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FRÉCHET SPACES WITH NUCLEAR KÖTHE QUOTIENTS[1]

BY

STEVEN F. BELLENOT AND ED DUBINSKY

ABSTRACT. Each separable Fréchet non-Banach space X with a continuous norm is shown to have a quotient Y with a continuous norm and a basis. If, in addition, Y can be chosen to be nuclear, we say that X has a nuclear Köthe quotient. We obtain a (slightly technical) characterization of those separable Fréchet spaces with nuclear Köthe quotients. In particular, separable reflexive Fréchet spaces which are not Banach (and thus Fréchet Montel spaces) have nuclear Köthe quotients.

The problem of determining what kinds of subspaces and quotients can be found in arbitrary Fréchet spaces arises not only from the aesthetic imperative to understand the internal structure of these spaces but also from certain applications. One recent example is the construction, in nuclear Fréchet spaces, of subspaces and quotients which do not have certain approximation properties such as basis, strong finite dimensional decomposition, etc. **[1]**.

In this paper we are specifically concerned with the determination of those Fréchet spaces which have quotients that are nuclear, admit continuous norm and have a basis. We call such spaces *nuclear Köthe* spaces. We obtain a characterization which permits us to answer most reasonable questions, but there are some fine points which we are unable to settle.

This problem has a history which relates to subspaces as well as quotients and also involves the (rather trivial) case in which the quotient is not required to have a continuous norm. Therefore we begin, in §1, with a very brief historical discussion of known results.

§2 lists the generally standard definitions and notations. In §3 we give the main construction that forms the basis of our results. The characterization is established in §4 and §5 is devoted to concrete situations and open questions.

1. History of the problem. The case of subspaces was solved some time ago. In 1957, C. Bessaga and A. Pełczyński **[2]** showed that *a Fréchet space fails to admit a continuous norm iff it has a subspace isomorphic to* ω. If a Fréchet space admits a continuous norm then so does every subspace, which simplifies the problem a little. In 1959, Bessaga, Pełczyński and S. Rolewicz showed that *a Fréchet space which admits continuous norm has a nuclear Köthe subspace iff it is not Banach* **[4]**. Thus it

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follows that, in general, *a Fréchet space has a nuclear Köthe subspace iff it has a non-Banach subspace which admits continuous norm.*

The situation with quotients is somewhat more complicated. In the first place, a Fréchet space which admits continuous norm can have a quotient which does not. Indeed, in 1936, M. Eidelheit showed that *any non-Banach Fréchet space has a quotient isomorphic to* ω [**9**]. Another complication is that in order to obtain results one usually has to assume that the original space is separable. The nonseparable case seems to be much more difficult—even for Banach spaces. Thus, for the rest of this paper we will be looking for nuclear Köthe quotients in separable Fréchet spaces.

Of course our space must be non-Banach and different from ω. Even so it might fail to have a nuclear Köthe quotient. For example, in §5 we will see that this is the case for countable products of Banach spaces and also for a slightly larger collection of Fréchet spaces called quojections.

On the other hand, if we consider only nuclear Fréchet spaces, then we are looking for Köthe quotients and here the problem has a positive solution: *every nuclear Fréchet space not isomorphic to* ω *has a Köthe quotient.* The proof of this fact which appears in [**7**, p. 42] is completely false. A correct proof is given in [**8**].

2. Definitions and notations. We use many concepts from the standard theory of locally convex spaces. For definitions and notations not explicitly explained we refer to the book of G. Köthe [**12**].

Generally we will be considering a locally convex space E which will be either normed, metrizable, Banach, or Fréchet. The completion of E will be denoted $(E)\hat{}$. The dual of E will be indicated by E' and, unless otherwise stated, will be considered to have the strong topology from E. The symbol E'' will stand for the dual of E'. If A is a subset of either E or E' we will denote its polar in E' or E by A°. We recall that if E is Fréchet, then in view of the uniform boundedness theorem, "bounded subset of E'" is an unambiguous phrase.

We recall that a *Fréchet space* E is a projective limit of operators $A_k\colon E_{k+1} \to E_k$ $(k = 1, 2, \ldots)$ on Banach spaces. That is, E is the set of all sequences $(x_k) \ni x_k = A_k x_{k+1}$ $(k = 1, 2, \ldots)$ with the product topology. The maps $P_j\colon E \to E_j$ $(j = 1, 2, \ldots)$ defined by $P_j((x_k)) = x_j$ are called the *canonical projections*. If E is such a projective limit with each A_k a one-to-one map, then we say that E is *countably normed*.

The topology of a metrizable locally convex space E is determined by a *fundamental sequence of seminorms* $(\|\cdot\|_k)$. That is, each $\|\cdot\|_k$ $(k = 1, 2, \ldots)$ is a seminorm on E and (x_ν) converges to 0 in E iff, for each k, $(\|x_\nu\|_k)$ converges to 0 as a sequence of real numbers. We always assume that $(\|\cdot\|_k)$ is "increasing", that is, $\exists$ a sequence of positive constants, $(C_k) \ni \|x\|_k \leq C_k \|x\|_{k+1}$ $(x \in E,\ k = 1, 2, \ldots)$. If E has the property that $(\|\cdot\|_k)$ can be chosen $\ni$ each $\|\cdot\|_k$ is a norm then we say that E *admits continuous norm*. Obviously this is the same as the existence of a single continuous norm defined on E.

If $\|\cdot\|$ is a seminorm on a locally convex space E its *dual norm* (not necessarily a seminorm) is a function $\|\cdot\|'$ on E' defined by

$$\|u\|' = \sup\{|u(x)| : x \in E,\ \|x\| \leq 1\} \qquad (u \in E').$$

Clearly $\|\cdot\|'$ is a norm if restricted to a vector subspace of E on which it is finite.

A sequence (x_n) in a locally convex space E is a *basis* if for each $x \in E$ $\exists$ unique expansion $x = \sum_n t_n x_n$, t_n scalars. In this case, the sequence $(f_n) \subset E'$ defined by $f_n(x_m) = \delta_{nm}$ (Kronecker delta) is called the *dual basis*. A *basic sequence* in E is a sequence that is a basis for the closed subspace it generates.

A sequence (x_n) in a normed space is *α-basic*, where $\alpha > 0$ if it is basic and the projection of the space generated by $x_1, \ldots, x_m$ onto the space generated by $x_1, \ldots, x_n$ along the space generated by $x_{n+1}, \ldots, x_m$ has norm $\leqslant \alpha$ for all $n \leqslant m$. The smallest such number is called the *basis constant* of (x_n).

Unless otherwise stated the term "subspace" means closed vector subspace. The reader is invited to insert the term infinite dimensional where appropriate to avoid trivial situations.

If E is a locally convex space and M is a vector subspace of E then E/M denotes the usual quotient with the usual quotient topology. The map T: $E \to E/M$ defined by $Tx = x + M$ will be called the canonical quotient map. If $\|\cdot\|$ is a seminorm on E then the *seminorm induced by* $\|\cdot\|$ *on* E/M will be the seminorm $|\cdot|$ given by

$$|Tx| = \inf\{\|y\| : Ty = Tx\}.$$

The expressions $[x_i]$, $[x_i]_{i=1}^n$ refer to the closed subspaces generated by the sequences $(x_i)_{i=1}^\infty$, $(x_i)_{i=1}^n$ respectively.

