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IRREDUCIBLE REPRESENTATIONS OF A_n WITH A 1-DIMENSIONAL WEIGHT SPACE

BY

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ABSTRACT. In this paper we classify all irreducible linear representations of the simple Lie algebra A_n which admit a one-dimensional weight space with respect to some Cartan subalgebra H of A_n. We first show that the problem is equivalent to determining all algebra homomorphisms from the centralizer of the Cartan subalgebra H in the universal enveloping algebra of A_n to the base field. We construct all such algebra homomorphisms and provide conditions under which two such algebra homomorphisms provide inequivalent irreducible representations of A_n.

1. Introduction. In this paper, we classify all irreducible representations (ρ, V) of $A_n = sl(n+1, F)$ having at least one 1-dimensional weight space. This classification includes all irreducible representations having a "highest" weight and hence all finite dimensional irreducible representations. A precise statement of our main result is given at the end of this section after we have developed the necessary terminology and background.

We use Humphreys [8] as our basic reference for notation, terminology and preliminary results. Throughout this paper, F denotes an algebraically closed field of characteristic 0, L a simple Lie algebra of rank n over F, H a fixed Cartan subalgebra, $U(L)$ the universal enveloping algebra of L, $C(L)$ the centralizer of H in $U(L)$ and (ρ, V) an irreducible representation of L having a weight space decomposition with at least one 1-dimensional weight space. If f is a function with domain D and $A \subseteq D$ then we denote by $f \downarrow A$ the restriction of the function f to the set A.

A weight space V^λ of (ρ, V) is a nonzero subspace of V corresponding to some element λ in the dual space H^* of H and is defined by

$$V^\lambda = \left\{ v \in V \mid (\rho(h) - \lambda(h))^n v = 0 \text{ for some positive integer } n \text{ and for all } h \in H \right\}.$$

By our assumptions of the existence of a 1-dimensional weight space and of the irreducibility of (ρ, V) we have that for each $\lambda \in H^*$, $V^\lambda = V_\lambda = \{v \in V \mid \rho(h)v = \lambda(h)v \text{ for all } h \in H\}$. Clearly both V^λ and V_λ are $\rho(H)$-invariant subspaces of V and

$$V = \sum_{\lambda \in H^*} \oplus V^\lambda = \sum_{\lambda \in H^*} \oplus V_\lambda.$$

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The adjoint representation (ad, L) of L induces a natural weight space decomposition of L called the root space decomposition. We write this as

$$L = L_0 \oplus \sum_{\gamma \in \Phi} \oplus L_\gamma,$$

where Φ denotes the set of nonzero roots. Then $L_0 = H$ and, for all $\gamma \in \Phi$, $\dim L_\gamma = 1$.

Let $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ denote a base of Φ and $\Phi^+ = \{\beta_1, \ldots, \beta_m\}$ the corresponding positive roots. For each $\beta_i \in \Phi^+$, we select elements X_{β_i} and $X_{-\beta_i}$ belonging to L_{β_i} and $L_{-\beta_i}$ respectively and for each $\alpha_i \in \Delta$ we select elements $h_{\alpha_i} \in H$ such that $\{X_{\beta_i}, X_{-\beta_i}, h_{\alpha_i} \mid \beta_i \in \Phi^+, \alpha_i \in \Delta\}$ forms a Chevalley basis of L. If we consider $U = U(L)$ as an L-module under the adjoint operation, U has a weight space decomposition $U = \sum_{\mu \in H^*} \oplus U_\mu$ with $U_\nu U_\mu \subseteq U_{\nu+\mu}$. The zero weight space U_0 is a subalgebra of U equal to $C(L)$. A Poincaré-Birkhoff-Witt basis for U is given by the set of all monomials

$$u(\bar{p}, \bar{l}, \bar{q}) = X_{-\beta_m}^{p_m} \cdots X_{-\beta_1}^{p_1} h_{\alpha_1}^{l_1} \cdots h_{\alpha_n}^{l_n} X_{\beta_1}^{q_1} \cdots X_{\beta_m}^{q_m}, \tag{1.1}$$

where $\bar{p} = (p_i)$ and $\bar{q} = (q_i)$ are m-tuples of nonnegative integers and $\bar{l} = (l_i)$ is an n-tuple of nonnegative integers. Using this basis we have that

$$C(L) = U_0 = \text{lin. span}\left\{u(\bar{p}, \bar{l}, \bar{q}) \mid \sum_{i=1}^{m} (q_i - p_i)\beta_i = 0\right\}. \tag{1.2}$$

The following theorem is the key to our classification of all pointed representations of A_n. Although a proof of this theorem appears in the literature [9], we include it for the convenience of the reader and to free it from Zorn's Lemma.

THEOREM 1.3. (i) *If V_λ is any weight space of the irreducible representation (ρ, V) of L then V_λ is an irreducible $C(L)$-module and if* $\dim V_\lambda = 1$ *and* $0 \neq v_\lambda \in V_\lambda$ *then the map ζ: $C(L) \to F$ defined by $\zeta(c)v_\lambda = \rho(c)v_\lambda$ is a nontrivial algebra homomorphism.*

(ii) *If ζ: $C(L) \to F$ is a nontrivial algebra homomorphism, then there exists a unique maximal left ideal M of U containing* $\ker \zeta$. *Moreover, the irreducible left regular L-module U/M has $(U_0 + M)/M$ as a* 1*-dimensional weight space with weight* $\lambda = \zeta \downarrow H$.

PROOF. (i) Let V_λ be a weight space of (ρ, V) and $0 \neq v_\lambda \in V_\lambda$. Consider (ρ, V) as a representation of U. Since $\rho(U)v_\lambda = V$ and $\rho(U_\mu)v_\lambda \subseteq V_{\lambda+\mu}$, the irreducibility of (ρ, V) implies that $\rho(C(L))v_\lambda = \rho(U_0)v_\lambda = V_\lambda$. Thus V_λ is an irreducible representation of $C(L)$. Suppose now that $\dim V_\lambda = 1$ and define ζ: $C(L) \to F$ by setting $\zeta(c)v_\lambda = \rho(c)v_\lambda$. Since $\rho(C(L))v_\lambda = V_\lambda$ and ρ is an associative representation of U, we have that ζ is a nontrivial algebra homomorphism.

(ii) Let ζ: $C(L) \to F$ be a nontrivial algebra homomorphism and set $\ker \zeta = A$. Let I be any proper left ideal of U containing A. There exists at least one such proper left ideal since $UA \cap C(L) = A \neq C(L)$ and hence $UA \neq U$. Let x be any nonzero element of I and write $x = \sum x_\mu$ where $x_\mu \in U_\mu$. We show that each $x_\mu \in I$ by induction on $k_x = |\{\mu \mid x_\mu \neq 0\}|$. The result is obviously true when $k_x = 1$ and

hence we assume that $k_x = n > 1$ and that the result is true for all $y \in I$ with $k_y < k_x$. Thus for each $h \in H$ we have

$$(1.4) \qquad \begin{aligned} hx &= \sum_\mu hx_\mu = \sum_\mu \big([h, x_\mu] + x_\mu h\big) \\ &= \sum_\mu \big(\mu(h)x_\mu + x_\mu \zeta(h)\big) \mod UA \\ &= \sum_\mu (\mu(h) + \zeta(h))x_\mu \mod UA. \end{aligned}$$

For each $\nu \in H^*$ with $x_\nu \neq 0$, define $y_{\nu,h} = \sum_{\mu \neq \nu}(\nu(h) - \mu(h))x_\mu$. Then, by (1.4), $y_{\nu,h} = (\nu(h) + \zeta(h))x - hx \mod UA$ and hence $y_{\nu,h} \in I$. Since $k_{y_{\nu,h}} < k_x$ we have that $(\nu(h) - \mu(h))x_\mu \in I$ for all $h \in H$ and all $\mu \neq \nu$. If $\mu \neq \nu$ then there exist $h \in H$ with $\nu(h) - \mu(h) \neq 0$ and hence $x_\mu \in I$ for all $\mu \neq \nu$ which in turn implies that $x_\mu \in I$ for all μ.

From (1.4), we see that $x_\mu + UA$ is a weight vector in U/UA with weight $\mu + \zeta \downarrow H$ and hence I/UA has a weight space decomposition given by

$$(1.5) \qquad I/UA = \sum_\mu \big((I \cap U_\mu) + UA\big)/UA.$$

Since I is a proper ideal, $I \cap C(L) = A$ and hence every proper left ideal I containing A has the property that

$$(1.6) \qquad I/(UA) \subseteq \sum_{\mu \neq 0} (U_\mu + UA)/(UA).$$

It follows that the sum of all proper left ideals containing A is the unique maximal left ideal M of U containing A.

Equation (1.4) implies that $(C(L) + M)/M$ is a weight space of U/M with weight $\lambda = \zeta \downarrow H$. Since $(C(L) + M)/M \cong C(L)/(C(L) \cap M) = C(L)/A$, this weight space is 1-dimensional. $\square$

From Theorem 1.3, it follows that in order to determine all irreducible representations (ρ, V) of L admitting a 1-dimensional weight space, it suffices to find all algebra homomorphisms ζ: $C(L) \to F$. An algebra homomorphism ζ: $C(L) \to F$ is called a *mass function*, and an irreducible representation admitting a 1-dimensional weight space is called *pointed*.

In this paper, we construct all mass functions for the algebras A_n and hence determine all pointed representations of A_n.

In order to illustrate how our results extend the standard results on classifying irreducible representations of highest weight we briefly review these results. If (ρ, V) denotes an irreducible representation with highest weight λ then the weight space V_λ is 1-dimensional and there exists a base Δ of roots such that $V_{\lambda+\alpha} = \{0\}$ for all $\alpha \in \Delta$. Let ζ: $C(L) \to F$ be the mass function associated with the 1-dimensional weight space V_λ. Let Ψ: $C(L) \to F[\lambda_1, \ldots, \lambda_n]$ be the Harish-Chandra homomorphism relative to the base Δ.

$$\Psi\big(u(\bar{p}, \bar{l}, \bar{q})\big) = \begin{cases} 0 & \text{if } u(\bar{p}, \bar{l}, \bar{q}) \in C(L) \text{ with } \bar{q} \neq \bar{0}, \\ \lambda_1^{l_1} \cdots \lambda_n^{l_n} & \text{if } u(\bar{p}, \bar{l}, \bar{q}) \in C(L) \text{ with } \bar{p} = \bar{q} = \bar{0}, \end{cases}$$

where the λ_i's are indeterminants. Now if ε: $F[\lambda_1,\ldots,\lambda_n] \to F$ is defined by $\varepsilon(\lambda_i) = \zeta(h_{\alpha_i})$ for each $\alpha_i \in \Delta$ we have $\zeta = \varepsilon \circ \Psi$. Thus for any base Δ we can construct all mass functions associated with highest weight spaces with respect to Δ by composing the Harish-Chandra homomorphism Ψ relative to Δ with any evaluation map ε.

Our aim is to generalize the preceding results for the case of $L = A_n$. For each pair (Δ', Δ) consisting of a base Δ of the roots of A_n and a subset Δ' of Δ we define a map $\Psi_{\Delta',\Delta}$ from $C(L)$ into a ring of polynomials over F in several commuting indeterminants. This map generalizes the Harish-Chandra homomorphism in the sense that the definition reduces to the standard Harish-Chandra homomorphism relative to Δ where $\Delta' = \varnothing$. Our main result is

THEOREM 1.7. *Let ζ: $C(L) \to F$ be a mass function. Then there exists a base Δ, a subset Δ' of Δ and an evaluation map ε such that $\zeta = \varepsilon \circ \Psi_{\Delta',\Delta}$.*

This paper is organized as follows. §2 contains the definition of the generalized Harish-Chandra homomorphisms referred to in Theorem 1.7 and the definition of a group of automorphisms on A_n which we use to change from one base of the root system of A_n to another and to compute the values of a given mass function relative to different bases. In §3 we prove Theorem 1.7 for the case A_1 and A_2. The proofs of these results are outlined in **[4, 15]** but we present them here in full detail since our approach is different and provides the key to the general result. §4 contains the bulk of the technical results concerning mass functions on A_3, A_4 and A_n which we use to establish Theorem 1.7 for the general case. Many of the results in this section depend heavily on the use of identities in $C(A_n)$. It should be noted that for the sake of simplicity and convenience these identities have not been expressed in any fixed linear basis of $C(A_n)$. In §5, we complete the proof of Theorem 1.7. Finally, §6 addresses the problem of selecting a set of mass functions on $C(A_n)$ which label the equivalence classes of pointed representations of A_n in a one-one fashion.

2. Generalized Harish-Chandra homomorphisms. In the remainder of this paper, we deal with $L = A_n = sl(n+1, F)$. The subalgebra of A_n consisting of all diagonal matrices in A_n is a Cartan subalgebra which we denote by H. The root system Φ of A_n relative to H is then given by

$$\Phi = \{w_i - w_j \mid i, j = 1,2,\ldots,n, i \neq j\},$$

where w_k is the map which projects an order $n+1$ matrix onto its (k,k)th component. A base Δ of Φ is given by

$$\Delta = \{w_i - w_{i+1} \mid i = 1,2,\ldots,n\}$$

and the set of positive roots Φ^+ of Φ relative to Δ is

$$\Phi^+ = \{w_i - w_j \mid 1 \leq i < j \leq n+1\}.$$

If $\{e_{ij}\}$ denotes the standard set of matrix units, then a Chevalley basis of A_n is given by $\{h_i, X_{w_k - w_l} \mid i = 1,2,\ldots,n;\ k, l = 1,2,\ldots,n+1 \text{ with } k \neq l\}$ where $h_i = e_{ii} - e_{i+1,i+1}$ and $X_{w_k - w_l} = e_{kl}$.

It should be noted that even if we select a different base of Φ we always choose a Chevalley basis of A_n corresponding to this base which has the same structure constants as those arising from the basis $\{e_{ii} - e_{i+1,i+1}, e_{k,l}\}$. This fact is used in the computation of identities in $C(A_n)$ which we need later in this paper.

Let $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ be some fixed base of Φ and let Φ^+ be the corresponding set of positive roots relative to Δ. We then select a Chevalley basis $\{h_i = h_{\alpha_i}, X_{\beta_j}, X_{-\beta_j} \mid \alpha_i \in \Delta, \beta_j \in \Phi^+\}$ as specified above. Assume that Δ' is either $\varnothing$ or $\Delta' = \bigcup_{i=1}^{l} \Delta_i \subseteq \Delta$ where the Δ_i's generate mutually orthogonal irreducible root subsystems Φ_i of Φ. The subalgebra of A_n generated by $\{X_\mu \mid \mu \in \Phi_i\}$ is isomorphic to A_{n_i} where $n_i = |\Delta_i|$. We identify A_{n_i} with this subalgebra. This identification extends naturally to the universal enveloping algebra $U(A_{n_i})$ and its cycle subalgebra $C(A_{n_i})$. The subalgebras of $U(A_n)$ identified in this manner with $U(A_{n_i})$ and $C(A_{n_i})$ are denoted by $U(\Delta_i)$ and $C(\Delta_i)$ respectively.

Let $\mathcal{H}$ denote the subalgebra of $C(A_n)$ with 1 generated by $\{h_i \mid \alpha_i \in \Delta, \alpha_i \notin \Delta'\}$ and let C' denote the linear subspace of $C(A_n)$ generated by all elements $u(\bar{p}, \bar{l}, \bar{q}) \in C(A_n)$ of the form (1.1) such that $\bar{q} = (q_i)$ and $q_j \neq 0$ for some $\beta_j \notin \bigcup_{i=1}^{l} \Phi_i$. It is shown in **[12]** that C' is an ideal of $C(A_n)$. It then follows from the proof of the Poincaré-Birkhoff-Witt Theorem that $C(A_n)$ can be written as a vector space direct sum $C(A_n) = \mathcal{H}C(\Delta_1)C(\Delta_2) \cdots C(\Delta_l) \oplus C'$. In case $\Delta' = \varnothing$, we have $C(A_n) = \mathcal{H} \oplus C'$ and $\mathcal{H}$ is the universal enveloping algebra of H.

