

## CONDITIONING BY $\langle$ EQUAL, LINEAR $\rangle$ <sup>1</sup>

BY

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**ABSTRACT.** We deal with a limit problem of regularity controlled probabilities in metric pattern theory. The probability on the generator space is given by a density function  $f(x, y)$  on which some integrability conditions are imposed. Let  $T$  denote the integral operator with kernel  $f$ . When  $n$  i.i.d. generators  $(X_k, Y_k)$  are connected together to form the configuration space  $\mathcal{C}_n$  via the regularity  $\langle$ EQUAL, LINEAR $\rangle$ , i.e. "conditioning" on  $X_{k+1} = Y_k$  for  $1 \leq k < n$ , an approximate identity is used to define the regularity controlled probability on  $\mathcal{C}_n$ . The probabilistic effect induced by the regularity conditions on some fixed subconfiguration of a larger configuration  $\mathcal{C}_n$  is described by its corresponding marginal probability within  $\mathcal{C}_n$ . When  $n$  goes to infinity in a suitable way, the above mentioned marginal probability converges weakly to a limit whose density can be expressed in terms of the largest eigenvalues and the corresponding eigenspaces of  $T$  and  $T^*$ . When  $f$  is bivariate normal, the eigenvalue problem is solved explicitly. The process determined by the limiting marginal probabilities is strictly stationary and Markovian.

**0. Introduction.** First let us recall the concept of regularity controlled probabilities introduced by Grenander in metric pattern theory [7, Chapter 5]. Generators are the building blocks in this pattern theory. With some given regular structures, generators are put together to form configurations. Suppose that  $n$  i.i.d. random generators are generated from a fixed generator space, then under reasonable conditions one should be able to induce naturally a probability on the configuration space  $\mathcal{C}_n$  by "conditioning" on the given regularity constraints.

If this is done and the probabilistic effect induced by the regularity constraints on some fixed subconfiguration of a larger configuration  $\mathcal{C}_n$  is denoted by its corresponding marginal probability within  $\mathcal{C}_n$ , then the following limiting problem arises: What is the ultimate effect on a particular fixed subconfiguration when the regularity constraints "grow"? In other words, as  $n \rightarrow \infty$  in a suitable way, what is the limiting marginal probability? And what can one say about the process determined by these limiting marginal probabilities?

In this article we deal with the regularity  $\langle$ EQUAL, LINEAR $\rangle$ . Using the terminology in pattern theory, the definition goes as follows: each generator has one

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in-bond and one out-bond; configuration consisting of  $n$  generators  $g_1, \dots, g_n$  satisfies the constraints: in-bond value of  $g_{k+1}$  equals out-bond value of  $g_k$  for  $1 \leq k < n$  (Grenander [4]).

To clarify the above definition, let us consider the following example. Let generators consist of linear functions over finite intervals  $[a, b]$ ,  $a \leq b$ , so that it can be completely specified by its in-bond value  $b_{\text{in}} = (a, h(a))$ , and out-bond value  $b_{\text{out}} = (b, h(b))$ . Any  $n$  generators  $(b_{\text{in}}^1, b_{\text{out}}^1), \dots, (b_{\text{in}}^n, b_{\text{out}}^n)$  do not necessarily form a configuration under the regularity  $\langle \text{EQUAL, LINEAR} \rangle$  unless  $b_{\text{out}}^k = b_{\text{in}}^{k+1}$ . I.e. a configuration is a continuous linear spline.

Conditioning by  $\langle \text{EQUAL, LINEAR} \rangle$  was first proposed by Grenander in [5]. Special cases have been considered in Plumeri [11] and Hwang [9]. We consider a generalized version in §§1 and 2. Approximate identity is used to induce the regularity controlled probability on  $\mathcal{C}_n$ . The density of the limiting marginal probability is given in Theorem 2.1 which involves the maximal eigenvalues (and the corresponding eigenspaces) of compact positive integral operators. Theorem 2.2 proves that the process is Markovian and strictly stationary. §§3 and 4 are devoted to open Problems 1 and 2 mentioned in Grenander [6]. In §3 the Gaussian case is settled by solving the eigenvalue problem of the integral equation with bivariate Gaussian density as its kernel. The relaxed regularity case is considered in §4. The assumptions about the initial density  $f$  on the generator space are expressed in functional-analytic terms. We try to give some probabilistic interpretation and motivation at the end of §4.

