IRREDUCIBLE REPRESENTATIONS OF $A_n$ WITH A 1-DIMENSIONAL WEIGHT SPACE

BY

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Abstract. In this paper we classify all irreducible linear representations of the simple Lie algebra $A_n$ which admit a one-dimensional weight space with respect to some Cartan subalgebra $H$ of $A_n$. We first show that the problem is equivalent to determining all algebra homomorphisms from the centralizer of the Cartan subalgebra $H$ in the universal enveloping algebra of $A_n$ to the base field. We construct all such algebra homomorphisms and provide conditions under which two such algebra homomorphisms provide inequivalent irreducible representations of $A_n$.

1. Introduction. In this paper, we classify all irreducible representations $(\rho, V)$ of $A_n = sl(n + 1, F)$ having at least one 1-dimensional weight space. This classification includes all irreducible representations having a "highest" weight and hence all finite dimensional irreducible representations. A precise statement of our main result is given at the end of this section after we have developed the necessary terminology and background.

We use Humphreys [8] as our basic reference for notation, terminology and preliminary results. Throughout this paper, $F$ denotes an algebraically closed field of characteristic 0, $L$ a simple Lie algebra of rank $n$ over $F$, $H$ a fixed Cartan subalgebra, $U(L)$ the universal enveloping algebra of $L$, $C(L)$ the centralizer of $H$ in $U(L)$ and $(\rho, V)$ an irreducible representation of $L$ having a weight space decomposition with at least one 1-dimensional weight space. If $f$ is a function with domain $D$ and $A \subseteq D$ then we denote by $f|_A$ the restriction of the function $f$ to the set $A$.

A weight space $V^\lambda$ of $(\rho, V)$ is a nonzero subspace of $V$ corresponding to some element $\lambda$ in the dual space $H^*$ of $H$ and is defined by

$$V^\lambda = \{ v \in V | (\rho(h) - \lambda(h))^n v = 0 \text{ for some positive integer } n \text{ and for all } h \in H \}.$$

By our assumptions of the existence of a 1-dimensional weight space and of the irreducibility of $(\rho, V)$ we have that for each $\lambda \in H^*$, $V^\lambda = V_\lambda = \{ v \in V | \rho(h)v = \lambda(h)v \text{ for all } h \in H \}$. Clearly both $V^\lambda$ and $V_\lambda$ are $\rho(H)$-invariant subspaces of $V$ and

$$V = \sum_{\lambda \in H^*} \oplus V^\lambda = \sum_{\lambda \in H^*} \oplus V_\lambda.$$
The adjoint representation $(\text{ad}, L)$ of $L$ induces a natural weight space decomposition of $L$ called the root space decomposition. We write this as

$$L = L_0 \oplus \bigoplus_{\gamma \in \Phi} \oplus L_{\gamma},$$

where $\Phi$ denotes the set of nonzero roots. Then $L_0 = H$ and, for all $\gamma \in \Phi$, $\dim L_{\gamma} = 1$.

Let $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ denote a base of $\Phi$ and $\Phi^+ = \{\beta_1, \ldots, \beta_m\}$ the corresponding positive roots. For each $\beta_i \in \Phi^+$, we select elements $X_{\beta_i}$ and $X_{-\beta_i}$ belonging to $L_{\beta_i}$ and $L_{-\beta_i}$ respectively and for each $\alpha_i \in \Delta$ we select elements $h_{\alpha_i} \in H$ such that

$$\{X_{\beta_i}, X_{-\beta_i}, h_{\alpha_i} | \beta_i \in \Phi^+, \alpha_i \in \Delta\}$$

forms a Chevalley basis of $L$. If we consider $U = U(L)$ as an $L$-module under the adjoint operation, $U$ has a weight space decomposition $U = \sum_{\mu \in H^*} \otimes U_{\mu}$ with $U_{\mu} \subseteq U_{\mu^*}$. The zero weight space $U_0$ is a subalgebra of $U$ equal to $C(L)$. A Poincaré-Birkhoff-Witt basis for $U$ is given by the set of all monomials

$$(1.1) \quad u(\bar{p}, \bar{l}, \bar{q}) = X_{p_{\beta_m}} \cdots X_{p_{\beta_1}} h_{l_1} \cdots h_{l_r} X_{q_1} \cdots X_{q_m},$$

where $\bar{p} = (p_i)$ and $\bar{q} = (q_i)$ are $m$-tuples of nonnegative integers and $\bar{l} = (l_i)$ is an $n$-tuple of nonnegative integers. Using this basis we have that

$$(1.2) \quad C(L) = U_0 = \text{lin. span}\left\{u(\bar{p}, \bar{l}, \bar{q}) | \sum_{i=1}^m (q_i - p_i)\beta_i = 0\right\}.$$

The following theorem is the key to our classification of all pointed representations of $A_n$. Although a proof of this theorem appears in the literature [9], we include it for the convenience of the reader and to free it from Zorn's Lemma.

**Theorem 1.3.** (i) If $V_\lambda$ is any weight space of the irreducible representation $(\rho, V)$ of $L$ then $V_\lambda$ is an irreducible $C(L)$-module and if $\dim V_\lambda = 1$ and $0 \neq v_\lambda \in V_\lambda$ then the map $\xi: C(L) \to F$ defined by $\xi(c)v_\lambda = \rho(c)v_\lambda$ is a nontrivial algebra homomorphism.

(ii) If $\xi: C(L) \to F$ is a nontrivial algebra homomorphism, then there exists a unique maximal left ideal $M$ of $U$ containing $\ker\xi$. Moreover, the irreducible left regular $L$-module $U/M$ has $(U_0 + M)/M$ as a 1-dimensional weight space with weight $\lambda = \xi^{-1}H$.

**Proof.** (i) Let $V_\lambda$ be a weight space of $(\rho, V)$ and $0 \neq v_\lambda \in V_\lambda$. Consider $(\rho, V)$ as a representation of $U$. Since $\rho(U_\lambda)v_\lambda = V$ and $\rho(U_\lambda)v_\lambda \subseteq V_{\lambda^+\mu}$, the irreducibility of $(\rho, V)$ implies that $\rho(C(L))v_\lambda = \rho(U_0)v_\lambda = V_\lambda$. Thus $V_\lambda$ is an irreducible representation of $C(L)$. Suppose now that $\dim V_\lambda = 1$ and define $\xi: C(L) \to F$ by setting $\xi(c)v_\lambda = \rho(c)v_\lambda$. Since $\rho(C(L))v_\lambda = V_\lambda$ and $\rho$ is an associative representation of $U$, we have that $\xi$ is a nontrivial algebra homomorphism.

(ii) Let $\xi: C(L) \to F$ be a nontrivial algebra homomorphism and set $\ker\xi = A$. Let $I$ be any proper left ideal of $U$ containing $A$. There exists at least one such proper left ideal since $UA \cap C(L) = A \neq C(L)$ and hence $UA \neq U$. Let $x$ be any nonzero element of $I$ and write $x = \sum x_{\mu}^a$ where $x_{\mu} \in U_{\mu}$. We show that each $x_{\mu} \in I$ by induction on $k_x = |\{\mu | x_{\mu} \neq 0\}|$. The result is obviously true when $k_x = 1$ and
hence we assume that \( k_x = n > 1 \) and that the result is true for all \( y \in I \) with \( k_y < k_x \). Thus for each \( h \in H \) we have

\[
(1.4) \quad hx = \sum_{\mu} h \mu = \sum_{\mu} \left( [h, \mu] + \mu h \right)
= \sum_{\mu} \left( \mu(h) \mu + \mu \xi(h) \right) \mod UA
= \sum_{\mu} \left( \mu(h) + \xi(h) \right) \mu \mod UA.
\]

For each \( \nu \in H^{*} \) with \( \nu \neq 0 \), define \( y_{\nu,h} = \sum_{\mu \neq \nu} (\nu(h) - \mu(h)) \mu \). Then, by (1.4),

\[
y_{\nu,h} = (\nu(h) + \xi(h))x - hx \mod UA \quad \text{and hence } y_{\nu,h} \in I.
\]

Since \( k_{\nu,h} < k_x \) we have that \( (\nu(h) - \mu(h)) \mu \in I \) for all \( h \in H \) and all \( \mu \neq \nu \). If \( \mu \neq \nu \) then there exist \( h \in H \) with \( \nu(h) - \mu(h) \neq 0 \) and hence \( \mu \in I \) for all \( \mu \neq \nu \) which in turn implies that \( \mu \in I \) for all \( \mu \).

From (1.4), we see that \( \mu + UA \) is a weight vector in \( \mathbf{U}/UA \) with weight \( \mu + \xi \downarrow H \) and hence \( \mathbf{I}/UA \) has a weight space decomposition given by

\[
(1.5) \quad \mathbf{I}/UA = \sum_{\mu} \left( (I \cap U_{\mu}) + UA \right)/UA.
\]

Since \( I \) is a proper ideal, \( I \cap C(L) = A \) and hence every proper left ideal \( I \)
containing \( A \) has the property that

\[
(1.6) \quad I / (UA) \subseteq \sum_{\mu \neq 0} \left( U_{\mu} + UA \right)/ (UA).
\]

It follows that the sum of all proper left ideals containing \( A \) is the unique maximal
left ideal \( M \) of \( \mathbf{U} \) containing \( A \).

Equation (1.4) implies that \( (C(L) + M)/M \) is a weight space of \( \mathbf{U}/M \) with weight \( \mu + \xi \downarrow H \). Since \( (C(L) + M)/M \cong C(L)/(C(L) \cap M) = C(L)/A \), this
weight space is 1-dimensional.

From Theorem 1.3, it follows that in order to determine all irreducible representa-
tions \((\rho, V)\) of \( L \) admitting a 1-dimensional weight space, it suffices to find all
algebra homomorphisms \( \xi : C(L) \rightarrow F \). An algebra homomorphism \( \xi : C(L) \rightarrow F \) is
called a mass function, and an irreducible representation admitting a 1-dimensional
weight space is called pointed.

In this paper, we construct all mass functions for the algebras \( A_n \) and hence
determine all pointed representations of \( A_n \).

In order to illustrate how our results extend the standard results on classifying
irreducible representations of highest weight we briefly review these results. If \((\rho, V)\)
denotes an irreducible representation with highest weight \( \lambda \) then the weight space \( V_{\lambda} \)
is 1-dimensional and there exists a base \( \Delta \) of roots such that \( V_{\lambda + \alpha} = \{0\} \) for all \( \alpha \in \Delta \). Let \( \xi : C(L) \rightarrow F \) be the mass function associated with the 1-dimensional
weight space \( V_{\lambda} \). Let \( \Psi : C(L) \rightarrow F[\lambda_1, \ldots, \lambda_n] \) be the Harish-Chandra homomor-
phism relative to the base \( \Delta \).

\[
\Psi(u(\bar{\rho}, \bar{l}, \bar{q})) = \begin{cases} 0 & \text{if } u(\bar{\rho}, \bar{l}, \bar{q}) \in C(L) \text{ with } \bar{q} \neq \bar{0}, \\ \lambda_1 \cdots \lambda_n & \text{if } u(\bar{\rho}, \bar{l}, \bar{q}) \in C(L) \text{ with } \bar{\rho} = \bar{q} = \bar{0}, \end{cases}
\]
where the $\lambda_i$'s are indeterminants. Now if $e: F[\lambda_1, \ldots, \lambda_n] \to F$ is defined by $e(\lambda_i) = \zeta(h_{\alpha_i})$ for each $\alpha_i \in \Delta$ we have $\zeta = e \circ \Psi$. Thus for any base $\Delta$ we can construct all mass functions associated with highest weight spaces with respect to $\Delta$ by composing the Harish-Chandra homomorphism $\Psi$ relative to $\Delta$ with any evaluation map $e$.

Our aim is to generalize the preceding results for the case of $L = A_n$. For each pair $(\Delta', \Delta)$ consisting of a base $\Delta$ of the roots of $A_n$ and a subset $\Delta'$ of $\Delta$ we define a map $\Psi_{\Delta', \Delta}$ from $C(L)$ into a ring of polynomials over $F$ in several commuting indeterminants. This map generalizes the Harish-Chandra homomorphism in the sense that the definition reduces to the standard Harish-Chandra homomorphism relative to $\Delta$ where $\Delta' = \emptyset$. Our main result is

**Theorem 1.7.** Let $\zeta: C(L) \to F$ be a mass function. Then there exists a base $\Delta$, a subset $\Delta'$ of $\Delta$ and an evaluation map $e$ such that $\zeta = e \circ \Psi_{\Delta', \Delta}$.

This paper is organized as follows. §2 contains the definition of the generalized Harish-Chandra homomorphisms referred to in Theorem 1.7 and the definition of a group of automorphisms on $A_n$ which we use to change from one base of the root system of $A_n$ to another and to compute the values of a given mass function relative to different bases. In §3 we prove Theorem 1.7 for the case $A_1$ and $A_2$. The proofs of these results are outlined in [4, 15] but we present them here in full detail since our approach is different and provides the key to the general result. §4 contains the bulk of the technical results concerning mass functions on $A_3$, $A_4$ and $A_n$ which we use to establish Theorem 1.7 for the general case. Many of the results in this section depend heavily on the use of identities in $C(A_n)$. It should be noted that for the sake of simplicity and convenience these identities have not been expressed in any fixed linear basis of $C(A_n)$. In §5, we complete the proof of Theorem 1.7. Finally, §6 addresses the problem of selecting a set of mass functions on $C(A_n)$ which label the equivalence classes of pointed representations of $A_n$ in a one-one fashion.

2. Generalized Harish-Chandra homomorphisms. In the remainder of this paper, we deal with $L = A_n = sl(n + 1, F)$. The subalgebra of $A_n$ consisting of all diagonal matrices in $A_n$ is a Cartan subalgebra which we denote by $H$. The root system $\Phi$ of $A_n$ relative to $H$ is then given by

$$\Phi = \{w_i - w_j \mid i, j = 1, 2, \ldots, n, i \neq j\},$$

where $w_k$ is the map which projects an order $n + 1$ matrix onto its $(k, k)$th component. A base $\Delta$ of $\Phi$ is given by

$$\Delta = \{w_i - w_{i+1} \mid i = 1, 2, \ldots, n\}$$

and the set of positive roots $\Phi^+$ of $\Phi$ relative to $\Delta$ is

$$\Phi^+ = \{w_i - w_j \mid 1 \leq i < j \leq n + 1\}.$$

If $\{e_{ij}\}$ denotes the standard set of matrix units, then a Chevalley basis of $A_n$ is given by $\{h_i, X_{w_k-w_l} \mid i = 1, 2, \ldots, n; k, l = 1, 2, \ldots, n + 1 \text{ with } k \neq l\}$ where $h_i = e_{ii} - e_{i+1,i+1}$ and $X_{w_k-w_l} = e_{kl}$.
It should be noted that even if we select a different base of \( \Phi \) we always choose a Chevalley basis of \( A_n \) corresponding to this basis which has the same structure constants as those arising from the basis \( \{ e_{ii} - e_{i+1,i+1}, e_{i,i} \} \). This fact is used in the computation of identities in \( C(A_n) \) which we need later in this paper.

Let \( \Delta = \{ \alpha_1, \ldots, \alpha_n \} \) be some fixed base of \( \Phi \) and let \( \Phi^+ \) be the corresponding set of positive roots relative to \( \Delta \). We then select a Chevalley basis \( \{ h_i = h_{\alpha_i}, X_{\beta_j}, X_{-\beta_j} \mid \alpha_i \in \Delta, \beta_j \in \Phi^+ \} \) as specified above. Assume that \( \Delta' \) is either \( \varnothing \) or \( \Delta' = \bigcup_{i=1}^l \Delta_i \subseteq \Delta \) where the \( \Delta_i \)'s generate mutually orthogonal irreducible root subsystems \( \Phi_i \) of \( \Phi \). The subalgebra of \( A_n \) generated by \( \{ X_\mu \mid \mu \in \Phi_\gamma \} \) is isomorphic to \( A_{n_i} \) where \( n_i = |\Delta_i| \). We identify \( A_{n_i} \) with this subalgebra. This identification extends naturally to the universal enveloping algebra \( U(A_{n_i}) \) and its cycle subalgebra \( C(A_{n_i}) \). The subalgebras of \( U(A_n) \) identified in this manner with \( U(A_{n_i}) \) and \( C(A_{n_i}) \) are denoted by \( U(\Delta_i) \) and \( C(\Delta_i) \) respectively.

Let \( \mathcal{K} \) denote the subalgebra of \( C(A_n) \) with 1 generated by \( \{ h_i \mid \alpha_i \in \Delta, \alpha_i \notin \Delta' \} \) and let \( C' \) denote the linear subspace of \( C(A_n) \) generated by all elements \( u(\bar{\rho}, \bar{l}, \bar{q}) \in C(A_n) \) of the form (1.1) such that \( \bar{q} = (q_i) \) and \( q_j \neq 0 \) for some \( \beta_j \notin \bigcup_{i=1}^l \Phi_i \). It is shown in [12] that \( C' \) is an ideal of \( C(A_n) \). It then follows from the proof of the Poincaré-Birkhoff-Witt Theorem that \( C(A_n) \) can be written as a vector space direct sum \( C(A_n) = \mathcal{K} C(\Delta_1) C(\Delta_2) \cdots C(\Delta_l) \oplus C' \). In case \( \Delta' = \varnothing \), we have \( C(A_n) = \mathcal{K} \oplus C' \) and \( \mathcal{K} \) is the universal enveloping algebra of \( H \).

