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RATIONAL HOMOTOPY OF THE SPACE OF SECTIONS OF A NILPOTENT BUNDLE

BY

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ABSTRACT. We show that an algebraic construction proposed by Sullivan is indeed a model for the rational homotopy type of the space of sections of a nilpotent bundle.

In his paper *L'homologie des espaces fonctionnels*, R. Thom studied the homotopy type of the space F_f^X of continuous maps of X into F homotopic to a given map f.

Starting from a Postnikov decomposition of F, he built the functional space F_f^X step by step. He also indicated how one could construct a differential graded algebra describing the rational homotopy type of F_f^X.

Later on, Sullivan gave an algebraic model which mirrors this construction in terms of a DG-algebra representing X and the minimal model of F.

The aim of this paper is to show, following the method of Thom, that the model of Sullivan is indeed a model for the functional space under suitable restrictions.

As in [3], we consider the slightly more general problem of the determination of the rational homotopy type of the space of sections Γ_s of a nilpotent fiber space $p: Y \to X$ homotopic to a given section s.

In §1 we explain Thom's geometric construction. In §2 we describe an algebraic model for an abelian Galois covering of a nilpotent space. In §3 we show how the model of Sullivan fits with the geometry.

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1. A Postnikov factorization of the space of sections.

1.1. Let G be a finitely generated abelian group and let X be a path connected space whose cohomology groups $H^k(X; G)$ are finitely generated for each k.

To avoid difficulties with the topologies (cf. [4]), we can work in the category of simplicial sets.

PROPOSITION (THOM [4]). *The space $K(G, m)^X$ of continuous maps of X in the Eilenberg-Mac Lane complex $K(G, m)$ is homotopically equivalent to the product $\prod_{i=0}^m K_i$ of the Eilenberg-Mac Lane spaces $K_i = K(H^{m-i}(X; G), i)$.*

More precisely, let $\chi \in H^m(K(G; m); G)$ be the fundamental class of $K(G; m)$. If

$$e: K(G, m)^X \times X \to K(G, m)$$

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is the evaluation map, we can write $e^*(\chi)$ *uniquely as* $\Sigma\chi_i$ *where* $\chi_i \in H^i(K(G, m)^X; H^{m-i}(X; G))$. *Then the projection* $K(G; m)^X \to K_i$ *is determined by the cohomology class* χ_i.

PROOF. As $K(G; m)$ is a Hopf space, all the connected components of $K(G, m)^X$ have the same homotopy type as the connected component $K(G, m)_0^X$ of the constant map. They are in bijection with $H^m(X; G) = K(H^m(X; G), 0)$.

If f: $S^i \times X \to K(G, m)$ represents an element of $\pi_i(K(G, m)_0^X)$, then its homotopy class is characterized by

$$f^*(\chi) \in H^0(S^i; Z) \otimes H^m(X; G) + H^i(S^i; Z) \otimes H^{m-i}(X; G).$$

The first component of $f^*(\chi)$ vanishes because f restricted to $S^i \times \{x\}$ is homotopic to a constant map. The second component is of the form $s \otimes u_f$, where s is the canonical generator of $H^i(S^i; \mathbf{Z})$, and $u_f \in H^{m-i}(X; G)$.

It is easy to see that the map $f \mapsto u_f$ induces an isomorphism of $\pi_i(K(G; m)_0^X)$ on $H^{m-i}(X; G) = \pi_i(\Pi K_j)$ and that the map $K(G; m)^X \to \Pi K_j$ described above induces an isomorphism on homotopy groups, so is a homotopy equivalence.

1.2. *Consequence*. Let Z be a topological space and let f: $Z \times X \to K(G; m)$ be a continuous map. It gives a map φ: $Z \to K(G, m)^X$; its composition with the projection on $K(H^{m-i}(X; G), i)$ will be denoted by φ_i. Then the homotopy class of φ_i is determined by the component of

$$f^*(\chi) \in H^m(Z \times X; G)$$

in $H^i(Z; H^{m-i}(X; G))$.

It follows that the map induced by φ_i on the rational cohomology can be described as follows.

Let $V = \mathrm{Hom}(G, Q) = H^m(K(G, m); Q)$. Then

$$H^i\big(K(H^{m-i}(X; G), i); Q\big) \approx H_{m-i}(X; Q) \otimes V,$$

where $H_{m-i}(X; Q)$ is identified with the dual of $H^{m-i}(X; Q)$.

The homomorphism φ_i^*: $H^i(K(H^{m-i}(X; G), i), Q) \to H^i(Z; Q)$ is given by

$$\varphi_i^*(a' \otimes v) = a' \cap f^*(v),$$

where $a' \in H_{m-i}(X; Q)$, $v \in V$ and $a' \cap (a \otimes b) = a'(a)b$, for $a \otimes b \in H^*(X; Q) \otimes H^*(Z; Q) \approx H^*(Z \times X; Q)$.