Let us mention some other related work in regular controlled probabilities. Plumeri discussed  $\langle \text{EQUAL, LINEAR} \rangle$ ,  $\langle \text{INCLUSION, LINEAR} \rangle$  and  $\langle \text{EQUAL, BINARY TREE} \rangle$  in [12], Chow and Wu studied some limit theorems for the totally symmetric case and relaxed  $\langle \text{EQUAL, LINEAR} \rangle$  in [1].

**1. Probability measure on the configuration space  $\mathcal{C}_n$ .** Generators with arity 2 are being connected with the regularity  $\langle \text{LINEAR, EQUAL} \rangle$ . The bond value space  $X$  is equipped with a  $\sigma$ -finite measure  $dx$ . There is a probability measure on the space of generators which is given by a density function  $f$  on  $X \times X$ . Two assumptions are imposed on  $f$ ,

$$(A1) f \in L^2(X \times X),$$

$$(A2) \int f(x, y) dy \text{ and } \int f(x, y) dx \text{ are in } L^2(X).$$

Let  $T$  denote the linear transform from  $L^2(X) \rightarrow L^2(X)$  defined by  $T(\phi)(x) = \int f(x, y)\phi(y) dy$ .

Let  $e$  denote the constant function 1. For brevity we write (A2) as  $T(e) \in L^2(X)$  and  $T^*(e) \in L^2(X)$ . Note that if (A2) holds then the eigenfunction of  $T$  is also in  $L^1(X)$ .

Clearly,  $T_u^n$  is defined by the kernel function  $f_n$ ,

$$f_n(x, y) = \int \cdots \int f(x, z_1)f(z_1, z_2) \cdots f(z_{n-2}, z_{n-1})f(z_{n-1}, y) dz_1 \cdots dz_{n-1}.$$

By (A2),  $f_n(x, y) \in L^1(X \times X)$  and  $\int \int f_n(x, y) dx dy = \langle (T^*)^{n-1}e, Te \rangle$ .

When we connect  $n$  i.i.d. random generators together via the regularity (EQUAL, LINEAR) to form configurations in the configuration space  $\mathcal{C}_n$ , the induced probability measure on  $\mathcal{C}_n$  is defined by the following density function w.r.t.  $dx_1 dx_2 \cdots dx_{n+1}$ ,

$$(1.1) \quad f(x_1, x_2, \dots, x_{n+1}) = \langle (T^*)^{n-1} e, Te \rangle^{-1} f(x_1, x_2) f(x_2, x_3) \cdots f(x_n, x_{n+1}).$$

(1.1) is well defined if

(D1)  $T$  is not nilpotent.

When the underlying measure  $dx$  is translation invariant, (1.1) is a natural choice. But in other cases, (1.1) might turn out to be too artificial, for example when  $dx$  is multiplication invariant. A similar situation has been discussed in Grenander [4, Chapter 2, §10]. In this article, we shall concentrate on density functions of the form (1.1).

The motivation of this definition comes from the following consideration. Let  $X = \mathbb{R}^d$  and  $dx$  be the Lebesgue measure. First let us consider configurations of two generators with densities  $f(x_1, x_2)$  and  $f(x_3, x_4)$ . To satisfy the regularity constraint means somehow we have to “condition” along the diagonal  $x_2 = x_3$ . To be able to do so we have to assume that  $\exists A$  with positive measure  $\exists f(x, y) > 0$  a.e. on  $A \times A$ . The measure  $P$  on  $\mathcal{C}_2$  is obtained by using approximate identity (a family of density functions  $\{\phi_\epsilon \mid \epsilon > 0\}$  is called an approximate identity if  $\phi_\epsilon$  converges weakly to the delta function at 0 as  $\epsilon \rightarrow 0$ ) as in Hwang and Anderson [8]. I.e. define