We denote by ω the Fréchet space of all sequences of scalars (product topology) and by c_0 the Banach space of all sequences of scalars which converge to 0 (sup topology).

The symbol $\mathbf{N}$ stands for the set of positive integers.

3. Basic construction. In this section we consider a separable Fréchet space E for which there exists a biorthogonal sequence (x_n, f_n) in $E \times E'$ satisfying certain conditions. We show (Theorem 1) that such a space has a nuclear Köthe quotient.

There are several approaches to constructing quotients with bases in a Fréchet space. One is to use the Mazur selection method **[3]** on the Banach spaces determined by the bounded sets in the dual. It was shown in **[8]** however that this does not lead to a basic sequence in the strong topology of the dual. Another method is to embed the dual in the dual of a Köthe space—in analogy with the embedding of a separable Banach space in $C[0, 1]$—and construct a basic sequence in the dual by considering block basic sequences and using stability theorems. This approach was used successfully in **[8]** but it seems to require that the original space be nuclear.

In this paper our approach is based on a variation of the construction of W. B. Johnson and H. P. Rosenthal **[10]**. This is worked out in Proposition 1. The main difference is that we work with a normed space rather than its completion and instead of requiring that the sequence (f_n) converge weakly (from E) to 0 we have it eventually vanish on certain finite sets whose union is dense. By this means we are able, in the proof of Theorem 1, to avoid serious difficulties that occur when one completes the canonical normed spaces determined by seminorms in a Fréchet space. With the help of Lemmas 2, 3 we are able to apply a diagonalization procedure to construct the desired quotient.

We begin with a selection theorem which follows from Helly's condition [**11**, p. 151].

LEMMA 1. *Let* $(X, \|\cdot\|)$ *be a normed space,* $(u_i)_{i=1}^n \subset X'$, $\phi \in ([u_i]_{i=1}^n)'$ *with* $\|\phi\|' = 1$ *and let* $\varepsilon > 0$. *Then* $\exists\, x \in X$ *with* $1 \leqslant \|x\| < 1 + \varepsilon$ *and* $\phi(u_i) = u_i(x)$ $(i = 1, \ldots, n)$.

The next result is our main tool in constructing quotients.

PROPOSITION 1. *Let* $(E_0, \|\cdot\|)$ *be a normed space,* (d_n) *a dense sequence whose span is* E_0, $(f_n) \subset E_0'$ *with* $f_m(d_n) = 0$ *for m sufficiently large (depending on n) and* $(x_n) \subset E_0$ *biorthogonal to* (f_n).

Then $\exists$ *a subsequence of indices* $(n(i))_i \ni$ *if* $M = \bigcap_i \ker f_{n(i)}$ *and* $T\colon E_0 \to E_0/M$ *is the canonical quotient map, it follows that* $(T(x_{n(i)}))_i$ *is* 4-*basic in* E_0/M *and spans* E_0/M.

PROOF. We may assume that $\|f_n\|' = 1$. We will choose sequences (ε_i) of positive numbers $(n(i))$ of indices and (D_i) of finite subsets of $E_0 \ni$

(1) $D_i \subset D_{i+1}$.

(2) $f_{n(i+1)}(D_i) = 0$.

(3) $d_i/\|d_i\| \in D_i$ and each element of D_i is one of $d_n/\|d_n\|$ $(n \in \mathbf{N})$.

(4) If $\phi \in ([f_{n(l)}]_{l=1}^i)'$ with $\|\phi\| = 1$ then $\exists\, x \in D_i \ni$

$$|\phi(f) - f(x)| < \varepsilon_i \|f\|' \quad \forall f \in [f_{n(l)}]_{l=1}^i.$$

(5) $0 < \varepsilon_i < 1$, $\prod_{l=1}^i (1 - \varepsilon_l) > \frac{1}{2}$ and $\varepsilon_i \sum_{l=1}^i \|x_{n(l)}\| \leqslant 1$.

Our construction is by induction. We begin by setting $n(1) = 1$ and $\varepsilon_1 = \min\{\frac{1}{4}, 1/\|x_1\|\}$. Let ϕ be as supposed in (4). By Lemma 1 (with $n = 1$) $\exists\, x \in E_0$ with $f_1(x) = \phi(f_1)$ and $1 \leqslant \|x\| < 1 + \varepsilon_1/3$. We choose $m \ni \|d_m - x\| \leqslant \varepsilon_1/3$. Now we calculate,

$$\left|\phi(f_1) - f_1\left(\frac{d_m}{\|d_m\|}\right)\right| = \left|f_1(x) - f_1\left(\frac{d_m}{\|d_m\|}\right)\right| \leqslant \left\|x - \frac{d_m}{\|d_m\|}\right\| \|f_1\|'$$
$$\leqslant (\|x - d_m\| + |\,\|d_m\| - 1\,|)\|f_1\|' \leqslant \left(\frac{\varepsilon_1}{3} + \frac{2\varepsilon_1}{3}\right)\|f_1\|' = \varepsilon_1 \|f_1\|'.$$

Thus if we set $D_1 = \{d_1/\|d_1\|, d_m/\|d_m\|\}$ we have (4) and also (3). Our choice of ε_1 gives (5) and (1), (2) are vacuous.

Now suppose the choice has been made for $1, 2, \ldots, i-1$. We choose $n(i)$ large enough so that (2) holds and ε_i so that (5) holds. Let $\phi_1, \ldots, \phi_k$ be an $\varepsilon_i/4$-net in the unit sphere of $([f_{n(l)}]_{l=1}^i)'$. We apply Lemma 1 to each ϕ_j and obtain $x_j \in E_0$ with $1 \leqslant \|x_j\| < 1 + \varepsilon_i/4$ and $\phi_j(f_{n(l)}) = f_{n(l)}(x_j)$ $(j = 1, \ldots, k;\ l = 1, \ldots, i)$. Then for each j we have $m(j) \ni \|x_j - d_{m(j)}\| \leqslant \varepsilon_i/4$. Let ϕ be as supposed in (4). We choose

$j \ni \|\phi - \phi_j\| \leqslant \varepsilon_i/4$. Then given $f \in [f_{n(l)}]_{l=1}^i$ we calculate

$$\left|\phi(f) - f\left(\frac{d_{m(j)}}{\|d_{m(j)}\|}\right)\right| \leqslant |\phi(f) - \phi_j(f)| + |\phi_j(f) - f(x_j)| + \left|f(x_j) - f\left(\frac{d_{m(j)}}{\|d_{m(j)}\|}\right)\right|$$

$$\leqslant \|\phi - \phi_j\| \, \|f\|' + \left\|x_j - \frac{d_{m(j)}}{\|d_{m(j)}\|}\right\| \, \|f\|'$$

$$\leqslant \big(\varepsilon_i/4 + \|x_j - d_{m(j)}\| + |\,\|d_{m(j)}\| - 1\,|\big)\|f\|' < \varepsilon_i \|f\|'.$$

Thus if we set $D_i = D_{i-1} \cup \{d_i/\|d_i\|\} \cup \{d_{m(j)}/\|d_{m(j)}\|\}_{j=1}^k$ we have (1), (3) and (4), which completes the induction.