1.3. *Space of sections of a nilpotent bundle*. Let p: $Y \to X$ be a bundle which admits a Moore-Postnikov factorization (in the sense of Spanier [**2**, pp. 437–444]). This means that the map p is, up to homotopy equivalence, the composition of a possibly infinite sequence of fibrations

$$X = Y_0 \overset{p_1}{\leftarrow} Y_1 \overset{p_2}{\leftarrow} Y_2 \leftarrow \cdots,$$

where p_r: $Y_r \to Y_{r-1}$ is a principal $K(G_r, n_r)$-bundle, $n_r \geq 1$. Also the sequence converges, i.e. for each positive integer k, then $n_r > k$ for r large enough. We also assume G_r abelian. Such a bundle will be called *nilpotent*.

We assume that the integral homology of X is finitely generated (in particular $H_j(X, \mathbf{Z}) = 0$ for j large) and that each G_r is finitely generated.

We want to describe the space Γ_s of continuous sections of p: $Y \to X$ which are homotopic to a given section s.

From s, we obtain a sequence s_r: $X \to Y_s$ of compatible sections of $Y_r \to X$, i.e. $s_{r-1} = p_r s_r$. The principal bundle p_r: $Y_r \to Y_{r-1}$ is induced from the path space bundle π: $P \to K(G_r, n_r + 1)$ by a map c_{r-1}: $Y_{r-1} \to K(G_r, n_r + 1)$. We can assume that $c_{r-1}s_{r-1}$ maps X on the base point.

Note that P^X is isomorphic to the path space of $K(G_r, n_r + 1)_0^X$, the space of maps of X in $K(G_r, n_r + 1)$ homotopic to a constant. The fiber above the constant map on the base point is $K(G_r, n_r)^X$, where $K(G_r, n_r)$ is the fiber of P above the base point.

Let Γ_r be the space of sections of the bundle $Y_r \to X$ which are homotopic to s_r. Let q_r: $\Gamma_r \to \Gamma_{r-1}$ be the map associating to a section σ the section $p_r \cdot \sigma$.

PROPOSITION. *Γ_s is homotopically equivalent to the limit of the convergent sequence of principal fibrations*

$$\Gamma_1 \leftarrow \Gamma_2 \leftarrow \cdots .$$

Let e_{r-1}: $\Gamma_{r-1} \times X \to Y_{r-1}$ be the evaluation map $e_{r-1}(\sigma, x) = \sigma(x)$. The bundle q_r: $\Gamma_r \to \Gamma_{r-1}$ is the connected component of s_r in the principal $K(G_r, n_r)^X$-bundle classified by the map $\bar{c}_{r-1}$: $\Gamma_{r-1} \to K(G_r, n_r + 1)_0^X$ corresponding to $c_{r-1} \circ e_{r-1}$.

Indeed let $\hat{\Gamma}_r$ be the space of sections of $Y_r \to X$ projecting on a section of Y_{r-1} homotopic to s_{r-1}. An element of $\hat{\Gamma}_r$ is given by a pair (f, g) where f: $X \to P$ and g: $X \to Y_{r-1}$, with $g \in \Gamma_{r-1}$ and $\pi \circ f = c_{r-1} \circ g$. This amounts to saying that $\hat{\Gamma}_r$ is the bundle over Γ_{r-1} induced by $\bar{c}_{r-1}$ from P^X.

Γ_r is just the connected component of s_r in $\hat{\Gamma}_r$. The fiber of $\Gamma_r \to \Gamma_{r-1}$ is isomorphic to

$$K(G_r, n_r)_0^X \times G',$$

where $G' \subset H^{n_r}(X; G_r)$ is the image of $\pi_1(\Gamma_r)$ by the homomorphism induced on π_1 by $\bar{c}_{r-1}$;

$$\lambda\colon \pi_1(\Gamma_{r-1}) \to \pi_1\big(K(G_r, n_r + 1)_0^X\big) = H^{n_r}(X; G).$$

Indeed, let $\tilde{\Gamma}_{r-1} \to \Gamma_{r-1}$ be the covering whose fibers are the set of connected components of the fibers of $\Gamma_r \to \Gamma_{r-1}$. It is a Galois covering with group G'. The fiber Γ_0 of $\Gamma_r \to \tilde{\Gamma}_{r-1}$ above the projection of s_r is the set of sections of $Y_r \to X$ projecting on s_{r-1} and homotopic to s_r by a homotopy whose projection in Γ_{r-1} is just the trivial path. So Γ_0 is canonically isomorphic to $K(G_r, n_r)_0^X$, because the bundle induced by s_{r-1} from $Y_r \to Y_{r-1}$ is canonically isomorphic to $X \times K(G_r, n_r)$ (we have assumed that $c_{r-1} \circ s_{r-1}$ is the constant map).

2. Model for a Galois covering of a nilpotent space.

2.1. *Notations.* All DG-algebras A (differential graded) will be defined over the field Q of rationals, commutative in the graded sense and positively graded ($A^q = 0$ for $q < 0$), unless otherwise specified.

