) we obtain

\[L_1(i, j) = C(i, j) \sum_{r=0}^{n} \frac{(q^{-n}),(q^{n+1}a_1 a_2 b_2 b_3)_r(b_3 q)^r}{(q),(a_2 b_3)_k(a_2 b_3 c q)_k(a_1 a_2 b_2 b_3 q^{N+2})_k},(b_2)_r,(b_2)_r,(b_2)_r, \]

\[(q-r)_k(b_3)_k(a_2 b_2 b_3 c q^{-i-r})_k(a_2 b_2 b_3 c q^{i-N-r})_k(-a_2^2 b_2 b_3 c q^{N+3})^k q^{3rk-3k(k+1)/2} \cdot (b_2)_{r-k}(a_2 b_3 c q^{1+j})_{r-k}(a_2 q^{N-j+1})_{r-k}.\]

Next we compute the following sum:

\[L_2(i) = \sum_{j=x}^{N} \frac{1 - cb_3 q^{2j-N}}{1 - cb_3 q^{-N}} \frac{(a_2 b_3 c q)_j(b_3 c q^{x-N})_j}{(b_3 c q)_j(b_1 b_2 b_3 c q^{1+x-N})_j} \]

\[\cdot (q^{j-x+1})_\infty(q^{N-j+1})_\infty(b_1 b_2 b_3 c q^{j-x})_\infty(b_1 b_2 b_3 c q^{j-x})_\infty \cdot C^{-1}(i, j)L_1(i, j).\]

Again, the structure of the r.h.s. is aimed at the summability over $j$ by using (1.6). Substituting (2.12) and (2.13), simplifying, summing, and again simplifying by means of (1.11), (2.5) and (2.6), we obtain the following form:

\[L_2(i) = A(x) \sum_{r=0}^{n} \frac{(q^{-n}),(q^{n+1}a_1 a_2 b_2 b_3)_r(b_3 q)^r}{(q),(a_2 b_3)_k(a_2 b_3 c q)_k(a_1 a_2 b_2 b_3 q^{N+2})_k},(b_2)_r,(b_2)_r, \]

\[\cdot \sum_{k=0}^{r} \frac{(q^{-r})_k(b_3)_k(a_2 q)_{r-k}(a_2 b_3 c q^{1+x})_{r-k}(a_2 b_2 b_3 b_1 c q^{N-x+1})_{r-k}(b_2)_{r-k}}{(q)_k(a_2 b_2 b_3 b_1 c q)_{r-k}} \cdot (-a_2^2 b_2 b_3 c q^{N+3})^k q^{3rk-3k(k+1)/2} \cdot (a_2^2 b_2 b_3 c q^{-r-i})_k(a_2^2 b_2 b_3 c q^{i-N-r})_k,\]
where

\[ A(x) = \frac{(q)_\infty^2 (b_3^{-1}c^{-1})_N (a_2 b_2 b_3 q)^N}{(a_2 q)_\infty (b_1 b_2 b_3) (q)_N (b_2^{-1}c^{-1})_N} \]

\[(2.15)\]

\[ \cdot \frac{(a_2 b_3 c q)_x (b_2^{-1} b_1 q^{N-x})_x (q^{-N})_x}{(b_3 c q^{-N})_x (b_1 b_2^{-1} c q)_x (b_1 a_2^{-1} b_2^{-1} b_3 q^{-N})_x} \left( \frac{b_1}{a_2 b_2 b_3} \right)^x. \]

Finally we compute the sum

\[ L(x) = \sum_{i=0}^{x} \frac{(cb^{-1} q^{-N})_x (q^{cb^{-1} q^{-N}})_x (-q^{cb^{-1} q^{-N}})_x (a_1 b_2^{-1} c q^{x-N})_x}{(a_1 q^{x-1})_x (b_1 q^{x-1})_x} (q a_1)^{-i} \cdot A^{-1}(x) L_2(i). \]

(2.16)

Substitution of (2.14) in (2.16) now leads to the following form:

\[ L(x) = \sum_{i=0}^{x} \frac{(q)_\infty^2 (b_1)_x}{(a_1 q)_\infty (q)_x} \sum_{r=0}^{N} \frac{(q^{-n})_r (q^{n+1} a_1 a_2 b_2 b_3)_r (b_3 q)^x}{(q)_r (a_2 b_3)_r (b_2)_r (a_1 a_2 b_2 b_3 q^{N+2})_r} \]

\[(2.17)\]

\[ \cdot \sum_{k=0}^{r} \frac{(q^{-r})_k (b_3)_k (b_2)_r-k (a_2 q)_r-k (a_2 b_3 c q^{1+x})_r-k (a_2 b_2 b_3 b_1 q^{-x-1})_r-k}{(q)_k (a_2 b_2 b_3 b_1^{-1} c q)_r-k} \]

\[ \cdot (-a_2^{-1} b_2 b_3 c q^{N+3})^k \cdot q^{3r k - 3(k+1)/2} \xi_{k,r}, \]

where

\[(2.18)\]

\[ \xi_{k,r} = (a_2^{-1} b_3^{-1} c^{-1} q^{-r}) (a_2^{-1} b_2^{-1} b_3^{-1} q^{-N-r})_k \]

\[ = \left\{ \begin{array}{l}
\frac{c b^{-1} q^{-N}}{q q^{cb^{-1} q^{-N}}, -q^{cb^{-1} q^{-N}}, a q, v q^{cb^{-1} q^{-N}}, \sqrt{q^{cb^{-1} q^{-N}}, q^{cb^{-1} q^{-N}}, a q^{-1} b_2^{-1} c q^{-N}},}
\frac{b_1 b_2^{-1} c q^{x-N}, a_2 b_2 c q^{1-x}, a_2^{-1} b_2^{-1} b_3^{-1} q^{k-N-r}, q^{-x}}{b_1^{-1} q^{-x}, a_2^{-1} b_2^{-1} b_3^{-1} q^{-N-r}, a_2 b_2 c q^{1-r-k}, q^{1-N-x} ; q, a^{-1} b_1^{-1} q^{-k}} \end{array} \right. \]

However, the \( s\Phi \) series on the right is very well-poised and has the structure that enables us to transform it to a balanced \( s\Phi \) by means of the formula [11, equation (3.4.1.5), p. 100]

\[ s\Phi \left[ \frac{a, q \sqrt{a}, -q \sqrt{a}, b, c, d, e, q^{-n}}{\sqrt{a}, -\sqrt{a}, q a b, a q c, a q d, a q e, q a q e ; q, a^{2} a^{-q+2}} \right] \]

\[(2.19)\]

\[ = \frac{(aq)_n (aq/de)_n}{(aq/d)_n (aq/e)_n} \Phi_{2q} \left[ \frac{q^{-n}, d, e, q a q b c}{aq b, a q c, d e q^{-n} / a ; q, q} \right]. \]
Thus we get
\[(2.20)\]
\[
\xi_{k,r} = \left( a_2^{-1} b_3^{-1} c^{-1} q^{-r} \right) k \left( a_2^{-1} b_2^{-1} b_3^{-1} q^{-N-r} \right) k \cdot \left( \frac{cb_2^{-1} q^{1-N}}{(a_1^{-1} b_1^{-1} q^{-x})}_x \left( a_1 b_1 q^{-x} \right) \left( a_2 b_2 c q^{-N} \right)_x \left( b_1 q^{-x} \right)_x \right) \times \\
\cdot \Phi^3 \left[ \begin{array}{c} q^{-k}, \quad a_1 q, \quad q^{-x}, \quad b_1 b_2 c q^{-N} \\ a_2 b_2 c q^{1+r-k}, \quad a_1 b_1 q, \quad a_2^{-1} b_2^{-1} b_3^{-1} q^{-N-r} \end{array} \right] \ ; q, q
\]
\[
= \left( b_1 \right)_k \left( a_2^{-1} b_3^{-1} c^{-1} q^{-r} \right) k \left( a_1^{-1} a_2 b_1^{-1} b_3^{-1} q^{-N-1-r} \right) k \\
\cdot \left( \frac{cb_2^{-1} q^{1-N}}{(a_1^{-1} b_1^{-1} q^{-x})}_x \left( a_1 b_1 q^{-x} \right) \left( a_2 b_2 c q^{-N} \right)_x \left( b_1 q^{-x} \right)_x \right) \times \\
\cdot \Phi^3 \left[ \begin{array}{c} q^{-k}, \quad a_1 q, \quad a_2 b_2 c q^{x+1+r-k}, \quad a_2 b_2 b_3^{-1} q^{-N-x+1+r-k} \\ a_2 b_2 c q^{1+r-k}, \quad b_1^{-1} q^{-1-k}, \quad a_1 a_2 b_2 b_3 q^{N+2+r-k} \end{array} \right] \ ; q, q
\]