(1.2)

$$s \frac{dP_\epsilon}{dx_1 dx_2 dx_3}(x_1, x_2, x_3) = \frac{\int \phi_\epsilon(x_2 - y) f(x_1, y) dy f(x_2, x_3)}{\int \int \int \phi_\epsilon(x_2 - y) f(x_1, y) dy f(x_2, x_3) dx_1 dx_2 dx_3}.$$

$P$  is the weak limit of  $P_\epsilon$  as  $\epsilon \rightarrow 0$ .

The denominator of (1.2) can be rewritten as

$$\int \int (\phi_\epsilon * T^*(e))(x_2) f(x_2, x_3) dx_2 dx_3 = \langle \phi_\epsilon * T^*(e), Te \rangle \rightarrow \langle T^*e, Te \rangle.$$

Similarly, for any measurable set  $A_1, A_2, A_3$

$$\int_{A_1} \int_{A_2} \int_{A_3} \left( \int \phi_\epsilon(x_2 - y) f(x_1, y) dy \right) f(x_2, x_3) dx_1 dx_2 dx_3 \\ \rightarrow \int_{A_1} \int_{A_2} \int_{A_3} f(x_1, x_2) f(x_2, x_3) dx_1 dx_2 dx_3.$$

Hence,  $P$  has density of the form (1.1).

Repeating the above method, we get the result for  $\mathcal{C}_n$  as shown by (1.1).

**2. Stochastic process induced by the regularity constraint.** We shall describe how the regularity induces the stochastic process by specifying its marginal densities. Of course, these marginal densities have to constitute a consistent family. Heuristically let us consider a configuration of  $n$  generators in  $\mathcal{C}_n$ , hence we have  $n + 1$  random variables. Then, consider the marginal distribution for some fixed random variables within  $\mathcal{C}_n$ . When we let  $n \rightarrow \infty$  and at the same time keep these fixed random

variables “inside”, the weak limit of these marginal distributions is defined to be the marginal distribution for these fixed random variables in the process.

Mathematically, we are considering a stochastic process  $\{X_k | k \in \mathbf{Z}\}$ . The marginal density of  $X_{k_1}, \dots, X_{k_m}$ ,  $k_1 < k_2 < \dots < k_m$ , is specified by a limiting procedure to be described. Using the same notations,  $X_{k_1}, \dots, X_{k_m}$  are regarded as among the  $n + 1$  random variables in  $\mathcal{C}_n$ .

Suppose that there are  $n_1$  ( $n_2$ ) random variables to the left (right) of  $X_{k_1}$  ( $X_{k_m}$ ) in the configuration,  $X_{k_1}, \dots, X_{k_m}$  kept inside the configuration as  $n \rightarrow \infty$  means  $n_1$  and  $n_2 \rightarrow \infty$ . One may regard  $\mathbf{Z}$  as a coordinate system and  $k_1, \dots, k_m$  are fixed positions (coordinates). We use consecutive coordinates  $k_1 - n_1, \dots, k_m + n_2$  to label the  $n + 1$  random variables in  $\mathcal{C}_n$ . The marginal density  $h_n$  of  $X_{k_1}, \dots, X_{k_m}$  within  $\mathcal{C}_n$  is obtained by integrating out other variables in (1.1).

$$(2.1) \quad h_n(X_{k_1}, \dots, X_{k_m}) = \langle (T^*)^{n-1} e, T e \rangle^{-1} (T^*)^{n_1} e(x_{k_1}) f_{k_2-k_1}(x_{k_1}, x_{k_2}) \\ \cdots f_{k_m-k_{m-1}}(x_{k_{m-1}}, x_{k_m}) T^{n_2} e(x_{k_m}).$$

The weak limit (if it exists) of the marginal distributions corresponding to (2.1) is defined to be the marginal distribution of  $X_{k_1}, \dots, X_{k_m}$ .