A morphism $A \to B$ of DG-algebras is a weak equivalence, abbreviated w.e., if it induces an isomorphism $H(A) \to H(B)$ in cohomology.

If A is a DG-algebra and V a graded vector space, then $A(V)$ will be a DG-algebra which is, as a graded algebra, the tensor product of A with the symmetric algebra (in the graded sense) over V, with a differential d extending the given differential on $A \subset A(V)$. We identify V to the vector subspace $1 \otimes V$ in $A(V)$.

A model for a space X is a DG-algebra A together with a w.e. α: $A \to \Omega^*(X)$, where $\Omega^*(X)$ denotes the DG-algebra of Q-polynomial forms on the singular complex of X.

For instance, let G be an abelian group such that $V = \mathrm{Hom}(G; Q)$ is a finite dimensional vector space over Q. Consider V as a graded vector space homogeneous of degree m. Then the algebra $Q(V)$ of polynomial or alternate forms on V according to the parity of m, with the zero differential, is a model for the Eilenberg-Mac Lane complex $K(G, m)$, $m > 0$.

We shall use repeatedly the following fact which follows easily from Grivel [**1**] (see also S. Halperin [**5**]).

PROPOSITION. *Let $X_2 \to X_1$ be a principal abelian fibration whose fiber X_0 is a product of connected Eilenberg-Mac Lane complexes. Let φ_1: $A \to \Omega^*(X_1)$ be a morphism of augmented* DG-*algebras which is a model for X_1. Let $A(V)$ be a* DG-*algebra such that $dV \subset A$. Assume we have a morphism φ_2: $A(V) \to \Omega^*(X_2)$ such that the diagram*

$$\begin{array}{ccc} A(W) & \overset{\varphi_2}{\to} & \Omega^*(X_2) \\ \uparrow & & \uparrow \\ A & \overset{\varphi_1}{\to} & \Omega^*(X_1) \end{array}$$

commutes.

Let φ_0: $Q(V) \to \Omega^*(X_0)$ be the induced morphism on the fiber. Then φ_2 (resp. φ_0) is a w.e. if φ_0 (resp. φ_2) is a w.e.d.

2.2. *Abelian Galois covering of a nilpotent space.* Let X be a connected nilpotent space (i.e. X considered as a bundle over a point is nilpotent as in 1.3). We assume that the homotopy groups $\pi_i(X)$, $i > 1$, are such that $\pi_i(X) \otimes Q$ are finite dimensional vector spaces, and the same property for the successive quotients in the lower central series of the nilpotent group $\pi_1(X)$.

Let p: $\tilde{X} \to X$ be a Galois covering with abelian Galois group G such that $G \otimes Q$ is finite dimensional. It is classified by a surjective homomorphism

$$\lambda: \pi_1(X) \to G.$$

We consider $V = \mathrm{Hom}(G, Q)$ as a graded vector space homogeneous of degree 0.

Let α: $A \to \Omega^*(X)$ be a model for X. There is an injective linear map h of V in the cocycles of degree 1 of A such that, when we pass to cohomology, h is the dual λ^*: $V \to \mathrm{Hom}(\pi_1(X), Q) = H^1(X; Q)$ of λ. Note that the kernel of p^*: $H^1(X; Q) \to H^1(\tilde{X}; Q)$ is the image of λ^*.

On $A(V)$, the algebra of polynomials on V with coefficients in A, consider the differential d extending the differential of A and such that $d(v) = h(v)$ for $v \in V$.

PROPOSITION. *$A(V)$ is a model of $\tilde{X}$. More precisely, any morphism $\tilde{\alpha}$: $A(V) \to \Omega^*(\tilde{X})$ extending α is a model.*

PROOF. Consider a minimal Postnikov tower of X,

$$X_1 \leftarrow X_2 \leftarrow X_{r-1} \leftarrow \cdots \leftarrow X_r \leftarrow \cdots \leftarrow X',$$

where each map is a principal abelian fibration, X_1 being $K(\pi_1(X)/[\pi_1(X), \pi_1(X)], 1)$ and $X' = \varprojlim X_r$ being homotopy equivalent to X by a map $X \to X'$.

Let $B = Q(W)$ be a minimal model for X (cf. [3]) reflecting the above decomposition: there is a filtration $W_1 \subset W_2 \subset \cdots \subset W$ such that $Q(W_r)$ is a DG-subalgebra of $Q(W_{r+1})$ with $dW_{r+1} \subset Q(W_r)$, and $W_1 = \mathrm{Hom}(\pi_1(X), Q) = H^1(X; Q)$. There is also a morphism $Q(W) \xrightarrow{\varphi} \Omega^*(X')$ which is a weak equivalence and whose restriction φ_r to $Q(W_r)$ gives a model $Q(W_r) \to \Omega^*(X_r)$ of X_r.

Consider the exact sequence

$$0 \to N \to \pi_1(X)/[\pi_1(X), \pi_1(X)] \to G \to 0$$

given by λ. It induces a fibration

$$X_1 = K(\pi_1(X)/[\pi_1(X), \pi_1(X)], 1) \to K(G, 1)$$

with fiber $K(N, 1)$.