In deriving the last line above we have made use of (1.5) once again. We now substitute (2.20) in (2.17), simplify the terms by means of (2.5) and (2.6) and replace the summation variable \( r \) by \( r - k \). This leads to
\[
L(x) = B(x) \sum_{r=0}^{\infty} \left( \frac{q^{-r}}{(a_2 b_3)_{r}} \right) \left( \frac{q^{r}}{b_2 b_3}_{r} \right) q^{r} \sum_{k=0}^{\infty} \sum_{l=0}^{r-k} \left[ \begin{array}{c} r \end{array} \right] \left( b_1 \right)_{r-k} \left( b_2 \right)_{k} \left( b_3 \right)_{r-k} \\
\cdot \left( \frac{1}{(a_1 q)_{l}} \right) \left( \frac{1}{(a_2 b_1 q)_{l}} \right) \left( \frac{1}{(a_2 b_2 c)_{l}} \right) \left( \frac{1}{(b_1 q^{-1})_{l}} \right) \left( \frac{1}{(b_2 b_3 q)_{l}} \right) \left( \frac{1}{(b_3 q)_{l}} \right) b_2 b_3 q^{l} \left( a_2 b_2 b_3 b_3 q^{N+2+r-k} \right)_{r-1} \\
\left( b_2 b_3 b_3 q\right)_{r-1} \left( b_3 q^{N+2+r-k} \right)_{r-1} \left( b_3 q^{N+2+r-k} \right)_{r-1} \left( a_2 b_2 b_3 q^{N+2+r-k} \right)_{r-1}
\]

where
\[
B(x) = \left( \frac{q}{a_1 q} \right)_{\infty} \left( \frac{a_1 b_1 q}{b_1 q} \right)_{\infty} \left( \frac{a_1 b_1 q}{a_1^{-1} b_1^{-1} q^{-x}} \right)_{x} \left( \frac{a_1 b_1 q}{a_1^{-1} b_1^{-1} q^{-x}} \right)_{x}
\]

By changing the order of summation in (2.21) one can easily see that
\[
B^{-1}(x) L(x) = \sum_{s=0}^{n} \left( \frac{a_2 b_3 c q^{1+x}}{a_2 b_3 c q} \right)_{s} \left( \frac{a_2 b_2 b_3 b_3 q^{N-x+1}}{a_2 b_2 b_3 q^{N+2}} \right)_{s} U_{n,s}
\]

where
\[
U_{n,s} = \sum_{r=s}^{n} \left( \frac{q^{-r}}{(a_2 b_3)_{r}} \right) \left( \frac{q^{r}}{(a_2 b_3)_{s}} \right) q^{2r} \sum_{k=0}^{r} \left[ \begin{array}{c} r \end{array} \right] \left( b_1 \right)_{r-k} \left( b_2 \right)_{k} \left( b_3 \right)_{r-k} \\
\cdot \left( \frac{1}{(a_1 b_1 q)_{l}} \right) \left( \frac{1}{(b_1 q^{-1})_{l}} \right) \left( \frac{1}{(b_2 b_3 q)_{l}} \right) \left( \frac{1}{(b_3 q)_{l}} \right) b_2 b_3 q^{l} \left( a_2 b_2 b_3 b_3 q^{N+2+r-k} \right)_{r-1} \\
\left( b_2 b_3 b_3 q\right)_{r-1} \left( b_3 q^{N+2+r-k} \right)_{r-1} \left( b_3 q^{N+2+r-k} \right)_{r-1} \left( a_2 b_2 b_3 q^{N+2+r-k} \right)_{r-1}
\]
Simplifying the terms in (2.24) by means of (2.5) and (2.6) and changing the summation variables, we get

\[ U_{n,s} = \frac{(q^{-n})_s(q^{n+1}a_1a_2b_2b_3)_s(b_2)_s(a_2q)_s(b_3q)^s}{(q)_s(a_2b_3)_s(b_2b_3)_s(a_2b_2b_3b_1q)_s} \]

\[ \cdot \sum_{k=0}^{n-s} \frac{(q^{s-n})_k(q^{s+n+1}a_1a_2b_2b_3)_k}{(q)_k(b_2b_2q^2)_k} \frac{(b_1)_k(b_3)_k}{(a_1b_1q)_k(a_2b_2q^{k+1})_k} q^k \]

\[ = 4\Phi_3 \left[ q^{-s}, a_1q, b_3q^k, b_1a_2^{-1}b_2^{-1}b_3^{-1}q^{-s} \right. \]

\[ \cdot \frac{(b_2b_3q^x)_s(a_2b_2q^{k+1})_s}{(b_2)_s(a_2q)_s} \frac{(b_3q^x)_s}{(a_2b_3)_s} \]

\[ \cdot \sum_{l=0}^{s} \frac{(q^{-l})_l(q^{s+k+1}a_1a_2b_2b_3)_l}{(q)_l(a_1b_1q^{k+1})_l(b_2b_2q^k)_l(a_2b_2q^{k+1})_l} q^l, \]

by virtue of (1.11). Substituting (2.26) in (2.25) and simplifying, we obtain

\[ U_{n,s} = \frac{(q^{-n})_s(q^{n+1}a_1a_2b_2b_3)_sq^s}{(q)_s(a_2b_2b_3b_1q)_s} \]

\[ \cdot \sum_{k=0}^{n-s} \frac{(q^{s-n})_k(q^{s+n+1}a_1a_2b_2b_3)_k}{(q)_k(q)_l(q^{s+n+1}a_1a_2b_2b_3)_k} \frac{(b_1)_k(b_3)_k}{(a_1b_1q)_k(a_2b_2q^{k+1})_k} q^k \]

\[ \cdot \sum_{l=0}^{s} \frac{(q^{-l})_l(q^{s+k+1}a_1a_2b_2b_3)_l}{(q)_l(q)_l(q^{s+n+1}a_1a_2b_2b_3)_l} q^l q^k q^{-l} \]

\[ = \frac{(q^{-n})_s(q^{n+1}a_1a_2b_2b_3)_sq^s}{(q)_s(a_2b_2b_3b_1q)_s} \]

\[ \cdot \sum_{m=0}^{n} \frac{(q^{s+n+1}a_1a_2b_2b_3)_m}{(a_1b_1q)_m(a_2b_2q)_m b_3)_m} q^m \]

\[ \cdot \sum_{k} \frac{(q^{s+n+1}a_1a_2b_2b_3)_k}{(q)_k(q)_k(q^{s+n+1}a_1a_2b_2b_3)_k} q^{-k} q^s. \]
However, by (2.5) and (2.3),

\[
\sum_k (q^{-z})_{m-k} \left( \frac{(q^{s+n+1}a_1a_2b_2b_3)_k (q^{s-n})_k}{(q)_k (q^s)_k (q^{s+n+1}a_1a_2b_2b_3)_k} \right) q^{-ks} 
\]

(2.28)

\[
= \left( \frac{q^{-z}}{q^s} \right)_m \phi_2 \left[ q^{-m}, \quad q^{s-n}, \quad q^{s+n+1}a_1a_2b_2b_3 \quad \frac{q^{s+1-m}}{q^{s+1}a_1a_2b_2b_3}; q, q \right]
\]

Thus, (2.27) and (2.28) give

\[
U_{n,s} = \left( \frac{q^{-z}}{q^s} \right)_m (q^{s+n+1}a_1a_2b_2b_3)_m q^z 
\]

(2.29)

\[
\cdot \phi_3 \left[ q^{-m}, \quad q^{s+n+1}a_1a_2b_2b_3, \quad b_1, \quad b_3 \quad \frac{q^{s+1-m}}{b_2b_3}; q, q \right]
\]