Next we show that $(f_{n(i)})_i$ is a basic sequence in E_0'. Suppose that $\|\Sigma_{l=1}^i \alpha_l f_{n(l)}\|' = 1$. Then $\exists\, \phi \in ([f_{n(l)}]_{l=1}^i)'$ with

$$1 = \|\phi\| = \phi\left(\sum_{l=1}^{i} \alpha_l f_{n(l)}\right)$$

so by (4) $\exists\, x \in D_i \ni$

$$\left|1 - \sum_{l=1}^{i} \alpha_l f_{n(l)}(x)\right| < \varepsilon_i.$$

Hence, in view of (2) and the fact that $x \in D_i$ implies $\|x\| = 1$, we have,

$$\left\|\sum_{l=1}^{i+1} \alpha_l f_{n(l)}\right\|' \geqslant \left|\sum_{l=1}^{i+1} a_l f_{n(l)}(x)\right| = \left|\sum_{l=1}^{i} \alpha_l f_{n(l)}(x)\right| > 1 - \varepsilon_i.$$

Therefore, by (5), $(f_{n(i)})$ is basic with constant $\leqslant 2$.

Now we define $S\colon E_0 \to ([f_{n(i)}]_i)'$ by $(Sx)(f) = f(x)$. Let (y_i) be the sequence in $([f_{n(i)}]_i)'$ biorthogonal to $(f_{n(i)})_i$ ad let Y be the vector subspace it generates. If $x \in E_0$, then since (d_n) spans E_0 it follows from (3) that x is in the span of D_i for some i and so by (1), (2) we have

$$Sx = \sum_{l=1}^{i} f_{n(l)}(x) y_l$$

so $S(E_0) \subset Y$. On the other hand $Sx_{n(i)} = y_i$ since they agree on each $f_{n(i)}$. Hence $S(E_0) = Y$. Moreover it follows from standard duality that (y_i) is 2-basic in Y.

It is clear that $\ker S = M$ and so we have the canonical map $S_0\colon E_0/M \to Y$ uniquely defined by the relation $S_0 T = S$. The map S_0 is 1-1, onto and continuous with norm 1. We will calculate $\|S_0^{-1}\|$.

Let $y \in Y \subset ([f_{n(l)}]_l)'$ and suppose $\|y\| = 1$. Fix $\delta > 0$. Since $(f_{n(l)})_l$ is 2-basic we can choose $i \ni y \in [y_l]_{l=1}^i$ and if we consider that $y \in ([f_{n(l)}]_{l=1}^i)'$ with operator norm $|||\cdot|||$, then $\frac{1}{2} - \delta \leqslant |||y||| \leqslant 1$. Thus we can choose $\lambda \in [1, 2/(1 - 2\delta)] \ni |||\lambda y||| = 1$. Applying (4) we have $x \in D_i$ with

$$|\lambda y(f_{n(l)}) - f_{n(l)}(x)| < \varepsilon_i \qquad (l = 1, \ldots, i).$$

Set $\beta_l = \lambda y(f_{n(l)}) - f_{n(l)}(x)$ and $w = x + \sum_{l=1}^{i} \beta_l x_{n(l)}$. Then it follows from (2) and the biorthogonality that $\lambda y(f_{n(l)}) = f_{n(l)}(w)$ ($l \in \mathbf{N}$) so $Sw = \lambda y$. Moreover, from (5), since $\|x\| = 1$,

$$\|w\| \leq 1 + \varepsilon_i \sum_{l=1}^{i} \|x_{n(l)}\| \leq 2.$$

Therefore we have $S_0 T(\frac{1}{\lambda}w) = y$ and $\|T(\frac{1}{\lambda}w)\| \leq 2$. This shows that $\|S_0^{-1}\| \leq 2$ and so $(S_0^{-1}y_i)_i$ is 4-basic in E_0/M, and it spans E_0/M. Since $y_i = Sx_{n(i)} = S_0 T x_{n(i)}$ we have $Tx_{n(i)} = S_0^{-1}y_i$ and the result is proved. □

The next two lemmas are simple facts that will be useful in Theorem 1 when we pass to subsequences. We include the straightforward proofs for completeness.

LEMMA 2. *Let X be a normed space and (x_n, f_n) a biorthogonal sequence $\ni (x_n)$ is K-basic and spans X. Let $(n(i))$ be a subsequence of indices, $M = \bigcap_i \ker f_{n(i)}$ and T: $X \to X/M$ the quotient map.*

Then $(T(x_{n(i)}))_i$ is K-basic in X/M and spans X/M.

PROOF. Let $x = \sum_{n=1}^{N} f_n(x)x_n$. Then $x - \sum_{n(i) \leq N} f_{n(i)}(x)x_{n(i)} \in M$ so applying T we see that the second statement holds.

Now we make the following calculations with the understanding that all sums are finitely nonzero.

Suppose $\|\sum_{i=1}^{m+p} \alpha_i T x_{n(i)}\| = 1$. Then for any $\varepsilon > 0\, \exists \sum_n \beta_n x_n \ni \|\sum_n \beta_n x_n\| \leq 1 + \varepsilon$ and $\sum_n \beta_n T x_n = \sum_{i=1}^{m+p} \alpha_i T x_{n(i)}$. The last equality means that $\sum_n \beta_n x_n - \sum_{i=1}^{m+p} \alpha_i x_{n(i)} \in M$ so in view of the biorthogonality, $\beta_n = 0$ if $n > n(m+p)$ or $n \neq n(i)$ for all i, and $\beta_n = \alpha_i$ for $n = n(i)$ ($i = 1, \ldots, m+p$). Therefore we have,

$$\left\|\sum_{i=1}^{m} \alpha_i T x_{n(i)}\right\| = \left\|\sum_{i=1}^{m} \beta_{n(i)} T x_{n(i)}\right\| \leq \left\|\sum_{i=1}^{m} \beta_{n(i)} x_{n(i)}\right\|$$
$$\leq K\left\|\sum_{i=1}^{m+p} \beta_{n(i)} x_{n(i)}\right\| = K\left\|\sum_n \beta_n x_n\right\| \leq K(1+\varepsilon).$$

Since this holds for all m, p, ε we have the desired result. □

LEMMA 3. *Let X be a normed space, (x_n, f_n) a biorthogonal sequence $\ni (x_n)$ spans X. Let $M = \bigcap_{n \geq k} \ker f_n$, T: $X \to X/M$ the quotient map and suppose that $(Tx_n)_{n \geq k}$ is K-basic in X/M.*

Then $\exists J \ni (x_n)$ is J-basic in X.