Now let us calculate the limit of (2.1). If

(D2) the spectrum  $\sigma(T)$  of  $T$  is not  $\{0\}$ ,

then (D2)  $\Rightarrow$  (D1). If  $\exists A$  with positive measure and  $\exists \varepsilon > 0$  such that  $f(x, y) \geq \varepsilon$  a.e. on  $A \times A$ , then (D2) is satisfied. In fact  $A$  can be chosen with  $dx(A) < \infty$  and

$$\|T^n\|^{1/n} \geq \|T^n(I_A)\|^{1/n} (dx(I_A))^{-1/2n} \geq \varepsilon (dx(A))^{1-1/2n} \rightarrow \varepsilon (dx(A)),$$

$\therefore \sigma(T) \neq \{0\}$ . Also, note that if  $T$  is normal, then (D2) holds by the following observations:  $T^*T \neq 0$  is s.a. and  $(T^*T)^n = (T^*)^n T^n$ ,  $\|(T^*T)^n\|^{1/n} \leq \|(T^*)^n\|^{1/n} \|T^n\|^{1/n}$ .

With the set of all  $L^2$ -nonnegative functions as a cone,  $T$  can be regarded as a compact positive linear operator. By Zabreyko et al. [13], the radius  $\lambda$  of  $\sigma(T)$  is an eigenvalue of  $T$ . We also assume

(A3)  $\lambda > 0$  is the only eigenvalue with magnitude equal to the radius of  $\sigma(T)$ .

Obviously (A3) and (D2) hold for  $T^*$ .

Let  $\pi(\bar{\pi})$  denote the projection onto  $M(\bar{M})$  which is the eigenspace of  $T(T^*)$  corresponding to eigenvalue  $\lambda(\lambda)$ .

If  $T$  is normal, then by a spectral theorem (Dunford and Schwartz [2])

$$(T^*/\lambda)^k \rightarrow \bar{\pi} \quad \text{and} \quad (T/\lambda)^k \rightarrow \pi \quad \text{as } k \rightarrow \infty.$$

To prove the existence of the limit of  $\{h_n(x_{k_1}, \dots, x_{k_m})\}$  in this case we need the following lemmas.

LEMMA 2.1. *If (A1) and (A3) hold and  $T$  is normal, then  $\xi \in M_\lambda$  implies  $|\xi| \in M_\lambda$ .*

PROOF.

$$|T^n(\xi)| = \left| \int f_n(x, y) \xi(y) dy \right| \leq \int f_n(x, y) |\xi(y)| dy = T^n(|\xi|).$$

$$\therefore \|T^n(|\xi|)/\lambda^n\| \geq \|T^n(\xi)/\lambda^n\| = \|\xi\| = \|(|\xi|)\|.$$

Let  $n \rightarrow \infty$ , then  $\|\pi(|\xi|)\| \geq \|(|\xi|)\|$ . Hence  $|\xi| \in M_\lambda$ .  $\square$

LEMMA 2.2. If (A1) to (A3) hold and  $T$  is normal, then  $\pi(Te)$  is not zero a.e. and  $\pi(Te) \geq 0$ .

PROOF. First we prove that  $\pi(Te) \neq 0$  a.s. If this is not the case, then  $Te \perp M_\lambda$ . From Lemma 2.1, we may choose  $0 \neq \xi \in M_\lambda$  and  $\xi \geq 0$ , then

$$\int \xi(x) \left( \int f(x, y) dy \right) dx = \int \left( \int f(x, y) \xi(x) dx \right) dy = 0.$$

Since  $\int f(x, y) \xi(x) dx \geq 0$  a.e.,  $\int f(x, y) \xi(x) dx = 0$  a.e. which implies  $T^*(\xi) = 0$ . But  $0 < \|\lambda \xi\| = \|T(\xi)\| = \|T^*(\xi)\| = 0 \rightarrow \leftarrow$ .