So we get a fibration $X' \to K(G, 1)$ by composition $X' \to X_1 \to K(G, 1)$. The inclusion of its fiber $\tilde{X}'$ in X' is homotopically equivalent to p: $\tilde{X} \to X$.

Let $W_0 \subset W_1$ be the image of V by λ^*: $V \to H^1(X; Q) = W_1$. Choose φ_1 so that its restriction to $Q(W_0)$ is a model $Q(W_0) \to \Omega^*(K(G, 1))$. Let $B/(W_0)$ be the DG-algebra quotient of $B = Q(W)$ by the ideal generated by W_0. The composition of $B \to \Omega^*(X')$ with the restriction to the fiber $\tilde{X}'$ vanishes on W_0. So we get a morphism $B/(W_0)$ in $\Omega^*(\tilde{X}')$ which is a model (in fact a minimal model) as follows from repeated applications of the proposition in 2.1.

On $B(V)$ define a differential d as above by $dv = \lambda^* v$, $v \in V$. The map $B(V) \to B/(W_0)$ obtained by mapping V on 0 and taking the quotient on B induces an isomorphism in cohomology (this can be proved by induction on r, using the filtration of W by the W_r).

So we can get a homotopy commutative diagram

$$\begin{array}{ccc} B(V) & \xrightarrow{\tilde{\beta}} & \Omega^*(\tilde{X}) \\ \uparrow & & \uparrow \\ B & \xrightarrow{\beta} & \Omega^*(X) \end{array}$$

where the horizontal maps are w.e. In fact, we can assume the diagram commutative, because B is a free nilpotent algebra.

Let α: $A \to \Omega^*(X)$ be a model for X and $\tilde{\alpha}$: $A(V) \to \Omega^*(\tilde{X})$ be a morphism extending α (such a morphism exists because $p^*\lambda^* = 0$). As B is the minimal model of A, there is a w.e. f: $B \to A$ extending to a morphism f: $B(V) \to A(V)$. It is easy to check that f also induces an isomorphism in cohomology (filter by the degree of the polynomials in V and use induction).

After changing β by a homotopy, we can assume that $\alpha f = \beta$. Now $\tilde{\alpha}\tilde{f}$ and $\tilde{\beta}$ differs only by a map in the constant functions in $\Omega^*(\tilde{X})$, so they are homotopic. Hence α is also a weak equivalence.

REMARK. If λ: $\pi_1(X) \to G$ is not surjective, then the same result is valid if $\tilde{X}$ is replaced by one of the connected components of the Galois G-covering defined by λ, and V replaced by its quotient $\overline{V}$ isomorphic to the image of λ^*.

3. The model of Sullivan and the main theorem.

3.1. *The algebraic model of Sullivan.* Let p: $Y \to X$ be a nilpotent bundle, i.e. admitting a Moore-Postnikov factorization through principal $K(G_r, n_r)$-fibrations as in 1.3. We assume that $V_r = \mathrm{Hom}(G_r, Q)$ is finite dimensional. It will be considered as a graded vector space homogeneous of degree n_r.

Let α: $A \to \Omega^*(X)$ be a model for X. Then a model for Y, reflecting this Postnikov decomposition, will be of the form $A(V)$, where $V = \oplus V_r$ (cf. [3]). Suppose that s: $X \to Y$ is a section. It gives a morphism σ: $A(V) \to A$ which is the identity on A. We can assume that σ is zero on V.

Indeed if this is not the case, let h be the A-algebra automorphism of $A(V)$ mapping v on $v - \sigma(v)$; define on $A(V)$ a new differential d' such that $dh = hd'$. Then h is a DG-automorphism and $\sigma \circ h$ maps V on zero.

We assume that A is finite dimensional. Denote by $\underline{A}$ the lower graded vector space whose ith component $\underline{A}_i$ is $\mathrm{Hom}(A^i, Q)$. Let $\underline{A} \otimes V$ be the graded vector space whose component of degree k is $\oplus_{-i+j=k} \underline{A}_i \otimes V^j$ (so in general we have components with negative degree).

There is a canonical A-algebra homomorphism

$$\varepsilon' \colon A(V) \to A \otimes Q(\underline{A} \otimes V)$$

defined by $\varepsilon'(a) = a \otimes 1$, $a' \cap \varepsilon'(v) = a' \otimes v$ for each $a' \in \underline{A}$, $v \in V$, where $a' \cap (a \otimes z) = a'(a)z$, for $z \in Q(\underline{A} \otimes V)$.

In terms of an additive basis a_i of A and the dual basis a_j' of $\underline{A}$, then $\varepsilon'(v) = \Sigma_i a_i \otimes (a_i' \otimes v)$.

On the algebra $Q(\underline{A} \otimes V)$, which is in general not positively graded, there is a unique differential d such that ε' is a morphism of DG-algebras. The natural augmentation $Q(\underline{A} \otimes V) \to Q$ which is the identity on Q and zero on $\underline{A} \otimes V$ commutes with the differentials.