But the \( \phi_3 \) on the right is balanced and so it transforms to

\[
\phi_3 \left[ q^{-m}, \quad q^{s+n+1}a_1a_2b_2b_3, \quad b_1, \quad b_3 \quad \frac{q^{s+1-m}}{b_2b_3}; q, q \right] \]

(2.30)

Thus we have

\[
B^{-1}(x)L(x) = \left( \frac{a_1b_2q}{a_1b_1q} \right)_n \left( \frac{a_2b_2b_3b_1^{-1}q}{a_2b_2b_3b_1^{-1}q} \right)_n \left( \frac{b_1}{b_2} \right)^n 
\]

\[
\cdot \lambda_n \phi_3 \left[ q^{-m}, \quad q^{s+n+1}a_1a_2b_2b_3, \quad a_2b_2b_3b_1^{1+x}, \quad a_2b_2b_3b_1^{N-x+1} \quad \frac{a_2b_2b_3b_1^{-1}q}{a_2b_2b_3b_1^{-1}q}; q, q \right] 
\]

(2.31)

\[
= \left( \frac{q^{-N}}{a_2b_2b_3b_1^{-1}q} \right)_n \left( \frac{a_1b_2q}{a_1a_2b_2b_3b_1^{N+2}q} \right)_n \left( a_2b_2b_3b_1^{-1}q \right)_n \lambda_n W^{(1)}_n(x; q).
\]

Note that \( \lambda_n \), as defined in (1.19), is the same as the \( \phi_3 \) in (2.30), and \( W^{(1)}(x; q) \) as defined in (1.16) is obtained from the \( \phi_3 \) of the first line in (2.31) by using the transformation (2.5) yet another time.
If we now follow the sequence of operations performed on $W_n^{(2)}(y; q)$ we find that

$$[B(x)A(x)]^{-1} \sum_{i=0}^{x} \left( \frac{cb_2^{-1}q^{-N}}{cb_2^{-1}q^{-N}} \right)_i \left( -q \frac{cb_2^{-1}q^{-N}}{cb_2^{-1}q^{-N}} \right)_i \left( b_1b_2^{-1}c q^{-N} \right)_i \left( qa_1^{-1}b_2^{-1}q^{-N} \right)_i$$

$$= \left( \frac{q^{i+1}}{a_1q^{i+1}} \right)_\infty \left( \frac{b_1q^{-i}}{q} \right)_\infty (aq_1)^{-i}$$

$$\cdot \frac{N}{j=x} \frac{1 - cb_3q^{-2j-N}}{1 - cb_3q^{-N}} \left( a_2b_3cq \right)_j \left( b_3cq^{-N} \right)_j$$

$$\cdot \frac{(q^{-i+1})_\infty (q^{-j+1})_\infty (b_1b_2^{-1}b_3^{-1})_j}{(b_1b_2^{-1}b_3^{-1})_\infty (a_2q^{-N-j+1})_\infty}$$

$$\cdot C^{-1}(i, j) \sum_{y=i}^{j} b_2^{-y} - \frac{cb_2^{-1}q^{-i-N}}{1 - cb_2^{-1}q^{-i-N}} \left( c q^{-i-N} \right)_y \left( b_3cq^{-N} \right)_y$$

$$\cdot \frac{(q^{-i+1})_\infty (q^{-j+1})_\infty W_n^{(2)}(y; q)}{(b_2q^{-y-i})_\infty (b_3q^{-y})_\infty}$$

$$= \lambda_n W_n^{(1)}(x; q),$$

where $A(x)$, $B(x)$ and $C(i, j)$ are defined by (2.15), (2.22) and (2.9), respectively. The reproducing kernel defined in (2.32) is the same as the one given in (1.20) and (1.21) once we express it in terms of basic hypergeometric coefficients and make repeated use of identities (1.11), (2.5) and (2.6). However, it may be pointed out that the nonnegativity of the kernel under conditions (1.22) or (1.23) is somewhat more obvious in (2.32) than in (1.20).

3. Some special cases.

Case 1: $c = 0$. As we have seen in §1 the $q$-Racah polynomials reduce to $q$-Hahn polynomials as $c$ approaches 0. So by taking the limit $c \to 0$ in (1.18) we should get a connection relation for $Q_n(x; q)$. Since $\lim_{c \to 0} (ac)_m = 1$ and $\lim_{c \to 0} (ac^{-1})_m/(bc^{-1})_m = (a/b)^m$, the connection relation (1.18) becomes

$$\sum_{y=0}^{N} L_n(x, y) Q_n^{(2)}(y; q) = \lambda_n Q_n^{(1)}(x; q)$$

where $\lambda_n$ is the same as in (1.19) and the $q$-Hahn polynomials are given by

$$Q_n^{(1)}(x; q) = \Phi_2 \begin{bmatrix} q^{-n}, q^{n+1}a_1a_2b_2b_3, q^{-x} \end{bmatrix}$$

$$a_1b_1q, q^{-N}; q, q \right)$$

$$Q_n^{(2)}(y; q) = \Phi_2 \begin{bmatrix} q^{-n}, q^{n+1}a_1a_2b_2b_3, q^{-y} \end{bmatrix}$$

$$a_1b_2q, q^{-N}; q, q \right)$$
while the kernel $L_N(x, y)$ is defined by

$$
L_N(x, y) = B_N(x, y) \sum_{i=0}^{\infty} \frac{(a, q)_i}{(q)_i} \frac{(b_2^{-1}b_3^{-1}q^{1-N})_i}{(q^{-N})_i} \frac{(q^{-y})_i}{(b_2^{-1}q^{1-y})_i} \left( \frac{b_3}{a, b_1} \right)^i
$$

(3.4)

$$
\Phi_3^{4\Phi_3} \left[ a_2 q, \frac{b_2^{-1}b_3^{-1}q^{1+N}, q^{-N}, q^{y-N}}{1, b_2^{-1}b_3^{-1}q^{1+y-N}, b_3^{-1}q^{1+y-N}, q^{-N}, q} \right],
$$

with

(3.5) $B_N(x, y) = \frac{(q)_N}{(b_2 b_3)_N} \frac{(b_2 b_3 b_1^{-1})_{N-x} (b_2)_{x-y} (a, q)^x}{(a, b_2 b_3 b_1^{-1} q)_{N-x} (a, b_1 q)_{x+y} (q)_{N-y}} b_{2N-y}^N.$

We would like to point out that this special case represents a $q$-extension of the results in [8].

Case II: $c = 0, a_2 q = 1$. Setting $a_2 q = 1$ in (1.19) we find that the eigenvalue reduces to a balanced $\Phi_2$. So, using the summation formula (2.3) we get

(3.6) $\lambda_n = 3\Phi_2 \left[ q^{-n}, q^n a_1 b_2 b_3, b_2, a_1 b_2 q, b_2 b_3; q, q \right] = \frac{(a, q)_n (b_1)_{1-N}}{(a, b_1 q)_{1-N}} = \frac{(a, q)_n (b_1)}{(a, b_1 q)_{1-N}} b_{2N-y}^N$

by (1.11). Hence the connection relation reduces to

(3.7) $\sum_{y=0}^{N} M_n(x, y) 3\Phi_2 \left[ q^{-n}, q^n a_1 b_2 b_3, b_2, a_1 b_2 q, b_2 b_3; q, q \right] = \frac{(a, q)_n (b_1)_{1-N}}{(a, b_1 q)_{1-N}} 3\Phi_2 \left[ q^{-n}, q^n a_1 b_2 b_3, b_2, a_1 b_2 q, b_2 b_3; q, q \right],$

where

(3.8) $M_n(x, y) = \frac{(q)_N (b_1)_{x} (b_2)_{y} (b_3)_{N-y}}{(b_2 b_3)_N (a, b_1 q)_x (q)_y (q)_{N-y}} (a, q)^x b_{2N-y}^N$

$$
\Phi_3^{4\Phi_3} \left[ a_2 q, \frac{b_2^{-1}b_3^{-1}q^{1+N}, q^{-N}, q^{y-N}}{b_1^{-1}q^{1-N}, b_2^{-1}b_3^{-1}q^{1+y-N}, b_3^{-1}q^{1+y-N}, q^{-N}, q} \right],
$$