PROOF. This is clear because M is finite dimensional so X/M is isomorphic to $[x_n]_{n \geq k}$ via T and $\{x_1, \ldots, x_{k-1}\}$ is linearly independent. □

Now we have the first major result of the paper which says that under certain technical conditions, a Fréchet space will have a nuclear Köthe quotient.

THEOREM 1. *Let E be a separable Fréchet space which admits continuous norm. Let $(\|\cdot\|_k)$ be a fundamental sequence of norms for E with dual norms $(\|\cdot\|'_k)$. Let (d_n) be a dense sequence in E and E_0 the vector subspace it generates. We suppose that there is a biorthogonal sequence (x_n, f_n) satisfying*

(a) $(x_n) \subset E_0$, $(f_n) \subset (E_0, \|\cdot\|_1)'$.

(b) $f_m(d_n) = 0$ *for* $m > n$.

(c) $\|f_n\|'_{k+1}/\|f_n\|'_k \leqslant 1/n^2$ *for* $k = 1, \ldots, n-1$.

Then E has a nuclear Köthe quotient.

PROOF. We shall write E_k for the normed space $(E_0, \|\cdot\|_k)$ and E'_k for its dual, $(E_0, \|\cdot\|_k)'$ equipped with the norm $\|\cdot\|'_k$. In view of (a) we may consider f_n to be a continuous linear functional on any E_k.

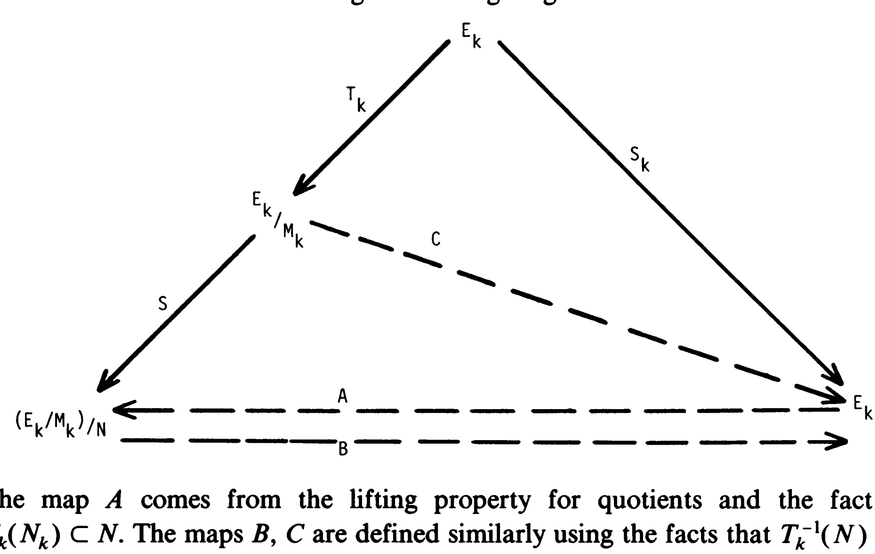
We apply Proposition 1 to E_1, (d_n), (x_n, f_n) to obtain a subsequence $(n_1(i))$ of indices $\ni$ if $M_1 = \bigcap_i \ker f_{n_1(i)}$ and T_1: $E_1 \to E_1/M_1$ is the quotient map, then $(T_1(x_{n_1(i)}))$ is 4-basic in E_1/M_1 and spans E_1/M_1.

Proceeding inductively we apply Proposition 1 to E_k, (d_n) and $(x_{n_{k-1}(i)}, f_{n_{k-1}(i)})_i$ to obtain a subsequence $(n_k(i))$ of $(n_{k-1}(i)) \ni$ if $M_k = \bigcap_i \ker f_{n_k(i)}$ and T_k: $E_k \to E_k/M_k$ is the quotient map then $(T_k(x_{n_k(i)}))$ is 4-basic in E_k/M_k and spans E_k/M_k.

Now we consider the diagonal sequence $(n_i(i))$. For each k, $(n_i(i))_{i \geqslant k}$ is a subsequence of $(n_k(i))_i$. Set $N_k = \bigcap_{i \geqslant k} \ker f_{n_i(i)}$ and let S_k: $E_k \to E_k/N_k$ be the quotient map. We claim that $(S_k(x_{n_i(i)}))_{i \geqslant k}$ is 4-basic in E_k/N_k and spans E_k/N_k.

To show this, we define $\tilde{f}_{n_k(i)} \in (E_k/M_k)'$ by $\tilde{f}_{n_k(i)}(T_k x) = f_{n_k(i)}(x)$ $(i = 1, 2, \ldots)$. It is immediate that $(T_k x_{n_k(i)}, \tilde{f}_{n_k(i)})_i$ is a biorthogonal sequence. Set $N = \bigcap_{i \geqslant k} \ker \tilde{f}_{n_i(i)}$ and let S: $E_k/M_k \to (E_k/M_k)/N$ be the quotient map. It follows from Lemma 2 that $(ST_k x_{n_i(i)})_{i \geqslant k}$ is 4-basic in $(E_k/M_k)/N$ and spans it.

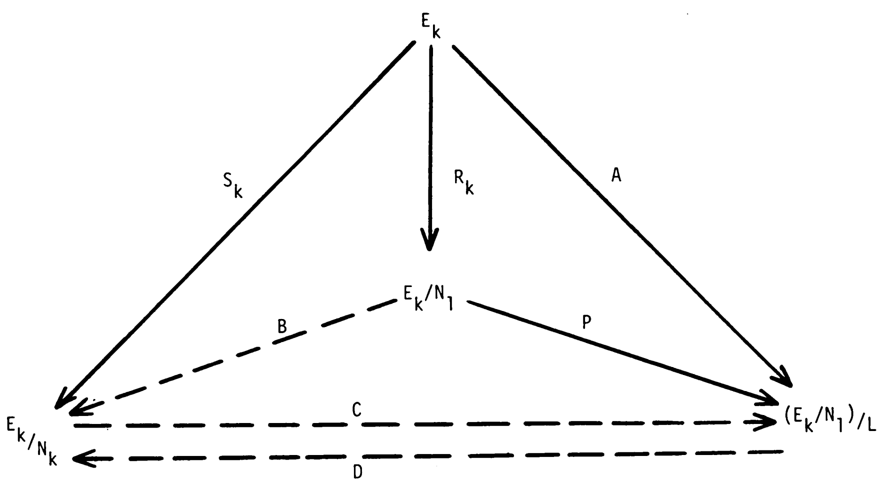
Next we consider the following commuting diagram:



The map A comes from the lifting property for quotients and the fact that $T_k(N_k) \subset N$. The maps B, C are defined similarly using the facts that $T_k^{-1}(N) \subset N_k$ and $M_k \subset N_k$. The commutativity follows from diagram chasing and the fact that T_k is onto. Also, from the facts that ST_k and S_k are onto it follows that AB and BA are the identities.

Therefore it follows that B is an isometry and $S_k x_{n_i(i)} + BST_k x_{n_i(i)}$ and the claim is proved.