LEMMA 2.1. *Let R be any commutative associative F-algebra with* 1. *If Ψ_0: $\mathcal{H} \to R$ and Ψ_i: $C(\Delta_i) \to R$ for $i = 1, 2, \ldots, l$ are F-algebra homomorphisms with $\Psi_i(1) = 1$ for $i = 0, 1, \ldots, l$, then there exists a unique F-algebra homomorphism Ψ: $C(A_n) \to R$ such that $\Psi \downarrow \mathcal{H} = \Psi_0$, $\Psi \downarrow C(\Delta_i) = \Psi_i$ and $C' \subseteq \ker \Psi$.*

PROOF. When $\Delta' = \varnothing$ the result follows trivially from the decomposition. Since the Δ_i's are mutually orthogonal we have that $[U(\Delta_i), U(\Delta_j)] = 0$ for all $i, j = 1, 2, \ldots, l$ with $i \neq j$. We also have that $[\mathcal{H}, C(\Delta_i)] = 0$ for $i = 1, 2, \ldots, l$ and hence every monomial $u(\bar{p}, \bar{l}, \bar{q}) \in C(A_n)$ of the form (1.1) is either in C' or can be expressed uniquely as a scalar times a product of monomials $hc_1 \cdots c_l$ where $h \in \mathcal{H}$ and $c_i \in C(\Delta_i)$. Thus if there exists an F-algebra homomorphism Ψ satisfying the condition above it must be unique. On the other hand, if we define Ψ on each basis element $u(\bar{p}, \bar{l}, \bar{q}) \in C(A_n)$ by setting $\Psi(u(\bar{p}, \bar{l}, \bar{q})) = 0$ if $u(\bar{p}, \bar{l}, \bar{q}) \in C'$ and $\Psi(u(\bar{p}, \bar{l}, \bar{q})) = k\Psi_0(h)\Psi_1(c_1) \cdots \Psi_l(c_l)$ where $u(\bar{p}, \bar{l}, \bar{q}) = khc_1 \cdots c_l$ with $h \in \mathcal{H}$, $c_i \in C(\Delta_i)$ then it readily follows that Ψ is an F-algebra homomorphism satisfying the required conditions. $\square$

Let $\lambda_1, \ldots, \lambda_n, s_1, \ldots, s_n$ be commuting indeterminates over F and

$$F[\lambda_1, \ldots, \lambda_n, s_1, \ldots, s_n]$$

be the corresponding polynomial ring. We use Lemma 2.1 to define the generalized Harish-Chandra homomorphism associated with the pair $\Delta' \subseteq \Delta$ which we write

$$\Psi_{\Delta', \Delta}: C(A_n) \to F[\lambda_1, \ldots, \lambda_n, s_1, \ldots, s_n].$$

Our first step is to note that using the Dynkin diagram associated with Δ, we can define an equivalence relation on Δ' by $\alpha \sim \beta$ iff the subgraph of the Dynkin

diagram connecting α to β has all of its vertices in Δ'. We denote the equivalence classes of this relation by Δ_i, $i = 1,\dots,l$. The root subsystems Φ_i generated by Δ_i, respectively are irreducible and mutually orthogonal. According to Lemma 2.1, we can define the map $\Psi_{\Delta',\Delta}$ with $C' \subseteq \ker \Psi_{\Delta',\Delta}$ by specifying its restrictions to $\mathcal{H}$, $C(\Delta_1),\dots,$ and $C(\Delta_l)$.

Let $\Delta = \{\alpha_1,\dots,\alpha_n\}$ be indexed so that $\Delta_i = \{\alpha_{k_i+1},\dots,\alpha_{k_i+n_i}\}$. Then $\mathcal{H}$ is merely the polynomial ring in the commuting indeterminates $h_i = h_{\alpha_i}$ for $\alpha_i \notin \Delta'$. Therefore, we can define the algebra homomorphism $\Psi_0\colon \mathcal{H} \to F[\lambda_1,\dots,\lambda_n, s_1,\dots,s_n]$ by extending the map sending h_i to λ_i. The restrictions to $C(\Delta_i)$ are more complicated. We generalize the construction of [**14**] and define $\Psi_i\colon C(\Delta_i) \to F[\lambda_1,\dots,\lambda_n, s_1,\dots,s_n]$ as follows. Let $V[i]$ denote the free $F[\lambda_{k_i+1},\dots,\lambda_{k_i+n_i}, s_i]$-module with basis $\{v(\bar{l}) \mid \bar{l} = (l_1,\dots,l_{n_i}) \in \mathbf{Z}^{n_i}\}$. We now define an F-algebra embedding ρ_i of the subalgebra A_{n_i} into the minus algebra of $\operatorname{End}_{F[\lambda_{k_i+1},\dots,\lambda_{k_i+n_i},s_i]} V[i]$ by describing the action of ρ_i on the basis of A_{n_i}. For $j = 1,2,\dots,n_i$ define $\rho_i(h_{k_i+j})$ by

$$\rho_i(h_{k_i+j})v(\bar{l}) = (\lambda_{k_i+j} - l_{j-1} + 2l_j - l_{j+1})v(\bar{l}) \tag{2.2}$$

and for $\xi = w_p - w_q \in \Phi_i^+ = \Phi_i \cap \Phi^+$ define $\rho_i(X_\xi)$ and $\rho_i(X_{-\xi})$ by

$$\begin{aligned} \rho_i(X_\xi)v(\bar{l}) &= (s_i - \lambda_{k_i+1} - \cdots - \lambda_{k_i+p-1} - l_{p-1} + l_p)v(\bar{l} + \bar{\xi}), \\ \rho_i(X_{-\xi})v(\bar{l}) &= (s_i - \lambda_{k_i+1} - \cdots - \lambda_{k_i+q-1} - l_{q-1} + l_q)v(\bar{l} - \bar{\xi}), \end{aligned} \tag{2.3}$$

where $\bar{\xi}$ is the n_i-tuple with 1 in the $p, p+1,\dots,q-1$ components and zero elsewhere. By convention we set λ_{k_i}, l_0 and l_{n_i+1} to be zero whenever they occur in these expressions.

THEOREM 2.4. *The map ρ_i is an injective F-algebra homomorphism.*

PROOF. Clearly the map ρ_i defined above is injective and F-linear. In order to see that it is also an algebra homomorphism it suffices to verify that ρ_i preserves the following multiplication relations in A_{n_i}.

$$\begin{aligned} [h_p, h_q] &= 0 && \text{for } k_i + 1 \leqslant p < q \leqslant k_i + n_i, \\ [h_p, X_\xi] &= \xi(h_p)X_\xi && \text{for } p = k_i + 1,\dots,k_i + n_i;\ \xi \in \Phi_i, \end{aligned} \tag{2.5}$$

$$[X_\xi, X_\eta] = \begin{cases} h_p + \cdots + h_{q-1} & \text{for } -\eta = \xi = w_p - w_q \in \Phi_i^+, \\ -(h_p + \cdots + h_{q-1}) & \text{for } \eta = -\xi = w_p - w_q \in \Phi_i^+, \\ (\delta_{qk} - \delta_{jp})X_{\xi+\eta} & \text{for } \xi = w_p - w_q \neq -\eta = -(w_k - w_j). \end{cases}$$

A direct computation completes the proof of this result. □

Let $\hat{\rho}_i$ be the extension of ρ_i to $U(A_{n_i})$. For each $c \in C(\Delta_i) \subseteq C(A_n)$, define $\Psi_i(c)$ by

$$\Psi_i(c)v(\bar{0}) = \hat{\rho}_i(c)v(\bar{0}). \tag{2.6}$$

We are using here the fact that the operator $\hat{\rho}_i(c)$ maps $v(\bar{0})$ to a $F[\lambda_{k_i+1},\dots,\lambda_{k_i+n_i}, s_i]$ multiple of $v(\bar{0})$, that is

$$\Psi_i(c) \in F[\lambda_{k_i+1},\dots,\lambda_{k_i+n_i}, s_i] \subseteq F[\lambda_1,\dots,\lambda_n, s_1,\dots,s_n].$$

Now, let $\Psi_0,\ldots,\Psi_l$ be the algebra homomorphisms just defined. Then the *generalized Harish-Chandra homomorphism* $\Psi_{\Delta',\Delta}$: $C(A_n) \to F[\lambda_1,\ldots,\lambda_n, s_1,\ldots,s_n]$ is that unique algebra homomorphism with $C' \subseteq \ker \Psi_{\Delta',\Delta}$ which extends all of $\Psi_0,\ldots,\Psi_l$. In the case $\Delta' = \varnothing$, $\Psi_{\varnothing,\Delta}$ differs from the usual Harish-Chandra homomorphism only in that normally the codomain is taken to be $F[\lambda_1,\ldots,\lambda_n]$ while here it is $F[\lambda_1,\ldots,\lambda_n, s_1,\ldots,s_n]$.

The following table is a complete list of all $\Psi_{\Delta',\Delta}$: $C(A_2) \to F[\lambda_1, \lambda_2, s_1, s_2]$ where $\Delta = \{\alpha, \beta\}$. The fact that $\Psi_{\Delta',\Delta}$ is determined by the values listed in this table is given in §3. We are writing $X_{-\alpha}$ as Y_α.

TABLE 2.7

	$\Psi_{\Delta,\Delta}$	$\Psi_{\{\alpha\},\Delta}$	$\Psi_{\{\beta\},\Delta}$	$\Psi_{\varnothing,\Delta}$
h_α	λ_1	λ_1	λ_1	λ_1
h_β	λ_2	λ_2	λ_2	λ_2
$Y_\alpha X_\alpha$	$s_1(s_1 - \lambda_1 - 1)$	$s_1(s_1 - \lambda_1 - 1)$	0	0
$Y_\beta X_\beta$	$(s_1 - \lambda_1)(s_1 - \lambda_1 - \lambda_2 - 1)$	0	$s_1(s_1 - \lambda_2 - 1)$	0
$Y_{\alpha+\beta} X_{\alpha+\beta}$	$s_1(s_1 - \lambda_1 - \lambda_2 - 1)$	0	0	0
$Y_{\alpha+\beta} X_\alpha X_\beta$	$s_1(s_1 - \lambda_1)(s_1 - \lambda_1 - \lambda_2 - 1)$	0	0	0

If $\Delta' = \varnothing$, then the image of $\Psi_{\Delta',\Delta} = F[\lambda_1,\ldots,\lambda_n] \subseteq F[\lambda_1,\ldots,\lambda_n, s_1,\ldots,s_n]$ and in this case the evaluation ε which we use in Theorem 1.7 is defined by convention to be 0 on $s_1,\ldots,s_n$. If Δ' is the union of nonempty $\Delta_1,\ldots,\Delta_l$ as described above, then the image of $\Psi_{\Delta',\Delta} \subseteq F[\lambda_1,\ldots,\lambda_n, s_1,\ldots,s_l] \subseteq F[\lambda_1,\ldots,\lambda_n, s_1,\ldots,s_n]$ and in this case ε is defined by convention to be 0 on $s_{l+1},\ldots,s_n$.

We close this section with the definition of a certain class of automorphisms which are associated in a natural way with the Weyl reflections and the Dynkin diagram reversing map.

Let $\alpha \in \Delta$ and σ_α be the usual Weyl reflection of the root system Φ through the hyperplane perpendicular to α. If we define, for $\beta \in \Delta$, $\sigma_\alpha(h_\beta) = h_{\sigma_\alpha(\beta)}$ and $\sigma_\alpha(X_{\pm\beta}) = X_{\sigma_\alpha \pm \beta}$, then by [**8**, Theorem 14.2], σ_α extends uniquely to an automorphism of A_n, which in turn extends uniquely to an automorphism of $U(A_n)$. Each of these automorphisms is denoted by σ_α. Since $\sigma_\alpha(H) = H$ and $\sigma_\alpha(X_\xi) = X_{\sigma_\alpha(\xi)}$ for all $\xi \in \Phi$, we have that σ_α acts as an automorphism on $U_0(A_n) = C(A_n)$. The set of automorphisms $\{\sigma_\alpha \mid \alpha \in \Delta\}$ on $U(A_n)$ generate a group which is isomorphic to the Weyl group of A_n. The elements in this group of automorphisms are called the *Weyl automorphisms*.

If we let σ_f denote the isomorphism on Φ corresponding to reversing the Dynkin diagram of A_n so that $\sigma_f(\alpha_i) = \alpha_{n+1-i}$ for $1 \leqslant i \leqslant n$, then the map on A_n determined by $\sigma_f(h_\beta) = h_{\sigma_f(\beta)}$ and $\sigma_f(X_{\pm\beta}) = X_{\pm\sigma_f(\beta)}$ for $\beta \in \Delta$ is an automorphism on A_n called the *flip map*.

Our use of these automorphisms σ is primarily to keep track of how a mass function $\zeta \circ \sigma$ with the property that $\zeta \circ \sigma = \varepsilon \circ \Psi_{\Delta',\Delta}$ can be used to give us Theorem 1.7. This is done by observing that in this case $\zeta = \varepsilon \circ \Psi_{\sigma\Delta',\sigma\Delta}$.

3. Mass functions on $C(A_1)$ and $C(A_2)$. In this section we prove Theorem 1.7 for the case $n = 1$ and 2. Lemire and Pap **[15]** completed the work begun by Bouwer **[4]** and exhibited a complete list of all mass functions on $C(A_2)$. This list is of central importance in our proof of Theorem 1.7 for arbitrary n. Because of this, we believe that it is worthwhile to devote the major portion of this section to reworking this material. Our presentation differs from **[15** and **4]** in that it is complete and self contained, it introduces the set of roots S_ζ which is the vehicle allowing us to generalize to arbitrary n, and its cases are simplified by making an appropriate choice of base Δ.

We begin by studying mass functions on $C(A_1)$. Let $\Delta = \{\alpha\}$ be a base of the root system of A_1, let $h_1 = h_\alpha$ and $Y_\alpha = X_{-\alpha}$. Then h_1, $Y_\alpha X_\alpha$ are algebraically independent over F and $C(A_1)$ is merely the polynomial ring $F[h_1, Y_\alpha X_\alpha]$. There are only two generalized Harish-Chandra homomorphisms associated with $\Delta = \{\alpha\}$, namely $\Psi_{\varnothing,\Delta}$ and $\Psi_{\Delta,\Delta}$. The first of these differs from the usual Harish-Chandra homomorphism only in that the codomain is normally taken to be $F[\lambda_1]$ while $\Psi_{\varnothing,\Delta}$ takes on values in $F[\lambda_1, s_1]$. The second map $\Psi_{\Delta,\Delta}$ is defined by

$$\Psi_{\Delta,\Delta}(h_1) = \lambda_1, \qquad \Psi_{\Delta,\Delta}(Y_\alpha X_\alpha) = s_1(s_1 - \lambda_1 - 1).$$

Since every mass function ζ: $C(A_1) \to F$ is uniquely determined by its values on h_1 and $Y_\alpha X_\alpha$, we see that there is an evaluation map ε: $F[\lambda_1, s_1] \to F$ such that $\zeta = \varepsilon \circ \Psi_{\Delta,\Delta}$. This proves Theorem 1.7 for $n = 1$.