The differential on $Q(\underline{A} \otimes V)$ induces on $\underline{A} \otimes V$ a differential d_0. Consider the quotient of $Q(\underline{A} \otimes V)$ by the ideal generated by elements of degree $\leqslant 0$ of $\underline{A} \otimes V$ and their differentials. It is isomorphic to the algebra $Q(W)$, where W is the quotient of $\underline{A} \otimes V$ by elements of degree $\leqslant 0$ and their images by d_0. Note that $Q(W)$ is positively graded and $Q(W)^0 = Q$.

From ε' we get a DG-map ε: $A(V) \to A \otimes Q(W)$. This is the model proposed by Sullivan for the evaluation map e: $X \times \Gamma_s \to Y$. It has the following universal property. Let D be a DG-algebra such that $D^0 = Q$, and let f: $A(V) \to A \otimes D$ be a morphism of augmented DG-algebras over A. Then there is a unique $\varphi\colon Q(W) \to D$ such that the diagram

$$\begin{tikzcd} A(V) \arrow[rr, "f"] \arrow[dr, "\varepsilon"'] & & A \otimes D \\ & A \otimes Q(W) \arrow[ur, "1\otimes\varphi"'] & \end{tikzcd}$$

commutes.

3.2. THEOREM. *Under the above assumptions* ($p\colon Y \to X$ *a nilpotent bundle and* X *admitting a finite dimensional model*),[1] *the* DG-*algebra* $Q(W)$ *is a model for the space* Γ_s *of sections of* $p\colon Y \to X$ *homotopic to a given section* s. *The morphism* ε *is a model for the evaluation map* e.

For the proof, we first show the theorem in the case of a trivial $K(G, m)$-bundle, using a model weakly equivalent to Sullivan's model. Then we assume by induction that the theorem is proved for the space Γ_{r-1} of sections of the bundle $Y_{r-1} \to X$ in the tower of $p\colon Y \to X$ (cf. 1.3). We then construct in 3.5 an algebraic model for the bundle $\Gamma_r \to \Gamma_{r-1}$ (remember that the fiber is not connected in general) weakly equivalent to Sullivan's model, and show in 3.6 that it is indeed a model using §2.

3.3. *Case of a* $K(G, m)$*-trivial bundle*. We assume that $Y = X \times K(G, m)$, with the section s corresponding to the constant map of X on the base point of $K(G, m)$.

In that case, $V = \mathrm{Hom}(G, Q)$ is homogeneous of degree m. The differential on $Q(\underline{A} \otimes V)$ is given by

$$d(a' \otimes v) = \pm \partial a' \otimes v,$$

where $\partial\colon \underline{A} \to \underline{A}$ is the transpose of d.

Let W be the quotient of $\underline{A} \otimes V$ by the subspace of elements of degree < 0 and cocycles in degree 0.

Let $H_*(\underline{A}) = \mathrm{Hom}(H^*(A), Q) = H_*(X; Q)$. We can construct a linear injection $j\colon H_*(A) \to \underline{A}$ mapping a homology class on a representative cycle.

Let $\overline{W}$ be the graded vector space defined by

$$\overline{W}^k = \bigoplus_{-i+j=k} H_i(\underline{A}) \otimes V^j \quad \text{for } k > 0,$$

$$\overline{W}^k = 0 \quad \text{for } k \leqslant 0.$$

j gives an inclusion of $\overline{W}$ in W and the corresponding inclusion $\bar{j}\colon Q(\overline{W}) \to Q(W)$ is a weak equivalence (the differential on $Q(\overline{W})$ is trivial).

There is a w.e.: $Q(V) \to \Omega^*(K(G, m))$ such that the induced map on cohomology gives the canonical isomorphism of $H^m(Q(V)) = V$ on $H^m(K(G, m); Q) = \mathrm{Hom}(G, Q)$.

[1] Cf. remark at the end for a less restrictive hypothesis.

Let Γ_0 be the space $K(G, m)_0^X$ of maps of X in $K(G, m)$ homotopic to the constant map on the base point, and let e: $X \times \Gamma_0 \to X \times K(G, m)$ defined by $e(x, g) = (x, g(x))$ be the evaluation map.

We can construct a homotopy commutative diagram

$$\begin{array}{ccc} \Omega^*(X \times \Gamma_0) & \overset{\Omega^*(e)}{\leftarrow} & \Omega^*(Y) \\ \uparrow & & \uparrow \\ A \otimes \Omega^*(\Gamma_0) & \overset{f}{\leftarrow} & A \otimes \mathbf{Q}(V) = A(V) \end{array}$$

where the vertical arrows are w.e., and f is a morphism of DG-algebras over A such that $f(v) \in A \otimes \Omega^*(\Gamma_0)^+$, where $\Omega^*(\Gamma_0)^+$ is the kernel of the augmentation given by the base point.