Now we make a similar argument to apply Lemma 3. Fix k and for each i we define $\bar{f}_{n_i(i)} \in (E_k/N_1)'$ by $\bar{f}_{n_i(i)}(R_k x) = f_{n_i(i)}(x)$ where R_k: $E_k \to E_k/N_1$ is the quotient map (equal to S_1 as a function). It is immediate that $(R_k x_{n_i(i)}, \bar{f}_{n_i(i)})_i$ is a biorthogonal sequence. Since $E_k/N_1 = E_1/N_1$ as a vector space and $R_k = S_1$ as a function, it follows that $(R_k(x_{n_i(i)}))_i$ spans E_k/N_1. We will show that it is basic. Set $L = \bigcap_{i \geq k} \ker \bar{f}_{n_i(i)}$ and let P: $E_k/N_1 \to (E_k/N_1)/L$ be the quotient map. We claim that $(PR_k(x_{n_i(i)}))_{i \geq k}$ is 4-basic. Consider the following commuting diagram:



The map $A = PR_k$. The maps B, C, D are defined using the quotient property and the facts that $N_1 \subset N_k$, $R_k(N_k) \subset L$ and $R_k^{-1}(L) \subset N_k$. Again the commutativity follows from the fact that R_k is onto and both CD, DC are the identity because P, B are onto.

Hence C is an isometry so $(CS_k x_{n_i(i)})_{i \geq k}$ is 4-basic. But $PR_k x_{n_i(i)} = CS_k x_{n_i(i)}$. This proves the claim and by Lemma 3 applied to E_k/N_1, $(R_k(x_{n_i(i)}), \bar{f}_{n_i(i)})$, L, P, k it follows that $\exists\, \alpha_k \ni (R_k(x_{n_i(i)}))_i$ is α_k-basic in E_k/N_1.

Now we are ready to construct the quotient. Consider the metric space E_0/N_1 with quotient map T: $E_0 \to E_0/N_1$ (as a function this is the same as R_k). The seminorms $|\cdot|_k$ induced by $\|\cdot\|_k$ are given by

$$|y|_k = \inf\{\|x\|_k : Tx = y\}.$$

Because of (a), N_1 is closed in E_0 with respect to each norm $\|\cdot\|_k$ and hence each $|\cdot|_k$ is a norm.

On the other hand, for each k we have, on the set $E_0/N_1 = E_k/N_1$, the other quotient norms $|||\cdot|||_k$ given by

$$|||y|||_k = \inf\{\|x\|_k : R_k x = y\} = \inf\{\|x\|_k : Tx = y\} = |y|_k$$

and it follows that $(Tx_{n_i(i)})_i = (R_k x_{n_i(i)})_i$ is α_k-basic in E_0/N_1 for each $|\cdot|_k$ so this sequence is basic in each $(E_0/N_1, |\cdot|_k)\hat{}$ so it is basic in $(E_0/N_1)\hat{}$. We also know that $(R_k x_{n_i(i)})_i$ spans $E_k/N_1 = E_0/N_1$ so $(Tx_{n_i(i)})_i$ is a basis for $(E_0/N_1)\hat{}$.

Since $T\colon E_0 \to E_0/N_1$ is a quotient map of metrizable spaces it follows that the completion $\hat{T}\colon E \to (E_0/N_1)\hat{}$ is a quotient map. Moreover since each $|\cdot|_k$ is a norm in E_0/N_1 the same holds for the completion.

Hence, $(E_0/N_1)\hat{}$ is a Köthe quotient of E. It remains to prove that it is nuclear.

Let us write (y_i, g_i) for $(x_{n_i(i)}, f_{n_i(i)})$ so that $(y_i) \subset E_0$ and $(\hat{T}y_i)$ is a basis for $(E_0/N_1)\hat{}$. Because of (c) and the fact that $(n_i(i))$ is a subsequence of (n) we have

$$\frac{\|g_i\|'_{k+1}}{\|g_i\|'_k} \leqslant 1/i^2 \quad \text{for } k < i.$$

For each i we define $\bar{g}_i \in (E_0/N_1)'$ by $\bar{g}_i(Tx) = g_i(x)$. This is valid because $N_1 = \bigcap_i \ker g_i$. If $|\cdot|'_k$ is the dual norm of the norm $|\cdot|_k$ induced by $\|\cdot\|_k$ on the quotient E_0/N_1 we have,

$$|\bar{g}_i|'_k = \sup_{|Tx|_k < 1} |\bar{g}_i(Tx)| = \sup_{\|x\|_k < 1} |g_i(x)| = \|g_i\|'_k.$$

Moreover $(Ty_i, \bar{g}_i)$ is biorthogonal and so $(\bar{g}_i)$ is the dual basis to (Ty_i). Hence, for each $k \,\exists\, C_k > 0 \ni$

$$1/|\bar{g}_i|'_k \leqslant |Ty_i|_k \leqslant C_k/|\bar{g}|'_k \qquad (i \in \mathbf{N}),$$

and so,

$$\frac{|Ty_k|_k}{|Ty_i|_{k+1}} \leqslant C_k \frac{|\bar{g}_i|'_{k+1}}{|\bar{g}_i|'_k} \leqslant \frac{C_k}{i^2} \quad \text{for } i > k.$$

Since (Ty_i) is a basis for $(E_0/N_1)\hat{}$ it follows that $(E_0/N_1)\hat{}$ is nuclear. □

4. The characterization. Our characterization of Fréchet spaces with nuclear Köthe quotients will be in terms of the following condition which we label $(*)$. If E is a Fréchet space and $(\|\cdot\|_k)$ a fundamental sequence of seminorms we denote by E'_k $(k \in \mathbf{N})$ the Banach space determined by the unit ball of the dual norm $\|\cdot\|'_k$ on E'.

$(*)$ There exists l such that for every k there exists j such that the $\|\cdot\|'_k$-closure of E'_l is not closed in E'_j.

It is easy to check that this condition is independent of the choice of $(\|\cdot\|_k)$. Moreover, in view of the open mapping theorem, the condition has the following equivalent formulation:

$$\exists\, l \ni \forall\, k \,\exists\, j \ni \sup\left\{\|u\|'_k : u \in E'_l,\ \|u\|'_j \leqslant 1\right\} = \infty.$$

In this form our condition is very close to being a dual to the following condition used by Bessaga, Pełczyński and Rolewicz **[4]** in thier determination of those Fréchet spaces which have nuclear Köthe subspaces:

$$\forall\, k \,\exists\, j \ni \sup\left\{\|x\|_j : x \in Y,\ \|x\|_k \leqslant 1\right\} = \infty$$

for every subspace Y of E with finite codimension.

The role of condition $(*)$ in our characterization is contained in the following result.

PROPOSITION 2. *A Fréchet space E satisfies condition* $(*)$ *iff it has a quotient which admits continuous norm and satisfies condition* $(*)$.

PROOF. Suppose that E satisfies condition $(*)$. We may assume for this condition that $l = 1$ and $j = k + 1$. Let $M = (E_1')^{\circ\circ}$ which, by the bipolar theorem, is the closure of E_1' in the weak topology from E. We will show that E/M° is the desired quotient.