Case III: $c = 0, a_1 q = 1$. A similar calculation leads to the connection relation

(3.9) $\sum_{y=0}^{N} P_n(x, y) 3\Phi_2 \left[ q^{-n}, q^n a_1 b_2 b_3, b_2, a_1 b_2 q, b_2 b_3; q, q \right] = \frac{(b_1)_n (a_1 q)}{(b_2 b_3)_n (a_2 b_2 b_3 b_1^{-1} q)_n} \frac{(b_2 b_3)_{1-N}}{(b_1)_n} 3\Phi_2 \left[ q^{-n}, q^n a_2 b_2 b_3, b_2, a_1 b_2 q, b_2 b_3; q, q \right],$
where

\[
P_N(x, y) = \frac{(q)_N(b_2b_3b_1^{-1})_{N-x}(b_2)_y(b_3)_{N-y} - b_2^{N-y}}{(b_2b_3)_N(a_2b_2b_3b_1^{-1}q)_{N-x}(q)_N} \cdot \phi_3\left[ a_2q, b_2^{-1}b_1^{-1}q^{1-N}, \begin{array}{c} q^{x-N} \\ b_1b_2^{-1}b_3^{-1}q^{1+x-N} \end{array}, \begin{array}{c} q^{y-N} \\ b_3^{-1}q^{1+y-N} \end{array}; q, b_2 \right].
\]

This case yields an interesting kernel for the little $q$-Jacobi polynomials defined in (1.15). If we take the limits $x, y, N \to \infty$ so that $N - x = r$ and $N - y = s$, $r, s$ being nonnegative integers, then (3.9) becomes

\[
\begin{align*}
(3.11) & \sum_{s=0}^{\infty} P_\infty(r, s)_2\phi_1\left[ q^{-n}, q^n a_2 b_2 b_3; q, q^{r+1} \right] \\
& = \frac{(b_1)_n(a_2 q)_n}{(b_2 b_3)_n(a_2 b_2 b_3 b_1^{-1} q)_n} \left( \frac{b_2 b_3}{b_1} \right)_n \cdot \phi_1\left[ q^{-n}, q^n a_2 b_2 b_3; q, q^{r+1} \right]
\end{align*}
\]

with

\[
(3.12) P_\infty(r, s) = \frac{(b_2)_s}{(b_2 b_3)_s} \left( \frac{b_2 b_3 b_1^{-1}}{a_2 b_2 b_3 b_1^{-1} q} \right)_s \cdot \phi_2\left[ q^{-r}, q^{-s}, a_2 q, b_1 b_2^{-1} b_3^{-1} q^{1-r}, b_3^{-1} q^{1-s}; q, b_2 b_3^{-1} q^2 \right].
\]

In the symmetric case $b_1 = b_2$, this is essentially the same kernel as found by Al-Salam and Ismail [1] by an entirely different method.

**Case IV**: $a_1 q = 1$, $c \neq 0$. The eigenvalue $\lambda_n$ is the same as in (3.9) while the “eigenfunctions” are:

\[
\begin{align*}
W_n^{(1)}(x; q) &= 4\phi_3\left[ q^{-n}, q^n a_2 b_2 b_3; q^{-x}, b_1 b_2^{-1} c q^{x-N} ; q, q \right] \\
W_n^{(2)}(y; q) &= 4\phi_3\left[ q^{-n}, q^n a_2 b_2 b_3; q^{-y}, c q^{y-N} ; q, q \right]
\end{align*}
\]

The kernel $K_N(x, y)$ becomes a multiple of a single $t_9\phi_9$ series:

\[
K_N(x, y) = \frac{(b_2 b_3 b_1^{-1})_{N-x}(b_3)_{N-y}}{(a_2 b_2 b_3 b_1^{-1} q)_{N-x}(c^{-1})_{N-y}} \cdot \phi_9\left[ b_3^{-1} c^{-1} q^{1-N}; q^{-x}, b_2 b_3^{-1} c q^{1-x-N} ; q, q \right]
\]

\[
(3.13)
\]

(3.13)
Although the above series is very well-poised, there is no known transformation formula for it unless it is also balanced, which requires that \( a_2q \) must equal \( b_1/b_3 \). Apart from Jackson’s formula for a very well-poised balanced \( 10_9 \) [11, equation (3.4.2.4), p. 102] the author recently derived a \( q \)-extension of Bailey’s transformation for a 2-balanced very well-poised \( \phi_8 \) which reads [10]:

\[
\begin{align*}
\phi_9 \left[ a, q\sqrt{a}, -q\sqrt{a}, b, c, d, \\
\sqrt{a}, -\sqrt{a}, qa/b, qa/c, qa/d, \\
B, C, D, q^{-k}, \\
aq/B, aq/C, aq/D, aq^{k+1}; q, q
\end{align*}
\]

\[
\begin{align*}
\frac{(aq)_k(aq/BC)_k(aq/BD)_k(aq/Bc)_k(aq/Bd)_k(b)_k}{(aq/c)_k(aq/d)_k(aq/B)_k(aq/C)_k(aq/D)_k(b/b)_k}
\begin{bmatrix}
Bb^{-1}q^{-k}, q\sqrt{Bb^{-1}q^{-k}}, -q\sqrt{Bb^{-1}q^{-k}}, B, qa/bc, qa/bd, \\
Bb^{-1}q^{-k}, -\sqrt{Bb^{-1}q^{-k}}, b^{-1}q^{-k}, a^{-1}Bcq^{-k}, a^{-1}Bdq^{-k}, \\
aq/bC, qa/bD, a^{-1}Bq^{-k}, a^{-1}Bq^{-k}, \\
a^{-1}BCq^{-k}, a^{-1}BDq^{-k}, b^{-1}aq, b^{-1}Bq; q, q
\end{bmatrix}
\end{align*}
\]

(3.14)

where \( k \) is a nonnegative integer, and balanced property requires

\[
a^2q^2 = bcdBCDq^{-k}.
\]

Setting \( a_2q = b_1/b_3 \) and choosing it as \( B, b_3^{-1}c^{-1}q \) as \( b \), then applying (3.14) in (3.13) and simplifying the coefficients by means of (1.11), (2.5) and (2.6) we then obtain

\[
\begin{align*}
K_N(x, y) &= \frac{(q)_N}{(b_2c)_N} \frac{(b_2c)_y}{(b_2)_y} \frac{c(q)_y}{(c)_y} \frac{b_2b_3b_1^{-1}c^{-1}}{(b_1)_N} \frac{b_3^{-1}c^{-1}q}{b_3^{-1}c^{-1}q-x} \frac{(q)_N}{(c)_N} \frac{b_3^{-1}c^{-1}q}{b_3^{-1}c^{-1}q-x} \frac{1}{(b_2)_y-x} \frac{1-cq^{2y-N}}{1-cq^{y-N}} \frac{b_1cq^{y-N-1}}{b_1cq^{y-N-1}} \frac{q\sqrt{b_1cq^{y-N-1}}}{\sqrt{b_1cq^{y-N-1}}} \frac{-q\sqrt{b_1cq^{y-N-1}}}{-\sqrt{b_1cq^{y-N-1}}} \frac{b_1b_3^{-1}}{b_3cq^{y-N}} \frac{b_1cqx+y-N}{b_1cq^{x+y-N}} \frac{b_1}{b_1c} \frac{b_2b_3q^{-1}}{b_1b_2^{-1}cq^{y-N+1}} \frac{b_1b_2^{-1}cq^{y-N+1}}{b_2q^{y-x}} \frac{q^{-N}}{q^{-N}} \frac{b_1}{b_1c}; q, q
\end{align*}
\]

(3.16)

where we have assumed, for the sake of definiteness that \( 0 \leq x \leq y \leq N \). A similar expression can be derived when \( x \geq y \).
An interesting situation arises if we set $b_1b_3^{-1} = q^{-m}$ where $m$ is a nonnegative integer, $0 \leq m \leq N$. Then

\begin{equation}
W_n^{(1)}(x; q) = 4\Phi_3 \left[ q^{-n}, \quad q^{n-1}b_1b_2, \quad q^{-x}, \quad b_1b_2^1cq^{x-N} \right],
\end{equation}