**Lemma 2.1.** Let \( R \) be any commutative associative \( F \)-algebra with 1. If \( \Psi_0: \mathcal{K} \to R \) and \( \Psi_i: C(\Delta_i) \to R \) for \( i = 1, 2, \ldots, l \) are \( F \)-algebra homomorphisms with \( \Psi_i(1) = 1 \) for \( i = 0, 1, \ldots, l \), then there exists a unique \( F \)-algebra homomorphism \( \Psi: C(A_n) \to R \) such that \( \Psi \circ \Psi_0 = \Psi \) and \( \Psi \circ C(\Delta_i) = \Psi_i \) for \( i = 0, 1, \ldots, l \). Moreover, \( \Psi \) is also a homomorphism of \( \mathcal{K} \).

**Proof.** When \( \Delta' = \varnothing \) the result follows trivially from the decomposition. Since the \( \Delta_i \)'s are mutually orthogonal we have that \( [U(\Delta_i), U(\Delta_j)] = 0 \) for all \( i, j = 1, 2, \ldots, l \) with \( i \neq j \). We also have that \( [\mathcal{K}, C(\Delta_i)] = 0 \) for \( i = 1, 2, \ldots, l \) and hence every monomial \( u(\bar{\rho}, \bar{l}, \bar{q}) \in C(A_n) \) of the form (1.1) is either in \( C' \) or can be expressed uniquely as a scalar times a product of monomials \( h c_1 \cdots c_i \) where \( h \in \mathcal{K} \) and \( c_i \in C(\Delta_i) \). Thus if there exists an \( F \)-algebra homomorphism \( \Psi \) satisfying the condition above it must be unique. On the other hand, if we define \( \Psi \) on each basis element \( u(\bar{\rho}, \bar{l}, \bar{q}) \in C(A_n) \) by setting \( \Psi(u(\bar{\rho}, \bar{l}, \bar{q})) = 0 \) if \( u(\bar{\rho}, \bar{l}, \bar{q}) \in C' \) and \( \Psi(u(\bar{\rho}, \bar{l}, \bar{q})) = k \Psi_0(h) \Psi_i(c_1) \cdots \Psi_l(c_i) \) where \( u(\bar{\rho}, \bar{l}, \bar{q}) = khc_1 \cdots c_i \) with \( h \in \mathcal{K}, c_i \in C(\Delta_i) \) then it readily follows that \( \Psi \) is an \( F \)-algebra homomorphism satisfying the required conditions. \( \square \)

Let \( \lambda_1, \ldots, \lambda_n, s_1, \ldots, s_n \) be commuting indeterminates over \( F \) and \( F[\lambda_1, \ldots, \lambda_n, s_1, \ldots, s_n] \) be the corresponding polynomial ring. We use Lemma 2.1 to define the generalized Harish-Chandra homomorphism associated with the pair \( \Delta' \subseteq \Delta \) which we write

\[ \Psi_{\Delta', \Delta}: C(A_n) \to F[\lambda_1, \ldots, \lambda_n, s_1, \ldots, s_n]. \]

Our first step is to note that using the Dynkin diagram associated with \( \Delta \), we can define an equivalence relation on \( \Delta' \) by \( \alpha \sim \beta \) iff the subgraph of the Dynkin
diagram connecting $\alpha$ to $\beta$ has all of its vertices in $\Delta'$. We denote the equivalence classes of this relation by $\Delta_i$, $i = 1, \ldots, l$. The root subsystems $\Phi_i$ generated by $\Delta_i$, respectively are irreducible and mutually orthogonal. According to Lemma 2.1, we can define the map $\Psi_{\Delta', \Delta}$ with $C' \subseteq \ker \Psi_{\Delta', \Delta}$ by specifying its restrictions to $\mathcal{R}$, $C(\Delta_1), \ldots, \text{and } C(\Delta_l)$.

Let $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ be indexed so that $\Delta_i = \{\alpha_{k_i+1}, \ldots, \alpha_{k_i+n_i}\}$. Then $\mathcal{R}$ is merely the polynomial ring in the commuting indeterminates $h_1 = h_{k_1}$ for $\alpha_i \notin \Delta'$. Therefore, we can define the algebra homomorphism $\Psi_0: \mathcal{R} \to F[\lambda_1, \ldots, \lambda_n, s_1, \ldots, s_n]$ by extending the map sending $h_i$ to $\lambda_i$. The restrictions to $C(\Delta_i)$ are more complicated. We generalize the construction of [14] and define $\Psi: C(\Delta_i) \to F[\lambda_1, \ldots, \lambda_n, s_1, \ldots, s_n]$ as follows. Let $V[i]$ denote the free $F[\lambda_{k_i+1}, \ldots, \lambda_{k_i+n_i}, s_i]$-module with basis $\{v(\tilde{i}) | \tilde{i} = (l_1, \ldots, l_{n_i}) \in \mathbb{Z}^{n_i}\}$. We now define an $F$-algebra embedding $\rho_i$ of the subalgebra $A_{n_i}$ into the minus algebra of $\text{End}_{F[\lambda_{k_i+1}, \ldots, \lambda_{k_i+n_i}, s_i]} V[i]$ by describing the action of $\rho_i$ on the basis of $A_{n_i}$. For $j = 1, 2, \ldots, n_i$, define $\rho_i(h_{k_i+j})$ by

\begin{equation}
\rho_i(h_{k_i+j})v(\tilde{i}) = (\lambda_{k_i+j} - l_{j-1} + 2l_j - l_{j+1})v(\tilde{i})
\end{equation}

and for $\xi = w_p - w_q \in \Phi_i^+ = \Phi_i \cap \Phi^+$ define $\rho_i(X_\xi)$ and $\rho_i(X_{-\xi})$ by

\begin{equation}
\rho_i(X_{\xi})v(\tilde{i}) = (s_i - \lambda_{k_i+1} - \cdots - \lambda_{k_i+p-1} - l_{p-1} + l_p)v(\tilde{i} + \xi),
\end{equation}

\begin{equation}
\rho_i(X_{-\xi})v(\tilde{i}) = (s_i - \lambda_{k_i+1} - \cdots - \lambda_{k_i+q-1} - l_{q-1} + l_q)v(\tilde{i} - \xi),
\end{equation}

where $\xi$ is the $n_i$-tuple with 1 in the $p$, $p+1$, $\ldots$, $q-1$ components and zero elsewhere. By convention we set $\lambda_{k_i}$, $l_0$ and $l_{n_i+1}$ to be zero whenever they occur in these expressions.

**Theorem 2.4.** The map $\rho_i$ is an injective $F$-algebra homomorphism.

**Proof.** Clearly the map $\rho_i$ defined above is injective and $F$-linear. In order to see that it is also an algebra homomorphism it suffices to verify that $\rho_i$ preserves the following multiplication relations in $A_{n_i}$:

\begin{align}
[h_p, h_q] &= 0 \quad \text{for } k_i + 1 \leq p < q \leq k_i + n_i, \\
[h_p, X_\xi] &= \xi(h_p)X_\xi \quad \text{for } p = k_i + 1, \ldots, k_i + n_i; \xi \in \Phi_i, \\
[X_\xi, X_\eta] &= \left\{ \begin{array}{ll}
(h_p + \cdots + h_{q-1})X_\xi & \text{for } -\eta = \xi = w_p - w_q \in \Phi_i^+, \\
-(h_p + \cdots + h_{q-1}) & \text{for } \eta = -\xi = w_p - w_q \in \Phi_i^+, \\
(\delta_{qk} - \delta_{jp})X_{\xi+\eta} & \text{for } \xi = w_p - w_q \neq -\eta = -(w_k - w_j).
\end{array} \right.
\end{align}

A direct computation completes the proof of this result. \(\square\)

Let $\hat{\rho}_i$ be the extension of $\rho_i$ to $U(A_{n_i})$. For each $c \in C(\Delta_i) \subseteq C(A_{n_i})$, define $\Psi_i(c)$ by

\begin{equation}
\Psi_i(c)v(\tilde{0}) = \hat{\rho}_i(c)v(\tilde{0}).
\end{equation}

We are using here the fact that the operator $\hat{\rho}_i(c)$ maps $v(\tilde{0})$ to a $F[\lambda_{k_i+1}, \ldots, \lambda_{k_i+n_i}, s_i]$ multiple of $v(\tilde{0})$, that is

$\Psi_i(c) \in F[\lambda_{k_i+1}, \ldots, \lambda_{k_i+n_i}, s_i] \subseteq F[\lambda_1, \ldots, \lambda_n, s_1, \ldots, s_n]$. 

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Now, let $\Psi_0, \ldots, \Psi_l$ be the algebra homomorphisms just defined. Then the \textit{generalized Harish-Chandra homomorphism} $\Psi_{\Delta, \Delta'}: C(A_n) \to F[\lambda_1, \ldots, \lambda_n, s_1, \ldots, s_n]$ is that unique algebra homomorphism with $C' \subseteq \ker \Psi_{\Delta, \Delta'}$ which extends all of $\Psi_0, \ldots, \Psi_l$.

In the case $\Delta' = \emptyset$, $\Psi_{\Delta, \Delta'}$ differs from the usual Harish-Chandra homomorphism only in that normally the codomain is taken to be $F[\lambda_1, \ldots, \lambda_n]$ while here it is $F[\lambda_1, \ldots, \lambda_n, s_1, \ldots, s_n]$.

The following table is a complete list of all $\Psi_{\Delta, \Delta'}: C(A_2) \to F[\lambda_1, \lambda_2, s_1, s_2]$ where $\Delta = \{ \alpha, \beta \}$. The fact that $\Psi_{\Delta, \Delta'}$ is determined by the values listed in this table is given in §3. We are writing $X_{-\alpha}$ as $Y_{\alpha}$.

<table>
<thead>
<tr>
<th>$\Delta, \Delta'$</th>
<th>$\Psi_{\Delta, \Delta}$</th>
<th>$\Psi_{(\alpha), \Delta}$</th>
<th>$\Psi_{(\beta), \Delta}$</th>
<th>$\Psi_{(\sigma), \Delta}$</th>
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<td>0</td>
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<tr>
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<td>0</td>
<td>$s_1(s_1 - \lambda_2 - 1)$</td>
<td>0</td>
</tr>
<tr>
<td>$Y_\alpha + \beta X_\alpha + \beta$</td>
<td>$s_1(s_1 - \lambda_1 - \lambda_2 - 1)$</td>
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<td>0</td>
</tr>
<tr>
<td>$Y_\alpha + \beta X_\alpha + \beta$</td>
<td>$s_1(s_1 - \lambda_1 - \lambda_2 - 1)$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

If $\Delta' = \emptyset$, then the image of $\Psi_{\Delta, \Delta'} = F[\lambda_1, \ldots, \lambda_n] \subseteq F[\lambda_1, \ldots, \lambda_n, s_1, \ldots, s_n]$ and in this case the evaluation $\epsilon$ which we use in Theorem 1.7 is defined by convention to be 0 on $s_1, \ldots, s_n$. If $\Delta'$ is the union of nonempty $\Delta_1, \ldots, \Delta_i$ as described above, then the image of $\Psi_{\Delta, \Delta'} \subseteq F[\lambda_1, \ldots, \lambda_n, s_1, \ldots, s_i] \subseteq F[\lambda_1, \ldots, \lambda_n, s_1, \ldots, s_n]$ and in this case $\epsilon$ is defined by convention to be 0 on $s_{i+1}, \ldots, s_n$.

We close this section with the definition of a certain class of automorphisms which are associated in a natural way with the Weyl reflections and the Dynkin diagram reversing map.

Let $\alpha \in \Delta$ and $\sigma_\alpha$ be the usual Weyl reflection of the root system $\Phi$ through the hyperplane perpendicular to $\alpha$. If we define, for $\beta \in \Delta$, $\sigma_{\alpha}(h_\beta) = h_{\sigma_{\alpha}(\beta)}$ and $\sigma_{\alpha}(X_\pm \beta) = X_{\pm \sigma_{\alpha}(\beta)}$, then by [8, Theorem 14.2], $\sigma_\alpha$ extends uniquely to an automorphism of $A_n$, which in turn extends uniquely to an automorphism of $U(A_n)$. Each of these automorphisms is denoted by $\sigma_\alpha$. Since $\sigma_\alpha(H) = H$ and $\sigma_\alpha(X_\xi) = X_{\sigma_\alpha(\xi)}$ for all $\xi \in \Phi$, we have that $\sigma_\alpha$ acts as an automorphism on $U_0(A_n) = C(A_n)$. The set of automorphisms $\{\sigma_\alpha| \alpha \in \Delta\}$ on $U(A_n)$ generate a group which is isomorphic to the Weyl group of $A_n$. The elements in this group of automorphisms are called the \textit{Weyl automorphisms}.

If we let $\sigma_\xi$ denote the isomorphism on $\Phi$ corresponding to reversing the Dynkin diagram of $A_n$ so that $\sigma_\xi(\alpha_i) = \alpha_{n+1-i}$ for $1 \leq i \leq n$, then the map on $A_n$ determined by $\sigma_\xi(h_\beta) = h_{\sigma_\xi(\beta)}$ and $\sigma_\xi(X_\pm \beta) = X_{\pm \sigma_\xi(\beta)}$ for $\beta \in \Delta$ is an automorphism on $A_n$ called the \textit{flip map}.

Our use of these automorphisms $\sigma$ is primarily to keep track of how a mass function $\xi \circ \sigma$ with the property that $\xi \circ \sigma = \epsilon \circ \Psi_{\Delta, \Delta'}$ can be used to give us Theorem 1.7. This is done by observing that in this case $\xi = \epsilon \circ \Psi_{\sigma(\Delta), \sigma(\Delta')}$. 

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3. Mass functions on $C(A_1)$ and $C(A_2)$. In this section we prove Theorem 1.7 for the case $n = 1$ and 2. Lemire and Pap [15] completed the work begun by Bouwer [4] and exhibited a complete list of all mass functions on $C(A_2)$. This list is of central importance in our proof of Theorem 1.7 for arbitrary $n$. Because of this, we believe that it is worthwhile to devote the major portion of this section to reworking this material. Our presentation differs from [15 and 4] in that it is complete and self contained, it introduces the set of roots $S$ which is the vehicle allowing us to generalize to arbitrary $n$, and its cases are simplified by making an appropriate choice of base $\Delta$.

We begin by studying mass functions on $C(A_1)$. Let $\Delta = \{\alpha\}$ be a base of the root system of $A_1$, let $h_1 = h_\alpha$ and $Y_\alpha = X_{-\alpha}$. Then $h_1$, $Y_\alpha X_\alpha$ are algebraically independent over $F$ and $C(A_1)$ is merely the polynomial ring $F[h_1, Y_\alpha X_\alpha]$. There are only two generalized Harish-Chandra homomorphisms associated with $A = \{\alpha\}$, namely $\Psi_{\phi, \Delta}$ and $\Psi_{\Delta, \Delta}$. The first of these differs from the usual Harish-Chandra homomorphism only in that the codomain is normally taken to be $F[\lambda_1]$ while $\Psi_{\phi, \Delta}$ takes on values in $F[\lambda_1, s_1]$. The second map $\Psi_{\Delta, \Delta}$ is defined by

$$\Psi_{\Delta, \Delta}(h_1) = \lambda_1, \quad \Psi_{\Delta, \Delta}(Y_\alpha X_\alpha) = s_1(s_1 - \lambda_1 - 1).$$

Since every mass function $\xi: C(A_1) \to F$ is uniquely determined by its values on $h_1$ and $Y_\alpha X_\alpha$, we see that there is an evaluation map $\epsilon: F[\lambda_1, s_1] \to F$ such that $\xi = \epsilon \circ \Psi_{\Delta, \Delta}$. This proves Theorem 1.7 for $n = 1$.

We now start our investigation of mass functions on $C(A_2)$. Let $\Delta = \{\alpha, \beta\}$ be a base of the root system $\Phi$ of $A_2$ and let

$$\begin{align*}
h_1 &= h_\alpha, \quad Y_\beta = X_{-\beta}, \\
h_2 &= h_\beta, \quad Y_\alpha = X_{-\alpha}, \\
X_\alpha &= X_{\alpha + \beta}, \\
Y_\beta &= X_{(\alpha + \beta)}
\end{align*}$$

be a Chevalley basis of $A_2$ relative to $\Delta$ with structure constants as indicated in §2. The elements $h_1$, $h_2$ and

$$\begin{align*}
c_1 &= Y_\alpha X_\alpha, & c_2 &= Y_\beta X_\beta, & c_3 &= Y_{\alpha + \beta} X_{\alpha + \beta}, \\
c_4 &= Y_{\alpha + \beta} X_\alpha X_\beta, & c_5 &= Y_\beta Y_\alpha X_{\alpha + \beta}
\end{align*}$$

are called the basic cycles of $C(A_2)$ relative to $\{\alpha, \beta\}$. When it is necessary to emphasize the base $\Delta = \{\alpha, \beta\}$ we speak of the basic cycles of $C(\alpha, \beta)$. Also, when we write $C(\alpha, \beta)$ for $C(A_2)$, we are using the basic cycles of $C(\alpha, \beta)$ unless otherwise specified. By direct computation, one can establish the following multiplication relations in $C(\alpha, \beta)$.