First we check that E/M° admits continuous norm. Let $x \in E$ and suppose that the seminorm in E/M° induced by $\|\cdot\|_1$ annihilates $x + M^\circ$. This means that $\exists (y_n) \subset M^\circ \ni \lim_n \|x + y_n\|_1 = 0$. Let $u \in M$. Then $\exists\, v \in E_1' \ni |u(x) - v(x)| \leq 1$. Hence we have

$$|u(x)| \leq |v(x)| + |u(x) - v(x)| \leq |v(x + y_n)| + 1.$$

Since $v \in E_1'$ it follows that $\lim v(x + y_n) = 0$ so $|u(x)| \leq 1$. This shows that $x \in M^\circ$, so the seminorm induced by $\|\cdot\|_1$ is a norm.

Now we verify condition $(*)$ for E/M°. Fix k and let V_k be the unit ball of $\|\cdot\|_k$ in E. Since E satisfies condition $(*)$ we have a sequence $(u_n) \subset E_1'$ with $\|u_n\|_k' > 1$ and $\|u_n\|_{k+1}' \leq \frac{1}{n}$. This implies that $u_n \in \frac{1}{n} V_{k+1}^\circ \sim V_k^\circ$ so a fortiori $u_n \notin (V_k + M^\circ)^\circ$. Moreover, if $x \in V_{k+1}, y \in M^\circ$ then, since $u_n \in M$,

$$|nu_n(x + y)| = |nu_n(x)| \leq 1.$$

Hence $u_n \in \frac{1}{n}(V_{k+1} + M^\circ)^\circ$. Thus we have shown that $u_n \in \frac{1}{n}(V_{k+1} + M^\circ)^\circ \sim (V_k + M^\circ)^\circ$. But, $(E/M^\circ)' = M$ and the unit ball of the dual norm of the norm in E/M° induced by $\|\cdot\|_k$ is $(V_k + M^\circ)^\circ$. This shows that E/M° satisfies condition $(*)$.

Conversely let E/H be a quotient of E. If (V_k) is a fundamental sequence of nbds of 0 for E then by general duality, $(E/H)'$ can be represented as a vector subspace of E' and a fundamental sequence of equicontinuous sets for $(E/H)'$ is given by $(V_k^\circ \cap H^\circ)_k$. Hence if E/H satisfies condition $(*)$ (say with $l = 1$ and $j = k + 1$) then $\exists (u_n) \subset H^\circ$ and a sequence of constants (C_n) with $u_n \in (C_n V_1^\circ \cap H^\circ) \cap (\frac{1}{n} V_{k+1}^\circ \cap H^\circ) \sim (V_k^\circ \cap H^\circ)$. It follows that $u_n \in E_1'$, $\|u_n\|_{k+1}' \leq \frac{1}{n}$ but since $u_n \in H^\circ$ then $u_n \notin V_k^\circ$ so $\|u_n\|_k' > 1$. Hence E satisfies condition $(*)$. □

We remark that in the second half of the proof of Proposition 2 we did not use the fact that the quotient admits continuous norm.

We are now ready for the main result of this paper.

THEOREM 2. *A separable Fréchet space E has a nuclear Köthe quotient iff it satisfies condition* $(*)$.

PROOF. Suppose that E satisfies condition $(*)$. By Proposition 2 we may suppose that E admits continuous norm. Let (d_n) be a dense sequence in E and E_0 the vector subspace it generates. We will establish the existence of a nuclear Köthe quotient by constructing a biorthogonal sequence (x_n, f_n) which satisfies conditions (a), (b), (c) of Theorem 1.

Let $(\|\cdot\|_k)$ be a fundamental sequence of norms for E. The space E_k' defined in the beginning of this section is the dual of the normed space $(E_0, \|\cdot\|_k)$ so the notation here is consistent with the notation in Theorem 1.

We may assume that condition $(*)$ is satisfied with $l = 1$ and $j = k + 1$. Then we have for each $k \in \mathbf{N}$,

$$\sup\{\|g\|'_k / \|g\|'_{k+1} : g \in E'_1\} = \infty.$$

Our construction is by induction. We can choose $f_1 \in E'_1$ with $f_1 \neq 0$ and so that (c) holds. Since E_0 is dense in $E \exists x_1 \in E_0 \ni f_1(x_1) = 1$ so we have (a). Condition (b) is vacuous.

Before passing to the induction step we must prove that the above statement of condition $(*)$ remains true if the requirement $g \in E'_1$ is replaced by the weaker requirement that $g \in G$ where $G = \{g \in E'_1\colon g(x) = 0 \,\forall\, x \in L\}$ for some finite $L \subset E$.

We may assume that L is a linearly independent set say $L = \{z_\nu\}_{\nu=1}^m$. Since $\|\cdot\|_1$ is a norm, E'_1 is dense in E' with respect to the weak topology from E so we can find $\{g_\nu\}_{\nu=1}^m \subset E'_1 \ni (z_\nu, g_\nu)$ is biorthogonal. Then the map $P\colon E'_1 \to E'_1$ defined by $Pf = \sum_{\nu=1}^m f(z_\nu) g_\nu$ is a projection whose kernel is G and P is continuous for every $\|\cdot\|'_k$. Hence if $H = P(E'_1)$ we have $C_k > 0$ $(k \in \mathbf{N})$ with

$$\|g\|'_k + \|h\|'_k \leqslant C_k \|g + h\|'_k \qquad (g \in G, h \in H)$$

and, since H is finite dimensional,

$$\|h\|'_k \leqslant C_k \|h\|'_{k+1} \qquad (h \in H).$$

Therefore we have, for each $k \in \mathbf{N}$,

$$\begin{aligned}
\infty = \sup\left\{\frac{\|f\|'_k}{\|f\|'_{k+1}} : f \in E'_1\right\} &= \sup\left\{\frac{\|g+h\|'_k}{\|g+h\|'_{k+1}} : g \in G, h \in H\right\} \\
&\leqslant C_{k+1} \sup\left\{\frac{\|g\|'_k + \|h\|'_k}{\|g\|'_{k+1} + \|h\|'_{k+1}} : g \in G, h \in H\right\} \\
&\leqslant C_{k+1} \sup\left\{\frac{\|g\|'_k}{\|g\|'_{k+1}} + \frac{\|h\|'_k}{\|h\|'_{k+1}} : g \in G, h \in H\right\} \\
&\leqslant C_{k+1} \sup\left\{\frac{\|g\|'_k}{\|g\|'_{k+1}} : g \in G\right\} + C_k C_{k+1}.
\end{aligned}$$

Hence we may conclude that $\sup\{\|g\|'_k / \|g\|'_{k+1} : g \in G\} = \infty$.