(3.17)

\begin{equation}
W_n^{(2)}(y; q) = 4\Phi_3 \left[ q^{-n}, \quad q^{n-1}b_1b_2, \quad q^{-y}, \quad cq^{y-N} \right],
\end{equation}

(3.18)

and

\begin{equation}
\lambda_n = \frac{(q^{-m})_n(b_1)_n}{(b_2)_n(b_1b_2q^m)_n} (b_2q^m)_n,
\end{equation}


\begin{equation}
K_N(x, y) = \frac{(b_1c)_N(b_1c)_N}{(b_2q^m)_N(b_2q^m)_N} (b_2q^m)_N.
\end{equation}

(3.19)

Case V: $a_2q = 1$, $c \neq 0$. Here the eigenvalue $\lambda_n$ is the same as in (3.6) and the eigenfunctions are

\begin{equation}
W_n^{(1)}(x; q) = 4\Phi_3 \left[ q^{-n}, \quad q^na_1b_2b_3, \quad q^{-x}, \quad b_1b_2^1cq^{x-N} \right],
\end{equation}

(3.20)

\begin{equation}
W_n^{(2)}(y, q) = 4\Phi_3 \left[ q^{-n}, \quad q^na_1b_2b_3, \quad q^{-y}, \quad cq^{y-N} \right],
\end{equation}

Also

\begin{equation}
K_N(x, y) = \frac{(q)_N(a_1b^2c^{-1}q)_N}{(b_2b_3)_N(b_2c^{-1})_N} \cdot \frac{(b_1)_N(b_2c^{-1})_N}{(a_1b^2c^{-1})_N(b_2c^{-1})_N}.
\end{equation}

\begin{equation}
\frac{1}{(b_2c^{-1})_N} \left[ q\sqrt{b_2c^{-1}q^N}, \quad -q\sqrt{b_2c^{-1}q^N}, \quad a_1q, \sqrt{b_2c^{-1}q^N}, \quad -\sqrt{b_2c^{-1}q^N}, \quad cb_2^{-1}q^{-N}, \quad b_1c, \quad b_2c^{-1}q^{-N}, \quad cb_2^{-1}q^{-N}, \quad b_3c, \quad q^{-N}, \quad b_2c^{-1}q^{-N} \right].
\end{equation}

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The \( q \) series is balanced if \( a_1q = b_3/b_1 \). As in the previous case it can be transformed by (3.14) if this condition is satisfied. If, further, \( b_3/b_1 = q^{-m} \), \( m \) a nonnegative integer, then the kernel \( K_N(x, y) \) above assumes a form similar to that in (3.19).

4. Bilinear formulas. Let us now multiply equation (1.18) by

\[
\frac{(a_1a_2b_2b_3q)_n(q/b_1q)_n}{(q)_n}\left(q/|a_1a_2b_2b_3q_2\right)_n(a_2b_3q)^n(a_2b_3q)^n(a_2b_3q)^n
\]

\[
\cdot (c^{-1}q)^nW_n^{(2)}(z; q)
\]

and sum over \( n \) from 0 to \( N \). Then, by the dual orthogonality relation (1.12) we obtain, after some simplification and replacing \( z \) by \( y \), the following bilinear formula:

\[
(a_2b_3cq)_N(a_1a_2b_3q_2)_N(a_1b_2c^{-1}q)_N
\]

\[
(b_3c)_N(b_2b_3)_N(b_2c^{-1})_N
\]

\[
(a_1a_2b_3q)_N(a_2b_3c)_N(a_1b_2c^{-1}q)_N
\]

\[
(a_1a_2b_3q)_N(a_2b_3c)_N(a_1b_2c^{-1}q)_N
\]

\[
\sum_{i=0}^{x/y} \left( \frac{(b_2b_3cq^{x-N})_i}{(q/b_1q^{x-N})_i}\frac{-q}{(b_2b_3cq^{x-N})_i} \right) \left( a_1b_1 \right)_i
\]

\[
\cdot \left( \frac{(b_1b_2c^{-1}q^{x-N})_i}{(b_2b_3c^{-1}q^{x-N})_i} \right) \left( b_3 \right)_i
\]

\[
\left( \frac{(b_1b_2c^{-1}q^{x-N})_i}{(b_2b_3c^{-1}q^{x-N})_i} \right) \left( b_3^{-1}c^{-1}q^{x-N} \right)_i
\]

\[
\left( (c^{-1}q)^nW_n^{(2)}(z; q) \right)
\]

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A number of identities can be deduced from this, but we shall be interested mainly in those that correspond to the special cases discussed in the previous section.

First of all, in the limit $c \to 0$, we get the following bilinear formula for the $q$-Hahn polynomials:

$$
\left( \frac{a_1 a_2 b_1 b_2 q^2}{b_2 b_3} \right)_N \left( \frac{b_1 x (b_2 b_1 b_3)}{a_1 b_1 q} \right)_{N-x} \left( \frac{b_2 b_1 b_3}{a_2 b_3 q} \right)_{N-y} \left( \frac{a_1 q}{x+y-N} \right)
$$

$$
\sum_{i=0}^{x+y} \left( a_1 q \right)_i \left( \frac{b_1 b_3 q^{i-N}}{b_2 b_3 q^{i-N}} \right) \left( \frac{b_1 b_3 q^{i-N}}{b_2 b_3 q^{i-N}} \right)_i \left( \frac{b_3}{a_1 b_1} \right)^i
$$

$$
\left[ \frac{a_2 q}{x+y-N} \right]_{a_1 b_1 q} \left[ \frac{q^{a_1 a_2 b_2 b_3}}{q^{a_1 a_2 b_2 b_3}} \right]_{a_1 b_1 q} \left[ \frac{q^{x+y-N}}{q^{a_1 a_2 b_2 b_3}} \right]_{a_1 b_1 q}
$$

$$
(4.2)
$$

This may be viewed as a $q$-extension of the bilinear formula that we found in [8] for Hahn polynomials. Setting $a_1 q = 1$ in (4.2) we get a much simpler formula:

$$
\left( \frac{a_2 b_2 b_3 q}{b_2 b_3} \right)_N \left( \frac{b_2 b_3 b_1 q}{a_2 b_3 q} \right)_{N-x} \left( \frac{b_2 b_1 b_3}{a_2 b_3 q} \right)_{N-y} \left( \frac{a_1 q}{x+y-N} \right)
$$

$$
\sum_{n=0}^{N} \frac{(a_2 b_2 b_3 q)_{n} (b_2 b_3 b_1 q_{n}) (a_2 b_3 q)_{n} (a_1 b_3 q)_{n} (q^{-n})_{n}}{(a_2 b_3 q)_{n} (b_2 b_3 b_1 q_{n}) (a_2 b_3 q)_{n} (a_2 b_3 q)_{n} (a_2 b_3 q)_{n} (q^{-n})_{n}}
$$

$$
(4.3)
$$

We may now take the Jacobi limit: $x, y, N \to \infty$ with $N - x = r, N - y = s, r, s = 0, 1, 2, \ldots$. Then (4.3) reduces to

$$
\left( \frac{a_2 b_2 b_3 q}{b_2 b_3} \right)_\infty \left( \frac{b_2 b_3 b_1 q}{a_2 b_3} \right)\left( \frac{b_2 b_3 q}{b_2 b_3} \right)_\infty \left( \frac{a_2 b_3 q}{b_2 b_3} \right)_s \left( \frac{a_1 q}{x+y-N} \right)
$$

$$
\left[ \frac{a_2 q}{b_2 b_3 q} \right]_{a_1 b_1 q} \left[ \frac{q^{a_1 a_2 b_2 b_3}}{q^{a_1 a_2 b_2 b_3}} \right]_{a_1 b_1 q} \left[ \frac{q^{x+y-N}}{q^{a_1 a_2 b_2 b_3}} \right]_{a_1 b_1 q}
$$

$$
(4.4)
$$
When $b_1 = b_2$, equation (4.4) leads to Al-Salam and Ismail's bilinear formula [1, equation (3.11)]. The validity of (4.4) cannot, of course, be taken for granted since we have an infinite series on the r.h.s. However, for $0 < b_1$, $b_2$, $b_3$, $b_2b_3/b_1 < 1$ and $0 < a_2q < 1$ one can prove in much the same way as done in [1] that the kernel on the l.h.s. is square-integrable, which ensures the validity of (4.4).