We now start our investigation of mass functions on $C(A_2)$. Let $\Delta = \{\alpha, \beta\}$ be a base of the root system Φ of A_2 and let

$$\begin{aligned} &h_1 = h_\alpha, && X_\beta, \\ &h_2 = h_\beta, && Y_\beta = X_{-\beta}, \\ &X_\alpha, && X_{\alpha+\beta}, \\ &Y_\alpha = X_{-\alpha}, && Y_{\alpha+\beta} = X_{-(\alpha+\beta)} \end{aligned} \tag{3.1}$$

be a Chevalley basis of A_2 relative to Δ with structure constants as indicated in §2. The elements h_1, h_2 and

$$\begin{gathered} c_1 = Y_\alpha X_\alpha, \quad c_2 = Y_\beta X_\beta, \quad c_3 = Y_{\alpha+\beta} X_{\alpha+\beta}, \\ c_4 = Y_{\alpha+\beta} X_\alpha X_\beta, \quad c_5 = Y_\beta Y_\alpha X_{\alpha+\beta} \end{gathered} \tag{3.2}$$

are called the *basic cycles* of $C(A_2)$ relative to $\{\alpha, \beta\}$. When it is necessary to emphasize the base $\Delta = \{\alpha, \beta\}$ we speak of the *basic cycles of* $C(\alpha, \beta)$. Also, when we write $C(\alpha, \beta)$ for $C(A_2)$, we are using the basic cycles of $C(\alpha, \beta)$ unless otherwise specified. By direct computation, one can establish the following multiplication relations in $C(\alpha, \beta)$.

$$
\begin{aligned}
&h_1h_2 = h_2h_1,\\
&h_ic_j = c_jh_i \quad \text{for } i = 1,2;\ 1 \leqslant j \leqslant 5,\\
&c_1c_2 = c_2c_1 + c_5 - c_4,\\
&c_1c_3 = c_3c_1 - c_5 + c_4,\\
&c_1c_4 = c_4c_1 - c_2c_1 + c_3c_1 - c_4h_1 + c_3h_1 - c_5 + c_4,\\
&c_1c_5 = c_5c_1 + c_2c_1 - c_3c_1 + c_5h_1 - c_3h_1 + c_5 - c_4,\\
&c_2c_3 = c_3c_2 + c_5 - c_4,\\
&c_2c_4 = c_4c_2 + c_2c_1 - c_3c_2 - c_4h_2 + c_5 - c_4,\\
&c_2c_5 = c_5c_2 - c_2c_1 + c_3c_2 + c_5h_2 + c_5 - c_4,\\
&c_3c_4 = c_4c_3 + c_3c_2 - c_3c_1 + c_4h_1 + c_4h_2 - c_3h_1,\\
&c_3c_5 = c_5c_3 + c_3c_1 - c_3c_2 - c_5h_1 - c_5h_2 + c_3h_1 - 2c_5 + 2c_4,\\
&c_4c_5 = c_3c_2c_1 + c_3c_2h_1 + c_3c_1h_2 + c_3h_2h_1 + 2c_3c_1 + 2c_3h_1 - 2c_3c_2\\
&\qquad + c_4c_2 - c_4c_1 - c_4h_1 - c_5h_1 - c_5h_2 - 2c_5 + 2c_4 + c_5c_3,\\
&c_5c_4 = c_3c_2c_1 + c_2c_1h_1 + c_2c_1h_2 + c_5c_3 - c_4h_2 + c_4c_2 + 2c_2c_1\\
&\qquad - c_4c_1 - 2c_3c_2.
\end{aligned}
\tag{3.3}
$$

LEMMA 3.5. *The set of elements of $C(\alpha, \beta)$ of the form*

$$(c_5 \text{ or } c_4)^{e_6} c_3^{e_5} c_2^{e_4} c_1^{e_3} h_2^{e_2} h_1^{e_1} \tag{3.6}$$

where the exponents e_i are nonnegative integers, constitutes a basis for $C(\alpha, \beta)$.

PROOF. In §1 we noted that a basis of $C(\alpha, \beta)$ is given by the set of all elements of the form (1.2)—i.e.

$$u(\bar{p}, \bar{l}, \bar{q}) = Y_{\alpha+\beta}^{p_3} Y_\beta^{p_2} Y_\alpha^{p_1} h_1^{l_1} h_2^{l_2} X_\alpha^{q_1} X_\beta^{q_2} X_{\alpha+\beta}^{q_3}$$

where $q_1 + q_3 - p_1 - p_3 = 0$ and $q_2 + q_3 - p_2 - p_3 = 0$. If $q_3 \geqslant p_3$ this element can be written as

$$u(\bar{p}, \bar{l}, \bar{q}) = c_5^{q_3 - p_3} c_3^{p_3} c_2^{p_2} c_1^{p_1} h_1^{l_1} h_2^{l_2} + \text{terms of lower degree.}$$

If $p_3 \geqslant q_3$ this element can be written as

$$u(\bar{p}, \bar{l}, \bar{q}) = c_4^{p_3 - q_3} c_3^{q_3} c_2^{p_2} c_1^{p_1} h_1^{l_1} h_2^{l_2} + \text{terms of lower degree.}$$

Hence, using induction on the degree, we have that the elements of the form (3.6) generate $C(\alpha, \beta)$ as a linear space.

It is clear that the element $c_4^{e_6} c_3^{e_5} c_2^{e_4} c_1^{e_3} h_2^{e_2} h_1^{e_1}$ can be written as

$$Y_{\alpha+\beta}^{e_6+e_5} Y_\beta^{e_4} Y_\alpha^{e_3} h_1^{e_1} h_2^{e_2} X_\alpha^{e_6+e_3} X_\beta^{e_6+e_4} X_{\alpha+\beta}^{e_5} + \text{terms of lower degree}$$

and similarly $c_5^{e_6} c_3^{e_5} c_2^{e_4} c_1^{e_3} h_2^{e_2} h_1^{e_1}$ can be written as

$$Y_{\alpha+\beta}^{e_5} Y_\beta^{e_6+e_4} Y_\alpha^{e_6+e_3} h_1^{e_1} h_2^{e_2} X_\alpha^{e_3} X_\beta^{e_4} X_{\alpha+\beta}^{e_6+e_5} + \text{terms of lower degree.}$$

Since the term of highest degree in the expansion of each element of the form (3.6) in the basis of $C(A_2)$ of the form (1.2) is unique, we have that the elements of the form (3.6) are linearly independent. □

REMARK 3.7. Let $\langle W\rangle$ be the free semigroup generated by $w_1,\ldots,w_7$, and $F\langle W\rangle$ the corresponding free algebra. Let E be the subset of $F\langle W\rangle$ obtained from the equations (3.3) by replacing h_1 by w_1, h_2 by w_2, and each c_i by w_{i+2}, and let I be the ideal of $F\langle W\rangle$ generated by E. Then Lemma 3.5 implies that $C(\alpha,\beta)\simeq F\langle W\rangle/I$, and that the problem of determining all mass functions on $C(\alpha,\beta)$ is equivalent to finding all algebra homomorphisms τ: $F\langle W\rangle\to F$ with $I\subseteq\ker\tau$. Such a map τ is uniquely determined by the 7-tuple $(\tau(w_1),\ldots,\tau(w_7))$ in F^7. We define $(a,b,z_1,\ldots,z_5)\in F^7$ to be *admissible* provided there exists an algebra homomorphism τ: $F\langle W\rangle\to F$ with $I\subseteq\ker\tau$ such that $(a,b,z_1,\ldots,z_5)=(\tau(w_1),\ldots,\tau(w_7))$. Equivalently, we could require that there exist a mass function ζ on $C(\alpha,\beta)$ such that $(a,b,z_1,\ldots,z_5)=(\zeta(h_1),\zeta(h_2),\zeta(c_1),\ldots,\zeta(c_5))$.

Let τ: $F\langle W\rangle\to F$ be an algebra homomorphism with $(\tau(w_1),\ldots,\tau(w_7))=(a,b,z_1,\ldots,z_5)$. Then if we apply τ to the elements of E, equate the result to zero and simplify, we obtain the condition that $(a,b,z_1,\ldots,z_5)$ is admissible if and only if its components satisfy the equations

$$z_5-z_4=0, \tag{3.8}$$

$$a(z_4-z_3)-z_1(z_3-z_2)=0, \tag{3.9}$$

$$bz_4-z_2(z_1-z_3)=0, \tag{3.10}$$

$$(z_4-z_3)(z_2-z_1-a-z_4)+z_3(z_2+b)(z_1+a)=0. \tag{3.11}$$

Let a, b, and p be arbitrary elements of F. Then the 7-tuple $(a,b,z_1,\ldots,z_5)$ is said to be T_i for $1\leqslant i\leqslant 6$ provided it equals the 7-tuple headed by T_i in Table 3.12, and it is said to be T_0 provided it equals the 7-tuple headed by T_0 in Table 3.12 and $0\neq z_1z_2(z_1+a)(z_2+b)$. A mass function ζ on $C(\alpha,\beta)$ is defined to be T_i provided $(\zeta(h_1),\zeta(h_2),\zeta(c_1),\ldots,\zeta(c_5))$ is T_i.

TABLE 3.12

	T_0	T_1	T_2	T_3	T_4	T_5	T_6
a	a	a	a	a	a	a	a
b	b	b	b	b	b	b	b
z_1	$p(p-a-1)$	p	0	$-a$	0	$-a$	p
z_2	$(p-a)(p-a-b-1)$	0	$-b$	p	p	0	$-b$
z_3	$p(p-a-b-1)$	0	p	$-a-b$	0	p	$-a-b$
z_4	$p(p-a)(p-a-b-1)$	0	p	p	0	0	$-a-b-p$
z_5	$p(p-a)(p-a-b-1)$	0	p	p	0	0	$-a-b-p$

We now prove a sequence of lemmas which establish

THEOREM 3.13. *The 7-tuple $(a,b,z_1,\ldots,z_5)$ in F^7 is admissible if and only if $(a,b,z_1,\ldots,z_5)$ is T_i for some $i=0,1,2,\ldots,6$.*

By direct computation one can verify that for arbitrary values of a, b and p, each of the 7-tuples listed in Table 3.12 satisfies equations (3.8) through (3.11) and hence, is admissible.

The proof of the converse amounts to the determination of all admissible 7-tuples $(a, b, z_1, \ldots, z_5)$. We break this investigation into two main cases defined by whether $z_1 z_2 (z_1 + a)(z_1 + b)(a + b)$ is nonzero or zero.

LEMMA 3.14. *Let $(a, b, z_1, \ldots, z_5)$ be an admissible 7-tuple such that*

$$z_1 z_2 (z_1 + a)(z_2 + b) \neq 0.$$

Then:

(i) *if $N = z_1 b + z_2 a + ab$, the equations*

$$z_5 - z_4 = 0, \tag{3.15}$$

$$N z_3 - (a + b) z_1 z_2 = 0, \tag{3.16}$$

$$N z_4 - (z_1 + a - z_2) z_1 z_2 = 0, \tag{3.17}$$

$$N(a + b) - (z_2 - z_1 + b)(z_2 - z_1 - a) = 0, \tag{3.18}$$

hold;

(ii) *if $a + b \neq 0$, then $(a, b, z_1, \ldots, z_5)$ is T_0 where the p needed to satisfy Table* 3.12 *is $p = (z_2 - z_1 + b)/(a + b)$.*

PROOF. Since $(a, b, z_1, \ldots, z_5)$ is admissible, it satisfies equations (3.8) through (3.11). These equations can be used to obtain equations (3.15) through (3.18) by following the outline given now.

(1) Equation (3.15) is identical to equation (3.8).

(2) Equation (3.16) is equal to a times equation (3.10) minus b times equation (3.9).

(3) Equation (3.17) is z_2 times equation (3.9) plus $(z_1 + a)$ times equation (3.10).

(4) Equation (3.18) is obtained by using (3.16) and (3.17) to simplify

$$N / \left(z_1 z_2 (z_1 + a)(z_2 + b) \right)$$

times equation (3.11).

This proves part (i). For part (ii), we may assume $(a, b, z_1, \ldots, z_5)$ satisfies (3.15) through (3.18), and $p = (z_2 - z_1 + b)/(a + b)$. Substituting $z_2 = p(a + b) + z_1 - b$ into equation (3.18), we find

$$(a + b)^2 (z_1 - p(p - a - 1)) = 0. \tag{3.19}$$

Since $a + b \neq 0$, we have $z_1 = p(p - a - 1)$. From the definition of p it follows that $z_2 = (p - a)(p - a - b - 1)$. Finally from equations (3.15) through (3.17), we have $z_3 = p(p - a - b - 1)$ and $z_4 = z_5 = p(p - a)(p - a - b - 1)$. Thus any 7-tuple $(a, b, z_1, \ldots, z_5) \in F^7$ satisfying (3.15) through (3.18) with

$$z_1 z_2 (z_1 + a)(z_2 + b)(a + b) \neq 0$$

is equal to one of the admissible elements of F^7 of the form T_0 in Table 3.12. □

LEMMA 3.20. *Let ζ be a mass function on $C(A_2)$, Φ the root system of A_2 and $S_\zeta = \{\gamma \in \Phi \mid \zeta(Y_\gamma X_\gamma) \neq 0$ or $-\zeta(h_j)\}$. Then*:

(i) $S_\zeta = \Phi$,

(ii) $S_\zeta = \{\pm\alpha\}$ *for some* $\alpha \in \Phi$ *or*

(iii) $S_\zeta = \varnothing$.

PROOF. Suppose $S_\zeta \neq \Phi$, $\varnothing$ or $\{\pm\alpha\}$ for some $\alpha \in \Phi$. Then since for each $\gamma \in S_\zeta$, $-\gamma \in S_\zeta$, it must be the case that $|S_\zeta| = 4$. From this it follows that S_ζ contains some base $\Delta = \{\alpha, \beta\}$ of Φ. Let h_α, h_β, c_i, $1 \leqslant i \leqslant 5$, denote the basic cycles of $C(\alpha, \beta)$, and let $a = \zeta(h_\alpha)$, $b = \zeta(h_\beta)$ and $z_i = \zeta(c_i)$, $1 \leqslant i \leqslant 5$. Then since $\alpha, \beta \in S_\zeta$ we have $z_1 z_2(z_1 + a)(z_2 + b) \neq 0$. By part (ii) of Lemma 3.14, if $a + b \neq 0$ then ζ is T_0, contrary to $\alpha + \beta \notin S_\zeta$. Therefore, $a = -b$.

Since $\zeta(h_\alpha + h_\beta) = a + b = 0$ and $\alpha + \beta \notin S_\zeta$, it must be the case that $z_3 = 0$. In this case (3.9) and (3.11) become

$$az_4 + z_1 z_2 = 0, \tag{3.21}$$

$$z_4(z_2 - z_1 - a - z_4) = 0 \tag{3.22}$$

respectively. From (3.22), we conclude that $a \neq 0 \neq z_4$ and hence (3.23) implies $z_2 - z_1 - a = z_4$. Substituting this into (3.22), we get

$$0 = a(z_2 - z_1 - a) + z_1 z_2 = az_2 - az_1 - a^2 + z_1 z_2 = (z_1 + a)(z_2 - a).$$

Therefore, $z_1 = -a$ or $z_2 = a = -b$ either of which is a contradiction. □

LEMMA 3.23. *Let ζ be a mass function on $C(A_2)$ such that $S_\zeta = \{\gamma \in \Phi \mid \zeta(Y_\gamma X_\gamma) \neq 0$ or $-\zeta(h_\gamma)\} = \Phi$. Then ζ is T_0 for every choice of base $\Delta = \{\alpha, \beta\}$ used to describe the basic cycles.*

PROOF. Let h_α, h_β, c_i, $1 \leqslant i \leqslant 5$, be the basic cycles of $C(\alpha, \beta)$ and let $(a, b, z_1, \ldots, z_5) = (\zeta(h_\alpha), \zeta(h_\beta), \zeta(c_1), \ldots, \zeta(c_5))$. If $a + b \neq 0$, this result becomes part (ii) of Lemma 3.14. Therefore, we may assume $a + b = 0$ and reduce equations (3.8) through (3.11) to

$$z_5 - z_4 = 0, \tag{3.24}$$

$$a(z_4 - z_3) - z_1(z_3 - z_2) = 0, \tag{3.25}$$

$$az_4 + z_2(z_1 - z_3) = 0, \tag{3.26}$$

$$(z_4 - z_3)(z_2 - z_1 - a - z_4) + z_3(z_2 - a)(z_1 + a) = 0. \tag{3.27}$$

Since $z_3 \neq 0$, equations (3.25) and (3.26) imply $z_2 = z_1 + a$. If $a = 0$ then $z_1 = z_2 = z_3$. From this and equation (3.27) one can show that $z_4 = ((1 \pm \sqrt{1 + 4z_1})/2)z_1$. Define p so that $z_1 p = z_4$. It is easy to show that this p is the one needed in column T_0 of Table 3.12.

If $a \neq 0$, then using (3.26) and $z_2 = z_1 + a$ we obtain $z_4 = \frac{1}{a}(z_3 - z_1)(z_1 + a)$. Define p so that $z_1 p = z_4$. it is easy to show that this p is the one needed in column T_0 of Table 3.12.

If $a \neq 0$, then using (3.26) and $z_2 = z_1 + a$ we obtain $z_4 = \frac{1}{a}(z_3 - z_1)(z_1 + a)$. Define $p = \frac{1}{a}(z_3 - z_1)$, so that $z_2 = z_1 + a$, $z_3 = ap + z_1$ and $z_4 = p(z_1 + a)$. Substituting these into (3.27) we obtain

$$(p - 1)z_1(-p(z_1 + a)) + (ap + z_1)z_1(z_1 + a) = 0$$

which simplifies to $z_1 = p(p - a - 1)$. Now, one can readily show using the expressions for z_i in terms of z_1 and a given above that ζ is T_0 on $C(\alpha, \beta)$. □

LEMMA 3.28. *If ζ is a mass function on $C(A_2)$ such that $S_\zeta = \{\pm\alpha\}$ or $\varnothing$ then $(a, b, z_1,\ldots,z_5) = (\zeta(h_\gamma), \zeta(h_\delta), \zeta(c_1),\ldots,\zeta(c_5))$ appears in Table* 3.12 *for some base $\{\gamma, \delta\}$ where the c_i's are basic cycles described using $\Delta = \{\gamma, \delta\}$.*

PROOF. Let γ be a root which is not in S_ζ. Then either $\zeta(Y_\gamma X_\gamma) = 0$ or $\zeta(X_\gamma Y_\gamma) = 0$. Hence we can pick $\gamma \notin S_\zeta$ such that $\zeta(Y_\gamma X_\gamma) = 0$. If $S_\zeta = \{\pm\alpha\}$ let $\delta = \alpha$ if $\delta + \alpha \in \Phi$ and let $\delta = -\alpha$ otherwise. In either case $\gamma + \delta \in \Phi$ and we can take $\{\gamma, \delta\}$ as a base of Φ. If $S = \varnothing$, let δ be any root such that $\{\gamma, \delta\}$ is a base.