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IRREDUCIBLE REPRESENTATIONS OF $A_n$

$h_1h_2 = h_2h_1,$
$h_1c_j = c_jh_1$ for $i = 1, 2; 1 \leq j \leq 5,$
$c_1c_2 = c_2c_1 + c_5 - c_4,$
$c_1c_3 = c_3c_1 - c_5 + c_4,$
$c_1c_4 = c_4c_1 - c_2c_1 + c_3c_1 - c_4h_1 + c_3h_1 - c_5 + c_4,$
$c_1c_5 = c_5c_1 + c_2c_1 - c_3c_1 + c_5h_1 - c_3h_1 + c_5 - c_4,$
$c_2c_3 = c_3c_2 + c_5 - c_4,$

(3.3) $c_2c_4 = c_4c_2 + c_2c_1 - c_3c_2 - c_4h_2 + c_5 - c_4,$
$c_2c_5 = c_5c_2 - c_2c_1 + c_3c_2 + c_5h_2 + c_5 - c_4,$
$c_3c_4 = c_4c_3 + c_3c_2 - c_3c_1 + c_4h_1 + c_4h_2 - c_3h_1,$
$c_3c_5 = c_5c_3 + c_3c_2 - c_5h_1 + c_5h_2 + c_5 - c_4,$
$c_4c_5 = c_5c_4 + c_4c_1 + c_5c_1h_1 + c_5c_1h_2 + c_5c_1h_1 + 2c_5c_1 + 2c_5h_1 - 2c_5c_2$

$+ c_4c_2 - c_4c_1 - c_4h_1 - c_5h_1 - c_5h_2 - 2c_5 + 2c_4 + c_5c_3,$
$c_5c_4 = c_4c_2 + c_2c_1 + c_2c_1h_1 + c_2c_1h_2 + c_5c_3 - c_4h_2 + c_4c_2 + 2c_5c_1$

Lemma 3.5. The set of elements of $C(\alpha, \beta)$ of the form

(3.6) \[(c_5 \text{ or } c_4)^{e_1}c_5^{e_2}c_2^{e_3}h_2^{e_4}h_1^{e_5}h_1^{e_6}\]

where the exponents $e_i$ are nonnegative integers, constitutes a basis for $C(\alpha, \beta)$.

Proof. In §1 we noted that a basis of $C(\alpha, \beta)$ is given by the set of all elements of the form (1.2)—i.e.

$u(p, \bar{q}, q) = Y_{a+\beta}^pY_{\alpha}^pY_{\beta}^pY_{a\beta}^pY_{a\beta}^pY_{a\beta}^pY_{a\beta}^pY_{a\beta}^pY_{a\beta}^pY_{a\beta}^pY_{a\beta}^pX_{a\beta}^qX_{a\beta}^qX_{a\beta}^q$

where $q_1 + q_3 - p_1 - p_3 = 0$ and $q_2 + q_3 - p_2 - p_3 = 0$. If $q_3 \geq p_3$ this element can be written as

$u(p, \bar{q}, q) = c_5^{q_3-p_3}c_5^{q_2}c_2^{q_1}h_1^{q_1}h_2^{q_2} + \text{terms of lower degree.}$

If $p_3 \geq q_3$ this element can be written as

$u(p, \bar{q}, q) = c_4^{q_3-q_1}c_4^{q_2}c_2^{q_1}c_1^{q_1}h_1^{q_1}h_2^{q_2} + \text{terms of lower degree.}$

Hence, using induction on the degree, we have that the elements of the form (3.6) generate $C(\alpha, \beta)$ as a linear space.

It is clear that the element $c_5^{e_1}c_5^{e_2}c_5^{e_3}c_5^{e_4}h_2^{e_5}h_1^{e_6}$ can be written as

$Y_{a+\beta}^{e_1}Y_{a\beta}^{e_2}Y_{a\beta}^{e_3}h_1^{e_1}h_2^{e_2}h_2^{e_3}h_1^{e_4}h_1^{e_5}h_1^{e_6} + \text{terms of lower degree}$

and similarly $c_4^{e_1}c_4^{e_2}c_4^{e_3}c_4^{e_4}h_2^{e_5}h_1^{e_6}$ can be written as

$Y_{a+\beta}^{e_1}Y_{a\beta}^{e_2}Y_{a\beta}^{e_3}h_1^{e_1}h_2^{e_2}h_2^{e_3}h_1^{e_4}h_1^{e_5}h_1^{e_6} + \text{terms of lower degree.$

Since the term of highest degree in the expansion of each element of the form (3.6) in the basis of $C(A_2)$ of the form (1.2) is unique, we have that the elements of the form (3.6) are linearly independent. □
Remark 3.7. Let \( \langle W \rangle \) be the free semigroup generated by \( w_1, \ldots, w_7 \), and \( F\langle W \rangle \) the corresponding free algebra. Let \( E \) be the subset of \( F\langle W \rangle \) obtained from the equations (3.3) by replacing \( h_1 \) by \( w_1 \), \( h_2 \) by \( w_2 \), and each \( c_i \) by \( w_{i+2} \), and let \( I \) be the ideal of \( F\langle W \rangle \) generated by \( E \). Then Lemma 3.5 implies that \( C(\alpha, \beta) \cong F\langle W \rangle / I \), and that the problem of determining all mass functions on \( C(\alpha, \beta) \) is equivalent to finding all algebra homomorphisms \( \tau: F\langle W \rangle \to F \) with \( I \subseteq \ker \tau \). Such a map \( \tau \) is uniquely determined by the 7-tuple \( (\tau(w_1), \ldots, \tau(w_7)) \) in \( F^7 \). We define \( (a, b, z_1, \ldots, z_5) \in F^7 \) to be admissible provided there exists an algebra homomorphism \( \tau: F\langle W \rangle \to F \) with \( I \subseteq \ker \tau \) such that \( (a, b, z_1, \ldots, z_5) = (\tau(w_1), \ldots, \tau(w_7)) \). Equivalently, we could require that there exist a mass function \( \xi \) on \( C(\alpha, \beta) \) such that \( (a, b, z_1, \ldots, z_5) = (\xi(h_1), \xi(h_2), \xi(c_1), \ldots, \xi(c_5)) \).

Let \( \tau: F\langle W \rangle \to F \) be an algebra homomorphism with \( (\tau(w_1), \ldots, \tau(w_7)) = (a, b, z_1, \ldots, z_5) \). Then if we apply \( \tau \) to the elements of \( E \), equate the result to zero and simplify, we obtain the condition that \( (a, b, z_1, \ldots, z_5) \) is admissible if and only if its components satisfy the equations

\[
\begin{align*}
(3.8) \quad & z_5 - z_4 = 0, \\
(3.9) \quad & a(z_4 - z_3) - z_1(z_3 - z_2) = 0, \\
(3.10) \quad & bz_4 - z_2(z_1 - z_3) = 0, \\
(3.11) \quad & (z_4 - z_3)(z_2 - z_1 - a - z_4) + z_3(z_2 + b)(z_1 + a) = 0.
\end{align*}
\]

Let \( a, b, \) and \( p \) be arbitrary elements of \( F \). Then the 7-tuple \( (a, b, z_1, \ldots, z_5) \) is said to be \( T_i \) for \( 1 \leq i \leq 6 \) provided it equals the 7-tuple headed by \( T_i \) in Table 3.12, and it is said to be \( T_0 \) if it equals the 7-tuple headed by \( T_0 \) in Table 3.12 and \( 0 \neq z_1 z_2 (z_1 + a)(z_2 + b) \). A mass function \( \xi \) on \( C(\alpha, \beta) \) is defined to be \( T_i \) provided \( (\xi(h_1), \xi(h_2), \xi(c_1), \ldots, \xi(c_5)) = T_i \).

**Table 3.12**

<table>
<thead>
<tr>
<th></th>
<th>( T_0 )</th>
<th>( T_1 )</th>
<th>( T_2 )</th>
<th>( T_3 )</th>
<th>( T_4 )</th>
<th>( T_5 )</th>
<th>( T_6 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a )</td>
<td>( a )</td>
<td>( a )</td>
<td>( a )</td>
<td>( a )</td>
<td>( a )</td>
<td>( a )</td>
<td></td>
</tr>
<tr>
<td>( b )</td>
<td>( b )</td>
<td>( b )</td>
<td>( b )</td>
<td>( b )</td>
<td>( b )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( z_1 )</td>
<td>( p(p - a - 1) )</td>
<td>( p )</td>
<td>( 0 )</td>
<td>( -a )</td>
<td>( 0 )</td>
<td>( -a )</td>
<td>( p )</td>
</tr>
<tr>
<td>( z_2 )</td>
<td>( (p - a)(p - a - b - 1) )</td>
<td>( 0 )</td>
<td>( -b )</td>
<td>( p )</td>
<td>( p )</td>
<td>( 0 )</td>
<td>( -b )</td>
</tr>
<tr>
<td>( z_3 )</td>
<td>( p(p - a - b - 1) )</td>
<td>( 0 )</td>
<td>( p )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( -a - b )</td>
<td></td>
</tr>
<tr>
<td>( z_4 )</td>
<td>( p(p - a)(p - a - b - 1) )</td>
<td>( 0 )</td>
<td>( p )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( -a - b - p )</td>
<td></td>
</tr>
<tr>
<td>( z_5 )</td>
<td>( p(p - a)(p - a - b - 1) )</td>
<td>( 0 )</td>
<td>( p )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( -a - b - p )</td>
<td></td>
</tr>
</tbody>
</table>

We now prove a sequence of lemmas which establish

**Theorem 3.13.** The 7-tuple \( (a, b, z_1, \ldots, z_5) \) in \( F^7 \) is admissible if and only if \( (a, b, z_1, \ldots, z_5) = T_i \) for some \( i = 0, 1, 2, \ldots, 6 \).

By direct computation one can verify that for arbitrary values of \( a, b \) and \( p \), each of the 7-tuples listed in Table 3.12 satisfies equations (3.8) through (3.11) and hence, is admissible.
The proof of the converse amounts to the determination of all admissible 7-tuples \((a, b, z_1, \ldots, z_5)\). We break this investigation into two main cases defined by whether \(z_1 z_2 (z_1 + a)(z_2 + b)(a + b)\) is nonzero or zero.

**Lemma 3.14.** Let \((a, b, z_1, \ldots, z_5)\) be an admissible 7-tuple such that
\[
z_1 z_2 (z_1 + a)(z_2 + b) \neq 0.
\]
Then:
(i) if \(N = z_1 b + z_2 a + ab\), the equations
\[
\begin{align*}
(3.15) & \quad z_5 - z_4 = 0, \\
(3.16) & \quad Nz_3 - (a + b)z_1 z_2 = 0, \\
(3.17) & \quad Nz_4 - (z_1 + a - z_2)z_1 z_2 = 0, \\
(3.18) & \quad N(a + b) - (z_2 - z_1 + b)(z_2 - z_1 - a) = 0,
\end{align*}
\]
hold;
(ii) if \(a + b \neq 0\), then \((a, b, z_1, \ldots, z_5)\) is \(T_0\) where the \(p\) needed to satisfy Table 3.12 is \(p = (z_2 - z_1 + b)/(a + b)\).

**Proof.** Since \((a, b, z_1, \ldots, z_5)\) is admissible, it satisfies equations (3.8) through (3.11). These equations can be used to obtain equations (3.15) through (3.18) by following the outline given now.

1. Equation (3.15) is identical to equation (3.8).
2. Equation (3.16) is equal to \(a\) times equation (3.10) minus \(b\) times equation (3.9).
3. Equation (3.17) is \(z_2\) times equation (3.9) plus \((z_1 + a)\) times equation (3.10).
4. Equation (3.18) is obtained by using (3.16) and (3.17) to simplify
\[
N/ (z_1 z_2 (z_1 + a)(z_2 + b))
\]
times equation (3.11).

This proves part (i). For part (ii), we may assume \((a, b, z_1, \ldots, z_5)\) satisfies (3.15) through (3.18), and \(p = (z_2 - z_1 + b)/(a + b)\). Substituting \(z_2 = p(a + b) + z_1 - b\) into equation (3.18), we find
\[
(a + b)^2 (z_1 - p(p - a - 1)) = 0.
\]
Since \(a + b \neq 0\), we have \(z_1 = p(p - a - 1)\). From the definition of \(p\) it follows that \(z_2 = (p - a)(p - a - b - 1)\). Finally from equations (3.15) through (3.17), we have \(z_3 = p(p - a - b - 1)\) and \(z_4 = z_5 = p(p - a)(p - a - b - 1)\). Thus any 7-tuple \((a, b, z_1, \ldots, z_5) \in F^7\) satisfying (3.15) through (3.18) with
\[
z_1 z_2 (z_1 + a)(z_2 + b)(a + b) \neq 0
\]
is equal to one of the admissible elements of \(F^7\) of the form \(T_0\) in Table 3.12. \(\square\)

**Lemma 3.20.** Let \(\xi\) be a mass function on \(C(A_2)\), \(\Phi\) the root system of \(A_2\) and \(S_\xi = \{ \gamma \in \Phi | \xi(\gamma, X_\gamma) \neq 0 \text{ or } -\xi(h_j) \}\). Then:
(i) \(S_\xi = \Phi\),
(ii) \(S_\xi = \{ \pm \alpha \} \) for some \(\alpha \in \Phi\) or
(iii) \(S_\xi = \emptyset\).
Proof. Suppose $S_f \neq \emptyset$, $\emptyset$ or $\{ \pm \alpha \}$ for some $\alpha \in \Phi$. Then since for each $\gamma \in S_f$, $-\gamma \in S_f$, it must be the case that $|S_f| = 4$. From this it follows that $S_f$ contains some base $\Delta = \{ \alpha, \beta \}$ of $\Phi$. Let $h_\alpha, h_\beta, c_i$, $1 \leq i \leq 5$, denote the basic cycles of $C(\alpha, \beta)$, and let $a = \xi(h_\alpha), b = \xi(h_\beta)$ and $z_i = \xi(c_i)$, $1 \leq i \leq 5$. Then since $\alpha, \beta \in S_f$ we have $z_1 z_2 (z_1 + a)(z_2 + b) \neq 0$. By part (ii) of Lemma 3.14, if $a + b \neq 0$ then $\xi$ is $T_0$, contrary to $\alpha + \beta \notin S_f$. Therefore, $a = -b$.

Since $\xi(h_\alpha + h_\beta) = a + b = 0$ and $\alpha + \beta \notin S_f$, it must be the case that $z_3 = 0$. In this case (3.9) and (3.11) become

\begin{align}
(3.21) & \quad az_4 + z_1 z_2 = 0, \\
(3.22) & \quad z_4(z_2 - z_1 - a - z_4) = 0
\end{align}

respectively. From (3.22), we conclude that $a \neq 0 \neq z_4$ and hence (3.23) implies $z_2 - z_1 - a = z_4$. Substituting this into (3.22), we get

$$
0 = a(z_2 - z_1 - a) + z_1 z_2 = az_2 - az_1 - a^2 + z_1 z_2 = (z_1 + a)(z_2 - a).
$$

Therefore, $z_1 = -a$ or $z_2 = a = -b$ either of which is a contradiction. □

Lemma 3.23. Let $\xi$ be a mass function on $C(A_2)$ such that $S_f = \{ \gamma \in \Phi \mid \xi(Y_\gamma X_\gamma) \neq 0 \text{ or } -\xi(Y_\gamma) \} = \Phi$. Then $\xi$ is $T_0$ for every choice of base $\Delta = \{ \alpha, \beta \}$ used to describe the basic cycles.

Proof. Let $h_\alpha, h_\beta, c_i$, $1 \leq i \leq 5$, be the basic cycles of $C(\alpha, \beta)$ and let $(a, b, z_1, \ldots, z_5) = (\xi(h_\alpha), \xi(h_\beta), \xi(c_1), \ldots, \xi(c_5))$. If $a + b \neq 0$, this result becomes part (ii) of Lemma 3.14. Therefore, we may assume $a + b = 0$ and reduce equations (3.8) through (3.11) to

\begin{align}
(3.24) & \quad z_3 - z_4 = 0, \\
(3.25) & \quad a(z_4 - z_3) - z_1(z_3 - z_2) = 0, \\
(3.26) & \quad az_4 + z_2(z_1 - z_3) = 0, \\
(3.27) & \quad (z_4 - z_3)(z_2 - z_1 - a - z_4) + z_3(z_2 - a)(z_1 + a) = 0.
\end{align}

Since $z_3 \neq 0$, equations (3.25) and (3.26) imply $z_2 = z_1 + a$. If $a = 0$ then $z_1 = z_2 = z_3$. From this and equation (3.27) one can show that $z_4 = ((1 \pm \sqrt{1 + 4z_1})/2)z_1$. Define $p$ so that $z_1^p = z_4$. It is easy to show that this $p$ is the one needed in column $T_0$ of Table 3.12.