There is a unique morphism φ: $Q(W) \to \Omega^*(\Gamma_0)$, mapping the class of $a' \otimes v$ on $a' \cap f(v)$, such that the diagram

$$\begin{array}{ccc} A \otimes \Omega^*(\Gamma_0) & \xleftarrow{\quad f \quad} & A(V) \\ & {}_{1 \otimes \varphi}\nwarrow \qquad \swarrow_{\varepsilon} & \\ & A \otimes Q(W) & \end{array}$$

commutes, where $a' \cap \varepsilon(v)$ is the class of $a' \otimes v$.

To check that φ (or $\varphi \circ \bar{j}$) is a weak equivalence, we pass to cohomology in the above diagram and use 1.2 which shows that $H^*(\Gamma_0; Q) = Q(\overline{W})$ and gives the precise description of the evaluation map.

3.4 *The induction hypothesis*. For the general case of a nilpotent bundle, we take the notations of 1.3.

To say that $A(V)$ is a model of Y reflecting the factorization $X = Y_0 \leftarrow Y_1 \leftarrow Y_2 \leftarrow \cdots$ means that

(a) each $B_r = A(\oplus_{k<r} V_k)$ is a DG-subalgebra of $A(V)$, and $dV_r \subset B_{r-1}$, so that we have an increasing sequence of DG-subalgebras

$$A = B_0 \subset B_1 \subset B_2 \subset \cdots;$$

(b) for each r, we have a weak equivalence α_r: $B_r \to \Omega^*(Y_r)$ such that the diagram

$$\begin{array}{ccc} B_r = B_{r-1}(V_r) & \overset{\alpha_r}{\to} & \Omega^*(Y_r) \\ \uparrow & & \uparrow \\ B_{r-1} & \overset{\alpha_{r-1}}{\to} & \Omega^*(Y_{r-1}) \end{array}$$

commutes and α_r induces a weak equivalence $Q(V_r) \to \Omega^*(K(G_r, n_r))$ which in cohomology gives the canonical isomorphism $V_r \to H^{n_r}(K(G_r, n_r); Q)$; and

(c) we can choose α_r such that the diagram

$$\begin{array}{ccc} B_r & \overset{\alpha_r}{\to} & \Omega^*(Y_r) \\ \downarrow{\scriptstyle \sigma_r} & & \downarrow{\scriptstyle \Omega^*(s_r)} \\ A & \to & \Omega^*(X) \end{array}$$

commutes, where σ_r is the restriction of σ to B_r.

Let C_{r-1} be the Sullivan model for Γ_{r-1}. We assume by induction that we have a w.e. $\varphi_{r-1}\colon C_{r-1} \to \Omega^*(\Gamma_{r-1})$ so that the diagram

$$\begin{tikzcd} A \otimes \Omega^*(\Gamma_{r-1}) & \\ & B_{r-1} \arrow[ul, "f_{r-1}"'] \arrow[dl, "\varepsilon_{r-1}"] \\ A \otimes C_{r-1} \arrow[uu, "1\otimes\varphi_{r-1}"] & \end{tikzcd}$$

commutes, where f_{r-1} is a model for the evaluation map $e_{r-1}\colon X \times \Gamma_{r-1} \to Y_{r-1}$. We also assume that σ_{r-1}, together with the augmentations in C_{r-1} and $\Omega^*(\Gamma_{r-1})$ (the latter given by s_{r-1}) give morphisms of the diagram in A so that everything is commutative.

3.5. *Construction of an algebraic model.* We construct as follows a model for the fibration $\Gamma_r \to \Gamma_{r-1}$.

There is an algebra homomorphism ε_r' such that

$$\begin{array}{ccc} A \otimes C_{r-1}(\underline{A} \otimes V_r) & \overset{\varepsilon_r'}{\leftarrow} & B_{r-1}(V_r) = B_r \\ \uparrow & & \uparrow \\ A \otimes C_{r-1} & \overset{\varepsilon_{r-1}}{\leftarrow} & B_{r-1} \end{array}$$

commutes (compare with 3.1), defined by $a' \cap \varepsilon_r'(v) = a' \otimes v$.

On $C_{r-1}(\underline{A} \otimes V_r)$, there is a unique differential such that ε_r' commutes with d;

$$d(a' \otimes v) = \pm \partial a' \otimes v + \pm a' \cap \varepsilon_{r-1}(dv).$$

Let $C_{r-1}(U)$ be the quotient of $C_{r-1}(\underline{A} \otimes V_r)$ by the ideal generated by elements of $\underline{A} \otimes V_r$ of degree < 0 and those elements of degree 0 whose boundary is in the ideal generated by elements of negative degree (this ideal is closed under d and $H^0(C_{r-1}(U)) = Q$ because $C^0_{r-1} = Q$).

We get from ε_r' a morphism

$$\varepsilon_r\colon B_{r-1}(V_r) \to A \otimes C_{r-1}(U)$$

extending ε_{r-1}. We shall prove that this is a model for e_r.