Define the basic cycles c_i, $1 \leqslant i \leqslant 5$, in terms of this base. In particular, $z_1 = \zeta(c_1) = \zeta(Y_\gamma X_\gamma) = 0$. The lemma follows, once we have shown that every admissible 7-tuple $(a, b, z_1,\ldots,z_5)$ for which $z_1 = 0$ appears in Table 3.12.

Under the condition $z_1 = 0$ equations (3.8) through (3.11) become

$$z_5 - z_4 = 0, \tag{3.29}$$

$$a(z_4 - z_3) = 0, \tag{3.30}$$

$$bz_4 + z_2 z_3 = 0, \tag{3.31}$$

$$(z_4 - z_3)(z_2 - a - z_4) + az_3(z_2 + b) = 0. \tag{3.32}$$

Our proof continues by studying two cases $a = 0$ and $a \neq 0$.

Case I. $a = 0$. Equation (3.32) then implies that $(z_4 - z_3)(z_2 - z_4) = 0$.

Subcase i. $z_4 = z_3$. Then equation (3.31) becomes $z_3(z_2 + b) = 0$. This leads to two families of solutions. If $z_3 = 0$ we have

$$(a, b, z_1,\ldots,z_5) = (0, b, 0, p, 0, 0, 0)$$

where b and p are arbitrary. If $z_2 = -b$ we have

$$(a, b, z_1,\ldots,z_5) = (0, b, 0, -b, p, p, p)$$

where b and p are arbitrary. Both of these families occur in Table 3.12.

Subcase ii. $z_4 = z_2$. The argument is similar to subcase i.

Case II. $a \neq 0$. From equation (3.30) we have $z_4 = z_3$. Using this in equation (3.32) we obtain $z_3(z_2 + b) = 0$. If $z_3 = 0$ we have $(a, b, z_1,\ldots,z_5) = (a, b, 0, p, 0, 0, 0)$ where a, b and p are arbitrary. If $z_2 = -b$ we have $(a, b, z_1,\ldots,z_5) = (a, b, 0, -b, p, p, p)$ where a, b and p are arbitrary. Both of these families of solutions occur in Table 3.12.

Thus every admissible 7-tuple $(a, b, z_1,\ldots,z_5)$ such that $z_1 = 0$ is listed in Table 3.12. □

The reader should note that the preceding lemmas establish Theorem 3.13.

THEOREM 3.33. *For mass functions ζ on $C(A_2)$, Theorem* 1.7 *holds. Moreover, if $S_\zeta = \Phi$ then Theorem* 1.7 *holds for every choice of base Δ used to describe the basic cycles.*

PROOF. According to Lemmas 3.20, 3.23 and 3.28, we need only consider T_i mass functions. Let ζ_i be a T_i mass function relative to $\Delta = \{\alpha, \beta\}$. The following table gives the type of mass function that ζ_i becomes relative to the bases $\sigma_\alpha\Delta$ and $\sigma_\beta\Delta$ where σ_α, σ_β denote Weyl reflections.

TABLE 3.34

	ζ_0	ζ_1	ζ_2	ζ_3	ζ_4	ζ_5	ζ_6
$\sigma_\alpha\Delta$	T_0	T_1	T_3	T_2	T_5	T_4	T_6
$\sigma_\beta\Delta$	T_0	T_2	T_1	T_3	T_4	T_6	T_5

From this table one sees that by an appropriate choice of base we may assume that the given mass function ζ is T_0, T_1 or T_4.

If ζ is T_0, then by Lemma 3.23 ζ is T_0 for every base Δ. From this and Table 2.7 it follows that $\zeta = \varepsilon \circ \Psi_{\Delta,\Delta}$ where $\varepsilon(s_1) = p$ and p is the value given in column T_0 of Table 3.12.

If ζ is T_1 relative to $\Delta = \{\alpha, \beta\}$, then $\zeta = \varepsilon \circ \Psi_{\{\alpha\},\Delta}$ where $\Psi_{\{\alpha\},\Delta}$ is given by Table 2.7 and $\varepsilon(s_1(s_1 - \lambda_1 - 1)) = p = \zeta(c_1)$. The proof in the case of T_4 mass functions is similar. □

COROLLARY 3.35. *If ζ is a mass function on $C(A_2)$ with $S_\zeta = \{\gamma \in \Phi \mid \zeta(Y_\gamma X_\gamma) \neq 0$ or $-\zeta(h_\gamma)\} = \varnothing$, then there is a base $\Delta = \{\alpha, \beta\}$ such that if Δ is used to describe the basic cycles then $\zeta = \varepsilon \circ \Psi_{\varnothing,\Delta}$ where ε is an evaluation map satisfying $\varepsilon(\lambda_1) = \zeta(h_\alpha)$ and $\varepsilon(\lambda_2) = \zeta(h_\beta)$.*

PROOF. By Lemmas 3.20 and 3.28 and Table 3.34, there is a base $\Delta = \{\alpha, \beta\}$ such that relative to Δ, ζ is T_1 with $p = 0$ or $-\zeta(h_\alpha)$ or ζ is T_4 with $p = -\zeta(h_\beta)$. Suppose ζ is T_1 relative to $\Delta = \{\alpha, \beta\}$ with $p \neq 0$. Then ζ relative to $\{-\alpha, \alpha + \beta\}$ is T_1 with the corresponding $p = 0$. A similar change of base for T_4 maps permits us to assume that ζ is T_1 relative to Δ and $p = 0$. The result then follows directly. □

4. Mass functions on $C(A_3)$ and $C(A_4)$. In this section, we prove a series of technical results concerning mass functions on $C(A_3)$ and $C(A_4)$. We make use of the convention laid down in §2 that $\varepsilon(s_i) = 0$ for $l + 1 \leqslant i \leqslant n$ if $\operatorname{Im}\Psi_{\Delta',\Delta} \subseteq F[\lambda_1, \ldots, \lambda_n, s_1, \ldots, s_l]$.

We continue to use $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ to denote a base of the root system Φ of A_n. A basic cycle of $C(A_n)$ relative to Δ is either an element of the form $h_i = h_{\alpha_i}$ or an element $u(\bar{p}, \bar{0}, \bar{q}) \in C(A_n)$ of the form (1.1) having the property that no proper factor (ignoring commutativity) of $u(\bar{p}, \bar{0}, \bar{q})$ is in $C(A_n)$. It follows immediately from the proof of the Poincaré-Birkhoff-Witt Theorem that the basic cycles generate $C(A_n)$ as an associative algebra with 1. Thus any mass function is completely determined by its values on the set of basic cycles relative to any fixed base Δ. In **[15]** it is shown that this statement can be considerably improved to read

PROPOSITION 4.1. *Any mass function ζ: $C(A_n) \to F$ is completely determined by its values on the set of all basic cycles of degree $\leqslant 3$ relative to a base Δ.*

We now suppose that we are given a mass function ζ: $C(A_n) \to F$ and wish to determine an appropriate Harish-Chandra homomorphism required in Theorem 1.7. We begin this determination by specifying a subset of the root system Φ associated with ζ, namely

$$S_\zeta = \{\alpha \in \Phi \mid \zeta(Y_\alpha X_\alpha) \neq 0, -\zeta(h_\alpha)\}. \tag{4.2}$$

PROPOSITION 4.3. (i) *S_ζ is a root subsystem of Φ*; (ii) *S_ζ is a disjoint union of a mutually orthogonal irreducible subsystem Φ_i' of Φ.*

PROOF. Let $(\ ,\ )$ denote the Killing form on Φ and define an equivalence relation "$\sim$" on S_ζ by setting $\alpha \sim \beta$ for $\alpha, \beta \in S_\zeta$ iff there exist $\gamma_1, \gamma_2, \ldots, \gamma_m \in S_\zeta$ such that $\gamma_1 = \alpha$, $\gamma_m = \beta$ and $(\gamma_i, \gamma_{i+1}) \neq 0$ for $1 \leqslant i \leqslant m-1$. Let Φ_i' denote the equivalence classes of S_ζ under "$\sim$". The proposition follows easily once we show that Φ_i' is a root subsystem of Φ. By [**8**, p. 42], it suffices to show that if $\alpha, \beta \in \Phi_i'$ then $\sigma_\alpha(\beta) \in \Phi_i'$ where σ_α is the Weyl reflection through the hyperplane perpendicular to α. Let $\alpha, \beta \in \Phi_i'$. If $(\alpha, \beta) = 0$ then $\sigma_\alpha(\beta) = \beta \in \Phi_i'$. If $(\alpha, \beta) \neq 0$ then

$$\sigma_\alpha(\beta) = \begin{cases} \mp\beta & \text{if } \beta = \pm\alpha, \\ \alpha + \beta & \text{if } \alpha + \beta \in \Phi, \\ \beta - \alpha & \text{if } \alpha - \beta \in \Phi. \end{cases}$$

Clearly in the first case $\sigma_\alpha(\beta) \in \Phi_i'$. If $\alpha + \beta \in \Phi$, then let $A_2(\alpha, \beta)$ be the subalgebra of A_n isomorphic to A_2 and generated by $\{X_\alpha, Y_\alpha, X_\beta, Y_\beta\}$. By employing Table 3.12 for ζ restricted to $C(\alpha, \beta)$ one verifies that $\sigma_\alpha(\beta) = \alpha + \beta \in \Phi_i'$. Similarly, by considering ζ restricted to $C(\alpha, -\beta)$ we have that $\sigma_\alpha(\beta) = \beta - \alpha \in \Phi_i'$ in the third case. This completes the proof that Φ_i' (and hence S_ζ) is a subsystem of Φ. □

Our proof of Theorem 1.7 in the general case requires a number of technical results concerning mass functions on $C(A_3)$. To facilitate the presentation of these results we introduce the following notation for $C(A_3)$. A base for the root system of A_3 will be denoted by $\Delta = \{\alpha, \beta, \gamma\}$ and the set of basic cycles of $C(A_3)$ relative to this base are given as follows.

(4.4)

$$\begin{array}{lll}
h_1 = h_\alpha, & h_2 = h_\beta, & h_3 = h_\gamma, \\
c_1 = Y_\alpha X_\alpha, & c_2 = Y_\beta X_\beta, & c_3 = Y_\gamma X_\gamma, \\
c_4 = Y_{\alpha+\beta} X_{\alpha+\beta}, & c_5 = Y_{\beta+\gamma} X_{\beta+\gamma}, & c_6 = Y_{\alpha+\beta+\gamma} X_{\alpha+\beta+\gamma}, \\
c_7 = Y_{\alpha+\beta} X_\alpha X_\beta, & c_8 = Y_{\beta+\gamma} X_\beta X_\gamma, & c_9 = Y_\beta Y_\alpha X_{\alpha+\beta}, \\
c_{10} = Y_\gamma Y_\beta X_{\beta+\gamma}, & c_{11} = Y_{\alpha+\beta+\gamma} X_{\alpha+\beta} X_\gamma, & c_{12} = Y_{\alpha+\beta+\gamma} X_\alpha X_{\beta+\gamma}, \\
c_{13} = Y_\gamma Y_{\alpha+\beta} X_{\alpha+\beta+\gamma}, & c_{14} = Y_{\beta+\gamma} Y_\alpha X_{\alpha+\beta+\gamma}, & c_{15} = Y_{\alpha+\beta+\gamma} X_\alpha X_\beta X_\gamma, \\
c_{16} = Y_\gamma Y_\beta Y_\alpha X_{\alpha+\beta+\gamma}, & c_{17} = Y_{\beta+\gamma} Y_{\alpha+\beta} X_{\alpha+\beta+\gamma} X_\beta, & \\
c_{18} = Y_\beta Y_{\alpha+\beta+\gamma} X_{\alpha+\beta} X_{\beta+\gamma}, & c_{19} = Y_\gamma Y_{\alpha+\beta} X_\alpha X_{\beta+\gamma}, & \\
c_{20} = Y_{\beta+\gamma} Y_\alpha X_{\alpha+\beta} X_\gamma. & &
\end{array}$$

The fact that this list exhausts the set of all basic cycles of $C(A_3)$ relative to Δ follows directly from the results of van den Hombergh [**16**] or from [**15**]. In the remainder of this paper whenever we refer to the basic cycles of $C(\alpha, \beta, \gamma)$ we mean

those listed above. The subalgebras of $C(A_3)$ which are isomorphic to $C(A_2)$ together with their basic cycles in this notation are

$$
\begin{aligned}
C(\alpha, \beta) &= \langle \{h_1, h_2, c_1, c_2, c_4, c_7, c_9\} \rangle, \\
C(\beta, \gamma) &= \langle \{h_2, h_3, c_2, c_3, c_5, c_8, c_{10}\} \rangle, \\
C(\alpha + \beta, \gamma) &= \langle \{h_1 + h_2, h_3, c_4, c_3, c_6, c_{11}, c_{13}\} \rangle, \\
C(\alpha, \beta + \gamma) &= \langle \{h_1, h_2 + h_3, c_1, c_5, c_6, c_{12}, c_{14}\} \rangle.
\end{aligned} \tag{4.5}
$$

By Proposition 4.3 the set S_ζ for any mass function ζ: $C(A_3) \to F$ must have order 12, 6, 4, 2 or 0.

Our first case is $|S_\zeta| = 12$. By using Lemma 3.23 we see that the restriction of ζ to each $C(A_2)$ of $C(A_3)$ must be a T_0 mass function relative to any base. Let $a = \zeta(h_1)$, $b = \zeta(h_2)$ and $c = \zeta(h_3)$. Then there must exist unique elements $r, t, u, v \in F$ such that

$$r(r - a - 1) = \zeta(c_1) = u(u - a - 1), \tag{4.6}$$

$$(r - a)(r - a - b - 1) = \zeta(c_2) = t(t - b - 1), \tag{4.7}$$

$$(t - b)(t - b - c - 1) = \zeta(c_3) = (v - a - b)(v - a - b - c - 1), \tag{4.8}$$

$$r(r - a - b - 1) = \zeta(c_4) = v(v - a - b - 1), \tag{4.9}$$

$$t(t - b - c - 1) = \zeta(c_5) = (u - a)(u - a - b - c - 1), \tag{4.10}$$

$$v(v - a - b - c - 1) = \zeta(c_6) = u(u - a - b - c - 1). \tag{4.11}$$

Direct substitution shows that the roots of these quadratic equations must satisfy the following conditions.

	Column A		Column B
(4.12)	$r = u$	or	$r = 1 + a - u$
(4.13)	$r = t + a$	or	$r = 1 + a + b - t$
(4.14)	$t = v - a$	or	$t = 1 + a + 2b + c - v$
(4.15)	$r = v$	or	$r = 1 + a + b - v$
(4.16)	$t = u - a$	or	$t = 1 + a + b + c - u$
(4.17)	$v = u$	or	$v = 1 + a + b + c - u$.

The possible choices for s, t, u and v are reduced by the following observations.

(4.18) If three equations from Column A are valid and all four of r, t, u and v are involved in these three equations then $r = u = v = t + a$. In particular, if four equations from Column A are valid, then $r = u = v = t + a$. If $r = u = v = t + a$ and some equations from Column B are valid then constraints are imposed on a, b and c.

(4.19) If five equations from Column B are valid, then all equations from Column B are valid and certain constraints are being forced on a, b and c.