Let us now discuss the special cases of (4.1) when $c \neq 0$ and either $a_1q = 1$ or $a_2q = 1$. To fix ideas let us take $a_1q = 1$, since the case $a_2q = 1$ will give essentially the same identities. Setting $a_1q = 1$ in (4.1) we obtain

\[(a_2b_2q)_{n}(a_2b_3q)_{n} (b_2c_{n})_{n} (b_3c_{n})_{n} (a_2b_2q)_{n}(a_2b_2q)_{n} \]

\[= \sum_{n=0}^{N} \left[ \frac{(a_2b_2q)_{n}(q^{n}a_2b_2q)_{n} (b_2c_{n})_{n} (b_3c_{n})_{n} (a_2b_2q)_{n}(a_2b_2q)_{n}}{(b_2c_{n})_{n}(a_2b_2q)_{n}(q^{n+1})_{n} (b_1c_{n})_{n} W_{n}^{(1)}(x;q)W_{n}^{(2)}(y;q)} \right],
\]

assuming that we have set $a_1q = 1$ in the expressions for $W_{n}^{(1)}(x;q)$ and $W_{n}^{(2)}(y;q)$.

This leads to a particularly interesting bilinear formula if we specialize the parameters $a_2$ and $b_3$ by setting

\[(4.6) \quad a_2q = q^{-m}, \quad b_3 = q^{m}b_4,
\]

where $b_4$ is an arbitrary parameter and $m$ a nonnegative integer. Then, whether or not $x$, $y$, $N$ are nonnegative integers as had been originally assumed, we get the identity

\[(b_2b_4)_{m}(cb_4q^{n})_{m}(cb_4q^{n})_{m}(b_2b_4q^{N-n})_{m}(b_4q^{N-n})_{m} \]

\[= \sum_{n=0}^{m} \left[ \frac{(b_2b_4q^{n})_{n}(q^{n}b_2b_4q^{n})_{n} (b_4c_{n})_{n} (b_4c_{n})_{n} (b_2b_4q^{n})_{n}(b_2b_4q^{n})_{n}}{(b_2b_4q^{n})_{n}(b_2b_4q^{n})_{n}(q^{n+1})_{n} (b_1c_{n})_{n} W_{n}^{(1)}(x;q)W_{n}^{(2)}(y;q)} \right],
\]

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By using the summation formula [11, p. 247]

\[ 2\phi_1[a, q^{-m}; b; q, q] = \frac{(b/a)_m a^m}{(b)_m}, \]

we can now invert this formula if we multiply both sides by \( q^m(q^{-k})_m(q^{k-1}b_2b_4)_{m/(q)_m(b_2b_4)_m} \) and sum over \( m \) from 0 to \( k \). Thus we obtain

\[ (4.8) \]

\[ \frac{(b_1)_k(b_2)_k(b_4c)_k(q^{-m})}{(b_1b^{-1}b_4)_k(b_2c^{-1})_k(b_2b_4q^{-m})} \frac{(b_2b_4)}{(b_1c)q^{-m}} W_k^{(1)}(x; q) W_k^{(2)}(y; q) = \sum_{m=0}^k \frac{(q^{-k})_m(q^{-k-1}b_2b_4)_m(b_4c^q)_m(b_2b_1^2b_4q^{-m})_m(b_2b_4q^{-x})_m(b_4c)^{-m}}{(q)_m(b_2b_1^2b_4)_m(b_4c)^{-m}} q^{-m}, \]

where \( 10\phi_0[ ] \) is the same as the one on the l.h.s. of (4.7). This formula simplifies even further if we apply the transformation formula (1.5) on both \( W_k^{(1)}(x; q) \) and \( W_k^{(2)}(y; q) \):

\[ W_k^{(1)}(x; q) \equiv 4\phi_3 \left[ \begin{array}{c} q^{-k}, \quad q^{-k-1}b_2b_4, \quad q^{-x}, \quad b_1b_2^{-1}c_{q^{-x}}(x)N; \quad q, q \end{array} \right] \]

\[ = \frac{(b_2b_1^{-1}b_4)_k(b_2^{-1}c)_k}{(b_2c)_k(b_4c)_k} \left[ \begin{array}{c} q^{-k}, \quad q^{-k-1}b_2b_4, \quad q^{-x}, \quad b_1b_2^{-1}c q^{-x} \end{array} \right]. \]

\[ W_k^{(2)}(y; q) \equiv 4\phi_3 \left[ \begin{array}{c} q^{-k}, \quad q^{-k-1}b_2b_4, \quad q^{-y}, \quad c_{q^{-y}}(y)N; \quad q, q \end{array} \right] \]

\[ = \frac{(b_2)_k(b_2c^{-1})_k}{(b_2)_k(b_2c)_k} c^{-k} \left[ \begin{array}{c} q^{-k}, \quad q^{-k-1}b_2b_4, \quad q^{-y}, \quad c q^{-y} \end{array} \right]. \]

We now substitute these in (4.9), replace \( x, y \) by \( N - x \) and \( N - y \), respectively, and improve the notation somewhat by replacing \( c^{-1} \) by \( b_3 \). The final form of the formula is

\[ (4.10) \]

\[ 4\phi_3 \left[ \begin{array}{c} q^{-n}, \quad q^{-n-1}b_2b_4, \quad q^{-x}, \quad b_1b_2b_3 g q^{-x} \end{array} \right] \]

\[ = \left[ \frac{(b_2b_3^{-1})_n(b_2b_4q^N)_n(b_4b_3^{-1}q^N)^{-n}}{(b_2b_3)_n(q^{-n})_n} \sum_{k=0}^n \frac{(q^{-k})_k(q^{-k-1}b_2b_4)_k(b_4b_3^{-1}q^{-x})_k(b_2b_1^{-1}b_4q^x)_k}{(q)_k(b_4)_k(b_2b_1^{-1}b_4)_k(b_4b_3^{-1})_k(b_4b_3^{-1}q^N)_k} \right]. \]
\[
\frac{(b_4 b_3^{-1} q^{N-y})_k (b_4 q^y)_n}{(b_2 b_4 q^N)_k} q^k
\]

\[
\binom{n}{k} \cdot 10_{\Phi_9} \left[ b_3 b_4^{-1} q^{-N-k} \right., \sqrt{b_3 b_4^{-1} q^{-N-k}}, \sqrt{b_3 b_4^{-1} q^{-N-k}}, b_2 b_3, b_3 b_4^{-1} q^{-1-k}, q^{-k}, b_1 b_2 b_3 q^{-x-N}, b_3 q^{-y-N}, q^{-N}, b_3 b_4^{-1} q^{-1-N}, b_1 b_2 b_4^{-1} q^{-1-x-k}, b_4^{-1} q^{-1-y-k}, q^{-x}, q^{-y}, b_3 b_4^{-1} q^{-1-N+x-k}, b_3 b_4^{-1} q^{-1-N+y-k}; q, b_1 b_4^{-1} q \right],
\]

where \( n \) is a nonnegative integer whose value is restricted by the requirement that none of the denominator parameters on the l.h.s. attains the value 1, if any of them do at all, before \( q^{-n} \) does.

5. **Product formulas for \( q \)-Racah and \( q \)-Wilson polynomials.** We would like to think of equation (4.10) as a master formula that leads to different types of product formulas for \( q \)-Racah polynomials defined in (1.2) as well as for \( q \)-Wilson polynomials [10] defined by

\[
(5.1) \quad p_n(x; a, b, c, d) = \phi_3 \left[ q^{n}, q^{n-1}abcd, ae^\theta, ae^{-i\theta}; ab, ac, ad; q, q \right],
\]

where \( x = \cos \theta, 0 \equiv \theta \equiv \pi \).