Turning to our induction step we assume that (x_i, f_i) $(i = 1, \ldots, n-1)$ have been selected. We apply the above condition with

$$G = \{g \in E'_1\colon g(x_i) = g(d_i) = 0, i = 1, \ldots, n-1\}.$$

Then we select $g_n, \ldots, g_1$ inductively so as to annihilate $x_1, \ldots, x_{n-1}, d_1, \ldots, d_{n-1}$ and satisfy, for $k = 1, \ldots, n-1$,

$$\|g_k\|'_{k+1} \leqslant 1/2^k, \qquad \|g_k\|'_k > (n^2+1)\left(1 + \|g_{k+1}\|'_k + \cdots + \|g_n\|'_k\right).$$

Next we set $f_n = g_1 + \cdots + g_n \in E_1'$ so f_n annihilates x_i, d_i ($i = 1,\ldots,n-1$). This gives (b). To obtain (c) we calculate, for $k = 1,\ldots,n-1$,

$$\frac{\|f_n\|'_{k+1}}{\|f_n\|'_k} \leqslant \frac{\|g_1\|'_{k+1} + \cdots + \|g_k\|'_{k+1} + \|g_{k+1}\|'_{k+1} + \cdots + \|g_n\|'_{k+1}}{-(\|g_1\|'_k + \cdots + \|g_{k-1}\|'_k) + \|g_k\|_k - (\|g_{k+1}\|'_k + \cdots + \|g_n\|'_k)}$$

$$< \frac{1/2 + \cdots + 1/2^k + \|g_{k+1}\|'_{k+1} + \cdots + \|g_n\|'_{k+1}}{-1 + (1+n^2)(1 + \cdots + \|g_{k+1}\|'_k + \cdots + \|g_n\|'_k) - (\|g_{k+1}\|'_k + \cdots + \|g_n\|'_k)}$$

$$< \frac{1 + \|g_{k+1}\|'_{k+1} + \cdots + \|g_n\|'_{k+1}}{n^2(1 + \|g_{k+1}\|'_k + \cdots + \|g_n\|'_k)} \leqslant \frac{1}{n^2}$$

as desired.

Finally since E_0 is dense in E and $f_n \neq 0$ we can find $x_n \in E_0 \ni f_n(x_n) = 1$ and $f_i(x_n) = 0$ ($i = 1,\ldots,n-1$), which gives (a) and the induction is completed.

The converse follows from Proposition 2 and the easily checked fact that a nuclear Köthe space satsifies condition (*). □

We remark that the construction of f_n in the above proof is based on the method of Bessaga, Pełczyński and Rolewicz [**4**, Lemma 2]. Also we note that in the proof of Theorem 2 we did not use separability to show that the existence of a nuclear Köthe quotient implies condition (*).

5. Special cases and open questions. The collection of Fréchet spaces which do not have nuclear Köthe quotients is somewhat more diverse than the set of those which do not have nuclear Köthe subspaces (see §1). We consider two general situations—countable products of Banach spaces and spaces which admit continuous norm. Using our characterization we obtain a fair amount of information, but there are still some unanswered questions. We close with a brief consideration of the weaker question of existence of quotients with continuous norm and basis.

Quojections. It is not hard to show directly that a countable product of Banach spaces cannot have a nuclear Köthe quotient. We obtain this fact below as a corollary of a more general result concerning an interesting class of Fréchet spaces. We say that a Fréchet space is a *quojection* if it is isomorphic to a projective limit of a sequence of surjective operators on Banach spaces. Obviously a countable product of Banach spaces is a quojection. The class of quojections was considered by V. B. Moscatelli [**13**] who gave examples of quojections which are not isomorphic to countable products of Banach spaces. In view of [**5**], such Fréchet spaces do not have unconditional bases.

PROPOSITION 3. *A Fréchet space E is a quojection iff every quotient of E which admits continuous norm is a Banach space.*

PROOF. Let E be a quojection and F a quotient of E. If E is the projective limit of the surjections $A_k\colon E_{k+1} \to E_k$ then it is easy to see from the definition of projective limit that the canonical projections $P_k\colon E \to E_k$ (given by $P_k((x_j)_j) = x_k$) are also surjections. Let $T\colon E \to F$ be the quotient map, $|\cdot|$ a continuous norm on F and $\|\cdot\|_k$ the norm on E_k.

By the continuity of $T \exists k$ and $C > 0 \ni |Tx| \leq C \| P_k x \|_k$, $x \in E$. Hence from an algebraic point of view $\exists$ a unique map S: $E_k \to F \ni SP_k = T$. Since T is a surjection, S is also. Moreover, it follows from the closed graph theorem and the fact that $|\cdot|$ is a norm that S is continuous. Hence F is a quotient of a Banach space so it is a Banach space.

Conversely let $(\|\cdot\|_k)$ be a fundamental sequence of seminorms for E, N_k the kernel of $\|\cdot\|_k$ and E/N_k the quotient Fréchet space. Since $(\|\cdot\|_k)$ is increasing we have the canonical maps $E/N_{k+1} \to E/N_k$ which are surjections and it is easy to see that E is isomorphic to the projective limit of this sequence of maps. On the other hand, the seminorm induced on E/N_k by $\|\cdot\|_k$ is clearly a norm so by assumption, E/N_k is a Banach space. Thus, E is a quojection. □

COROLLARY 1. *If E is a quojection then E does not have a nuclear Köthe quotient.*

COROLLARY 2. *If E is isomorphic to a countable product of Banach spaces then E does not have a nuclear Köthe quotient.*

It would be nice to know that the quojections are precisely those Fréchet spaces (amongst separable spaces) which fail to have nuclear Köthe quotients. Unfortunately, we are unable to prove the converse of Corollary 1 so this remains open. We can obtain this result, however, if we restrict our considerations to reflexive spaces.

In order to investigate this situation we consider, for an arbitrary Fréchet space E, the vector space E'^b of all linear functionals on E' which are bounded on bounded sets. Obviously $\langle E', E'^b \rangle$ is a dual system and we indicate the polar of a set A by A^b. We will consider the topology $\mathfrak{T}$ on E'^b of uniform convergence on bounded sets, that is, a fundamental system of neighborhoods of 0 is given by the sets B^b, B a bounded subset of E'.

PROPOSITION 4. *If E is a separable Fréchet space which does not have a nuclear Köthe quotient then E'^b is quojection.*

PROOF. We can apply Theorem 2 to conclude that E does not satisfy condition (*) and so we can find a fundamental sequence of seminorms $(\|\cdot\|_k)$ for $E \ni$ for every k, the $\|\cdot\|'_{k+1}$ closure of E'_k is closed in each Banach space E'_j, $j > k+1$. This implies that if F_k is the $\|\cdot\|'_{k+1}$-closure of E'_k in E'_{k+1} and each F_k is equipped with the norm $\|\cdot\|'_{k+1}$ then F_k is closed subspace of F_{k+1}.