These two observations aid us in the case study which makes up the proof of

LEMMA 4.20. *For any mass function ζ: $C(A_3) \to F$ with $|S_\zeta| = 12$, there is an evaluation map ε: $F[\lambda_1, \lambda_2, \lambda_3, s_1, s_2, s_3] \to F$ and an automorphism σ such that $\zeta \circ \sigma = \varepsilon \circ \Psi_{\Delta,\Delta}$.*

PROOF. If the condition of remark (4.18) holds, then define ε by $\varepsilon(\lambda_1) = \zeta(h_1) = a$, $\varepsilon(\lambda_2) = \zeta(h_2) = b$, $\varepsilon(\lambda_3) = \zeta(h_3) = c$ and $\varepsilon(s_1) = r = u = v = t + a$. In this case then $\zeta = \varepsilon \circ \Psi_{\Delta,\Delta}$ for all basic cycles of degree $\leqslant 3$ and Theorem 4.1 implies that $\zeta = \varepsilon \circ \Psi_{\Delta,\Delta}$ as required. Therefore, in the remainder of the proof, we assume that at most three equations from Column A hold.

Case I. The condition of (4.18) fails and exactly three equations from Column A hold. This is equivalent to the following subcases.

Subcase i. (4.12), (4.13), (4.16) Column A hold.

Subcase ii. (4.12), (4.15), (4.17) Column A hold.

Subcase iii. (4.13), (4.14), (4.15) Column A hold.

Subcase iv. (4.14), (4.16), (4.17) Column A hold.

Table 4.21 indicates the action of ζ on the basic cycles of degree $\leqslant 3$ relative to the base $\{\alpha, \beta, \gamma\}$ in each of these subcases.

The analysis of these four subcases can be simplified by observing that if ζ satisfies subcase ii then $\zeta \circ \sigma_\alpha$ together with a change of variable replacing r by $r - a$ yields a map which satisfies subcase i. Similarly with an appropriate change of variable, we again arrive at subcase i by using $\zeta \circ \sigma_f$ to alter subcase iii and $\zeta \circ \sigma_f \circ \sigma_\beta$ to change subcase iv where σ_f is the flip map defined in §2. Thus, it suffices to prove Case I of Lemma 4.20 assuming subcase i holds.

Let ζ: $C(\alpha, \beta, \gamma) \to F$ be a mass function satisfying subcase i and consider the following identities in $C(\alpha, \beta, \gamma)$.

TABLE 4.21

	Subcase i	Subcase ii	Subcase iii	Subcase iv
	$b = -a$		$a = 0$	$a = b = 0$
	$c = 0$	$b = c = 0$	$c = -b$	
	$t = r - a$	$t = 1 + a - s$	$t = r$	$t = 1 - r$
	$u = r$	$u = r$	$u = 1 - r$	$u = 1 - r$
	$v = 1 - r$	$v = r$	$v = r$	$v = 1 - r$
$\zeta(h_1)$	a	a	0	0
$\zeta(h_2)$	$-a$	0	b	0
$\zeta(h_3)$	0	0	$-b$	c
$\zeta(c_1)$	$r(r - a - 1)$	$r(r - a - 1)$	$r(r - 1)$	$r(r - 1)$
$\zeta(c_2)$	$(r - 1)(r - a)$	$(r - a)(r - a - 1)$	$r(r - b - 1)$	$r(r - 1)$
$\zeta(c_3)$	$r(r - 1)$	$(r - a)(r - a - 1)$	$(r - 1)(r - b)$	$(r - 1)(r + c)$
$\zeta(c_4)$	$r(r - 1)$	$r(r - a - 1)$	$r(r - b - 1)$	$r(r - 1)$
$\zeta(c_5)$	$(r - 1)(r - a)$	$(r - a)(r - a - 1)$	$r(r - 1)$	$(r - 1)(r + c)$
$\zeta(c_6)$	$r(r - 1)$	$r(r - a - 1)$	$r(r - 1)$	$(r - 1)(r + c)$
$\zeta(c_7) = \zeta(c_9)$	$r(r - 1)(r - a)$	$r(r - a)(r - a - 1)$	$r^2(r - b - 1)$	$r^2(r - 1)$
$\zeta(c_8) = \zeta(c_{10})$	$r(r - 1)(r - a)$	$-(r - a)(r - a - 1)^2$	$r(r - 1)(r - b)$	$-(r - 1)^2(r + c)$
$\zeta(c_{11}) = \zeta(c_{13})$	$-r(r - 1)^2$	$r(r - a)(r - a - 1)$	$r(r - 1)(r - b)$	$-(r - 1)^2(r + c)$
$\zeta(c_{12}) = \zeta(c_{14})$	$r(r - 1)(r - a)$	$r(r - a)(r - a - 1)$	$-r(r - 1)^2$	$-(r - 1)^2(r + c)$

$$c_1 c_{17} = (c_{14} - c_5)(c_7 - c_4) + (c_8 - c_5)c_1, \tag{4.22}$$

$$c_{17}(c_1 + h_1) = (c_7 - c_4)(c_{14} - c_5) + (c_{13} - c_6)(c_1 + h_1). \tag{4.23}$$

(4.22) and (4.23) are obtained by rewriting $c_{14}c_7$ and c_7c_{14}. Since $\zeta(c_1)$ and $\zeta(c_1 + h_1)$ are nonzero we can use each of (4.22) and (4.23) to find $\zeta(c_{17})$. This yields

$$\zeta(c_{17}) = r(r-1)^2(r-a), \tag{4.24}$$

$$\zeta(c_{17}) = r(r-1)^2(r^2 - ra - 3r + a + 1). \tag{4.25}$$

From this, it follows that $r = \frac{1}{2}$. In this case, we select an evaluation map ε satisfying $\varepsilon(\lambda_1) = a$, $\varepsilon(\lambda_2) = -a$, $\varepsilon(\lambda_3) = 0$ and $\varepsilon(s_1) = \frac{1}{2}$. Then $\zeta = \varepsilon \circ \Psi_{\Delta,\Delta}$ on all basic cycles of $C(\alpha, \beta, \gamma)$ of degree $\leqslant 3$ and hence by Theorem 4.1, $\zeta = \varepsilon \circ \Psi_{\Delta,\Delta}$.

Case II. Exactly two equations from Column A are valid. One can verify that there are only three subcases.

Subcase i. (4.12), (4.14) Column A hold.

Subcase ii. (4.13), (4.17) Column A hold.

Subcase iii. (4.15), (4.16) Column A hold.

This is established quickly by noting that any other pair of equations from Column A implies a third equation from Column A.

We list these solutions in the following table.

TABLE 4.26

	Subcase i	Subcase ii	Subcase iii
	$a = c = 0$	$c = -a$	$b = -a$
		$b = 0$	$c = a$
	$t = 1 + b - r$	$t = r - a$	$t = 1 - r$
	$u = r$	$u = 1 + a - r$	$u = 1 + a - r$
	$v = 1 + b - r$	$v = 1 + a - r$	$v = r$
$\zeta(h_1)$	0	a	a
$\zeta(h_2)$	b	0	$-a$
$\zeta(h_3)$	0	$-a$	a
$\zeta(c_1)$	$r(r-1)$	$r(r-a-1)$	$r(r-a-1)$
$\zeta(c_2)$	$r(r-b-1)$	$(r-a)(r-a-1)$	$(r-1)(r-a)$
$\zeta(c_3)$	$r(r-1)$	$(r-1)(r-a)$	$r(r-a-1)$
$\zeta(c_4)$	$r(r-b-1)$	$r(r-a-1)$	$r(r-1)$
$\zeta(c_5)$	$r(r-b-1)$	$(r-1)(r-a)$	$r(r-1)$
$\zeta(c_6)$	$r(r-b-1)$	$(r-a)(r-a-1)$	$r(r-a-1)$
$\zeta(c_7) = \zeta(c_9)$	$r^2(r-b-1)$	$r(r-a)(r-a-1)$	$r(r-1)(r-a)$
$\zeta(c_8) = \zeta(c_{10})$	$-r(r-1)(r-b-1)$	$(r-1)(r-a)^2$	$-r(r-1)(r-a-1)$
$\zeta(c_{11}) = \zeta(c_{13})$	$-r(r-1)(r-b-1)$	$-(r-1)(r-a)(r-a-1)$	$r^2(r-a-1)$
$\zeta(c_{12}) = \zeta(c_{14})$	$r^2(r-b-1)$	$-(r-1)(r-a)(r-a-1)$	$-r(r-1)(r-a-1)$

Once again we can use the Weyl automorphisms to simplify our analysis. If ζ satisfies subcase ii, then $\zeta \circ \sigma_\alpha \circ \sigma_\beta$ with r replaced by $r - a$ satisfies subcase i and if ζ satisfies subcase iii then $\zeta \circ \sigma_\beta$ satisfies subcase i. Hence, we may assume the ζ is a mass function on $C(\alpha, \beta, \gamma)$ satisfying this subcase i. By applying ζ to equations (4.22) and (4.23), we obtain

$$\zeta(c_{17}) = -(r-1)(r-a)(r-a-1)^2, \tag{4.27}$$

$$\zeta(c_{17}) = -r(r-a)^2(r-a-1), \tag{4.28}$$

and hence $r = \frac{1}{2}(1 + a)$. Select an evaluation map ε: $F[\lambda_1, \lambda_2, \lambda_3, s_1, s_2] \to F$ which satisfies $\varepsilon(\lambda_1) = -\varepsilon(\lambda_3) = a$, $\varepsilon(\lambda_2) = 0$ and $\varepsilon(s_1) = \frac{1}{2}(1 + a)$, and observe that

$\zeta = \varepsilon \circ \Psi_{\Delta,\Delta}$ on all basic cycles of $C(\alpha, \beta, \gamma)$ of degree $\leqslant 3$. Thus, by Theorem 4.1, we have $\zeta = \varepsilon \circ \Psi_{\Delta,\Delta}$.

Case III. Exactly one equation from Column A is valid. Remark 4.19 reduces this case to

Case IV. Every equation from Column B is valid. For the remainder of this proof the numbers (4.12) through (4.17) refer to equations (4.12) Column B through (4.17) Column B. By adding (4.12) and (4.13) and then using equation (4.16), we get

$$2r = 2 + 2a + b - t - u = 1 + a - c. \tag{4.29}$$

Similarly, (4.14), (4.16) and (4.17) give us

$$u = \tfrac{1}{2}(1 + a + c), \tag{4.30}$$

and

$$t = v = \tfrac{1}{2}(1 + a + 2b + c). \tag{4.31}$$

Therefore, the mass functions described by this case are uniquely determined by the set of equations

$$\zeta(h_1) = a, \tag{4.32}$$

$$\zeta(h_2) = b, \tag{4.33}$$

$$\zeta(h_3) = c, \tag{4.34}$$

$$\zeta(c_1) = -\tfrac{1}{4}(1 + a - c)(1 + a + c), \tag{4.35}$$

$$\zeta(c_2) = -\tfrac{1}{4}(1 - a - c)(1 + a + 2b + c), \tag{4.36}$$

$$\zeta(c_3) = -\tfrac{1}{4}(1 + a + c)(1 - a + c), \tag{4.37}$$

$$\zeta(c_4) = -\tfrac{1}{4}(1 + a - c)(1 + a + 2b + c), \tag{4.38}$$

$$\zeta(c_5) = -\tfrac{1}{4}(1 - a + c)(1 + a + 2b + c), \tag{4.39}$$

$$\zeta(c_6) = -\tfrac{1}{4}(1 + a + c)(1 + a + 2b + c), \tag{4.40}$$

$$\zeta(c_7) = \zeta(c_9) = -\tfrac{1}{8}(1 + a - c)(1 - a - c)(1 + a + 2b + c), \tag{4.41}$$

$$\zeta(c_8) = \zeta(c_{10}) = -\tfrac{1}{8}(1 - a + c)(1 + a + c)(1 + a + 2b + c), \tag{4.42}$$

$$\zeta(c_{11}) = \zeta(c_{13}) = -\tfrac{1}{8}(1 - a + c)(1 + a + c)(1 + a + 2b + c), \tag{4.43}$$

$$\zeta(c_{12}) = \zeta(c_{14}) = -\tfrac{1}{8}(1 - a + c)(1 + a + c)(1 + a + 2b + c). \tag{4.44}$$

From identities (4.22) and (4.23), we may deduce that

$$\zeta(c_{17}) = \tfrac{1}{16}(1 - a + c)(1 - a - c)(1 + a + 2b + c)(1 - a - 2b - c), \tag{4.45}$$

$$\zeta(c_{17}) = \tfrac{1}{16}(1 + a - c)(1 + a + c)(1 + a + 2b + c)(1 - a - 2b - c) \tag{4.46}$$

and hence we must have one of

$$(1 + a + 2b + c)(1 - a - 2b - c) = 0, \quad \text{or} \quad a = 0.$$

If the first of these holds then (4.36) implies either $\zeta(c_2) = 0$ or $\zeta(c_2) = -\zeta(h_2)$ but this contradicts our assumption that $|S_\zeta| = 12$. Thus we must conclude that $a = 0$.

We may assume that $\zeta \circ \sigma$ is of the type described in Case IV for each Weyl automorphism σ for otherwise $\zeta \circ \sigma$ falls into one of the previous cases. Here we are using the easily established fact that $|S_\zeta| = 12$ implies $|S_{\zeta \circ \sigma}| = 12$ for each Weyl automorphism σ.

The above assumption along with the fact that $a = 0$ quickly yields $a = b = c = 0$. From this and equations (4.29) through (4.31), we obtain $r = t = u = v = \frac{1}{2}$. Therefore, $\zeta = \varepsilon \circ \Psi_{\Delta,\Delta}$ where ε is defined by $\varepsilon(\lambda_i) = 0$, $1 \leqslant i \leqslant 3$, and $\varepsilon(s_1) = \frac{1}{2}$.

This concludes the proof of Lemma 4.20. $\square$

The key to the proof of Lemma 4.20 also unlocks the generalization of this lemma as stated below in Corollary 4.47. Therefore, we emphasize this point now. If ζ: $C(A_2) \to F$ is a mass function with S_ζ equal to the entire root system of A_2 then $\zeta = \varepsilon \circ \Psi_{\Delta,\Delta}$ for some evaluation map ε. Thus the choice of ε corresponding to ζ is uniquely determined by Δ. One can check this easily by consulting Table 3.12.

COROLLARY 4.47. *Let Δ be any base of the root system Φ_n of A_n, and ζ be a mass function with $S_\zeta = \Phi_n$. Then there is an evaluation map ε: $F[\lambda_1,\ldots,\lambda_n, s_1,\ldots,s_n] \to F$ such that $\zeta = \varepsilon \circ \Psi_{\Delta,\Delta}$.*

PROOF. If $n = 1$, then $C(A_n) \simeq F[\lambda_1, s_1]$ and the result is clear. For $n = 2, 3$, we use Corollary 3.32 and Lemma 4.20 respectively to conclude that there is an automorphism and an evaluation map ε' such that $\zeta \circ \sigma = \varepsilon' \circ \Psi_{\Delta,\Delta}$ or equivalently $\zeta = \varepsilon' \circ \Psi_{\Delta,\Delta} \circ \sigma^{-1}$. In both of these cases, define ν: $\operatorname{Im}\Psi_\Delta \to F[\lambda_1,\ldots,\lambda_n, s_1,\ldots,s_n]$ by $\nu(\Psi_\Delta(c)) = \Psi_\Delta(\sigma^{-1}(c))$ for all $c \in C(A_n)$ where Ψ_Δ denotes $\Psi_{\Delta,\Delta}$. Clearly ν is a homomorphism which makes the following diagram commute.

$$\begin{tikzcd}
C(A_n) \arrow[r, "\sigma^{-1}"] \arrow[d, "\Psi_\Delta"'] & C(A_n) \arrow[r, "\Psi_\Delta"] & F[\lambda_1,\ldots,\lambda_n, s_1,\ldots,s_n] \arrow[d, "\varepsilon'"] \\
\operatorname{Im}\Psi_\Delta \arrow[urr, "\nu"'] & & F
\end{tikzcd}$$

The lemma for $n = 2, 3$ follows after one defines ε to be $\varepsilon' \circ \nu$.