If $a \neq 0$, then using (3.26) and $z_2 = z_1 + a$ we obtain $z_4 = a(z_3 - z_1)(z_1 + a)$. Define $p$ so that $z_1^p = z_4$. It is easy to show that this $p$ is the one needed in column $T_0$ of Table 3.12.

If $a \neq 0$, then using (3.26) and $z_2 = z_1 + a$ we obtain $z_4 = a(z_3 - z_1)(z_1 + a)$. Define $p = (z_3 - z_1)$, so that $z_2 = z_1 + a, z_3 = ap + z_1$ and $z_4 = p(z_1 + a)$. Substituting these into (3.27) we obtain

$$
(p - 1)z_1(-p(z_1 + a)) + (ap + z_1)z_1(z_1 + a) = 0
$$

which simplifies to $z_1 = p(p - a - 1)$. Now, one can readily show using the expressions for $z_i$ in terms of $z_1$ and $a$ given above that $\xi$ is $T_0$ on $C(\alpha, \beta)$. □
Lemma 3.28. If $\xi$ is a mass function on $C(A_2)$ such that $S_\xi = \{ \pm \alpha \}$ or $\emptyset$ then $(a, b, z_1, \ldots, z_5) = (\xi(h_\gamma), \xi(h_\delta), \xi(c_1), \ldots, \xi(c_5))$ appears in Table 3.12 for some base $(\gamma, \delta)$ where the $c_i$'s are basic cycles described using $\Delta = (\gamma, \delta)$.

Proof. Let $\gamma$ be a root which is not in $S_\xi$. Then either $\xi(Y, X_\gamma) = 0$ or $\xi(X, Y_\gamma) = 0$. Hence we can pick $\gamma \in S_\xi$ such that $\xi(Y, X_\gamma) = 0$. If $S_\xi = \{ \pm \alpha \}$ let $\delta = \alpha$ if $\delta + \alpha \in \Phi$ and let $\delta = -\alpha$ otherwise. In either case $\gamma + \delta \in \Phi$ and we can take $(\gamma, \delta)$ as a base of $\Phi$. If $S = \emptyset$, let $\delta$ be any root such that $(\gamma, \delta)$ is a base.

Define the basic cycles $c_i$, $1 \leq i \leq 5$, in terms of this base. In particular, $z_1 = \xi(c_1) = \xi(Y, X_\gamma) = 0$. The lemma follows, once we have shown that every admissible 7-tuple $(a, b, z_1, \ldots, z_5)$ for which $z_1 = 0$ appears in Table 3.12.

Under the condition $z_1 = 0$ equations (3.8) through (3.11) become

\begin{align}
(3.29) \quad z_5 - z_4 &= 0, \\
(3.30) \quad a(z_4 - z_3) &= 0, \\
(3.31) \quad bz_4 + z_2 z_3 &= 0, \\
(3.32) \quad (z_4 - z_3)(z_2 - a - z_4) + az_3(z_2 + b) &= 0.
\end{align}

Our proof continues by studying two cases $a = 0$ and $a \neq 0$.

Case I. $a = 0$. Equation (3.32) then implies that $(z_4 - z_3)(z_2 - z_4) = 0$.

Subcase i. $z_4 = z_3$. Then equation (3.31) becomes $z_3(z_2 + b) = 0$. This leads to two families of solutions. If $z_3 = 0$ we have

$$(a, b, z_1, \ldots, z_5) = (0, b, 0, p, 0, 0)$$

where $b$ and $p$ are arbitrary. If $z_2 = -b$ we have

$$(a, b, z_1, \ldots, z_5) = (0, b, 0, -b, p, p)$$

where $b$ and $p$ are arbitrary. Both of these families occur in Table 3.12.

Subcase ii. $z_4 = z_2$. The argument is similar to subcase i.

Case II. $a \neq 0$. From equation (3.30) we have $z_4 = z_3$. Using this in equation (3.32) we obtain $z_3(z_2 + b) = 0$. If $z_3 = 0$ we have $(a, b, z_1, \ldots, z_5) = (a, b, 0, p, 0, 0, 0)$ where $a$, $b$ and $p$ are arbitrary. If $z_2 = -b$ we have $(a, b, z_1, \ldots, z_5) = (a, b, 0, -b, p, p, p)$ where $a$, $b$ and $p$ are arbitrary. Both of these families of solutions occur in Table 3.12.

Thus every admissible 7-tuple $(a, b, z_1, \ldots, z_5)$ such that $z_1 = 0$ is listed in Table 3.12. \square

The reader should note that the preceding lemmas establish Theorem 3.13.

Theorem 3.33. For mass functions $\xi$ on $C(A_2)$, Theorem 1.7 holds. Moreover, if $S_\xi = \Phi$ then Theorem 1.7 holds for every choice of base $\Delta$ used to describe the basic cycles.

Proof. According to Lemmas 3.20, 3.23 and 3.28, we need only consider $T_i$ mass functions. Let $\xi_i$ be a $T_i$ mass function relative to $\Delta = (\alpha, \beta)$. The following table gives the type of mass function that $\xi_i$ becomes relative to the bases $\sigma_\alpha \Delta$ and $\sigma_\beta \Delta$ where $\sigma_\alpha$, $\sigma_\beta$ denote Weyl reflections.
From this table one sees that by an appropriate choice of base we may assume that the given mass function $\xi$ is $T_0$, $T_1$ or $T_4$.

If $\xi$ is $T_0$, then by Lemma 3.23 $\xi$ is $T_0$ for every base $\Delta$. From this and Table 2.7 it follows that $\xi = e \circ \Psi_{\Delta,\Delta}$ where $e(s_1) = p$ and $p$ is the value given in column $T_0$ of Table 3.12.

If $\xi$ is $T_1$ relative to $\Delta = \{\alpha, \beta\}$, then $\xi = e \circ e_{\{\alpha\},\Delta}$ where $e_{\{\alpha\},\Delta}$ is given by Table 2.7 and $e(s_1(s_1 - \lambda_1 - 1)) = p = \xi(c_1)$. The proof in the case of $T_4$ mass functions is similar.\end{proof}

\begin{corollary} If $\xi$ is a mass function on $C(A_2)$ with $S_1 = \{\gamma \in \Phi \mid \xi(Y_\gamma X_\gamma) \neq 0$ or $-\xi(h_\gamma)) = \varnothing$, then there is a base $\Delta = \{\alpha, \beta\}$ such that if $\Delta$ is used to describe the basic cycles then $\xi = e \circ \Psi_{\Delta,\Delta}$ where $e$ is an evaluation map satisfying $e(\lambda_1) = \xi(h_\alpha)$ and $e(\lambda_2) = \xi(h_\beta)$.
\end{corollary}

\begin{proof} By Lemmas 3.20 and 3.28 and Table 3.34, there is a base $\Delta = \{\alpha, \beta\}$ such that relative to $\Delta$, $\xi$ is $T_1$ with $p = 0$ or $-\xi(h_\alpha)$ or $\xi$ is $T_4$ with $p = -\xi(h_\beta)$. Suppose $\xi$ is $T_1$ relative to $\Delta = \{\alpha, \beta\}$ with $p \neq 0$. Then $\xi$ relative to $\{-\alpha, \alpha + \beta\}$ is $T_1$ with the corresponding $p = 0$. A similar change of base for $T_4$ maps permits us to assume that $\xi$ is $T_1$ relative to $\Delta$ and $p = 0$. The result then follows directly.\end{proof}

4. Mass functions on $C(A_3)$ and $C(A_4)$. In this section, we prove a series of technical results concerning mass functions on $C(A_3)$ and $C(A_4)$. We make use of the convention laid down in §2 that $e(s_i) = 0$ for $l + 1 \leq i \leq n$ if $\text{Im} \Psi_{\Delta,\Delta} \subseteq F[\lambda_1, \ldots, \lambda_n, s_1, \ldots, s_l]$.

We continue to use $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ to denote a base of the root system $\Phi$ of $A_n$. A basic cycle of $C(A_n)$ relative to $\Delta$ is either an element of the form $h_i = h_{\alpha_i}$ or an element $u(\vec{p}, \vec{0}, \vec{q}) \in C(A_n)$ of the form (1.1) having the property that no proper factor (ignoring commutativity) of $u(\vec{p}, \vec{0}, \vec{q})$ is in $C(A_n)$. It follows immediately from the proof of the Poincaré-Birkhoff-Witt Theorem that the basic cycles generate $C(A_n)$ as an associative algebra with 1. Thus any mass function is completely determined by its values on the set of basic cycles relative to any fixed base $\Delta$. In [15] it is shown that this statement can be considerably improved to read

\begin{proposition} Any mass function $\xi$: $C(A_n) \to F$ is completely determined by its values on the set of all basic cycles of degree $\leq 3$ relative to a base $\Delta$.
\end{proposition}

We now suppose that we are given a mass function $\xi$: $C(A_n) \to F$ and wish to determine an appropriate Harish-Chandra homomorphism required in Theorem 1.7. We begin this determination by specifying a subset of the root system $\Phi$ associated with $\xi$, namely

$$S_1 = \{\alpha \in \Phi \mid \xi(Y_\alpha X_\alpha) \neq 0, -\xi(h_\alpha)\}.$$
Proposition 4.3. (i) $S_t$ is a root subsystem of $\Phi$; (ii) $S_t$ is a disjoint union of a mutually orthogonal irreducible subsystem $\Phi'$ of $\Phi$.

Proof. Let $(\ , \ )$ denote the Killing form on $\Phi$ and define an equivalence relation $\sim$ on $S_t$ by setting $\alpha \sim \beta$ for $\alpha, \beta \in S_t$ iff there exist $\gamma_1, \gamma_2, \ldots, \gamma_m \in S_t$ such that $\gamma_1 = \alpha, \gamma_m = \beta$ and $(\gamma_i, \gamma_{i+1}) \neq 0$ for $1 \leq i \leq m - 1$. Let $\Phi'$ denote the equivalence classes of $S_t$ under $\sim$. The proposition follows easily once we show that $\Phi'$ is a root subsystem of $\Phi$. By [8, p. 42], it suffices to show that if $\alpha, \beta \in \Phi'$ then $\sigma_a(\beta) \in \Phi'$ where $\sigma_a$ is the Weyl reflection through the hyperplane perpendicular to $a$. Let $\alpha, \beta \in \Phi'$. If $(\alpha, \beta) = 0$ then $\sigma_a(\beta) = \beta \in \Phi'$. If $(\alpha, \beta) \neq 0$ then

$$
\sigma_a(\beta) = \begin{cases} 
\beta & \text{if } \beta = \pm \alpha, \\
\alpha + \beta & \text{if } \alpha + \beta \in \Phi, \\
\beta - \alpha & \text{if } \alpha - \beta \in \Phi.
\end{cases}
$$

Clearly in the first case $\sigma_a(\beta) \in \Phi'$. If $\alpha + \beta \in \Phi$, then let $A_2(\alpha, \beta)$ be the subalgebra of $A_n$ isomorphic to $A_2$ and generated by $(X_\alpha, Y_\alpha, X_\beta, Y_\beta)$. By employing Table 3.12 for $\xi$ restricted to $C(\alpha, \beta)$ one verifies that $\sigma_a(\beta) = \alpha + \beta \in \Phi'$. Similarly, by considering $\xi$ restricted to $C(\alpha, -\beta)$ we have that $\sigma_a(\beta) = \beta - \alpha \in \Phi'$ in the third case. This completes the proof that $\Phi'$ (and hence $S_t$) is a subsystem of $\Phi$. \Box

Our proof of Theorem 1.7 in the general case requires a number of technical results concerning mass functions on $C(A_3)$. To facilitate the presentation of these results we introduce the following notation for $C(A_3)$. A base for the root system of $A_3$ will be denoted by $A = \{\alpha, \beta, \gamma\}$ and the set of basic cycles of $C(A_3)$ relative to this base are given as follows.

\begin{align}
(4.4) \\
h_1 &= h_\alpha, \\
h_2 &= h_\beta, \\
h_3 &= h_\gamma, \\
c_1 &= Y_\alpha X_\alpha, \\
c_2 &= Y_\beta X_\beta, \\
c_3 &= Y_\gamma X_\gamma, \\
c_4 &= Y_\alpha + \beta X_\alpha + \beta, \\
c_5 &= Y_\beta + \gamma X_\beta + \gamma, \\
c_6 &= Y_\alpha + \beta + \gamma X_\alpha + \beta + \gamma, \\
c_7 &= Y_\alpha + \beta X_\alpha X_\beta, \\
c_8 &= Y_\beta + \gamma X_\beta X_\gamma, \\
c_9 &= Y_\beta Y_\alpha X_\alpha + \beta, \\
c_{10} &= Y_\beta Y_\beta X_\beta + \gamma, \\
c_{11} &= Y_\alpha + \beta + \gamma X_\alpha + \beta X_\gamma, \\
c_{12} &= Y_\alpha + \beta + \gamma X_\alpha X_\beta X_\gamma, \\
c_{13} &= Y_\alpha + \beta + \gamma X_\alpha + \beta + \gamma, \\
c_{14} &= Y_\beta + \gamma Y_\alpha X_\alpha + \beta + \gamma, \\
c_{15} &= Y_\alpha + \beta + \gamma X_\alpha X_\beta X_\gamma, \\
c_{16} &= Y_\beta Y_\beta Y_\alpha X_\alpha + \beta + \gamma, \\
c_{17} &= Y_\beta + \gamma Y_\alpha + \beta X_\alpha + \beta + \gamma X_\beta, \\
c_{18} &= Y_\beta Y_\alpha + \beta + \gamma X_\alpha + \beta X_\beta + \gamma, \\
c_{19} &= Y_\gamma Y_\alpha + \beta X_\alpha X_\beta + \gamma, \\
c_{20} &= Y_\beta + \gamma Y_\alpha X_\alpha + \beta X_\gamma.
\end{align}

The fact that this list exhausts the set of all basic cycles of $C(A_3)$ relative to $\Delta$ follows directly from the results of van den Hombergh [16] or from [15]. In the remainder of this paper whenever we refer to the basic cycles of $C(\alpha, \beta, \gamma)$ we mean...
those listed above. The subalgebras of \( C(A_3) \) which are isomorphic to \( C(A_2) \) together with their basic cycles in this notation are

\[
\begin{align*}
C(\alpha, \beta) &= \langle \{h_1, h_2, c_1, c_2, c_4, c_7, c_9\} \rangle, \\
C(\beta, \gamma) &= \langle \{h_2, h_3, c_2, c_3, c_5, c_8, c_{10}\} \rangle, \\
C(\alpha + \beta, \gamma) &= \langle \{h_1 + h_2, h_3, c_4, c_3, c_6, c_{11}, c_{13}\} \rangle, \\
C(\alpha, \beta + \gamma) &= \langle \{h_1, h_2 + h_3, c_1, c_5, c_6, c_{12}, c_{14}\} \rangle.
\end{align*}
\]

By Proposition 4.3 the set \( S_i \) for any mass function \( \xi : C(A_3) \to F \) must have order 12, 6, 4, 2 or 0.

Our first case is \(| S_i | = 12\). By using Lemma 3.23 we see that the restriction of \( \xi \) to each \( C(A_2) \) of \( C(A_3) \) must be a \( T_0 \) mass function relative to any base. Let \( a = \xi(h_1), b = \xi(h_2) \) and \( c = \xi(h_3) \). Then there must exist unique elements \( r, t, u, v \in F \) such that

\[
\begin{align*}
\xi(c_1) &= r(r - a - 1) = u(u - a - 1), \\
\xi(c_2) &= (r - a)(r - a - b - 1) = t(t - b - 1), \\
\xi(c_3) &= (t - b)(t - b - c - 1) = \xi(c_3) = (v - a - b - c - 1), \\
\xi(c_4) &= r(r - a - b - 1) = v(v - a - b - 1), \\
\xi(c_5) &= t(t - b - c - 1) = \xi(c_5) = (u - a)(u - a - b - c - 1), \\
\xi(c_6) &= v(v - a - b - c - 1) = \xi(c_6) = u(u - a - b - c - 1).
\end{align*}
\]

Direct substitution shows that the roots of these quadratic equations must satisfy the following conditions.

\[
\begin{align*}
\text{Column A} & \quad \text{Column B} \\
(4.12) \quad r &= u \quad \text{or} \quad r = 1 + a - u \\
(4.13) \quad r &= t + a \quad \text{or} \quad r = 1 + a + b - t \\
(4.14) \quad t &= v - a \quad \text{or} \quad t = 1 + a + 2b + c - v \\
(4.15) \quad r &= v \quad \text{or} \quad r = 1 + a + b - v \\
(4.16) \quad t &= u - a \quad \text{or} \quad t = 1 + a + b + c - u \\
(4.17) \quad v &= u \quad \text{or} \quad v = 1 + a + b + c - u.
\end{align*}
\]

The possible choices for \( s, t, u \) and \( v \) are reduced by the following observations.

(4.18) If three equations from Column A are valid and all four of \( r, t, u \) and \( v \) are involved in these three equations then \( r = u = v = t + a \). In particular, if four equations from Column A are valid, then \( r = u = v = t + a \). If \( r = u = v = t + a \) and some equations from Column B are valid then constraints are imposed on \( a, b \) and \( c \).

(4.19) If five equations from Column B are valid, then all equations from Column B are valid and certain constraints are being forced on \( a, b \) and \( c \).