For the \( q \)-Racah case let us specialize the parameters in the following way:

\[
(5.2) \quad b_2 = q^{M+a+b+2}, \quad b_3 = q^{-(M+b+1)}, \quad b_4 = q^{-M},
\]

\[
\binom{n}{k} \cdot 10_{\Phi_9} \left[ q^{-n}, q^{n+a+b+1}, q^{-x}, q^{x-M+\gamma}, q^{a+1}, q^{-M}, q^{-y+\gamma+1}; q, q \right],
\]

where \( M \) and \( M' \) are nonnegative integers. (4.10) then gives

\[
(5.3) = (q^{b+1}_x)_k (q^{a+y+1}_x)_k (q^{a+1}_x)_k (q^{a+y+1}_x)_k \sum_{k=0}^{n} \frac{(q)^n_k (q^{a+a+b+1}_x)_k (q^{-y-x})_k (q^{y-M})_k (q^{y})_k (q^{-M})_k}{(q)^k_M (q^{-y+1})_k (q^{y})_k (q^{a-y+1})_k}
\]

\[
\binom{k}{10_{\Phi_9}} \left[ q^{-k}, q\sqrt{q^{-k}}, -q\sqrt{q^{-k}}, q^{y-a-k}, q^{-y-a-k}, q^{-b-k}, q^{y-b-k}, q^{x-M}, q^{a+1}, q^{y+1}, q^{y+1}, q^{M'-x-1-k}, q^{y+1}, q^{M'-y+1-k}, q^{y+1}, q^{y+1}, q^{y+1}; q, q^{M+M'+a+b+3} \right].
\]
Unless $M + M' + \alpha + \beta + 2 = 0$ the $1_{\Phi_9}$ series on the right cannot be transformed to another $1_{\Phi_9}$ which does not mean, however, that the sum on the r.h.s. cannot be transformed to any other form. All one needs to do is to use various choices of the parameters $b_1, b_2, b_3, b_4, N$ so that the l.h.s. of (4.10) remains essentially the same while the r.h.s. undergoes a transformation. For example, an alternative to (5.2) is the choice

$$b_1 = q^{a+\gamma+1}, \ b_2 = q^{a-\gamma+1}, \ b_3 = q^\gamma, \ b_4 = q^{\beta+\gamma+1}$$

which leads to

$$\begin{align*}
\phi_3^4 &\left[ q^{-n}, \ q^{n+a+\beta+1}, \ q^{-x}, \ q^{x-N-\gamma} \right. \\
&\left. q^{a+1}, \ q^{-N}, \ q^{\beta+\gamma+1} \right] \phi_3^4 \left[ q^{-n}, \ q^{n+a+\beta+1}, \ q^{-y}, \ q^{y-N+\gamma} \right. \\
&\left. q^{a+1}, \ q^{-N}, \ q^{\beta+\gamma+1} \right]
\end{align*}$$

$$\frac{(q^{\beta+1})_n(q^{N+a+\beta+2})_n}{(q^{a+1})_n(q^{-N})_n} q^{-(N+\beta+1)n} \sum_{k=0}^{n} \frac{(q^{-n})_k(q^{n+a+\beta+1})_k(q^{N+\beta+1-\gamma})_k(q^{\beta-\gamma+1+x})_k(q^{\beta+\gamma+1+y})_k}{(q^{a+1})_k(q^{N+\beta+1})_k(q^{\beta+\gamma+1})_k} q^k$$

$$\Phi_9 10 \left[ q^{-N-\beta-1-k}, \ q^{-N-\beta-1-k}, q^{-N-a-\beta-1-k}, q^{-\beta-k}, \\
q^{-k}, \ q^{x-N-\gamma}, q^{\gamma-N+\gamma}, q^{-x}, q^{-N}, \\
q^{-\beta-N}, q^{-\beta-x-k}, q^{-\gamma-\beta-y-k}, q^{x-N-\beta-k}, q^{N-N-\beta-k} \right].$$

One can derive a number of things from (5.3) and (5.5). First of all we will show how (5.3) leads to a Watson-type product formula for the $4\Phi_3$ polynomials. Using (2.5) one can easily show that

$$\frac{(q^{-\gamma-x})_k-l(q^{-\gamma-y})_k-l(q^{-x-M'})_k-l(q^y-M')_k-l}{(q)_k-l(q^{-y+1})_k-l(q^{-y})_k-l(q^{\beta+1})_k-l}$$

$$\begin{align*}
&\left( q^{-\gamma-x})_k(q^{-\gamma-y})_k(q^{x-M'})_k(q^{y-M'})_k \\
&\left( q)_k(q^{-y+1})_k(q^{-y})_k(q^{\beta+1})_k
\frac{(q^{\gamma-a-k})_l(q^{y+1-k})_l(q^{\beta-k})_l(q^k)_l}{(q^{a-x+1-k})_l(q^{y+1+y-k})_l(q^{M+1-x-k})_l(q^{M+1-y-k})_l} q^{M+M'+\alpha+\beta+3} l.
\end{align*}$$
Using this in (5.3) and replacing the summation variables \( k, l \) by \( r + s \) and \( r \), respectively, we get

\[
\Phi_3^{(4)} \left[ q^{-n}, q^{n+a+\beta+1}, q^{-x}, q^{x-M' + y} \right]
\]

\[
\Phi_3^{(4)} \left[ q^{-n}, q^{n+a+\beta+1}, q^{-y}, q^{y-M} \right]
\]

\[
\frac{(q^{\beta+1})_n(q^{\alpha-y+1})_n}{(q^{\alpha+1})_n(q^{\beta+y+1})_n}
\]

This is an interesting formula which is likely to have important applications to convolution structures of all orthogonal \( \Phi_3 \) polynomials. We hope to report on these applications in a forthcoming paper.

It is obvious that a similar formula can be obtained from (5.5), but its main advantage is revealed when we take the limit \( y \to \infty \) and obtain the formula for the \( q \)-Hahn polynomials:

\[
\Phi_2 \left[ q^{-n}, q^{n+a+\beta+1}, q^{-x}, q^{x-N-y} \right] \Phi_2 \left[ q^{-n}, q^{n+a+\beta+1}, q^{-y}, q^{y-N} \right]
\]

\[
(q^{\beta+1})_n(q^{N+a+\beta+2})_n/(q^{\alpha+1})_n(q^{-N})_n
\]

\[
\Phi_7 \left[ q^{-N-\beta-1-k}, q^{x-N-\beta-1-k}, q^{-y}, q^{x-N-\beta-k}, q^{-x}, q^{y} \right]
\]

\[
\Phi_7 \left[ q^{-N-\beta-k}, q^{-N-\beta-k}, q^{-x}, q^{x-N-\beta-k}, q^{y-N-\beta-k} ; q, q^{a+1-N+x+y+k} \right]
\]

However, use of (2.19) now gives

\[
\Phi_7 \left[ (q^{N+\beta+1})_k(q^{N+\beta+1-x})_k(q^{N+\beta+1+y})_k \Phi_3^{(4)} \left[ q^{-k}, q^{k+a+\beta+1}, q^{-k}, q^{k+a+\beta+1} \right] \right]
\]

\[
(q^{N+\beta+1})_k(q^{N+\beta+1-x})_k(q^{N+\beta+1+y})_k \Phi_3^{(4)} \left[ q^{-k}, q^{k+a+\beta+1}, q^{-k}, q^{k+a+\beta+1} \right]
\]
by (1.11). So equation (5.8) reduces to

\[
3\Phi_2 \left[ \begin{array}{c} \alpha_1, \\ \beta_1, \\ \gamma_1 \end{array} \begin{array}{c} q^{-n}, \\ q^{n+a+b+1}, \\ q^{-x} \end{array}; q, q^{x-N-\beta} \right] 3\Phi_2 \left[ \begin{array}{c} \alpha_1, \\ \beta_1, \\ \gamma_1 \end{array} \begin{array}{c} q^{-n}, \\ q^{n+a+b+1}, \\ q^{-y} \end{array}; q, q^{x-N-\beta} \right] \\
= \frac{(q^{a+1})_n}{(q^{a})_n} q^{-n} \left[ \begin{array}{c} \alpha_1, \\ \beta_1, \\ \gamma_1 \end{array} \begin{array}{c} q^{-n}, \\ q^{n+a+b+1}, \\ q^{-y} \end{array}; q, q^n \right] \\
= (q^{a+1})_n (q^{n+a+b+2})_n (q^{a})_n (q^{a+1})_n (q^{a+1})_n (q^{a+1})_n (q^{a+1})_n (q^{a+1})_n (q^{a+1})_n (q^{a+1})_n \\
(5.9)
\]