We assume now that $n \geqslant 4$ and proceed by induction. Let $A_{n-1}^{(i)}$ be the subalgebra of A_n which is isomorphic to A_{n-1} and has $\Delta^{(i)}$ as a base for its root system where

$$\begin{aligned}
\Delta^{(1)} &= \{\alpha_1,\ldots,\alpha_{n-1}\}, \\
\Delta^{(2)} &= \{\alpha_1,\ldots,\alpha_{n-2}, \alpha_{n-1} + \alpha_n\} \\
\Delta^{(3)} &= \{\alpha_1,\ldots,\alpha_{n-3}, \alpha_{n-2} + \alpha_{n-1}, \alpha_n\}, \\
&\vdots \\
\Delta^{(n)} &= \{\alpha_1 + \alpha_2, \alpha_3,\ldots,\alpha_n\}, \\
\Delta^{(n+1)} &= \{\alpha_2,\ldots,\alpha_n\}.
\end{aligned} \tag{4.48}$$

Let $\varepsilon^{(i)}$ be the evaluation map given to us by the induction hypothesis so that $\zeta \downarrow C(A_{n-1}^{(i)}) = \varepsilon^{(i)} \circ \Psi_{\Delta^{(i)},\Delta^{(i)}}$. Now define ε: $F[\lambda_1,\ldots,\lambda_n, s_1,\ldots,s_n] \to F$ by $\varepsilon(\lambda_i) = \zeta(h_i)$ and $\varepsilon(s_1) = \varepsilon^{(1)}(s_1)$. Since $A_2\{\alpha_1, \alpha_2\}$ is a subalgebra of both $A_{n-1}^{(1)}$ and $A_{n-1}^{(2)}$, we have $\varepsilon^{(1)}(s_1) = \varepsilon^{(2)}(s_1)$. In a similar manner $\varepsilon^{(k-1)}(s_1) = \varepsilon^{(k)}(s_1)$, $k = 2,\ldots,n$.

Viewing $A_2\{\alpha_2, \alpha_3\}$ as a subalgebra of both $A_{n-1}^{(1)}$ and $A_{n-1}^{(n+1)}$, one can conclude from Table 3.12 that $\varepsilon^{(n)}(s_1) = \varepsilon^{(n+1)}(s_1) + \zeta(h_1)$. Define $\varepsilon(\lambda_i) = \zeta(h_i)$ and $\varepsilon(s_1) = \varepsilon^{(1)}(s_1)$ and extend this to an algebra homomorphism on $F[\lambda_1,\ldots,\lambda_n, s_1,\ldots,s_n]$. To show that $\zeta = \varepsilon \circ \Psi_\Delta$ we introduce a set of maps $\pi^{(i)}$: $F[\lambda_1,\ldots,\lambda_{n-1}, s_1,\ldots,s_{n-1}] \to F[\lambda_1,\ldots,\lambda_n, s_1,\ldots,s_n]$ defined by

$$\pi^{(i)}(s_j) = s_j \quad \text{for } i = 1,\ldots,n+1 \text{ and } j = 1,2,\ldots,n-1,$$

$$\pi^{(i)}(\lambda_j) = \begin{cases} \lambda_j & j < n+1-i, \\ \lambda_j + \lambda_{j+1} & j = n+1-i, \\ \lambda_{j+1} & j > n+1-i. \end{cases}$$

The $\pi^{(i)}$'s have the property that $\Psi_\Delta \downarrow C(A_{n-1}^{(i)}) = \pi^{(i)} \circ \Psi_{\Delta^{(i)},\Delta^{(i)}}$ and ε has the property that

$$\varepsilon \circ \Psi_\Delta \downarrow C\left(A_{n-1}^{(i)}\right) = \varepsilon \circ \pi^{(i)} \circ \Psi_{\Delta^{(i)},\Delta^{(i)}} = \varepsilon^{(i)} \circ \Psi_{\Delta^{(i)},\Delta^{(i)}} = \zeta \downarrow\left(A_{n-1}^{(i)}\right).$$

Since every basic cycle of $C(A_n)$ having degree $\leqslant 3$ is contained in at least one of $C(A_{n-1}^{(i)})$, Theorem 4.1 implies $\zeta = \varepsilon \circ \Psi_\Delta$. $\square$

In the case of $S_\zeta = \Phi$, the preceding corollary proves that Theorem 1.7 holds for every choice of base Δ. However, in general this is not the case, and we must choose a base Δ corresponding to ζ. We do this by using Weyl automorphisms as indicated in §2.

We now proceed to investigate the mass function ζ on $C(A_3)$ such that $|S_\zeta| = 6$. Without loss of generality, we may assume that the base $\Delta = \{\alpha, \beta, \gamma\}$ is such that $S_\zeta = \{\pm\beta, \pm\gamma, \pm(\beta+\gamma)\}$.

LEMMA 4.49. *Let ζ be as above, then the restrictions of ζ to the three subalgebras $C(\alpha, \beta)$, $C(\alpha+\beta, \gamma)$ and $C(\alpha, \beta+\gamma)$ are either all T_4 or all T_3.*

PROOF. By using Table 3.12, one can verify that the restrictions of ζ to each of $C(\alpha, \beta)$, $C(\alpha+\beta, \gamma)$ and $C(\alpha, \beta+\gamma)$ is either T_3 or T_4. Therefore, it suffices to show that if the restriction of ζ to one of these algebras is T_4, then the remaining two are also T_4.

Assume first that $\zeta \downarrow C(\alpha, \beta)$ is T_4 and $\zeta \downarrow C(\alpha+\beta, \gamma)$ is T_3. Then the following equations hold.

$$\zeta(c_1) = \zeta(c_4) = \zeta(c_7) = \zeta(c_9) = 0,$$
$$\zeta(c_2) = p \neq 0, \quad -\zeta(h_2),$$
$$\zeta(c_3) = \zeta(c_{11}) = \zeta(c_{13}) = q \neq 0, -\zeta(h_3),$$
$$\zeta(h_1 + h_2) = 0, \qquad \zeta(c_6) = -\zeta(h_3).$$

Applying ζ to the identities

$$[c_1, c_5] = c_{14} - c_{12}, \tag{4.50}$$

$$[c_1, c_8] = c_{20} - c_{15}, \tag{4.51}$$

$$[c_3, c_7] = c_{15} - c_{19}, \tag{4.52}$$

$$[c_3, c_9] = c_{20} - c_{16}, \tag{4.53}$$

$$\begin{aligned}[c_1, c_{17}] = (c_5 - c_6)(c_7 - c_4) + (c_{14} - c_6)(c_4 - c_2) \\ + c_{20} - c_{14} - c_{15} + c_{12},\end{aligned} \tag{4.54}$$

$$c_{10}c_7 = c_3c_7 + (c_{19} - c_{13})c_2, \tag{4.55}$$

$$[c_7, c_{10}] = (h_2 + 1)(c_{19} - c_{12}) + (c_{13} - c_{12})c_2 - c_{15} + c_{12} \tag{4.56}$$

one sees that (4.55) implies $\zeta(c_{19}) = q$ and that (4.54) implies $\zeta(c_{14}) = -\zeta(h_3)$. Then, from (4.56), we obtain $(p + \zeta(h_2))(q + \zeta(h_3)) = 0$, which contradicts the assumptions on p and q. Thus whenever $\zeta \downarrow C(\alpha, \beta)$ is T_4, then $\zeta \downarrow C(\alpha + \beta, \gamma)$ is also T_4.

We may now use Weyl automorphisms to complete the proof as follows. By direct computation one sees that $\zeta \downarrow C(\alpha + \beta, \gamma)$ is T_4 if and only if $\zeta \circ \sigma_\gamma \circ \sigma_\beta \circ \sigma_\gamma \downarrow C(\alpha, \beta)$ is T_4. From this and the first part of the proof we have the implications

$$\begin{aligned}\zeta \downarrow C(\alpha + \beta, \gamma) \text{ is } T_4 &\Leftrightarrow \zeta \circ \sigma_\gamma \circ \sigma_\beta \circ \sigma_\gamma \downarrow C(\alpha, \beta) \text{ is } T_4 \\ &\Leftrightarrow \zeta \circ \sigma_\gamma \circ \sigma_\beta \circ \sigma_\gamma \downarrow C(\alpha + \beta, \gamma) \text{ is } T_4 \\ &\Leftrightarrow \zeta \downarrow C(\alpha, \beta) \text{ is } T_4.\end{aligned}$$

Combining this with the first part gives us that $\zeta \downarrow C(\alpha, \beta)$ and $\zeta \downarrow C(\alpha + \beta, \gamma)$ are both T_4 or both T_3. From this and direct computation, we obtain the following chain of implications.

$$\begin{aligned}\zeta \downarrow C(\alpha, \beta) \text{ is } T_4 &\Rightarrow \zeta \circ \sigma_\beta \downarrow C(\alpha, \beta) \text{ is } T_4 \\ &\Rightarrow \zeta \circ \sigma_\beta \downarrow C(\alpha + \beta, \gamma) \text{ is } T_4 \\ &\Rightarrow \zeta \downarrow C(\alpha, \beta + \gamma) \text{ is } T_4.\end{aligned}$$

By combining the above arguments, we see that if any one of the restrictions is T_4 then they all are. □

COROLLARY 4.57. *If ζ is a mass function on $C(\alpha, \beta, \gamma)$ with*

$$S_\zeta = \{\pm\alpha, \pm\beta, \pm(\alpha + \beta)\},$$

then the restrictions of ζ to each of $C(\beta, \gamma)$, $C(\alpha, \beta + \gamma)$, and $C(\alpha + \beta, \gamma)$ are all T_1 or all T_6.

PROOF. By using Table 3.12, one can verify that the restriction of ζ to each of $C(\beta, \gamma)$, $C(\alpha, \beta + \gamma)$ and $C(\alpha + \beta, \gamma)$ is either T_1 or T_6. According to Table 3.27, corresponding to each such restriction there is some Weyl automorphism σ such that the same restriction of $\zeta \circ \sigma$ is either T_3 or T_4. The corollary now follows from Lemma 4.49. □

We postpone the proof that mass functions ζ on $C(A_3)$ with $|S_\zeta| = 0, 2, 4$ or 6 satisfy Theorem 1.7 until we are treating the general case. We conclude this section

with an analysis of the structure of mass functions ζ: $C(A_3) \to F$ with $|S_\zeta| = 2$ and 4 which is similar to the analysis for $|S_\zeta| = 6$ and we establish two technical results about the structure of mass functions ζ: $C(A_4) \to F$.

LEMMA 4.58. *Let ζ be a mass function on $C(\alpha, \beta, \gamma)$ such that $S_\zeta = \{\pm\alpha, \pm\gamma\}$, then the following are equivalent.*

(i) $\zeta \downarrow C(\alpha, \beta)$ *is* T_1.
(ii) $\zeta \downarrow C(\beta, \gamma)$ *is* T_4.
(iii) $\zeta \downarrow C(\alpha, \beta + \gamma)$ *is* T_1.
(iv) $\zeta \downarrow C(\alpha + \beta, \gamma)$ *is* T_4.

PROOF. First we prove the implication

$$\zeta \downarrow C(\alpha, \beta) \text{ is } T_1 \Rightarrow \zeta \downarrow C(\beta, \gamma) \text{ is } T_4.$$

Assume $\zeta \downarrow C(\alpha, \beta)$ is T_1 so that $\zeta(c_1) = p \neq 0, -\zeta(h_1)$, $\zeta(c_2) = \zeta(c_4) = \zeta(c_7) = \zeta(c_9) = 0$. If $\zeta \downarrow C(\beta, \gamma)$ is not T_4, then according to Table 3.12, it must be T_3, and in this case we have

$$\zeta(c_3) = \zeta(c_8) = \zeta(c_{10}) = q \neq 0, \ -\zeta(h_3),$$
$$\zeta(c_5) = -\zeta(h_3), \qquad \zeta(h_2) = 0.$$

Applying ζ to the identities

$$[c_7, c_8] = c_7(c_5 - c_3) + (c_4 - c_1)(c_8 - c_5) + c_{19} - c_{15}, \tag{4.59}$$

$$[c_3, c_7] = c_{15} - c_{19} \tag{4.60}$$

and using the above values of ζ, we get $\zeta(c_8) = \zeta(c_5)$, contrary to $q \neq -\zeta(h_3)$.

Next we prove $\zeta \downarrow C(\beta, \gamma)$ is $T_4 \Rightarrow \zeta \downarrow C(\alpha, \beta)$ is T_1. The combined assumptions of $\zeta \downarrow C(\beta, \gamma)$ being T_4 and $\zeta \downarrow C(\alpha, \beta)$ being not T_1, imply that $\zeta \downarrow C(\alpha, \beta)$ is T_6 and the following hold.

$$\zeta(c_2) = \zeta(c_5) = \zeta(c_8) = \zeta(c_{10}) = 0,$$
$$\zeta(c_3) = q \neq 0, \ -\zeta(h_3),$$
$$\zeta(c_1) = p \neq 0, \ -\zeta(h_1),$$
$$\zeta(c_4) = -\zeta(h_1),$$
$$\zeta(c_7) = \zeta(c_9) = -p - \zeta(h_1).$$

When ζ is applied to the identities (4.59) and (4.60) we obtain the contradiction $(p + \zeta(h_1))q = 0$. This proves the implication we are after.

The remaining equivalences can be obtained from this and direct computation through the use of appropriate Weyl automorphisms. We outline this by the following chain of equivalences.

$$\begin{aligned}\zeta \downarrow C(\alpha+\beta,\gamma) \text{ is } T_4 &\Leftrightarrow \zeta \circ \sigma_\alpha \downarrow C(\beta,\gamma) \text{ is } T_4,\\ &\Leftrightarrow \zeta \circ \sigma_\alpha \downarrow C(\alpha,\beta) \text{ is } T_1,\\ &\Leftrightarrow \zeta \downarrow C(\alpha,\beta) \text{ is } T_1,\\ &\Leftrightarrow \zeta \downarrow C(\beta,\gamma) \text{ is } T_4,\\ &\Leftrightarrow \zeta \circ \sigma_\gamma \downarrow C(\beta,\gamma) \text{ is } T_4,\\ &\Leftrightarrow \zeta \circ \sigma_\gamma \downarrow C(\alpha,\beta) \text{ is } T_1,\\ &\Leftrightarrow \zeta \downarrow C(\alpha,\beta+\gamma) \text{ is } T_1.\end{aligned}$$

This concludes the proof of Lemma 4.58. □

LEMMA 4.61. *If* ζ: $C(\alpha,\beta,\gamma) \to F$ *is a mass function where*
(i) $S_\zeta = \varnothing$,
(ii) $\zeta(h_1) \neq 0$ *and* $\zeta(h_3) \neq 0$, *and*
(iii) $\zeta(c_i) = 0$ *for* $i = 1,2,\ldots,5$
then $\zeta(c_6) = 0$.

PROOF. Assume ζ: $C(\alpha,\beta,\gamma) \to F$ is a mass function satisfying (i) through (iii) and $\zeta(c_6) \neq 0$. Since $S_\zeta = \varnothing$, and $\zeta(c_6) \neq 0$ we must have

$$\zeta(c_6) = -\zeta(h_1 + h_2 + h_3) \neq 0.$$

Since $\zeta \downarrow C(\alpha,\beta+\gamma)$ is a mass function with $\zeta(h_1) \neq 0$ then by Table 3.12 we must have $\zeta(h_2 + h_3) = 0$. Similarly, considering the restriction of ζ to $C(\alpha+\beta,\gamma)$, we have $\zeta(h_1 + h_2) = 0$ and hence $\zeta(h_1) = -\zeta(h_2) = \zeta(h_3) \neq 0$.

Table 3.12 and the conditions that $\zeta(c_1) = \zeta(c_2) = \zeta(c_4) = 0$ imply that $\zeta(c_7) = \zeta(c_9) = 0$. Similarly considering the restrictions of ζ to $C(\beta,\gamma)$, $C(\alpha,\beta+\gamma)$ and $C(\alpha+\beta,\gamma)$ yields $\zeta(c_8) = \zeta(c_{10}) = \zeta(c_{11}) = \zeta(c_{13}) = 0$ and $\zeta(c_{12}) = \zeta(c_{14}) = -\zeta(h_2) \neq 0$.

Then, using these values for ζ, in the identities

$$[c_1, c_8] = c_{20} - c_{15}, \tag{4.62}$$

$$[c_3, c_7] = c_{15} - c_{19}, \tag{4.63}$$

$$[c_1, c_{15}] = c_1(c_{11} - c_8) - (h_1 + 1)(c_{15} - c_{11}) + c_{20} - c_{11}, \tag{4.64}$$

$$[c_7, c_{10}] = (h_2 + 1)(c_{19} - c_{12}) + (c_{13} - c_{12})c_2 - c_{15} + c_{12} \tag{4.65}$$

we obtain $\zeta(h_2) = 0$ which is a contradiction. □

LEMMA 4.66. *If* ζ: $C(\alpha,\beta,\gamma) \to F$ *is a mass function where* $S_\zeta = \{\pm\alpha\}$, $\zeta \downarrow C(\alpha,\beta)$ *is* T_1, $\zeta \downarrow C(\alpha,\beta+\gamma)$ *is* T_6 *and* $\zeta(c_3) = 0$ *then* $\zeta(h_3) = 0$.