These two observations aid us in the case study which makes up the proof of

**Lemma 4.20.** For any mass function \( \xi : C(A_3) \to F \) with \(| S_i | = 12\), there is an evaluation map \( \varepsilon : F[\lambda_1, \lambda_2, \lambda_3, s_1, s_2, s_3] \to F \) and an automorphism \( \sigma \) such that \( \xi \circ \sigma = \varepsilon \circ \Psi_{\Delta, \Delta} \).
PROOF. If the condition of remark (4.18) holds, then define \( e \) by \( e(\lambda_1) = \xi(h_1) = a \), 
\( e(\lambda_2) = \xi(h_2) = b \), \( e(\lambda_3) = \xi(h_3) = c \) and \( e(s_1) = r = u = v = t + a \). In this case 
then \( \xi = e \circ \Psi_{\Delta, \Delta} \) for all basic cycles of degree \( \leq 3 \) and Theorem 4.1 implies 
that \( \xi = e \circ \Psi_{\Delta, \Delta} \) as required. Therefore, in the remainder of the proof, we assume that 
at most three equations from Column A hold.

Case I. The condition of (4.18) fails and exactly three equations from Column A 
hold. This is equivalent to the following subcases.

Subcase i. (4.12), (4.13), (4.16) Column A hold.
Subcase ii. (4.12), (4.15), (4.17) Column A hold.
Subcase iii. (4.13), (4.14), (4.15) Column A hold.

Table 4.21 indicates the action of \( \xi \) on the basic cycles of degree \( \leq 3 \) relative to 
the base \( \{a, \beta, \gamma\} \) in each of these subcases.

The analysis of these four subcases can be simplified by observing that if \( \xi \) satisfies 
subcase ii then \( \xi \circ \sigma_a \) together with a change of variable replacing \( r \) by \( r - a \) 
yields a map which satisfies subcase i. Similarly with an appropriate change of 
variable, we again arrive at subcase i by using \( \xi \circ \sigma_f \) to alter subcase iii and \( \xi \circ \sigma_f \circ \sigma_g \) 
to change subcase iv where \( \sigma_f \) is the flip map defined in \$2\). Thus, it suffices to prove 
Case I of Lemma 4.20 assuming subcase i holds.

Let \( \xi : C(\alpha, \beta, \gamma) \to F \) be a mass function satisfying subcase i and consider the 
following identities in \( C(\alpha, \beta, \gamma) \).

<table>
<thead>
<tr>
<th>Subcase i</th>
<th>Subcase ii</th>
<th>Subcase iii</th>
<th>Subcase iv</th>
</tr>
</thead>
<tbody>
<tr>
<td>( b = -a )</td>
<td>( c = 0 )</td>
<td>( a = 0 )</td>
<td>( a = b = 0 )</td>
</tr>
<tr>
<td>( c = 0 )</td>
<td>( t = r - a )</td>
<td>( c = -b )</td>
<td>( t = 1 - r )</td>
</tr>
<tr>
<td>( t = r - a )</td>
<td>( u = r )</td>
<td>( t = r )</td>
<td>( u = 1 - r )</td>
</tr>
<tr>
<td>( u = r )</td>
<td>( v = r )</td>
<td>( u = 1 - r )</td>
<td>( v = r )</td>
</tr>
<tr>
<td>( v = r )</td>
<td>( v = 0 )</td>
<td>( v = 1 - r )</td>
<td>( v = 0 )</td>
</tr>
<tr>
<td>( \xi(h_1) )</td>
<td>( a )</td>
<td>( -b )</td>
<td>( c )</td>
</tr>
<tr>
<td>( \xi(h_2) )</td>
<td>( -a )</td>
<td>( 0 )</td>
<td>( a )</td>
</tr>
<tr>
<td>( \xi(h_3) )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( b )</td>
</tr>
<tr>
<td>( \xi(c_1) )</td>
<td>( r(r - a - 1) )</td>
<td>( r(r - a - 1) )</td>
<td>( r(r - 1) )</td>
</tr>
<tr>
<td>( \xi(c_2) )</td>
<td>( (r - 1)(r - a) )</td>
<td>( (r - 1)(r - a - 1) )</td>
<td>( r(r - 1) )</td>
</tr>
<tr>
<td>( \xi(c_3) )</td>
<td>( (r - 1)(r - a) )</td>
<td>( (r - 1)(r - a - 1) )</td>
<td>( (r - 1)(r - c) )</td>
</tr>
<tr>
<td>( \xi(c_4) )</td>
<td>( r(r - a - 1) )</td>
<td>( (r - 1)(r - a - 1) )</td>
<td>( (r - 1)(r - a - 1) )</td>
</tr>
<tr>
<td>( \xi(c_5) )</td>
<td>( (r - 1)(r - a) )</td>
<td>( (r - 1)(r - a - 1) )</td>
<td>( (r - 1)(r - a - 1) )</td>
</tr>
<tr>
<td>( \xi(c_6) )</td>
<td>( r(r - 1) )</td>
<td>( (r - 1)(r - a - 1) )</td>
<td>( (r - 1)(r - a - 1) )</td>
</tr>
<tr>
<td>( \xi(c_7) = \xi(c_9) )</td>
<td>( r(r - 1)(r - a) )</td>
<td>( r(r - 1)(r - a - 1) )</td>
<td>( r(r - 1)(r - a - 1) )</td>
</tr>
<tr>
<td>( \xi(c_8) = \xi(c_{10}) )</td>
<td>( r(r - 1)(r - a) )</td>
<td>( -r(a - r - a - 1)^2 )</td>
<td>( -r(a - r - a - 1)^2 )</td>
</tr>
<tr>
<td>( \xi(c_{11}) = \xi(c_{13}) )</td>
<td>( -r(r - 1)^2 )</td>
<td>( (r - 1)(r - a - 1) )</td>
<td>( (r - 1)(r - a - 1) )</td>
</tr>
<tr>
<td>( \xi(c_{12}) = \xi(c_{14}) )</td>
<td>( r(r - 1)(r - a) )</td>
<td>( -r(r - 1)^2 )</td>
<td>( -r(r - 1)^2 )</td>
</tr>
</tbody>
</table>

\[(4.22) \quad c_1c_{17} = (c_{14} - c_5)(c_7 - c_4) + (c_8 - c_3)c_1,\]
\[(4.23) \quad c_{17}(c_1 + h_1) = (c_7 - c_4)(c_{14} - c_5) + (c_{13} - c_6)(c_1 + h_1).\]
(4.22) and (4.23) are obtained by rewriting \( c_1 c_7 \) and \( c_7 c_{14} \). Since \( \xi(c_1) \) and \( \xi(c_1 + h) \) are nonzero we can use each of (4.22) and (4.23) to find \( \xi(c_{17}) \). This yields
\[(4.24)\]
\[\xi(c_{17}) = r(r - 1)^2(r - a),\]
\[(4.25)\]
\[\xi(c_{17}) = r(r - 1)^2(r^2 - ra - 3r + a + 1).\]

From this, it follows that \( r = \frac{1}{2} \). In this case, we select an evaluation map \( \epsilon \) satisfying \( \epsilon(\lambda_1) = a, \epsilon(\lambda_2) = -a, \epsilon(\lambda_3) = 0 \) and \( \epsilon(s_1) = \frac{1}{2} \). Then \( \xi = \epsilon \circ \Psi_{\Delta, \Delta} \) on all basic cycles of \( C(\alpha, \beta, \gamma) \) of degree \( \leq 3 \) and hence by Theorem 4.1, \( \xi = \epsilon \circ \Psi_{\Delta, \Delta} \).

**Case II.** Exactly two equations from Column A are valid. One can verify that there are only three subcases.

- **Subcase i.** (4.12), (4.14) Column A hold.
- **Subcase ii.** (4.13), (4.17) Column A hold.
- **Subcase iii.** (4.15), (4.16) Column A hold.

This is established quickly by noting that any other pair of equations from Column A implies a third equation from Column A.

We list these solutions in the following table.

<table>
<thead>
<tr>
<th>Subcase i</th>
<th>Subcase ii</th>
<th>Subcase iii</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a = c = 0 )</td>
<td>( c = -a )</td>
<td>( b = -a )</td>
</tr>
<tr>
<td>( t = 1 + b - r )</td>
<td>( b = 0 )</td>
<td>( c = a )</td>
</tr>
<tr>
<td>( u = r )</td>
<td>( t = r - a )</td>
<td>( t = 1 - r )</td>
</tr>
<tr>
<td>( v = 1 + b - r )</td>
<td>( u = 1 + a - r )</td>
<td>( u = 1 + a - r )</td>
</tr>
<tr>
<td>( \xi(h_1) = 0 )</td>
<td>( a )</td>
<td>( a )</td>
</tr>
<tr>
<td>( \xi(h_2) = b )</td>
<td>( 0 )</td>
<td>( -a )</td>
</tr>
<tr>
<td>( \xi(h_3) = 0 )</td>
<td>( -a )</td>
<td>( a )</td>
</tr>
<tr>
<td>( \xi(c_1) = r(r - 1) )</td>
<td>( r(r - a - 1) )</td>
<td>( r(r - a - 1) )</td>
</tr>
<tr>
<td>( \xi(c_2) = r(r - b - 1) )</td>
<td>( (r - a)(r - a - 1) )</td>
<td>( (r - 1)(r - a) )</td>
</tr>
<tr>
<td>( \xi(c_3) = r(r - 1) )</td>
<td>( (r - 1)(r - a) )</td>
<td>( r(r - a - 1) )</td>
</tr>
<tr>
<td>( \xi(c_4) = r(r - b - 1) )</td>
<td>( r(r - a - 1) )</td>
<td>( r(r - a - 1) )</td>
</tr>
<tr>
<td>( \xi(c_5) = r(r - b - 1) )</td>
<td>( (r - 1)(r - a) )</td>
<td>( r(r - a - 1) )</td>
</tr>
<tr>
<td>( \xi(c_6) = r(r - b - 1) )</td>
<td>( (r - a)(r - a - 1) )</td>
<td>( r(r - a - 1) )</td>
</tr>
<tr>
<td>( \xi(c_7) = \xi(c_9) = r^2(r - b - 1) )</td>
<td>( r(r - a)(r - a - 1) )</td>
<td>( r(r - 1)(r - a) )</td>
</tr>
<tr>
<td>( \xi(c_8) = \xi(c_{10}) = -(r - 1)(r - b - 1) )</td>
<td>( (r - 1)(r - a)^2 )</td>
<td>( -r(r - 1)(r - a - 1) )</td>
</tr>
<tr>
<td>( \xi(c_{11}) = \xi(c_{13}) = -(r - 1)(r - b - 1) )</td>
<td>( -(r - 1)(r - a)(r - a - 1) )</td>
<td>( r^2(r - a - 1) )</td>
</tr>
<tr>
<td>( \xi(c_{12}) = \xi(c_{14}) = r^2(r - b - 1) )</td>
<td>( -(r - 1)(r - a)(r - a - 1) )</td>
<td>( -(r - 1)(r - a - 1) )</td>
</tr>
</tbody>
</table>

Once again we can use the Weyl automorphisms to simplify our analysis. If \( \xi \) satisfies subcase ii, then \( \xi \circ \sigma_a \circ \sigma_b \) with \( r \) replaced by \( r - a \) satisfies subcase i and if \( \xi \) satisfies subcase iii then \( \xi \circ \sigma_b \) satisfies subcase i. Hence, we may assume the \( \xi \) is a mass function on \( C(\alpha, \beta, \gamma) \) satisfying this subcase i. By applying \( \xi \) to equations (4.22) and (4.23), we obtain
\[(4.27)\]
\[\xi(c_{17}) = -(r - 1)(r - a)(r - a - 1)^2,\]
\[(4.28)\]
\[\xi(c_{17}) = -r(r - a)^2(r - a - 1),\]
and hence \( r = \frac{1}{2}(1 + a) \). Select an evaluation map \( \epsilon: F[\lambda_1, \lambda_2, \lambda_3, s_1, s_2] \to F \) which satisfies \( \epsilon(\lambda_1) = -\epsilon(\lambda_3) = a, \epsilon(\lambda_2) = 0 \) and \( \epsilon(s_1) = \frac{1}{2}(1 + a) \), and observe that
\( \zeta = e \circ \Psi_{\Delta, \Delta} \) on all basic cycles of \( C(\alpha, \beta, \gamma) \) of degree \( \leq 3 \). Thus, by Theorem 4.1, we have \( \zeta = e \circ \Psi_{\Delta, \Delta} \).

**Case III.** Exactly one equation from Column A is valid. Remark 4.19 reduces this case to

**Case IV.** Every equation from Column B is valid. For the remainder of this proof the numbers (4.12) through (4.17) refer to equations (4.12) Column B through (4.17) Column B. By adding (4.12) and (4.13) and then using equation (4.16), we get

\[
2r = 2 + 2a + b - t - u = 1 + a - c.
\]

Similarly, (4.14), (4.16) and (4.17) give us

\[
u = \frac{1}{2}(1 + a + c),
\]

and

\[
t = v = \frac{1}{2}(1 + a + 2b + c).
\]

Therefore, the mass functions described by this case are uniquely determined by the set of equations

\[
\begin{align*}
(4.32) & \quad \zeta(h_1) = a, \\
(4.33) & \quad \zeta(h_2) = b, \\
(4.34) & \quad \zeta(h_3) = c, \\
(4.35) & \quad \zeta(c_1) = -\frac{1}{4}(1 + a - c)(1 + a + c), \\
(4.36) & \quad \zeta(c_2) = -\frac{1}{4}(1 - a - c)(1 + a + 2b + c), \\
(4.37) & \quad \zeta(c_3) = -\frac{1}{4}(1 + a + c)(1 - a + c), \\
(4.38) & \quad \zeta(c_4) = -\frac{1}{4}(1 + a - c)(1 + a + 2b + c), \\
(4.39) & \quad \zeta(c_5) = -\frac{1}{4}(1 - a + c)(1 + a + 2b + c), \\
(4.40) & \quad \zeta(c_6) = -\frac{1}{4}(1 + a + c)(1 + a + 2b + c), \\
(4.41) & \quad \zeta(c_7) = \zeta(c_9) = -\frac{1}{8}(1 + a - c)(1 - a - c)(1 + a + 2b + c), \\
(4.42) & \quad \zeta(c_8) = \zeta(c_{10}) = -\frac{1}{8}(1 - a - c)(1 + a + c)(1 + a + 2b + c), \\
(4.43) & \quad \zeta(c_{11}) = \zeta(c_{13}) = -\frac{1}{8}(1 - a + c)(1 + a + c)(1 + a + 2b + c), \\
(4.44) & \quad \zeta(c_{12}) = \zeta(c_{14}) = -\frac{1}{8}(1 - a + c)(1 + a + c)(1 + a + 2b + c).
\end{align*}
\]

From identities (4.22) and (4.23), we may deduce that

\[
(4.45) \quad \zeta(c_{17}) = \frac{1}{16}(1 - a + c)(1 - a - c)(1 + a + 2b + c)(1 - a - 2b - c),
\]

\[
(4.46) \quad \zeta(c_{17}) = \frac{1}{16}(1 + a - c)(1 + a + c)(1 + a + 2b + c)(1 - a - 2b - c)
\]

and hence we must have one of

\[(1 + a + 2b + c)(1 - a - 2b - c) = 0, \quad \text{or} \quad a = 0.\]

If the first of these holds then (4.36) implies either \( \zeta(c_2) = 0 \) or \( \zeta(c_2) = -\zeta(h_2) \) but this contradicts our assumption that \( |S_\zeta| = 12 \). Thus we must conclude that \( a = 0 \).
We may assume that \( \xi \circ \sigma \) is of the type described in Case IV for each Weyl automorphism \( \sigma \) for otherwise \( \xi \circ \sigma \) falls into one of the previous cases. Here we are using the easily established fact that \(| S_\xi| = 12 \) implies \(| S_{\xi \circ \sigma}| = 12 \) for each Weyl automorphism \( \sigma \).

The above assumption along with the fact that \( a = 0 \) quickly yields \( a = b = c = 0 \). From this and equations (4.29) through (4.31), we obtain \( r = t = u = v = \frac{1}{2} \). Therefore, \( \xi = \epsilon \circ \Psi_{\Delta, \Delta} \) where \( \epsilon \) is defined by \( \epsilon(\lambda_i) = 0 \), \( 1 \leq i \leq 3 \), and \( \epsilon(s_1) = \frac{1}{2} \).

This concludes the proof of Lemma 4.20. \( \square \)

The key to the proof of Lemma 4.20 also unlocks the generalization of this lemma as stated below in Corollary 4.47. Therefore, we emphasize this point now. If \( \xi : C(A_2) \to F \) is a mass function with \( S_\xi \) equal to the entire root system of \( A_2 \) then \( \xi = \epsilon \circ \Psi_{\Delta, \Delta} \) for some evaluation map \( \epsilon \). Thus the choice of \( \epsilon \) corresponding to \( \xi \) is uniquely determined by \( \Delta \). One can check this easily by consulting Table 3.12.