If we let \( q \to 1 \) this becomes the Bateman-type product formula for the Hahn polynomials that was obtained in [8]. Unfortunately (5.9) cannot be strictly regarded as its proper \( q \)-analogue because of the argument \( q^{x-N-\beta} \) in the first \( 3\Phi_2 \) on the l.h.s. However, using (1.5) we may easily derive the following transformation formula:

\[
3\Phi_2 \left[ \begin{array}{c} \alpha_1, \\ \beta_1, \\ \gamma_1 \end{array} \begin{array}{c} q^{-n}, \\ q^{n+a+b+1}, \\ q^{-x} \end{array}; q, q^{x-N-\beta} \right] \\
= (-1)^n \left( \frac{q^{a+1}}{q^{a}} \right)_n \left[ \begin{array}{c} \alpha_1, \\ \beta_1, \\ \gamma_1 \end{array} \begin{array}{c} q^{-n}, \\ q^{n+a+b+1}, \\ q^{-x} \end{array}; q, q^n \right]. \\
(5.10)
\]

Substituting (5.10) in (5.9) and then replacing \( x \) by \( N - x \) we obtain

\[
3\Phi_2 \left[ \begin{array}{c} \alpha_1, \\ \beta_1, \\ \gamma_1 \end{array} \begin{array}{c} q^{-n}, \\ q^{n+a+b+1}, \\ q^{-y} \end{array}; q, q \right] 3\Phi_2 \left[ \begin{array}{c} \alpha_1, \\ \beta_1, \\ \gamma_1 \end{array} \begin{array}{c} q^{-n}, \\ q^{n+a+b+1}, \\ q^{-y} \end{array}; q, q \right] \\
= \frac{(q^{N+a+b+2})_n (q^{N})_n}{(q^{-N})_n} (-1)^n q^{n(n-1)/2-Nn} \left[ \begin{array}{c} \alpha_1, \\ \beta_1, \\ \gamma_1 \end{array} \begin{array}{c} q^{-n}, \\ q^{n+a+b+1}, \\ q^{-y} \end{array}; q, q \right] \\
= \frac{(q^{N+a+b+2})_n (q^{N})_n}{(q^{-N})_n} \left[ \begin{array}{c} \alpha_1, \\ \beta_1, \\ \gamma_1 \end{array} \begin{array}{c} q^{-n}, \\ q^{n+a+b+1}, \\ q^{-y} \end{array}; q, q \right] \\
(5.11)
\]

The two \( q \)-Hahn polynomials on the l.h.s. have the same parameters only when \( \alpha = \beta \), which seems to imply that a product formula of Bateman type does not exist for this particular \( q \)-analogue of the Hahn polynomials except in this special case. However, a Watson-type product formula for these polynomials always exists, as can be seen by taking the limit \( \gamma \to \infty \) in (5.7):

\[
3\Phi_2 \left[ \begin{array}{c} \alpha_1, \\ \beta_1, \\ \gamma_1 \end{array} \begin{array}{c} q^{-n}, \\ q^{n+a+b+1}, \\ q^{-x} \end{array}; q, q \right] 3\Phi_2 \left[ \begin{array}{c} \alpha_1, \\ \beta_1, \\ \gamma_1 \end{array} \begin{array}{c} q^{-n}, \\ q^{n+a+b+1}, \\ q^{-y} \end{array}; q, q \right] \\
= (-1)^{n(n+1)/2} \left( \frac{q^{a+1}}{q^{a}} \right)_n \sum_{r=0}^{n-r} \sum_{s=0}^{n} \left( \frac{q^{a}}{q} \right)_{r+s} \left( \frac{q^{a}}{q} \right)_{r+s} \left( \frac{q^{a}}{q} \right)_{r+s} \left( \frac{q^{a}}{q} \right)_{r+s} \\
(5.12)
\]

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This is a straight generalization of Gasper's product formula for Hahn polynomials [6].

Let us now set $M = M'$ in (5.7) (although this specialization is not necessary) and introduce parameters $a, b, c, d$ and real angles $\theta, \phi$ through the correspondence:

\begin{align}
    q^{-x} &= ae^{i\theta}, \quad q^{-y} = ae^{i\phi}, \quad q^{-M} = ad,
    
    q^{a+1} = ab, \quad q^{b+1} = cd, \quad q^{y} = a/d.
\end{align}

Since (5.7) remains valid no matter what values we choose for the parameters as long as $n$ is a nonnegative integer, substitution of (5.13) in (5.7) now leads to an interesting formula for $q$-Wilson polynomials defined in (5.1):

\begin{align}
    p_n(x; a, b, c, d)p_n(y; a, b, c, d) &= \frac{(cd)_n(bd)_n(a)}{(ab)_n(ac)_n} \\
    &\cdot \sum_{r=0}^{n-r} \sum_{s=0}^{r} \frac{(q^n)_r(s)(q^{n+1-\alpha+\beta+1})_r(q^{n+1})_{r+s}(ae^{-i\theta})_r(ae^{i\theta})_r(ae^{i\phi})_r(ae^{-i\phi})_r}{(q)_r(q)_s(ad)_r(ab)_s(ac)_r(ad)_s(ab)_s(ac)_s} \\
    &\cdot \frac{(de^{i\theta})_s(de^{-i\theta})_s(de^{i\phi})_s(de^{-i\phi})_s}{(cd)_s(bd)_s(d/a)_s} \cdot \frac{1 - q^{-r-s}d}{1 - q^{-r-s}d} q^{r+s}.
\end{align}

Note that the reality of the r.h.s. is self-evident as is its symmetry in $\theta, \phi$ and $b, c$. The continuous $q$-Jacobi polynomials, as defined in [10], are a special case of $q$-Wilson polynomials with $\alpha = \beta = -d, b = q^{a+1/2}, c = -q^{b+1/2}$. We show the corresponding formula explicitly:

\begin{align}
    \Phi_3^{(4)} \left[ q^{-n}, \quad q^{n+\alpha+\beta+1}, \quad \sqrt{q}e^{i\theta}, \quad \sqrt{q}e^{-i\theta}, \quad q^{\alpha+1}, \quad -q^{\beta+1}, \quad -q \right] \\
    \Phi_3^{(4)} \left[ q^{-n}, \quad q^{n+\alpha+\beta+1}, \quad \sqrt{q}e^{i\phi}, \quad \sqrt{q}e^{-i\phi}, \quad q^{\alpha+1}, \quad -q^{\beta+1}, \quad -q \right] \\
    (5.15) &= \frac{(q^{\beta+1})_n(-q^{\alpha+1})_n}{(q^{\alpha+1})_n(-q^{\beta+1})_n} \\
    &\cdot (1)^n \sum_{r=0}^{n-r} \sum_{s=0}^{r} \frac{(q^{-n})_r(q^{n+\alpha+\beta+1})_r(q^{\alpha+1})_r(q^{-\alpha+1})_r}{(q)_r(q)_s(-q)_r(-q)_s(q^{\alpha+1})_r} \\
    &\cdot \frac{(\sqrt{q}e^{i\theta})_r(\sqrt{q}e^{-i\theta})_r(-\sqrt{q}e^{i\phi})_s(-\sqrt{q}e^{-i\phi})_s}{(q^{-\alpha+1})_s(-q)_r(q^{\alpha+1})_s(-q^{\beta+1})_s(-1)_{s}} \cdot \frac{1 + q^{-r-s}}{1 + q^{-r-s}} q^{r+s}.
\end{align}

In the ultraspherical case $\alpha = \beta$ one can derive a Bateman-type formula by setting $\alpha = \beta$ in (5.5), transforming the $\Phi_{10}$ by a formula of type (3.14) so that the top left-hand parameter becomes free of $k$. But the resulting formula seems to be less interesting than, and strictly speaking, only a special case of equation (5.15) and so does not seem worth separate display.
References