PROOF. Assume ζ: $C(\alpha,\beta,\gamma) \to F$ satisfies the hypotheses and $\zeta(h_3) \neq 0$. The restriction of ζ to $C(\beta,\gamma)$ then must be either T_3 with $p = 0$ and $\zeta(h_2) = 0$ or T_5 with $\zeta(h_2) = 0$ and $p = -\zeta(h_3)$. In either case $\zeta(c_5) = -\zeta(h_3)$ and $\zeta(c_8) = \zeta(c_{10}) = 0$. A similar analysis of $\zeta \downarrow C(\alpha+\beta,\gamma)$ yields $\zeta(h_1 + h_2) = 0$ and $\zeta(c_{11}) = \zeta(c_{13}) = 0$.

Since $\zeta \downarrow C(\alpha, \beta)$ is T_1 and $\zeta \downarrow C(\alpha, \beta + \gamma)$ is T_6, we have that $\zeta(c_7) = \zeta(c_9) = 0$ and $\zeta(c_{12}) = \zeta(c_{14}) = -\zeta(h_1 + h_2 + h_3) - \zeta(c_1)$. Now applying ζ to the identities

$$[c_7, c_8] = c_7(c_5 - c_3) + c_4(c_8 - c_5) - (c_8 - c_5)c_1 - c_{12} + c_{14} + c_{19} - c_{20}, \tag{4.67}$$

$$[c_1, c_8] = c_{20} - c_{15}, \tag{4.68}$$

$$[c_3, c_7] = c_{15} - c_{19} \tag{4.69}$$

we have that $\zeta(c_1)\zeta(h_3) = 0$. This contradicts our assumptions and hence we may conclude that $\zeta(h_3) = 0$. □

LEMMA 4.70. *If ζ: $C(\alpha, \beta, \gamma) \to F$ is a mass function where $S_\zeta = \{\pm\beta\}$, $\zeta \downarrow C(\alpha, \beta)$ is T_4 and $\zeta \downarrow C(\beta, \gamma)$ is T_1 then $\zeta(c_6) = 0$.*

PROOF. Assume that ζ satisfies the conditions and $\zeta(c_6) \neq 0$. Since $S_\zeta = \{\pm\beta\}$ we have that $\zeta(c_6) = -\zeta(h_1 + h_2 + h_3) \neq 0$. Now ζ satisfies the conditions of Lemma 4.66 relative to the base $\{\beta, \gamma, -\alpha - \beta - \gamma\}$ which implies that $\zeta(h_1 + h_2 + h_3) = 0$ contrary to our assumption. Therefore $\zeta(c_6) = 0$. □

LEMMA 4.71. *There does not exist a mass function ζ: $C(A_4) \to F$ such that, relative to a base $\{\alpha, \beta, \gamma, \delta\}$ of the root system of A_4, we have:*

(i) $S_\zeta = \{\pm\alpha, \pm\gamma\}$,
(ii) $\zeta \downarrow C(\alpha, \beta)$, $C(\alpha, \beta + \gamma)$ *and* $C(\gamma, \delta)$ *are* T_1,
(iii) $\zeta \downarrow C(\alpha + \beta, \gamma)$ *and* $C(\beta, \gamma)$ *are* T_4, *and*
(iv) $\zeta \downarrow C(\alpha, \beta + \gamma + \delta)$ *is* T_6.

PROOF. Assume that ζ: $C(\alpha, \beta, \gamma, \delta) \to F$ is a mass function which satisfies conditions (i) through (iv) and seek a contradiction. Since $\zeta \downarrow C(\alpha, \beta + \gamma, \delta)$ and $\zeta \downarrow C(\alpha, \beta, \gamma + \delta)$ satisfy the conditions of Lemma 4.66 we have that

$$\zeta(h_\delta) = \zeta(h_\gamma + h_\delta) = 0.$$

Also since $\zeta \downarrow C(\alpha + \beta, \gamma, \delta)$ and $\zeta \downarrow C(\beta, \gamma, \delta)$ satisfy the conditions of Lemma 4.70 we have that $\zeta(h_\alpha + h_\beta + h_\gamma + h_\delta) = \zeta(h_\beta + h_\gamma + h_\delta) = 0$ and hence we may assume that $\zeta(h_\alpha) = \zeta(h_\beta) = \zeta(h_\gamma) = \zeta(h_\delta) = 0$.

Let $c = (Y_{\gamma+\delta}Y_{\alpha+\beta}X_{\beta+\gamma+\delta}X_\alpha)(Y_\delta Y_\gamma X_{\gamma+\delta}) \in C(\alpha, \beta, \gamma, \delta)$. Since $\zeta \downarrow C(\gamma, \delta)$ is T_1 we have $\zeta(Y_\delta Y_\gamma X_{\gamma+\delta}) = 0$ and hence $\zeta(c) = 0$. On the other hand we have

$$\begin{aligned} c = {} & (Y_{\gamma+\delta}X_{\gamma+\delta})(Y_{\alpha+\beta}X_{\beta+\gamma+\delta}X_\alpha Y_\delta Y_\gamma) \\ & + (Y_{\gamma+\delta}Y_{\alpha+\beta}X_\alpha X_{\beta+\gamma+\delta} - Y_{\gamma+\delta}Y_{\alpha+\beta}X_{\alpha+\beta+\gamma+\delta}) \\ & \cdot (Y_\delta X_\delta - Y_\gamma X_\gamma - h_\gamma). \end{aligned}$$

Since $\zeta \downarrow C(\gamma, \delta)$ is T_1 we have $\zeta(Y_{\gamma+\delta}X_{\gamma+\delta}) = \zeta(Y_\delta X_\delta) = 0$. Also by considering

$\zeta \downarrow C(\alpha+\beta, \gamma+\delta)$ we have that $\zeta(Y_{\gamma+\delta}Y_{\alpha+\beta}X_{\alpha+\beta+\gamma+\delta}) = 0$ and hence $0 = \zeta(c) = -\zeta(Y_{\gamma+\delta}Y_{\alpha+\beta}X_\alpha X_{\beta+\gamma+\delta})\zeta(Y_\gamma X_\gamma)$. Finally by applying ζ to the following identities

$$(4.72)\quad \left[Y_\alpha X_\alpha, Y_{\gamma+\delta}Y_\beta X_{\beta+\gamma+\delta}\right] = Y_{\gamma+\delta}Y_\beta Y_\alpha X_{\alpha+\beta+\gamma+\delta} - Y_{\gamma+\delta}Y_{\alpha+\beta}X_\alpha X_{\beta+\gamma+\delta},$$

$$(4.73)\quad \begin{aligned}\left[Y_\beta X_\beta, Y_{\beta+\gamma+\delta}Y_\alpha X_{\alpha+\beta+\gamma+\delta}\right] &+ Y_{\beta+\gamma+\delta}Y_{\alpha+\beta}X_{\alpha+\beta+\gamma+\delta}X_\beta \\ &- Y_{\gamma+\delta}Y_\beta Y_\alpha X_{\alpha+\beta+\gamma+\delta} + Y_{\beta+\gamma+\delta}Y_\alpha X_{\alpha+\beta+\gamma+\delta},\end{aligned}$$

(4.74)

$$\begin{aligned}(Y_{\beta+\gamma+\delta}Y_\alpha X_{\alpha+\beta+\gamma+\delta}) \cdot (Y_{\alpha+\beta}X_\alpha X_\beta) &= (Y_{\beta+\gamma+\delta}Y_\alpha X_{\alpha+\beta+\gamma+\delta})(Y_{\alpha+\beta}X_{\alpha+\beta}) \\ &+ (Y_{\beta+\gamma+\delta}X_{\beta+\gamma+\delta})(Y_{\alpha+\beta}X_\alpha X_\beta - Y_{\alpha+\beta}Y_{\alpha+\beta}) \\ &+ (Y_{\beta+\gamma+\delta}Y_{\alpha+\beta}X_{\alpha+\beta+\gamma+\delta}X_\beta - Y_{\beta+\gamma+\delta}X_\beta X_{\gamma+\delta} + Y_{\beta+\gamma+\delta}X_{\beta+\gamma+\delta})(Y_\alpha X_\alpha)\end{aligned}$$

we have that $\zeta(Y_{\gamma+\delta}Y_{\alpha+\beta}X_\alpha X_{\beta+\gamma+\delta}) = -\zeta(Y_\alpha X_\alpha)$ and hence $0 = \zeta(c) = \zeta(Y_\alpha X_\alpha)\zeta(Y_\gamma X_\gamma)$ which contradicts our assumption that $S_\zeta = \{\pm\alpha, \pm\gamma\}$. □

LEMMA 4.75. *There does not exist a mass function ζ: $C(A_4) \to F$ such that, relative to a base $\{\alpha, \beta, \gamma, \delta\}$ of the root system of A_4, we have*:

(i) $S_\zeta = \{\pm\alpha, \pm\delta\}$,

(ii) $\zeta \downarrow C(\alpha, \beta)$ *is* T_1,

(iii) $\zeta \downarrow C(\alpha, \beta+\gamma)$ *is* T_6 *and*

(iv) $\zeta \downarrow C(\gamma, \delta)$ *is* T_4.

PROOF. If ζ: $C(\alpha, \beta, \gamma, \delta) \to F$ were a mass function satisfying conditions (i) through (iv) then ζ satisfies the conditions of Lemma 4.71 relative to the base $\{\alpha, -\alpha-\beta-\gamma-\delta, \delta, \gamma\}$ and this is impossible. □

5. Proof of main theorem. In this section, we complete the proof of Theorem 1.7. First, we introduce some notation and terminology.

Let ζ be a mass function on $C(A_n)$, Φ a root system of A_n and $S_\zeta = \{\gamma \in \Phi \mid \zeta(Y_\gamma X_\gamma) = 0 \text{ or } -\zeta(h_\gamma)\}$. By Proposition 4.3, $S_\zeta = \bigcup_{i=1}^{l} \Phi_i'$ where Φ_i' are mutually orthogonal irreducible root subsystems of Φ. Let Δ_i' be a base of Φ_i'. By [6, Theorem 5.2], we can extend $\bigcup_{i=1}^{l} \Delta_i'$ to Δ as a base of Φ. By permitting some Δ_i's to be void if necessary, we may present the Dynkin diagram for A_n by

$$(5.1)\quad \begin{array}{ccccccccc} \Delta_1 & \alpha_{k_1} & \Delta_2 & \alpha_{k_2} & & & \alpha_{k_t} & \Delta_{t+1} \\ \circ & \text{---}\cdot\text{---} & \circ & \text{---}\cdot\text{---} & \cdots & \text{---}\cdot\text{---} & & \circ \end{array}$$

where $\Delta_i = \varnothing$ or Δ_j' for some j and $|\{i \mid \Delta_i \neq \varnothing\}| = l$. A base with such a presentation is called a *ζ-base*. By convention, we set $k_0 = 0$, $\alpha_0 = 0$ and $k_{t+1} = n + 1$. If Δ is a ζ-base of the form (5.1) and $\Delta_i \neq \varnothing$ then Δ is assumed to be labeled such that $\Delta_i = \{\alpha_{k_{i-1}+1}, \alpha_{k_{i-1}+2}, \ldots, \alpha_{k_i-1}\}$.

Let Δ be a ζ-base of the form (5.1), Φ^+ be the positive roots relative to this base, Φ_i be the set of roots generated by Δ_i, and $\Phi_i^+ = \Phi_i \cap \Phi^+$. For $1 \leqslant i < j \leqslant t+1$, we define

$$B_{ij}(\Delta) = \{\alpha_\mu + \cdots + \alpha_\nu \mid \alpha_\mu \in \Delta_i \cup \{\alpha_{k_i}\} \text{ and } \alpha_\nu \in \Delta_j \cup \{\alpha_{k_{j-1}}\}\}, \tag{5.2}$$

$$I_{ij}(\Delta) = \{(\alpha, \beta) \mid \alpha \in \Phi_i^+, \beta \in B_{ij}(\Delta) \text{ and } \alpha + \beta \in \Phi^+\}, \tag{5.3}$$

$$J_{ij}(\Delta) = \{(\beta, \gamma) \mid \beta \in B_{ij}(\Delta), \gamma \in \Phi_j^+ \text{ and } \beta + \gamma \in \Phi^+\}. \tag{5.4}$$

These definitions allow us to get control over the value of ζ on entire blocks of basic cycles as illustrated by

PROPOSITION 5.5. *If there exists a pair $(\alpha_0, \beta_0) \in I_{ij}(\Delta)$ such that $\zeta \downarrow C(\alpha_0, \beta_0)$ is T_1 or a pair $(\beta_0, \gamma_0) \in J_{ij}(\Delta)$ such that $\zeta \downarrow C(\beta_0, \gamma_0)$ is T_4 then $\zeta \downarrow C(\alpha, \beta)$ is T_1 for all $(\alpha, \beta) \in I_{ij}(\Delta)$ and $\zeta \downarrow C(\beta, \gamma)$ is T_4 for all $(\beta, \gamma) \in J_{ij}(\Delta)$.*

PROOF. Suppose first that $\Delta_i = \varnothing$ and $(\beta_0, \gamma_0) \in J_{ij}(\Delta)$ with $\zeta \downarrow C(\beta_0, \gamma_0)$ being T_4. Then $\beta_0 = \alpha_{k_i} + \cdots + \alpha_\mu$ and $\gamma_0 = \alpha_{\mu+1} + \cdots + \alpha_\nu$ where $\alpha_{\mu+1}, \ldots, \alpha_\nu \in \Delta_j$ and hence $\alpha_\mu \in \Delta_j \cup \{\alpha_{k_{j-1}}\}$. By Lemma 4.49, we have that

$$\begin{aligned} \zeta \downarrow C(\beta_0, \gamma_0) \text{ is } T_4 &\Leftrightarrow \zeta \downarrow C\big(\alpha_{k_i} + \cdots + \alpha_{k_{j-1}}, \alpha_{k_{j-1}+1} + \cdots + \alpha_\nu\big) \text{ is } T_4, \\ &\Leftrightarrow \zeta \downarrow C\big(\alpha_{k_i} + \cdots + \alpha_{k_{j-1}}, \alpha_{k_{j-1}+1}\big) \text{ is } T_4. \end{aligned}$$

Thus $\zeta \downarrow C(\beta_0, \gamma_0)$ is T_4 iff $\zeta \downarrow C(\beta, \gamma)$ is T_4 for all $(\beta, \gamma) \in J_{ij}(\Delta)$. Since $\Delta_i = \varnothing$ implies that $I_{ij}(\Delta) = \varnothing$ we have that the proposition is valid for $\Delta_i = \varnothing$.

A similar argument, applying Corollary 4.57, proves the proposition if $\Delta_j = \varnothing$. Thus we may assume that $\Delta_i \neq \varnothing \neq \Delta_j$. In this case the proposition follows directly from

LEMMA 5.6. *With notation as above and assuming $\Delta_i \neq \varnothing \neq \Delta_j$ we have:*
(i) *for $(\alpha, \beta) \in I_{ij}(\Delta)$, $\zeta \downarrow C(\alpha, \beta)$ is T_1 iff $\zeta \downarrow C(\alpha_{k_i-1}, \alpha_{k_i} + \cdots + \alpha_{k_{j-1}})$ is T_1,*
(ii) *for $(\beta, \gamma) \in J_{ij}(\Delta)$, $\zeta \downarrow C(\beta, \gamma)$ is T_4 iff $\zeta \downarrow C(\alpha_{k_i-1}, \alpha_{k_i} + \cdots + \alpha_{k_{j-1}})$ is T_1.*

PROOF. (i) Set $\bar{\beta} = \alpha_{k_i} + \cdots + \alpha_{k_{j-1}}$ and write $\beta = \alpha' + \bar{\beta} + \gamma'$ where $\alpha' \in \Phi_i \cap \Phi^+$ or $\alpha' = 0$ and $\gamma' \in \Phi_j \cap \Phi^+$ or $\gamma' = 0$. Applying Corollary 4.57 and Lemma 4.58, we have the sequence of equivalent statements

$$\begin{aligned} \zeta \downarrow C(\alpha, \beta) \text{ is } T_1 &\Leftrightarrow^{(4.57)} \zeta \downarrow C(\alpha', \bar{\beta} + \gamma') \text{ is } T_1, \\ &\Leftrightarrow^{(4.58)} \zeta \downarrow C(\alpha', \bar{\beta}) \text{ is } T_1, \\ &\Leftrightarrow^{(4.57)} \zeta \downarrow C\big(\alpha_{k_i-1}, \bar{\beta}\big) \text{ is } T_1. \end{aligned}$$

(Note that if $\alpha' = 0$ we may start at the second statement replacing α' by α to obtain the required result.)