**Corollary 4.47.** Let \( \Delta \) be any base of the root system \( \Phi_n \) of \( A_n \), and \( \xi \) be a mass function with \( S_\xi = \Phi_n \). Then there is an evaluation map \( \epsilon : F[\lambda_1, \ldots, \lambda_n, s_1, \ldots, s_n] \to F \) such that \( \xi = \epsilon \circ \Psi_{\Delta, \Delta} \).

**Proof.** If \( n = 1 \), then \( C(A_n) = F[\lambda_1, s_1] \) and the result is clear. For \( n = 2, 3 \), we use Corollary 3.32 and Lemma 4.20 respectively to conclude that there is an automorphism and an evaluation map \( \epsilon' \) such that \( \xi \circ \sigma = \epsilon' \circ \Psi_{\Delta, \Delta} \) or equivalently \( \xi = \epsilon' \circ \Psi_{\Delta, \Delta} \circ \sigma^{-1} \). In both of these cases, define \( \nu : \text{Im} \Psi_{\Delta} \to F[\lambda_1, \ldots, \lambda_n, s_1, \ldots, s_n] \) by \( \nu(\Psi_{\Delta}(c)) = \Psi_{\Delta}(\sigma^{-1}(c)) \) for all \( c \in C(A_n) \) where \( \Psi_{\Delta} \) denotes \( \Psi_{\Delta, \Delta} \). Clearly \( \nu \) is a homomorphism which makes the following diagram commute.

\[
\begin{array}{ccc}
C(A_n) & \xrightarrow{\sigma^{-1}} & C(A_n) \\
\downarrow \Psi_{\Delta} & & \downarrow \Psi_{\Delta} \\
\text{Im} \Psi_{\Delta} & \xrightarrow{\nu} & F
\end{array}
\]

The lemma for \( n = 2, 3 \) follows after one defines \( \epsilon \) to be \( \epsilon' \circ \nu \).

We assume now that \( n \geq 4 \) and proceed by induction. Let \( A_{n-1}^{(i)} \) be the subalgebra of \( A_n \) which is isomorphic to \( A_{n-1} \) and has \( \Delta^{(i)} \) as a base for its root system where

\[
\begin{align*}
\Delta^{(1)} &= \{ \alpha_1, \ldots, \alpha_{n-1} \}, \\
\Delta^{(2)} &= \{ \alpha_1, \ldots, \alpha_{n-2}, \alpha_{n-1} + \alpha_n \}, \\
\Delta^{(3)} &= \{ \alpha_1, \ldots, \alpha_{n-3}, \alpha_{n-2} + \alpha_n, \alpha_n \}, \\
& \vdots \\
\Delta^{(n)} &= \{ \alpha_1 + \alpha_2, \alpha_3, \ldots, \alpha_n \}, \\
\Delta^{(n+1)} &= \{ \alpha_2, \ldots, \alpha_n \}.
\end{align*}
\]

Let \( \epsilon^{(i)} \) be the evaluation map given to us by the induction hypothesis so that \( \xi \downarrow C(A_{n-1}^{(i)}) = \epsilon^{(i)} \circ \Psi_{\Delta^{(i)}, \Delta^{(0)}} \). Now define \( \epsilon : F[\lambda_1, \ldots, \lambda_n, s_1, \ldots, s_n] \to F \) by \( \epsilon(\lambda_i) = \xi(h_i) \) and \( \epsilon(s_1) = \epsilon^{(1)}(s_1) \). Since \( A_2(\alpha_1, \alpha_2) \) is a subalgebra of both \( A_{n-1}^{(i)} \) and \( A_{n-1}^{(2)} \), we have \( \epsilon^{(1)}(s_1) = \epsilon^{(2)}(s_1) \). In a similar manner \( \epsilon^{(k-1)}(s_1) = \epsilon^{(k)}(s_1), \) \( k = 2, \ldots, n \).
Viewing $A_2(\alpha_2, \alpha_3)$ as a subalgebra of both $A_{n-1}^{(1)}$ and $A_{n+1}^{(1)}$, one can conclude from Table 3.12 that $e^{(n)}(s_1) = e^{(n+1)}(s_1) + \xi(h_1)$. Define $e(\lambda_i) = \xi(h_i)$ and $e(s_1) = e^{(1)}(s_1)$ and extend this to an algebra homomorphism on $F[\lambda_1, \ldots, \lambda_n, s_1, \ldots, s_n]$. To show that $\xi = e \circ \Psi_\Delta$ we introduce a set of maps $\pi^{(i)}$: $F[\lambda_1, \ldots, \lambda_n, s_1, \ldots, s_{n-1}] \to F[\lambda_1, \ldots, \lambda_n, s_1, \ldots, s_{n-1}]$ defined by

$$
\pi^{(i)}(s_j) = s_j \quad \text{for } i = 1, \ldots, n + 1 \text{ and } j = 1, 2, \ldots, n - 1,
$$

$$
\pi^{(i)}(\lambda_j) = \begin{cases} 
\lambda_j & j < n + 1 - i, \\
\lambda_j + \lambda_{j+1} & j = n + 1 - i, \\
\lambda_{j+1} & j > n + 1 - i.
\end{cases}
$$

The $\pi^{(i)}$'s have the property that $\Psi_\Delta \downarrow C(A_{n-1}^{(1)}) = \pi^{(i)} \circ \Psi_{\Delta^{(i)}, \Delta^{(i)}}$ and $e$ has the property that

$$
e \circ \Psi_\Delta \downarrow C(A_{n-1}^{(1)}) = e \circ \pi^{(i)} \circ \Psi_{\Delta^{(i)}, \Delta^{(i)}} = e^{(i)} \circ \Psi_{\Delta^{(i)}, \Delta^{(i)}} = \xi \downarrow (A_{n-1}^{(1)}).
$$

Since every basic cycle of $C(A_n)$ having degree $\leq 3$ is contained in at least one of $C(A_{n-1}^{(1)})$, Theorem 4.1 implies $\xi = e \circ \Psi_\Delta$. □

In the case of $S_3 = \Phi$, the preceding corollary proves that Theorem 1.7 holds for every choice of base $\Delta$. However, in general this is not the case, and we must choose a base $\Delta$ corresponding to $\xi$. We do this by using Weyl automorphisms as indicated in §2.

We now proceed to investigate the mass function $\xi$ on $C(A_3)$ such that $|S_5| = 6$. Without loss of generality, we may assume that the base $\Delta = \{\alpha, \beta, \gamma\}$ is such that $S_5 = \{\pm \beta, \pm \gamma, \pm (\beta + \gamma)\}$.

**Lemma 4.49.** Let $\xi$ be as above, then the restrictions of $\xi$ to the three subalgebras $C(\alpha, \beta)$, $C(\alpha + \beta, \gamma)$ and $C(\alpha, \beta + \gamma)$ are either all $T_4$ or all $T_3$.

**Proof.** By using Table 3.12, one can verify that the restrictions of $\xi$ to each of $C(\alpha, \beta)$, $C(\alpha + \beta, \gamma)$ and $C(\alpha, \beta + \gamma)$ is either $T_3$ or $T_4$. Therefore, it suffices to show that if the restriction of $\xi$ to one of these algebras is $T_4$, then the remaining two are also $T_4$.

Assume first that $\xi \downarrow C(\alpha, \beta)$ is $T_4$ and $\xi \downarrow C(\alpha + \beta, \gamma)$ is $T_3$. Then the following equations hold.

$$
\xi(c_1) = \xi(c_4) = \xi(c_7) = \xi(c_9) = 0,
$$

$$
\xi(c_2) = p \neq 0, \quad -\xi(h_2),
$$

$$
\xi(c_3) = \xi(c_{11}) = \xi(c_{13}) = q \neq 0, \quad -\xi(h_3),
$$

$$
\xi(h_1 + h_2) = 0, \quad \xi(c_6) = -\xi(h_3).
$$
Applying $\xi$ to the identities

\begin{align}
(4.50) \quad [c_1, c_3] &= c_{14} - c_{12}, \\
(4.51) \quad [c_1, c_4] &= c_{20} - c_{15}, \\
(4.52) \quad [c_3, c_7] &= c_{15} - c_{19}, \\
(4.53) \quad [c_3, c_9] &= c_{20} - c_{16}, \\
(4.54) \quad [c_1, c_{17}] &= (c_5 - c_6)(c_7 - c_4) + (c_{14} - c_6)(c_4 - c_2) \\
&\quad + c_{20} - c_{14} - c_{15} + c_{12}, \\
(4.55) \quad c_{10}c_7 &= c_3c_9 + (c_{19} - c_{13})c_2, \\
(4.56) \quad [c_7, c_{10}] &= (h_2 + 1)(c_{19} - c_{12}) + (c_{13} - c_{12})c_2 - c_{15} + c_{12}
\end{align}

one sees that (4.55) implies $\xi(c_{19}) = q$ and that (4.54) implies $\xi(c_{14}) = -\xi(h_3)$. Then, from (4.56), we obtain $(p + \xi(h_2))(q + \xi(h_3)) = 0$, which contradicts the assumptions on $p$ and $q$. Thus whenever $\xi \downarrow C(\alpha, \beta)$ is $T_4$, then $\xi \downarrow C(\alpha + \beta, \gamma)$ is also $T_4$.

We may now use Weyl automorphisms to complete the proof as follows. By direct computation one sees that $\xi \downarrow C(\alpha + \beta, \gamma)$ is $T_4$ if and only if $\xi \circ \sigma \circ \sigma_\beta \circ \sigma_\gamma \downarrow C(\alpha, \beta)$ is $T_4$. From this and the first part of the proof we have the implications

\begin{align}
\xi \downarrow C(\alpha + \beta, \gamma) &\Rightarrow \xi \circ \sigma \circ \sigma_\beta \circ \sigma_\gamma \downarrow C(\alpha, \beta) \Rightarrow T_4 \\
&\Rightarrow \xi \downarrow C(\alpha + \beta, \gamma) \Rightarrow T_4
\end{align}

Combining this with the first part gives us that $\xi \downarrow C(\alpha, \beta)$ and $\xi \downarrow C(\alpha + \beta, \gamma)$ are both $T_4$ or both $T_3$. From this and direct computation, we obtain the following chain of implications.

\begin{align}
\xi \downarrow C(\alpha, \beta) &\Rightarrow \xi \circ \sigma_\beta \downarrow C(\alpha, \beta) \Rightarrow T_4 \\
&\Rightarrow \xi \circ \sigma_\beta \downarrow C(\alpha + \beta, \gamma) \Rightarrow T_4 \\
&\Rightarrow \xi \downarrow C(\alpha, \beta + \gamma) \Rightarrow T_4
\end{align}

By combining the above arguments, we see that if any one of the restrictions is $T_4$ then they all are. $\square$

**Corollary 4.57.** If $\xi$ is a mass function on $C(\alpha, \beta, \gamma)$ with

$$S_\xi = \{ \pm \alpha, \pm \beta, \pm (\alpha + \beta) \},$$

then the restrictions of $\xi$ to each of $C(\beta, \gamma)$, $C(\alpha, \beta + \gamma)$, and $C(\alpha + \beta, \gamma)$ are all $T_1$ or all $T_6$.

**Proof.** By using Table 3.12, one can verify that the restriction of $\xi$ to each of $C(\beta, \gamma)$, $C(\alpha, \beta + \gamma)$ and $C(\alpha + \beta, \gamma)$ is either $T_1$ or $T_6$. According to Table 3.27, corresponding to each such restriction there is some Weyl automorphism $\sigma$ such that the same restriction of $\xi \circ \sigma$ is either $T_3$ or $T_4$. The corollary now follows from Lemma 4.49. $\square$

We postpone the proof that mass functions $\xi$ on $C(A_3)$ with $|S_\xi| = 0, 2, 4$ or 6 satisfy Theorem 1.7 until we are treating the general case. We conclude this section.
with an analysis of the structure of mass functions \( \zeta: C(A_3) \to F \) with \( |S_\zeta| = 2 \) and 4 which is similar to the analysis for \( |S_\zeta| = 6 \) and we establish two technical results about the structure of mass functions \( \zeta: C(A_4) \to F \).

**Lemma 4.58.** Let \( \zeta \) be a mass function on \( C(\alpha, \beta, \gamma) \) such that \( S_\zeta = \{ \pm \alpha, \pm \gamma \} \), then the following are equivalent.

(i) \( S_\zeta C(\alpha, \beta) \) is \( T_1 \).

(ii) \( S_\zeta C(\beta, \gamma) \) is \( T_4 \).

(iii) \( S_\zeta C(\alpha + \beta, \gamma) \) is \( T_1 \).

(iv) \( S_\zeta C(\alpha + \beta, \gamma) \) is \( T_4 \).

**Proof.** First we prove the implication

\[ \zeta \downarrow C(\alpha, \beta) \text{ is } T_1 \Rightarrow \zeta \downarrow C(\beta, \gamma) \text{ is } T_4. \]

Assume \( \zeta \downarrow C(\alpha, \beta) \) is \( T_1 \) so that \( \zeta(c_1) = p \neq 0, -\zeta(h_1), \zeta(c_2) = \zeta(c_4) = \zeta(c_7) = \zeta(c_9) = 0 \). If \( \zeta \downarrow C(\beta, \gamma) \) is not \( T_4 \), then according to Table 3.12, it must be \( T_3 \), and in this case we have

\[ \zeta(c_3) = \zeta(c_8) = \zeta(c_{10}) = q \neq 0, -\zeta(h_3), \]

\[ \zeta(c_5) = -\zeta(h_3), \quad \zeta(h_2) = 0. \]

Applying \( \zeta \) to the identities

\[ (4.59) \quad [c_7, c_8] = c_7(c_5 - c_3) + (c_4 - c_1)(c_8 - c_5) + c_{19} - c_{15}, \]

\[ (4.60) \quad [c_3, c_7] = c_{15} - c_{19} \]

and using the above values of \( \zeta \), we get \( \zeta(c_8) = \zeta(c_3) \), contrary to \( q \neq -\zeta(h_3) \).

Next we prove \( \zeta \downarrow C(\beta, \gamma) \) is \( T_4 \Rightarrow \zeta \downarrow C(\alpha, \beta) \) is \( T_1 \). The combined assumptions of \( \zeta \downarrow C(\beta, \gamma) \) being \( T_4 \) and \( \zeta \downarrow C(\alpha, \beta) \) being not \( T_1 \), imply that \( \zeta \downarrow C(\alpha, \beta) \) is \( T_6 \) and the following hold.

\[ \zeta(c_2) = \zeta(c_5) = \zeta(c_8) = \zeta(c_{10}) = 0, \]

\[ \zeta(c_3) = q \neq 0, -\zeta(h_3), \]

\[ \zeta(c_1) = p \neq 0, -\zeta(h_1), \]

\[ \zeta(c_4) = -\zeta(h_1), \]

\[ \zeta(c_7) = \zeta(c_9) = -p - \zeta(h_1). \]

When \( \zeta \) is applied to the identities (4.59) and (4.60) we obtain the contradiction \( (p + \zeta(h_1))q = 0 \). This proves the implication we are after.

The remaining equivalences can be obtained from this and direct computation through the use of appropriate Weyl automorphisms. We outline this by the following chain of equivalences.
\[ \zeta \downarrow C(\alpha + \beta, \gamma) \text{ is } T_4 \iff \zeta \circ \sigma_\alpha \downarrow C(\beta, \gamma) \text{ is } T_4, \]
\[ \iff \zeta \circ \sigma_\alpha \downarrow C(\alpha, \beta) \text{ is } T_1, \]
\[ \iff \zeta \downarrow C(\alpha, \beta) \text{ is } T_1, \]
\[ \iff \zeta \downarrow C(\beta, \gamma) \text{ is } T_4, \]
\[ \iff \zeta \circ \sigma_\gamma \downarrow C(\beta, \gamma) \text{ is } T_4, \]
\[ \iff \zeta \circ \sigma_\gamma \downarrow C(\alpha, \beta) \text{ is } T_1, \]
\[ \iff \zeta \downarrow C(\alpha, \beta + \gamma) \text{ is } T_1. \]

This concludes the proof of Lemma 4.58. \( \Box \)

**Lemma 4.61.** If \( \zeta: C(\alpha, \beta, \gamma) \to F \) is a mass function where

(i) \( S_\zeta = \emptyset \),

(ii) \( \zeta(h_1) \neq 0 \) and \( \zeta(h_3) \neq 0 \), and

(iii) \( \zeta(c_i) = 0 \) for \( i = 1, 2, \ldots, 5 \)

then \( \zeta(c_6) = 0 \).