But the unit balls of the F_k ($k \in \mathbf{N}$) form a fundamental sequence of bounded sets for E' and it is easy to check that E'^b is isomorphic to the projective limit of the sequence of maps $F'_{k+1} \to F'_k$, adjoint to the inclusions $F_k \to F_{k+1}$ ($k \in \mathbf{N}$). Since F_k is a closed subspace of F_{k+1} it follows that the maps $F'_{k+1} \to F'_k$ are surjections so E'^b is a quojection. □

COROLLARY 3. *If E is a separable reflexive Fréchet space, then E has a nuclear Köthe quotient iff E is not a quojection.*

PROOF. If E is reflexive, the strong dual is bornological [**12**, p. 400] so $E'^b = E'' = E$ (algebraically and topologically) and the result follows from Corollary 1 and Proposition 4. □

COROLLARY 4. *Every Fréchet Montel space not isomorphic to ω has a nuclear Köthe quotient.*

PROOF. If E is a Fréchet Montel space then, as is well known, E is separable and reflexive. We will show that E is not a quojection. Suppose that E is the projective limit of the surjections $A_k: E_{k+1} \to E_k$. We may assume that A_k maps the unit ball of E_{k+1} onto the unit ball of E_k. Also, since E is not isomorphic to ω we may assume that E_1 is infinite dimensional. Then, viewing E as the projective limit, it is easy to see that $\{(x_k) \in E: x_k \text{ is in the unit ball of } E_k \forall\, k\}$ is a closed, bounded subset of E. But the projection of this set in E_1 is the unit ball so it is not compact. Hence the set is not compact in E which is a contradiction. □

Fréchet spaces which admit continuous norm. Again the situation with quotients seems more difficult than in the case of subspaces. We do not know whether a separable non-Banach Fréchet space which admits a continuous norm (or is even countable normed) necessarily has a nuclear Köthe quotient. Actually, this is the same as the above question regarding the converse of Corollary 1. In fact, if E is not a quojection, then by Proposition 3, E has a non-Banach quotient which admits a continuous norm, so if this question had a positive answer, E would have a nuclear Köthe quotient and thus the converse of Corollary 1 would hold. Conversely, if E is a non-Banach Fréchet space which admits continuous norm then clearly E is not a quojection so if the converse of Corollary 1 held, E would have a nuclear Köthe quotient.

In particular, we have the following simple conclusions.

COROLLARY 5. *If E is a non-Banach, separable, reflexive Fréchet space which admits a continuous norm, then E has a nuclear Köthe quotient.*

PROOF. In view of Proposition 3, E is clearly not a quojection so the result follows from Corollary 3. □

COROLLARY 6. *If E is a non-Banach separable Fréchet space $\ni E'^b$ admits a continuous norm then E has a nuclear Köthe quotient.*

PROOF. Clearly E is a subset of E'^b and since E is barreled the topology induced on E by E'^b is the same as the original. On the other hand, by Proposition 4 if E does not have a nuclear Köthe quotient then E'^b is a quojection so by Proposition 3, E'^b is a Banach space. This is impossible because its subspace E is not a Banach space. □

Unfortunately, if E admits continuous norm, or is even countably normed, it can still happen that E'^b does not admit continuous norm. We establish this in the following example.

Let T: $c_0 \to c_0$ by $T(\xi) = (\xi_i - \xi_{i+1})_i$. Clearly T is continuous, one-to-one and has dense range. Write $X_i = c_0$ ($i \in \mathbf{N}$) and let X be the Banach space (isomorphic to c_0) given by

$$X = \left\{(x_i): x_i \in X_i, \lim_i \|x_i\| = 0\right\}, \qquad \|(x_i)\| = \sup_i \|x_i\|.$$

For each k we take $E_k = X$ and define S_k: $E_{k+1} \to E_k$ by setting $S_k((x_i)) = (y_i)$ where

$$y_i = \begin{cases} Tx_k & \text{if } i = k, \\ x_i & \text{if } i \neq k. \end{cases}$$

It was proved in [**6**] that since the S_k are one-to-one and have dense range, the projective limit of (S_k) is a Fréchet space E with a fundamental sequence of norms $(\|\cdot\|_k) \ni$ for each of the Banach spaces $F_k = (E, \|\cdot\|_k)\hat{}$ there is an isomorphism U_k: $F_k \to E_k \ni S_k U_{k+1} = U_k A_k$ where A_k: $F_{k+1} \to F_k$ is the canonical map. Thus E is countably normed.

On the other hand, E'^b is the projective limit of the maps S_k'': $E_{k+1}'' \to E_k'$. It is not hard to check that for each k_0 $\exists (x_k)$ in this projective limit $\ni x_k = 0$ for $k < k_0$ but $x_{k_0} \neq 0$. This is because the map T'' is not one-to-one and S_k'' consists of T'' on one coordinate and the identity on the others. Hence E'^b does not admit continuous norm.

Quotients with continuous norm and basis. If we do not require that our quotient be nuclear but only that it admit continuous norm and have a basis, then the problem becomes a little easier. For example if E is a separable Fréchet space which admits continuous norm it is easy to show (as in the proof of Theorem 2) that the hypotheses of Theorem 1 are satisfied except for (c). Since condition (c) was only used to prove that the quotient is nuclear, we can obtain the following result which is analogous to the result of Johnson and Rosenthal [**10**, Theorem IV.1].

PROPOSITION 5. *If E is a separable Fréchet space which admits a continuous norm, then E has a quotient which admits a continuous norm and has a basis.*

Of course, one cannot go further with Proposition 5 and take the quotient with basis and look for a further quotient which is nuclear. In fact, the quotient provided by Proposition 5 could be Banach. There is one case in which this difficulty does not occur.

PROPOSITION 6. *If E is a separable Fréchet space which admits a continuous norm and has the property that no quotient of E is a Banach space, then E has a nuclear Köthe quotient.*

PROOF. By Proposition 5, E has a quotient F with a basis (x_n). Since E has continuous norm we may assume that its topology is defined by a sequence of norms $(\|\cdot\|_k)$ and that $\|x_n\|_1 = 1$ ($n \in \mathbf{N}$). Then if (f_n) is the dual basis of (x_n) and G is the set of all finite linear combinations of (f_n) it follows that $G \subset F_1'$. We may assume that (x_n) is 1-basic with respect to each norm $\|\cdot\|_k$. It then follows from standard Banach space techniques that for $k \in \mathbf{N}$ and $x \in F$,

$$\|x\|_k = \sup\{|g(x)| : g \in G, \|g\|'_k \leq 1\}.$$

Now we can verify condition (*) with $l = 1$. Fix $k \in \mathbf{N}$ and suppose that for every $j\ \exists C_j > 0 \ni \|u\|'_k \leq C_j \|u\|'_j$ for all $u \in F'_1$.

In particular this will hold for $u \in G$ so we have for $x \in F$,

$$\|x\|_j = \sup\{|g(x)| : g \in G, \|g\|'_j \leq 1\} \leq \sup\{|g(x)| : g \in G, \|g\|'_k \leq C_k\}$$
$$= C_j \sup\{|g(x)| : g \in G, \|g\|'_k \leq 1\} = C_j \|x\|_k.$$

Since this is to hold for all $j \in \mathbf{N}$ it follows that F is a Banach space which contradicts our hypothesis.

Hence F satisfies condition (*) so by Theorem 2, F and hence E has a nuclear Köthe quotient.

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Department of Mathematics and Computer Science, Florida State University, Tallahassee, Florida 32306

Department of Mathematics and Computer Science, Clarkson College of Technology, Potsdam, New York 13676