(ii) Applying Lemmas 4.49 and 4.58, we have the sequence of equivalent statements

$$\begin{aligned}\zeta \downarrow C(\beta,\gamma) \text{ is } T_4 &\Leftrightarrow^{(4.49)} \zeta \downarrow C(\alpha' + \bar{\beta}, \gamma') \text{ is } T_4,\\ &\Leftrightarrow^{(4.58)} \zeta \downarrow C(\bar{\beta}, \gamma') \text{ is } T_4,\\ &\Leftrightarrow^{(4.58)} \zeta \downarrow C(\alpha_{k_i}, \bar{\beta}) \text{ is } T_1.\end{aligned}$$

(Note that if $\gamma' = 0$ we may start at the second statement replacing γ' by γ to obtain the required result.) □

DEFINITION 5.7. Let Δ be a ζ-base given by (5.1). Then the pair (i, j) with $1 \leqslant i < j \leqslant t+1$ is *bad relative to* Δ if

(i) when $|B_{ij}(\Delta)| = 1$ we have

$$\zeta\left(Y_{\alpha_{k_i}+\cdots+\alpha_{k_{j-1}}} X_{\alpha_{k_i}+\cdots+\alpha_{k_{j-1}}}\right) = -\zeta\left(h_{\alpha_{k_i}} + \cdots + h_{\alpha_{k_{j-1}}}\right) \neq 0$$

or

(ii) when $|B_{ij}(\Delta)| > 1$ we have $\zeta \downarrow C(\alpha, \beta)$ is T_6 for all $(\alpha, \beta) \in I_{ij}(\Delta)$ and $\zeta \downarrow C(\beta, \gamma)$ is T_3 for all $(\beta, \gamma) \in J_{ij}(\Delta)$.

LEMMA 5.8. *For any mass function ζ: $C(A_n) \to F$, $n \geqslant 3$, there exists a ζ-base Δ such that there are no bad pairs relative to Δ.*

PROOF. Our proof is by induction on t. For $t = 0$, the lemma is vacuously satisfied. Let ζ be a mass function on $C(A_n)$ and Δ be a ζ-base having the property that among all such bases the number of bad pairs is minimal relative to ζ. Evidently we must show that this number is zero when $t \geqslant 0$.

Assume first that there is a bad pair (i, j) relative to Δ with $i < j < t+1$. If $\zeta_1 = \zeta \downarrow C(\alpha_1, \ldots, \alpha_{k_t - 1})$ then by our induction, there exists a ζ_1-base $\Delta^{(1)}$ of the root subsystem generated by $\{\alpha_1, \ldots, \alpha_{k_t-1}\}$ such that there are no bad pairs relative to $\Delta^{(1)}$. Also, there is some product σ of Weyl reflections σ_α with $\alpha \in \{\alpha_1, \ldots, \alpha_{k_t-1}\}$ such that $\sigma(\alpha_1, \ldots, \alpha_{k_t-1}) = \Delta^{(1)}$. From the form of σ, we have that σ leaves Δ_{t+1} pointwise fixed and maps $B_{l,t+1}(\Delta)$ to some $B_{k,t+1}(\Delta)$. Therefore, the number of bad pairs of ζ relative to $\sigma\Delta$ is strictly less than the number of bad pairs of ζ relative to Δ. This contradiction implies that there are no bad pairs (i, j) of ζ relative to Δ with $i < j < t+1$.

By a similar argument, there are no bad pairs (i, j) of ζ relative to Δ with $1 < i < j$. Thus, if ζ has a minimum number of bad pairs relative to Δ, then the only possible bad pair must be $(1, t+1)$.

We now assume that $(1, t+1)$ is the only bad pair of ζ relative to Δ and seek a contradiction. Our argument splits into two cases.

Case I. One of Δ_1 or Δ_{t+1} is nonempty. If $\Delta_1 = \varnothing$ and $\Delta_{t+1} \neq \varnothing$ we can replace the ζ-base $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ by the ζ-base $\hat{\Delta} = \{\alpha_n, \ldots, \alpha_1\}$. From Table 3.12 we have that $\zeta \downarrow C(\alpha, \beta)$ is T_1 iff $\zeta \downarrow C(\beta, \alpha)$ is T_4. It follows then that $(1, t+1)$ is the only bad pair of ζ relative to $\hat{\Delta}$ and moreover $\hat{\Delta}_1 \neq \varnothing$. Therefore, without loss of generality we may assume $\Delta_1 \neq \varnothing$.

If $t = 1$ then there is only one pair $(1, 2)$,

$$\tilde{\Delta} = \left\{\alpha_{k_1-1}, \ldots, \alpha_1, -(\alpha_1 + \cdots + \alpha_n), \alpha_n, \ldots, \alpha_{k_1-1}\right\}$$

is a ζ-base, and $\zeta \downarrow C(\alpha_1, -(\alpha_1 + \cdots + \alpha_n))$ is T_1. Hence by Proposition 5.5 the only pair relative to $\tilde{\Delta}$ is not bad. This contradiction completes Case I for $t = 1$.

If $t = 2$ we consider two subcases.

Subcase (i). $\Delta_t \neq \varnothing$. In this case, ζ restricted to

$$C\left(\alpha_1, \alpha_2 + \cdots + \alpha_{k_t-2}, \alpha_{k_t-1}, \alpha_{k_t} + \cdots + \alpha_n\right)$$

satisfies the hypothesis of Lemma 4.71 and hence there is no such mass function.

Subcase (ii). $\Delta_t = \varnothing$. If $\Delta_{t+1} \neq \varnothing$ then ζ restricted to

$$C\left(\alpha_1, \alpha_2 + \cdots + \alpha_{k_t-1}, \alpha_{k_t}, \alpha_{k_t+1} + \cdots + \alpha_n\right)$$

satisfies the hypothesis of Lemma 4.75 and no such mass function exists. Thus we may assume that $\Delta_t = \Delta_{t+1} = \varnothing$. Then ζ restricted to $C(\alpha_1, \alpha_2 + \cdots + \alpha_{n-1}, \alpha_n)$ satisfies the hypothesis of Lemma 4.66 and hence $\zeta(h_{\alpha_n}) = 0$. Now $\sigma_{\alpha_n}\Delta$ is a ζ-base and ζ admits exactly one bad pair namely $(1, t)$ (and hence a minimum number) relative to $\sigma_{\alpha_n}\Delta$. This contradicts our previous argument again and hence completes this subcase.

Case II. $\Delta_1 = \varnothing = \Delta_{t+1}$. If, in addition, there exists some i with $\Delta_i \neq \varnothing$ the restriction of ζ to $C(\alpha_1 + \cdots + \alpha_{l-1}, \alpha_l, \alpha_{l+1} + \cdots + \alpha_n)$ where $\alpha_l \in \Delta_i$ satisfies the hypothesis of Lemma 4.70 and hence

$$\zeta\left(Y_{\alpha_1 + \cdots + \alpha_n} X_{\alpha_1 + \cdots + \alpha_n}\right) = 0.$$

Since $|B_{1,t+1}(\Delta)| = 1$, this implies that the pair $(1, t+1)$ is not bad contrary to our assumption. Thus we may assume that $\Delta_i = \varnothing$ for $1 \leqslant i \leqslant t+1$. This implies that $t = n$, $S_\zeta = \varnothing$, $|B_{ij}(\Delta)| = 1$ for $1 \leqslant i < j \leqslant n+1$ and $\zeta(Y_\beta X_\beta) = 0$ for all positive roots β except $\alpha_1 + \cdots + \alpha_n$.

If $\zeta(h_{\alpha_n}) = 0$ or $\zeta(h_{\alpha_1}) = 0$ then by an argument similar to subcase (ii) of Case I we arrive at a contradiction and hence we may assume that $\zeta(h_{\alpha_1}) \neq 0 \neq \zeta(h_{\alpha_n})$. From this we see that $\zeta \downarrow C(\alpha_1, \alpha_2 + \cdots + \alpha_{n-1}, \alpha_n)$ satisfies the hypothesis of Lemma 4.61 and hence $\zeta(Y_{a_1 + \cdots + \alpha_n} X_{\alpha_1 + \cdots + \alpha_n}) = 0$—i.e. the $(1, t+1)$ pair is not bad. This contradiction completes this case and hence the entire theorem. $\square$

We now have all the facts needed to prove our main theorem.

PROOF OF THEOREM 1.7. In §3, we proved this result for $n = 1$ and 2. Let Δ be a ζ-base as given to us by Lemma 5.8 so that Δ has form (5.1). By Corollary 4.47, there is an evaluation map ε: $F[\lambda_1, \ldots, \lambda_n, s_1, \ldots, s_n] \to F$ such that

$$\zeta \downarrow C(\Delta_i) = \varepsilon \circ \Psi_{\Delta', \Delta} \downarrow C(\Delta_i).$$

Since Δ is a ζ-base described by Lemma 5.8, ζ is zero on all degree 2 and 3 basic cycles of $C(A_n)$ relative to Δ which are not in $\bigcup_1^{t+1} C(\Delta_i)$. Therefore, $\zeta(c) = \varepsilon \circ \Psi_{\Delta', \Delta}(c)$ for all basic cycles c of degree $\leqslant 3$. By Proposition 4.1, this implies that $\zeta = \varepsilon \circ \Psi$. $\square$

6. Pointed representations of A_n. In the introduction we proved that corresponding to each mass function ζ: $C(A_n) \to F$ there exists a unique (up to equivalence) pointed representation of A_n. Unfortunately this correspondence is not one-one since a pointed representation of A_n may admit many 1-dimensional weight spaces. In this section we indicate how one may select a set of mass functions which label the pointed representations of A_n in a one-one fashion. A first step in this direction is provided by the following theorem.

THEOREM 6.1 [**11**, THEOREM 4]. *If ζ_1, ζ_2: $C(A_n) \to F$ are mass functions then the pointed representations corresponding to ζ_1 and ζ_2 are equivalent iff there exist elements $x, y \in U(A_n)$ such that $yx \in C(A_n)$, $\zeta_2(yx) = 1$ and $\zeta_1(c) = \zeta_2(ycx)$ for all $c \in C(A_n)$.*

To motivate our approach to the general labelling problem, we first review the known results on highest weight (Verma) representations and show how these can be interpreted in the setting of pointed representations. Fix a base Δ of the root system of A_n and let Ψ: $C(A_n) \to F[\lambda_1,\ldots,\lambda_n]$ be the usual Harish-Chandra homomorphism relative to Δ. For any evaluation map ε: $F[\lambda_1,\ldots,\lambda_n] \to F$, $\zeta = \varepsilon \circ \Psi$ is a mass function and moreover it is clear that the left ideal I_ζ of $U(A_n)$ generated by $\{X_\beta \mid \beta \in \Phi^+\} \cup \{h_\alpha - \zeta(h_\alpha) \cdot 1 \mid \alpha \in \Delta\}$ contains the kernel of ζ. Then the regular representation of A_n on $U(A_n)/I_\zeta$ is the Verma representation with highest weight $\lambda = \zeta \downarrow H$ relative to the base Δ. By the argument given in the proof of Theorem 1.3(ii), I_ζ is contained in the unique maximal left ideal M_ζ of $U(A_n)$ which contains $\ker \zeta$. Thus, for any mass function ζ, as described above, the associated pointed representation of A_n admits $\zeta \downarrow H$ as its highest weight relative to the base Δ. Conversely, if (ρ, V) is an irreducible representation which admits $\lambda \in H^*$ as a highest weight relative to Δ then the mass function corresponding to this 1-dimensional weight space V_λ is equal to $\varepsilon \circ \Psi$ where ε: $F[\lambda_1,\ldots,\lambda_n] \to F$ with $\varepsilon(\lambda_i) = \lambda(h_{\alpha_i})$ for all $\alpha_i \in \Delta$.

If we restrict our attention to irreducible representations having a highest weight relative to a fixed base Δ, it is well known that such representations are labelled in a one-one fashion by their highest weight. To address the analogous question for pointed representations we first restrict our attention to those pointed representations which are associated with mass functions of the form $\varepsilon \circ \Psi_{\Delta',\Delta}$ for some fixed base Δ. These representations will be called Δ-pointed representations. Theorem 1.7 states that every pointed representation is a Δ-pointed representation for some base Δ. We now specify a set of mass functions which label the Δ-pointed representations in a one-one fashion.

DEFINITION 6.2. A mass function ζ: $C(A_n) \to F$ is said to be Δ-*complete* for some base Δ of the root system of A_n iff $\zeta = \varepsilon \circ \Psi_{\Delta,\Delta}$ where the evaluation map ε: $F[\lambda_1,\ldots,\lambda_n, s_1,\ldots,s_n] \to F$ satisfies the two conditions

(i) $\varepsilon(s_1) \notin Z \cup (Z + \zeta(h_{\alpha_1})) \cup \cdots \cup (Z + \zeta(h_{\alpha_1} + \cdots + h_{\alpha_n}))$ where $\Delta = \{\alpha_1,\ldots,\alpha_n\}$ and

(ii) $\varepsilon \circ \Psi_{\Delta,\Delta} \downarrow H = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n$ where the coefficients a_i are normalized as follows. Select a linear basis of F as a rational vector space where the first basis vector is 1. Each a_i, when expressed in terms of this basis must have first coordinate greater than or equal to 0 and strictly less than 1.

In [**14**] it is proven that the pointed representation of A_n associated with a mass function $\zeta = \varepsilon \circ \Psi_{\Delta,\Delta}$ where $\varepsilon(s_1)$ satisfies condition (i) in Definition 6.2 has the property that its set of weight functions consist of $\{\zeta \downarrow H + \sum_{i=1}^{n} n_i\alpha_i \mid n_i \in Z\}$ and each of these weight spaces is one-dimensional. The role of condition (ii) in Definition 6.2 is to provide a method to select a unique mass function for such representations. In a sense we are selecting the unique mass function which is closest to the center in the lattice of weights of such representations.

DEFINITION 6.3. A mass function ζ: $C(A_n) \to F$ is said to be Δ-*extreme* for a base Δ iff there exists a subset Δ' of Δ such that $\zeta = \varepsilon \circ \Psi_{\Delta',\Delta}$ where the evaluation map ε satisfies the property that for each nonempty connected component Δ_i of Δ', the restriction of ζ to $C(\Delta_i)$ is Δ_i-complete.

Under the assumption that Theorem 1.7 holds, it has been shown in [**15**] that for each pointed representation (ρ, V) of A_n there exists a base Δ such that V admits a Δ-extreme mass function. It is also shown that distinct Δ-extreme mass functions yield inequivalent pointed representations. Thus the Δ-extreme mass functions play the same role for Δ-pointed representations as the elements of H^* play for irreducible representations admitting a highest weight with respect to Δ. In particular it should be noted that for any $\lambda \in H^*$ the mass function $\zeta = \varepsilon \circ \Psi_{\varnothing,\Delta}$ with $\varepsilon(\lambda_i) = \lambda(h_{\alpha_i})$ for all $\alpha_i \in \Delta$ is Δ-extreme and yields the irreducible representation of A_n admitting λ as a highest weight relative to Δ.

The general problem of classifying all irreducible representations of A_n is as yet an unsolved problem. Block [**3**] has recently completed a classification of all irreducible representations for the case $n = 1$. In order to place the category of all pointed representations of A_n in the perspective of all irreducible representations of A_n we summarize some of the known types of such representations. In [**1, 13**] it is shown that there exist irreducible representations of A_n (even of A_1) which do not admit a weight space decomposition with respect to any Cartan subalgebra. It is known [**10**] that if an irreducible representation of A_n admits a weight space decomposition then either all weight spaces are infinite dimensional or all are finite dimensional. There are examples [**7**] of irreducible representations of A_n having weight space decompositions with all weight spaces being infinite dimensional. More recently, Benkart [**2**] has constructed an irreducible representation of A_2 with all weight spaces being 2-dimensional by choosing an irreducible submodule of the tensor product of a fundamental representation of A_2 and a pointed representation coming from an $\{\alpha, \beta\}$-complete mass function.

Combining the results of this paper with the general remarks in [**14**] we can construct large families of mass functions for the other finite dimensional simple Lie algebras over F and hence construct pointed representations for these simple Lie algebras. The question of whether these exhaust all pointed representations is an open question.

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