**Proof.** Assume \( \zeta: C(\alpha, \beta, \gamma) \to F \) is a mass function satisfying (i) through (iii) and \( \zeta(c_6) \neq 0 \). Since \( S_\zeta = \emptyset \), and \( \zeta(c_6) \neq 0 \) we must have

\[ \zeta(c_6) = -\zeta(h_1 + h_2 + h_3) \neq 0. \]

Since \( \zeta \downarrow C(\alpha, \beta + \gamma) \) is a mass function with \( \zeta(h_1) \neq 0 \) then by Table 3.12 we must have \( \zeta(h_2 + h_3) = 0. \) Similarly, considering the restriction of \( \zeta \) to \( C(\alpha + \beta, \gamma) \), we have \( \zeta(h_1 + h_2) = 0 \) and hence \( \zeta(h_1) = -\zeta(h_2) = \zeta(h_3) \neq 0. \)

Table 3.12 and the conditions that \( \zeta(c_1) = \zeta(c_2) = \zeta(c_4) = 0 \) imply that \( \zeta(c_7) = \zeta(c_6) = 0. \) Similarly considering the restrictions of \( \zeta \) to \( C(\beta, \gamma) \), \( C(\alpha, \beta + \gamma) \) and \( C(\alpha + \beta, \gamma) \) yields \( \zeta(c_8) = \zeta(c_{10}) = \zeta(c_{11}) = \zeta(c_{13}) = 0 \) and \( \zeta(c_{12}) = \zeta(c_{14}) = -\zeta(h_2) \neq 0. \)

Then, using these values for \( \zeta \) in the identities

\begin{align*}
(4.62) & \quad [c_1, c_8] = c_{20} - c_{15}, \\
(4.63) & \quad [c_3, c_7] = c_{15} - c_{19}, \\
(4.64) & \quad [c_1, c_{15}] = c_1(c_{11} - c_8) - (h_1 + 1)(c_{15} - c_{11}) + c_{20} - c_{11}, \\
(4.65) & \quad [c_7, c_{10}] = (h_2 + 1)(c_{19} - c_{12}) + (c_{13} - c_{12})c_2 - c_{15} + c_{12}
\end{align*}

we obtain \( \zeta(h_2) = 0 \) which is a contradiction. \( \Box \)

**Lemma 4.66.** If \( \zeta: C(\alpha, \beta, \gamma) \to F \) is a mass function where \( S_\zeta = \{ \pm \alpha \} \), \( \zeta \downarrow C(\alpha, \beta) \) is \( T_5 \), \( \zeta \downarrow C(\alpha, \beta + \gamma) \) is \( T_6 \) and \( \zeta(c_3) = 0 \) then \( \zeta(h_3) = 0. \)

**Proof.** Assume \( \zeta: C(\alpha, \beta, \gamma) \to F \) satisfies the hypotheses and \( \zeta(h_3) \neq 0 \). The restriction of \( \zeta \) to \( C(\beta, \gamma) \) then must be either \( T_3 \) with \( p = 0 \) and \( \zeta(h_2) = 0 \) or \( T_5 \) with \( \zeta(h_2) = 0 \) and \( p = -\zeta(h_3) \). In either case \( \zeta(c_2) = -\zeta(h_3) \) and \( \zeta(c_8) = \zeta(c_{10}) = 0. \) A similar analysis of \( \zeta \downarrow C(\alpha + \beta, \gamma) \) yields \( \zeta(h_1 + h_2) = 0 \) and \( \zeta(c_{11}) = \zeta(c_{13}) = 0. \)
Since \( \zeta \downarrow C(\alpha, \beta) \) is \( T_1 \) and \( \zeta \downarrow C(\alpha, \beta + \gamma) \) is \( T_6 \), we have that \( \zeta(c_7) = \zeta(c_9) = 0 \) and 
\[ \zeta(c_{12}) = \zeta(c_{14}) = -\zeta(h_1 + h_2 + h_3) = \zeta(c_1). \]
Now applying \( \zeta \) to the identities
\[
\begin{align*}
[c_7, c_8] &= c_7(c_5 - c_3) + c_4(c_8 - c_3) - (c_8 - c_5)c_1 \\
- c_{12} + c_{14} + c_{19} - c_{20},
\end{align*}
\]
\[
\begin{align*}
[c_1, c_8] &= c_{20} - c_{15},
\end{align*}
\]
\[
\begin{align*}
[c_3, c_7] &= c_{15} - c_{19}
\end{align*}
\]
we have that \( \zeta(c_1)\zeta(h_3) = 0 \). This contradicts our assumptions and hence we may conclude that \( \zeta(h_3) = 0 \).

**Lemma 4.70.** If \( \zeta: C(\alpha, \beta, \gamma) \to F \) is a mass function where \( S_\zeta = \{ \pm \beta \} \), \( \zeta \downarrow C(\alpha, \beta) \) is \( T_4 \) and \( \zeta \downarrow C(\beta, \gamma) \) is \( T_4 \) then \( \zeta(c_6) = 0 \).

**Proof.** Assume that \( \zeta \) satisfies the conditions and \( \zeta(c_6) \neq 0 \). Since \( S_\zeta = \{ \pm \beta \} \) we have that \( \zeta(c_6) = -\zeta(h_1 + h_2 + h_3) \neq 0 \). Now \( \zeta \) satisfies the conditions of Lemma 4.66 relative to the base \( \{ \beta, \gamma, -\alpha - \beta - \gamma \} \) which implies that \( \zeta(h_1 + h_2 + h_3) = 0 \) contrary to our assumption. Therefore \( \zeta(c_6) = 0 \).

**Lemma 4.71.** There does not exist a mass function \( \zeta: C(A_4) \to F \) such that, relative to a base \( \{ \alpha, \beta, \gamma, \delta \} \) of the root system of \( A_4 \), we have:

(i) \( S_\zeta = \{ \pm \alpha, \pm \beta \} \),
(ii) \( \zeta \downarrow C(\alpha, \beta) \), \( C(\alpha, \beta + \gamma) \) and \( C(\gamma, \delta) \) are \( T_1 \),
(iii) \( \zeta \downarrow C(\alpha + \beta, \gamma) \) and \( C(\beta, \gamma) \) are \( T_4 \), and
(iv) \( \zeta \downarrow C(\alpha, \beta + \gamma + \delta) \) is \( T_6 \).

**Proof.** Assume that \( \zeta: C(\alpha, \beta, \gamma, \delta) \to F \) is a mass function which satisfies conditions (i) through (iv) and seek a contradiction. Since \( \zeta \downarrow C(\alpha, \beta + \gamma, \delta) \) and \( \zeta \downarrow C(\alpha, \beta, \gamma + \delta) \) satisfy the conditions of Lemma 4.66 we have that
\[
\zeta(h_\delta) = \zeta(h_\gamma + h_\delta) = 0.
\]
Also since \( \zeta \downarrow C(\alpha + \beta, \gamma, \delta) \) and \( \zeta \downarrow C(\beta, \gamma, \delta) \) satisfy the conditions of Lemma 4.70 we have that \( \zeta(h_\alpha + h_\beta + h_\gamma + h_\delta) = \zeta(h_\beta + h_\gamma + h_\delta) = 0 \) and hence we may assume that \( \zeta(h_\alpha) = \zeta(h_\beta) = \zeta(h_\gamma) = \zeta(h_\delta) = 0 \).

Let \( c = (Y_{\gamma+\delta}Y_{\alpha+\beta}X_{\beta+\gamma+\delta}X_{\alpha}Y_\delta Y_\gamma Y_{\gamma+\delta}) \in C(\alpha, \beta, \gamma, \delta) \). Since \( \zeta \downarrow C(\gamma, \delta) \) is \( T_1 \) we have \( \zeta(Y_\delta Y_\gamma Y_{\gamma+\delta}) = 0 \) and hence \( \zeta(c) = 0 \). On the other hand we have
\[
c = (Y_{\gamma+\delta}X_{\gamma+\delta})(Y_{\alpha+\beta}X_{\beta+\gamma+\delta}X_{\alpha}Y_\delta Y_\gamma)
+ (Y_{\gamma+\delta}Y_{\alpha+\beta}X_{\alpha}X_{\beta+\gamma+\delta} - Y_{\gamma+\delta}Y_{\alpha+\beta}X_{\alpha+\beta+\gamma+\delta})
\cdot (Y_\delta X_\delta - Y_\gamma X_\gamma - h_\gamma).
\]
Since \( \zeta \downarrow C(\gamma, \delta) \) is \( T_1 \) we have \( \zeta(Y_{\gamma+\delta}X_{\gamma+\delta}) = \zeta(Y_\delta X_\delta) = 0 \). Also by considering
\[ \xi \downarrow C(\alpha + \beta, \gamma + \delta) \] we have that \( \xi(Y_{\gamma + \delta} Y_{\alpha + \beta} X_{\alpha + \beta + \gamma + \delta}) = 0 \) and hence \( 0 = \xi(c) = -\xi(Y_{\gamma + \delta} Y_{\alpha + \beta} X_{\alpha + \beta + \gamma + \delta}) \xi(Y, X_{\gamma}). \) Finally by applying \( \xi \) to the following identities

\[
\begin{align*}
(4.72) & \quad [Y_{\alpha} X_{\alpha}, Y_{\gamma + \delta} Y_{\beta} X_{\beta + \gamma + \delta}] = Y_{\gamma + \delta} Y_{\beta} Y_{\alpha} X_{\alpha + \beta + \gamma + \delta} - Y_{\gamma + \delta} Y_{\alpha} X_{\alpha} X_{\beta + \gamma + \delta}, \\
(4.73) & \quad [Y_{\beta} X_{\beta}, Y_{\beta + \gamma + \delta} Y_{\alpha} X_{\alpha + \beta + \gamma + \delta}] = Y_{\beta + \gamma + \delta} Y_{\alpha} X_{\alpha + \beta + \gamma + \delta} X_{\beta} - Y_{\gamma + \delta} Y_{\beta} Y_{\alpha} X_{\alpha + \beta + \gamma + \delta}, \\
(4.74) & \quad (Y_{\beta + \gamma + \delta} Y_{\alpha} X_{\alpha + \beta + \gamma + \delta}) \cdot (Y_{\alpha + \beta} X_{\alpha}) = (Y_{\beta + \gamma + \delta} Y_{\alpha} X_{\alpha + \beta + \gamma + \delta}) (Y_{\alpha + \beta} X_{\alpha + \beta}) + (Y_{\beta + \gamma + \delta} X_{\beta + \gamma + \delta}) (Y_{\alpha + \beta} X_{\alpha} - Y_{\alpha + \beta} Y_{\alpha} + Y_{\beta + \gamma + \delta} Y_{\alpha} X_{\alpha + \beta} + Y_{\beta + \gamma + \delta} X_{\alpha + \beta} X_{\beta}).
\end{align*}
\]

we have that \( \xi(Y_{\gamma + \delta} Y_{\alpha + \beta} X_{\alpha + \beta + \gamma + \delta}) = -\xi(Y_{\alpha} X_{\alpha}) \) and hence \( 0 = \xi(c) = \xi(Y_{\alpha} X_{\alpha}) \xi(Y, X_{\gamma}) \) which contradicts our assumption that \( S_{\xi} = \{ \pm \alpha, \pm \gamma \}. \)

**Lemma 4.75.** There does not exist a mass function \( \xi : C(A_4) \rightarrow F \) such that, relative to a base \( \{ \alpha, \beta, \gamma, \delta \} \) of the root system of \( A_4, \) we have:

(i) \( S_{\xi} = \{ \pm \alpha, \pm \delta \}, \)

(ii) \( \xi \downarrow C(\alpha, \beta) \) is \( T_1, \)

(iii) \( \xi \downarrow C(\alpha, \beta + \gamma) \) is \( T_6 \) and

(iv) \( \xi \downarrow C(\gamma, \delta) \) is \( T_4. \)

**Proof.** If \( \xi : C(\alpha, \beta, \gamma, \delta) \rightarrow F \) were a mass function satisfying conditions (i) through (iv) then \( \xi \) satisfies the conditions of Lemma 4.71 relative to the base \( \{ \alpha, -\alpha - \beta - \gamma - \delta, \delta, \gamma \} \) and this is impossible. \( \square \)

**5. Proof of main theorem.** In this section, we complete the proof of Theorem 1.7. First, we introduce some notation and terminology.

Let \( \xi \) be a mass function on \( C(A_n), \) \( \Phi \) a root system of \( A_n \) and \( S_{\xi} = \{ \gamma \in \Phi | \xi(Y_{\gamma} X_{\gamma}) = 0 \ or \ -\xi(h_{\gamma}) \}. \) By Proposition 4.3, \( S_{\xi} = \bigcup_{i=1}^{l} \Phi_i \) where \( \Phi_i \) are mutually orthogonal irreducible root subsystems of \( \Phi. \) Let \( \Delta_i \) be a base of \( \Phi_i. \) By [6, Theorem 5.2], we can extend \( \bigcup_{i=1}^{l} \Delta_i \) to \( \Delta \) as a base of \( \Phi. \) By permitting some \( \Delta_i \)'s to be void if necessary, we may present the Dynkin diagram for \( A_n \) by

\[
\Delta_1 \alpha_{k_1} \Delta_2 \alpha_{k_2} \ldots \alpha_{k_{r-1}} \Delta_{r+1}
\]

where \( \Delta_j = \emptyset \) or \( \Delta_j' \) for some \( j \) and \( |\{ i | \Delta_i \neq \emptyset \}| = l. \) A base with such a presentation is called a \( \xi \)-base. By convention, we set \( k_0 = 0, \alpha_0 = 0 \) and \( k_{r+1} = n + 1. \) If \( \Delta \) is a \( \xi \)-base of the form (5.1) and \( \Delta \neq \emptyset \) then \( \Delta \) is assumed to be labeled such that \( \Delta_i = \{ \alpha_{k_{i-1}+1}, \alpha_{k_{i-1}+2}, \ldots, \alpha_{k_i-1} \}. \)
Let $\Delta$ be a $\zeta$-base of the form (5.1), $\Phi^+$ be the positive roots relative to this base, $\Phi_i$ be the set of roots generated by $\Delta_i$, and $\Phi^+_i = \Phi_i \cap \Phi^+$. For $1 \leq i < j \leq t + 1$, we define

\begin{align*}
(5.2) \quad B_{ij}(\Delta) &= \{ \alpha_1 + \cdots + \alpha_r | \alpha_i \in \Delta_i \cup \{ \alpha_k \} \text{ and } \alpha_r \in \Delta_j \cup \{ \alpha_k \} \}, \\
(5.3) \quad I_{ij}(\Delta) &= \{ (\alpha, \beta) | \alpha \in \Phi^+_i, \beta \in B_{ij}(\Delta) \text{ and } \alpha + \beta \in \Phi^+ \}, \\
(5.4) \quad J_{ij}(\Delta) &= \{ (\beta, \gamma) | \beta \in B_{ij}(\Delta), \gamma \in \Phi^+_j \text{ and } \beta + \gamma \in \Phi^+ \}.
\end{align*}

These definitions allow us to get control over the value of $\zeta$ on entire blocks of basic cycles as illustrated by

**Proposition 5.5.** If there exists a pair $(\alpha_0, \beta_0) \in I_{ij}(\Delta)$ such that $\zeta \downarrow C(\alpha_0, \beta_0)$ is $T_1$, or a pair $(\beta_0, \gamma_0) \in J_{ij}(\Delta)$ such that $\zeta \downarrow C(\beta_0, \gamma_0)$ is $T_4$, then $\zeta \downarrow C(\alpha, \beta)$ is $T_1$ for all $(\alpha, \beta) \in I_{ij}(\Delta)$ and $\zeta \downarrow C(\beta, \gamma)$ is $T_4$ for all $(\beta, \gamma) \in J_{ij}(\Delta)$.

**Proof.** Suppose first that $\Delta_i = \emptyset$ and $(\beta_0, \gamma_0) \in J_{ij}(\Delta)$ with $\zeta \downarrow C(\beta_0, \gamma_0)$ being $T_4$. Then $\beta_0 = \alpha_k + \cdots + \alpha_m$ and $\gamma_0 = \alpha_{m+1} + \cdots + \alpha_p$ where $\alpha_{m+1}, \ldots, \alpha_p \in \Delta_j$ and hence $\alpha_m \in \Delta_j \cup \{ \alpha_k \}$. By Lemma 4.49, we have that

$$\zeta \downarrow C(\beta_0, \gamma_0) \text{ is } T_4 \iff \zeta \downarrow C(\beta, \alpha) \text{ is } T_4.$$ 

Thus $\zeta \downarrow C(\beta_0, \gamma_0)$ is $T_4$ iff $\zeta \downarrow C(\beta, \gamma)$ is $T_4$ for all $(\beta, \gamma) \in J_{ij}(\Delta)$. Since $\Delta_i = \emptyset$ implies that $I_{ij}(\Delta) = \emptyset$ we have that the proposition is valid for $\Delta_i = \emptyset$.