$C_{r-1}(U)$ verifies the following universal property: let D be a DG-algebra with an augmentation and such that $H^0(D) = Q$. Given a commutative diagram

$$\begin{array}{ccccc} A \otimes D & & \overset{f}{\leftarrow} & & B_{r-1}(V_r) \\ & \searrow & & \swarrow \sigma_r & \\ \uparrow 1 \otimes h & & A & & \uparrow \\ & \nearrow & & \nwarrow \sigma_{r-1} & \\ A \otimes C_{r-1} & & \underset{\varepsilon_{r-1}}{\leftarrow} & & B_{r-1} \end{array}$$

where the left-hand maps in A are the tensor product of the identity on A with the augmentation, there is a unique morphism $\varphi\colon C_{r-1}(U) \to D$ of augmented DG-algebras over C_{r-1} such that

$$\begin{tikzcd} A \otimes D & & \\ & & B_{r-1}(V_r) \arrow[ull, "f"'] \arrow[dll, "\varepsilon_r"] \\ A \otimes C_{r-1}(U) \arrow[uu, "1 \otimes \varphi"] & & \end{tikzcd}$$

commutes. On the class u of $a' \otimes v$, φ is given by

$$\varphi(u) = a' \cap f(v).$$

We get the Sullivan model C_r by taking the quotient of $C_{r-1}(U)$ by the ideal generated by elements of U of degree 0 and their differentials.

It will be sufficient to prove that ε_r is a model for e_r, once we have checked that the quotient map $C_{r-1}(U) \to C_r$ is a weak equivalence.

C_{r-1} is of the form $Q(S)$ with $S^p = 0$ for $p \leqslant 0$. Hence the differential d maps U^0 injectively in $U^1 \oplus S^1$. The DG-algebra $C_{r-1}(U)$ is nilpotent, hence we can choose elements $x_1, \ldots, x_k$ of $U \oplus S$ so that $C_{r-1}(U) = Q(U^0, dU^0)(x_1, \ldots, x_k)$ and so that $dx_{i+1} \in Q(U^0, dU^0)(x_1, \ldots, x_i)$. By induction on i, the projection of $Q(U^0, dU^0)(x_1, \ldots, x_i)$ on its quotient by the ideal generated by U^0 and dU^0 is a w.e. When $i = k$ this is the projection of $C_{r-1}(U)$ on C_r.

3.6. *Proof of the induction step*. One can find a morphism f_r which is a model for the evaluation map $e_r\colon X \times \Gamma_r \to Y_r$ and such that

$$\begin{tikzcd} A \otimes \Omega^*(\Gamma_r) \arrow[dr] & & B_{r-1}(V_r) = B_r \arrow[ll, "f_r"'] \arrow[dl] \\ & A & \\ A \otimes \Omega^*(\Gamma_{r-1}) \arrow[uu] \arrow[ur] & & B_{r-1} \arrow[ll, "f_{r-1}"] \arrow[uu] \arrow[ul] \end{tikzcd}$$

commutes.

By the universal property, we get a morphism

$$\varphi_r\colon C_{r-1}(U) \to \Omega^*(\Gamma_r)$$

such that the diagram

$$\begin{tikzcd} A \otimes \Omega^*(\Gamma_r) & & \\ & A \otimes C_{r-1}(U) \arrow[ul, "1 \otimes \varphi_r"'] & B_{r-1}(V_r) \arrow[l] \arrow[ull, "f_r"'] \\ & A \otimes C_{r-1} \arrow[u] \arrow[dl, "1 \otimes \varphi_{r-1}"] & B_{r-1} \arrow[l] \arrow[u] \arrow[dll, "f_{r-1}"] \\ A \otimes \Omega^*(\Gamma_{r-1}) \arrow[uuu] & & \end{tikzcd} \tag{1}$$

commutes.

We have to show that φ_r is a weak equivalence. For $u \in U$, $du = d_0u + d_1u$, where $d_0u \in U$ and $d_1u \in C_{r-1}$ (cf. 3.5).

Let $\bar{U}^0$ be the kernel of d_0: $U^0 \to U^1$. Then $C_{r-1}(\bar{U}^0)$ is a DG-subalgebra of $C_{r-1}(U)$ and d maps $\bar{U}^0$ injectively into the 1-cocycles of C_{r-1}. When we pass to cohomology d: $\bar{U}^0 \to H^1(C_{r-1}) = H^1(\Gamma_{r-1}; Q)$ is still injective because $C^0_{r-1} = Q$. The image is generated by cohomology classes of elements of the form $a' \cap \varepsilon_{r-1}(dv)$, where $v \in V_r$, $a' \in \underline{A}_{n_r}$ with $\partial a' = 0$.