A similar argument, applying Corollary 4.57, proves the proposition if $\Delta_j = \emptyset$. Thus we may assume that $\Delta_i \neq \emptyset \neq \Delta_j$. In this case the proposition follows directly from

**Lemma 5.6.** With notation as above and assuming $\Delta_i \neq \emptyset \neq \Delta_j$, we have:

i) for $(\alpha, \beta) \in I_{ij}(\Delta)$, $\zeta \downarrow C(\alpha, \beta)$ is $T_1$ iff $\zeta \downarrow C(\alpha_{k-1}, \alpha_k + \cdots + \alpha_{j-1})$ is $T_1$,

ii) for $(\beta, \gamma) \in J_{ij}(\Delta)$, $\zeta \downarrow C(\beta, \gamma)$ is $T_4$ iff $\zeta \downarrow C(\beta, \alpha)$ is $T_4$ for all $\alpha \in \Phi_i \cap \Phi^+$ or $\alpha' = 0$ and $\gamma' \in \Phi_j \cap \Phi^+$ or $\gamma' = 0$. Applying Corollary 4.57 and Lemma 4.58, we have the sequence of equivalent statements

\[ \zeta \downarrow C(\alpha, \beta) \text{ is } T_1 \iff (4.57) \zeta \downarrow C(\alpha', \tilde{\beta} + \gamma') \text{ is } T_1, \]

\[ \iff (4.58) \zeta \downarrow C(\alpha', \tilde{\beta}) \text{ is } T_1, \]

\[ \iff (4.57) \zeta \downarrow C(\alpha_{k-1}, \tilde{\beta}) \text{ is } T_1. \]

(Note that if $\alpha' = 0$ we may start at the second statement replacing $\alpha'$ by $\alpha$ to obtain the required result.)
(ii) Applying Lemmas 4.49 and 4.58, we have the sequence of equivalent statements

$$\xi \downarrow C(\beta, \gamma) is T_4 \Leftrightarrow (4.49) \xi \downarrow C(\alpha' + \overline{\beta}, \gamma') is T_4,$$

$$\Leftrightarrow (4.58) \xi \downarrow C(\overline{\beta}, \gamma') is T_4,$$

$$\Leftrightarrow (4.58) \xi \downarrow C(\alpha, \overline{\beta}) is T_1.$$

(Nota that if $\gamma' = 0$ we may start at the second statement replacing $\gamma'$ by $\gamma$ to obtain the required result.)

**Definition 5.7.** Let $\Delta$ be a $\xi$-base given by (5.1). Then the pair $(i, j)$ with $1 \leq i < j \leq t + 1$ is **bad relative to** $\Delta$ if

(i) when $|A, /A)| = 1$ we have

$$\xi(Y_{\alpha_{k_1} + \ldots + \alpha_{k_j-1}} X_{\alpha_{k_1} + \ldots + \alpha_{k_j-1}}) = -\xi(h_{\alpha_{k_1} + \ldots + h_{\alpha_{k_j-1}}} \neq 0$$

or

(ii) when $|B_{i,j}(\Delta)| > 1$ we have $\xi \downarrow C(\alpha, \beta)$ is $T_6$ for all $(\alpha, \beta) \in I_{i,j}(\Delta)$ and $\xi \downarrow C(\beta, \gamma)$ is $T_3$ for all $(\beta, \gamma) \in J_{i,j}(\Delta)$.

**Lemma 5.8.** For any mass function $\xi: C(A_n) \rightarrow F, n \geq 3$, there exists a $\xi$-base $\Delta$ such that there are no bad pairs relative to $\Delta$.

**Proof.** Our proof is by induction on $t$. For $t = 0$, the lemma is vacuously satisfied. Let $\xi$ be a mass function on $C(A_n)$ and $\Delta$ be a $\xi$-base having the property that among all such bases the number of bad pairs is minimal relative to $\xi$. Evidently we must show that this number is zero when $t > 0$.

Assume first that there is a bad pair $(i, j)$ relative to $\Delta$ with $i < j < t + 1$. If $\xi_1 = \xi \downarrow C(\alpha_1, \ldots, \alpha_{k_i-1})$ then by our induction, there exists a $\xi_1$-base $\Delta^{(1)}$ of the root subsystem generated by $\{\alpha_1, \ldots, \alpha_{k_i-1}\}$ such that there are no bad pairs relative to $\Delta^{(1)}$. Also, there is some product $\sigma$ of Weyl reflections $\sigma_\alpha$ with $\alpha \in \{\alpha_1, \ldots, \alpha_{k_i-1}\}$ such that $\sigma(\alpha_1, \ldots, \alpha_{k_i-1}) = \Delta^{(1)}$. From the form of $\sigma$, we have that $\sigma$ leaves $\Delta_{i+1}$ pointwise fixed and maps $B_{i,t+1}(\Delta)$ to some $B_{k_{i+1},i+1}(\Delta)$. Therefore, the number of bad pairs of $\xi$ relative to $\Delta$ is strictly less than the number of bad pairs of $\xi$ relative to $\Delta$. This contradiction implies that there are no bad pairs $(i, j)$ of $\xi$ relative to $\Delta$ with $i < j < t + 1$.

By a similar argument, there are no bad pairs $(i, j)$ of $\xi$ relative to $\Delta$ with $1 < i < j$. Thus, if $\xi$ has a minimum number of bad pairs relative to $\Delta$, then the only possible bad pair must be $(1, t + 1)$.

We now assume that $(1, t + 1)$ is the only bad pair of $\xi$ relative to $\Delta$ and seek a contradiction. Our argument splits into two cases.

**Case I.** One of $\Delta_i$ or $\Delta_{i+1}$ is nonempty. If $\Delta_i = \emptyset$ and $\Delta_{i+1} \neq \emptyset$ we can replace the $\xi$-base $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ by the $\xi$-base $\hat{\Delta} = \{\alpha_n, \ldots, \alpha_1\}$. From Table 3.12 we have that $\xi \downarrow C(\alpha, \beta)$ is $T_1$ iff $\xi \downarrow C(\beta, \alpha)$ is $T_4$. It follows then that $(1, t + 1)$ is the only bad pair of $\xi$ relative to $\hat{\Delta}$ and moreover $\hat{\Delta} \neq \emptyset$. Therefore, without loss of generality we may assume $\Delta_i \neq \emptyset$. 

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If $t = 1$ then there is only one pair $(1, 2)$,
\[ \tilde{A} = \{ \alpha_{k_1-1}, \ldots, \alpha_1, -(\alpha_1 + \cdots + \alpha_n), \alpha_n, \ldots, \alpha_{k_1-1} \} \]
is a $\zeta$-base, and $\zeta \in C(\alpha_1, -(\alpha_1 + \cdots + \alpha_n))$ is $T_1$. Hence by Proposition 5.5 the only pair relative to $\tilde{A}$ is not bad. This contradiction completes Case I for $t = 1$.

If $t = 2$ we consider two subcases.

**Subcase (i).** $\Delta_t \neq \emptyset$. In this case, $\zeta$ restricted to
\[ C(\alpha_1, \alpha_2 + \cdots + \alpha_{k_1-2}, \alpha_{k_1} + \cdots + \alpha_n) \]
satisfies the hypothesis of Lemma 4.71 and hence there is no such mass function.

**Subcase (ii).** $\Delta_t = \emptyset$. If $\Delta_{t+1} \neq \emptyset$ then $\zeta$ restricted to
\[ C(\alpha_1, \alpha_2 + \cdots + \alpha_{k_1-1}, \alpha_{k_1} + \cdots + \alpha_n) \]
satisfies the hypothesis of Lemma 4.75 and no such mass function exists. Thus we may assume that $\Delta_t = \Delta_{t+1} = \emptyset$. Then $\zeta$ restricted to $C(\alpha_1, \alpha_2 + \cdots + \alpha_{n-1}, \alpha_n)$ satisfies the hypothesis of Lemma 4.66 and hence $\zeta(h_{\alpha}) = 0$. Now $\alpha_{n-1}$ has a $\zeta$-base and $\zeta$ admits exactly one bad pair namely $(1, t)$ (and hence a minimum number) relative to $\sigma_{n-1}$. This contradicts our previous argument again and hence completes this subcase.

**Case II.** $\Delta_1 = \emptyset = \Delta_{t+1}$. If, in addition, there exists some $i$ with $\Delta_i \neq \emptyset$ the restriction of $\zeta$ to $C(\alpha_1 + \cdots + \alpha_i-1, \alpha_i, \alpha_{i+1} + \cdots + \alpha_n)$ satisfies the hypothesis of Lemma 4.70 and hence
\[ \zeta(Y_1, \ldots + \alpha_i, X_1, \ldots + \alpha_n) = 0. \]
Since $|B_{t+1}(\Delta)| = 1$, this implies that the pair $(1, t + 1)$ is not bad contrary to our assumption. Thus we may assume that $\Delta_t = \emptyset$ for $1 \leq i \leq t + 1$. This implies that $t = n$, $S_t = \emptyset$, $|B_{t+1}(\Delta)| = 1$ for $1 \leq i < j \leq n + 1$ and $\zeta(Y_\beta X_\beta) = 0$ for all positive roots $\beta$ except $\alpha_1 + \cdots + \alpha_n$.

If $\zeta(h_{\alpha_i}) = 0$ or $\zeta(h_{\alpha_j}) = 0$ then by an argument similar to subcase (ii) of Case I we arrive at a contradiction and hence we may assume that $\zeta(h_{\alpha_i}) \neq 0 \neq \zeta(h_{\alpha_j})$. From this we see that $\zeta \in C(\alpha_1, \alpha_2 + \cdots + \alpha_{n-1}, \alpha_n)$ satisfies the hypothesis of Lemma 4.61 and hence $\zeta(Y_1, \ldots + \alpha_n X_1, \ldots + \alpha_n) = 0$—i.e. the $(1, t + 1)$ pair is not bad. This contradiction completes this case and hence the entire theorem. \(\square\)

We now have all the facts needed to prove our main theorem.

**Proof of Theorem 1.7.** In §3, we proved this result for $n = 1$ and 2. Let $\Delta$ be a $\zeta$-base as given to us by Lemma 5.8 so that $\Delta$ has form (5.1). By Corollary 4.47, there is an evaluation map $\varepsilon: F[\lambda_1, \ldots, \lambda_n, s_1, \ldots, s_n] \to F$ such that
\[ \zeta \downarrow C(\Delta_i) = \varepsilon \circ \Psi_{\Delta, \Delta} \downarrow C(\Delta_i). \]
Since $\Delta$ is a $\zeta$-base described by Lemma 5.8, $\zeta$ is zero on all degree 2 and 3 basic cycles of $C(A_n)$ relative to $\Delta$ which are not in $\cup_{i=1}^{t+1} C(\Delta_i)$. Therefore, $\zeta(c) = \varepsilon \circ \Psi_{\Delta, \Delta}(c)$ for all basic cycles $c$ of degree $\leq 3$. By Proposition 4.1, this implies that $\zeta = \varepsilon \circ \Psi$. \(\square\)
6. Pointed representations of $A_n$. In the introduction we proved that corresponding to each mass function $\xi: C(A_n) \to F$ there exists a unique (up to equivalence) pointed representation of $A_n$. Unfortunately this correspondence is not one-one since a pointed representation of $A_n$ may admit many 1-dimensional weight spaces. In this section we indicate how one may select a set of mass functions which label the pointed representations of $A_n$ in a one-one fashion. A first step in this direction is provided by the following theorem.

**Theorem 6.1** [11, Theorem 4]. If $\xi_1, \xi_2: C(A_n) \to F$ are mass functions then the pointed representations corresponding to $\xi_1$ and $\xi_2$ are equivalent iff there exist elements $x, y \in U(A_n)$ such that $yx \in C(A_n)$, $\xi_2(yx) = 1$ and $\xi_1(c) = \xi_2(ycx)$ for all $c \in C(A_n)$.

To motivate our approach to the general labelling problem, we first review the known results on highest weight (Verma) representations and show how these can be interpreted in the setting of pointed representations. Fix a base $\Delta$ of the root system of $A_n$ and let $\Psi: C(A_n) \to F[\lambda_1, \ldots, \lambda_n]$ be the usual Harish-Chandra homomorphism relative to $\Delta$. For any evaluation map $\varepsilon: F[\lambda_1, \ldots, \lambda_n] \to F$, $\xi = \varepsilon \circ \Psi$ is a mass function and moreover it is clear that the left ideal $I_\xi$ of $U(A_n)$ generated by $\{X_\beta | \beta \in \Phi^+\} \cup \{h_a - \xi(h_a) \cdot 1 | a \in \Delta\}$ contains the kernel of $\xi$. Then the regular representation of $A_n$ on $U(A_n)/I_\xi$ is the Verma representation with highest weight $\lambda = \xi \downarrow \Delta$ relative to the base $\Delta$. By the argument given in the proof of Theorem 1.3(ii), $I_\xi$ is contained in the unique maximal left ideal $M_\xi$ of $U(A_n)$ which contains $\ker \xi$. Thus, for any mass function $\xi$, as described above, the associated pointed representation of $A_n$ admits $\xi \downarrow \Delta$ as its highest weight relative to the base $\Delta$. Conversely, if $(\rho, V)$ is an irreducible representation which admits $\lambda \in H^*$ as a highest weight relative to $\Delta$ then the mass function corresponding to this 1-dimensional weight space $V_\lambda$ is equal to $\varepsilon \circ \Psi$ where $\varepsilon: F[\lambda_1, \ldots, \lambda_n] \to F$ with $\varepsilon(\lambda_i) = \lambda(h_{\alpha_i})$ for all $\alpha_i \in \Delta$.

If we restrict our attention to irreducible representations having a highest weight relative to a fixed base $\Delta$, it is well known that such representations are labelled in a one-one fashion by their highest weight. To address the analogous question for pointed representations we first restrict our attention to those pointed representations which are associated with mass functions of the form $\varepsilon \circ \Psi_{\Delta, \Delta}$ for some fixed base $\Delta$. These representations will be called $\Delta$-pointed representations. Theorem 1.7 states that every pointed representation is a $\Delta$-pointed representation for some base $\Delta$. We now specify a set of mass functions which label the $\Delta$-pointed representations in a one-one fashion.

**Definition 6.2.** A mass function $\xi: C(A_n) \to F$ is said to be $\Delta$-complete for some base $\Delta$ of the root system of $A_n$ iff $\xi = \varepsilon \circ \Psi_{\Delta, \Delta}$ where the evaluation map $\varepsilon: F[\lambda_1, \ldots, \lambda_n, s_1, \ldots, s_n] \to F$ satisfies the two conditions

(i) $\varepsilon(s_i) \notin Z \cup (Z + \xi(h_{\alpha_i})) \cup \cdots \cup (Z + \xi(h_{\alpha_1} + \cdots + h_{\alpha_n}))$ where $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ and
(ii) $\epsilon \circ \Psi_{\Delta, \Delta} \downarrow H = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n$ where the coefficients $a_i$ are normalized as follows. Select a linear basis of $F$ as a rational vector space where the first basis vector is 1. Each $a_i$, when expressed in terms of this basis must have first coordinate greater than or equal to 0 and strictly less than 1.

In [14] it is proven that the pointed representation of $A_n$ associated with a mass function $\xi = \epsilon \circ \Psi_{\Delta, \Delta}$ where $\epsilon(s_1)$ satisfies condition (i) in Definition 6.2 has the property that its set of weight functions consist of $\{\xi \downarrow H + \sum_{i=1}^{n} n_i\alpha_i | n_i \in \mathbb{Z}\}$ and each of these weight spaces is one-dimensional. The role of condition (ii) in Definition 6.2 is to provide a method to select a unique mass function for such representations. In a sense we are selecting the unique mass function which is closest to the center in the lattice of weights of such representations.

**Definition 6.3.** A mass function $\xi : C(A_n) \rightarrow F$ is said to be $\Delta$-extreme for a base $\Delta$ iff there exists a subset $\Delta'$ of $\Delta$ such that $\xi = \epsilon \circ \Psi_{\sigma, \Delta}$ where the evaluation map $\epsilon$ satisfies the property that for each nonempty connected component $\Delta_i$ of $\Delta'$, the restriction of $\xi$ to $C(\Delta_i)$ is $\Delta_i$-complete.

Under the assumption that Theorem 1.7 holds, it has been shown in [15] that for each pointed representation $(\rho, V)$ of $A_n$ there exists a base $\Delta$ such that $V$ admits a $\Delta$-extreme mass function. It is also shown that distinct $\Delta$-extreme mass functions yield inequivalent pointed representations. Thus the $\Delta$-extreme mass functions play the same role for $\Delta$-pointed representations as the elements of $\mathfrak{h}^*$ play for irreducible representations admitting a highest weight with respect to $\Delta$. In particular it should be noted that for any $\lambda \in \mathfrak{h}^*$ the mass function $\xi = \epsilon \circ \Psi_{\sigma, \Delta}$ with $\epsilon(\lambda_i) = \lambda(h_{\alpha_i})$ for all $\alpha_i \in \Delta$ is $\Delta$-extreme and yields the irreducible representation of $A_n$ admitting $\lambda$ as a highest weight relative to $\Delta$.

The general problem of classifying all irreducible representations of $A_n$ is as yet an unsolved problem. Block [3] has recently completed a classification of all irreducible representations for the case $n = 1$. In order to place the category of all pointed representations of $A_n$ in the perspective of all irreducible representations of $A_n$ we summarize some of the known types of such representations. In [1, 13] it is shown that there exist irreducible representations of $A_n$ (even of $A_1$) which do not admit a weight space decomposition with respect to any Cartan subalgebra. It is known [10] that if an irreducible representation of $A_n$ admits a weight space decomposition then either all weight spaces are infinite dimensional or all are finite dimensional. There are examples [7] of irreducible representations of $A_n$ having weight space decompositions with all weight spaces being infinite dimensional. More recently, Benkart [2] has constructed an irreducible representation of $A_2$ with all weight spaces being 2-dimensional by choosing an irreducible submodule of the tensor product of a fundamental representation of $A_2$ and a pointed representation coming from an $(\alpha, \beta)$-complete mass function.

Combining the results of this paper with the general remarks in [14] we can construct large families of mass functions for the other finite dimensional simple Lie algebras over $F$ and hence construct pointed representations for these simple Lie algebras. The question of whether these exhaust all pointed representations is an open question.
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