As in 1.3, we consider the factorization $\Gamma_r \to \tilde{\Gamma}_{r-1} \to \Gamma_{r-1}$. We get a commutative diagram

$$\begin{array}{ccc} C_{r-1}(U) & \overset{\varphi_r}{\to} & \Omega^*(\Gamma_r) \\ \uparrow & & \uparrow \\ C_{r-1}(\bar{U}^0) & \overset{\tilde{\varphi}_{r-1}}{\to} & \Omega^*(\tilde{\Gamma}_{r-1}) \\ \uparrow & & \uparrow \\ C_{r-1} & \overset{\varphi_{r-1}}{\to} & \Omega^*(\Gamma_{r-1}) \end{array}$$

because ε_r maps elements of $\bar{U}^0$ on functions on the singular complex of Γ_r whose differential comes from $\Omega^*(\Gamma_{r-1})$, hence are constant on the connected components of the fibers of $\Gamma_r \to \Gamma_{r-1}$.

We claim that $\tilde{\varepsilon}_{r-1}$ is a weak equivalence. Indeed, $\tilde{\Gamma}_{r-1}$ is a connected component of the Galois covering defined by the homomorphism λ: $\pi_1(\Gamma_r) \to H^{n_r}(X, G_r)$ induced by $\bar{c}_{r-1}$: $\Gamma_{r-1} \to K(G_r, n_r + 1)^X_0$ (cf. 1.3). Its dual gives a map

$$\lambda^*: \operatorname{Hom}\big(H^{n_r}(X; G_r), Q\big) = H_{n_r}(X; Q) \otimes V_r \to H^1(\Gamma_{r-1}; Q)$$

whose image is isomorphic to $d\bar{U}^0$ by the remark above and 1.2. So we can apply the proposition and the last remark of 2.2 to deduce that $\tilde{\varepsilon}_{r-1}$ is a weak equivalence (Γ_{r-1} is a nilpotent space).

$C_{r-1}(U)$ can be considered as an algebraic bundle over $C_{r-1}(\bar{U}^0)$, with fiber $Q(W) = C_{r-1}(U) \otimes_{C_{r-1}(\bar{U}^0)} Q$ where Q is considered as a $C_{r-1}(\bar{U}^0)$-module via the augmentation. The vector space W is isomorphic to the quotient of $\underline{A} \otimes V_r$ by the subspace of elements of degree < 0 and d_0-cycles in degree 0. We can express $C_{r-1}(U)$ as $C_{r-1}(\bar{U}^0) \otimes Q(W)$, where the differential of $w \in W$ has a component in W and another one of degree at least 2 in C_{r-1}.

Recall that the fiber Γ_0 of $\Gamma_r \to \tilde{\Gamma}_{r-1}$ containing s_r is isomorphic to $K(G_r, n_r)^X_0$, where $K(G_r, n_r)$ is the fiber of $Y_r \to Y_{r-1}$.

As $\tilde{\varphi}_{r-1}$ preserves the augmentation, we get a morphism φ^0_r of the algebraic fiber $Q(W)$ in the DG-algebra of forms $\Omega^*(\Gamma_0)$ of the geometric fiber, and the diagram (1) gives the commutative diagram

$$\begin{tikzcd} A \otimes \Omega^*(\Gamma_0) & \\ A \otimes Q(W) \arrow[u] & A \otimes Q(V_r) \arrow[l] \arrow[ul] \end{tikzcd}$$

and we show as in 3.3 that φ^0_r is a weak equivalence.

As φ_r induces a weak equivalence on the base and the fiber, we can apply the proposition in 2.1 to show that φ_r is also a weak equivalence.

Consider the commutative diagram over A

$$\begin{tikzcd}
A \otimes \Omega^*(\Gamma_r) & \\
A \otimes C_{r-1}(U) \arrow[u, "1\otimes\varphi_r"] \arrow[d, "1\otimes\rho"'] & B_r \arrow[ul, "f_r"'] \arrow[l, "\varepsilon_r"'] \arrow[dl, "\varepsilon"] \\
A \otimes C_r &
\end{tikzcd}$$

when ρ: $C_{r-1}(U) \to C_r$ is the w.e. described in 3.5 and ε the morphism defined in 3.1.

As ρ is a surjective w.e., then there is a w.e. i: $C_r \to C_{r-1}(U)$ such that $\rho \circ i = \mathrm{id}$. Then $\varphi = \varphi_r \circ i$: $C_r \to \Omega^*(\Gamma_r)$ is a w.e. and $f = (1 \otimes \varphi) \circ \varepsilon$ is homotopic over A to f_r, hence is a model for the evaluation map $X \times \Gamma_r \to Y_r$.

Remark. In the construction 3.1 of the Sullivan model, we only need to assume A finite dimensional in each degree provided V is finite dimensional, because the DG-algebra $Q(W)$ depends only on elements of A whose degree is not more than the maximal degree of the elements in V (same remark for 3.5).

Hence the main theorem 3.2, with essentially the same proof, is valid under the hypothesis that either

(i) the rational homotopy of the fiber of the nilpotent bundle p: $Y \to X$ is finite dimensional and X has a model A which is finite dimensional in each degree, or

(ii) X has a model A which is finite dimensional in each degree and such that $\dim H^*(A)$ is finite.

Indeed in that case, we can prove the induction step 3.5 using A; to pass to the limit when r tends to infinity, we can replace A by a finite dimensional algebra.

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