Now, let us prove  $\pi(Te) \geq 0$  a.e.

$$\int |Te - \pi(Te)|^2 = \int_{\pi(Te) \geq 0} |Te - \pi(Te)|^2 + \int_{\pi(Te) < 0} |Te - \pi(Te)|^2,$$

$$\int |Te - |\pi(Te)||^2 = \int_{\pi(Te) \geq 0} |Te - \pi(Te)|^2 + \int_{\pi(Te) < 0} |Te + \pi(Te)|^2.$$

Since  $Te \geq 0$  a.e.

$$\int_{\pi(Te) < 0} |Te - \pi(Te)|^2 \geq \int_{\pi(Te) < 0} |Te + \pi(Te)|^2$$

and thus  $\|Te - |\pi(Te)|\| \leq \|Te - \pi(Te)\|$ . By Lemma 1,  $|\pi(Te)| \in M_\lambda$ , hence  $|\pi(Te)| = \pi(Te)$ , and  $\pi(Te) \geq 0$  a.e.  $\square$

If  $T$  is normal, then

$$\langle \bar{\pi} T^* e, Te \rangle = \langle T^* e, \pi Te \rangle,$$

$$\langle T^* e, \pi Te \rangle = \int \pi(Te)(y) \left( \int f(x, y) dx \right) dy = \int \lambda \pi(Te)(x) dx > 0.$$

For the nonnormal case,  $T^n/\lambda^n$  and  $(T^*)^n/\lambda^n$  may not converge. If  $M_\lambda$  is one-dimensional (hence so in  $\bar{M}_\lambda$ ), then  $\exists \phi \in M_\lambda$  and  $\phi^* \in \bar{M}_\lambda$  such that  $\langle \phi, \phi^* \rangle = 1$  (Friedman [3]). Moreover, with (A1) and (A3), it can be shown that (Koopmans [10])

$$\frac{(T^*)^n}{\lambda^n} \rightarrow \langle \cdot, \phi \rangle \phi^*, \quad \frac{T^n}{\lambda^n} \rightarrow \langle \cdot, \phi^* \rangle \phi.$$

Clearly we may choose  $\phi^*$  with  $(\phi^*)^+ \neq 0$  a.e.  $T^n(\phi^*)^+/\lambda^n \rightarrow (f((\phi^*)^+)^2)\phi$ , hence  $\phi \geq 0$  a.e. Similarly,  $\phi^* \geq 0$  a.e.

Now let us write down the main theorem.

THEOREM 2.1. Suppose that (A1), (A2) and (A3) are satisfied. If  $T$  is normal, then the marginal density of  $X_{k_1}, \dots, X_{k_m}$  is

$$(2.2) \quad \lambda^{k_1 - k_m} \langle \bar{\pi} T^* e, Te \rangle^{-1} \bar{\pi}(T^* e)(x_{k_1}) f_{k_2 - k_1}(x_{k_1}, x_{k_2}) \cdots f_{k_m - k_{m-1}}(x_{k_{m-1}}, x_{k_m}) \pi(Te)(x_{k_m}).$$

If

(D3)  $M_\lambda$  is one dimensional, then the marginal density is

$$(2.3) \quad \lambda^{k_1 - k_m} \phi^*(x_{k_1}) f_{k_2 - k_1}(x_{k_1}, x_{k_2}) \cdots f_{k_m - k_{m-1}}(x_{k_{m-1}}, x_{k_m}) \phi(x_{k_m}).$$

PROOF. Let us prove the normal case first.

$$\begin{aligned}
& \int \left| \frac{(T^*)^{n_1}}{\lambda^{n_1-1}} e(x_{k_1}) f_{k_2-k_1}(x_{k_1}, x_{k_2}) \cdots f_{k_m-k_{m-1}}(x_{k_{m-1}}, x_{k_m}) \frac{T^{n_2}}{\lambda^{n_2-1}} e(x_{k_m}) \right. \\
& \quad \left. - \bar{\pi}(T^*e)(x_{k_1}) f_{k_2-k_1}(x_{k_1}, x_{k_2}) \cdots f_{k_m-k_{m-1}}(x_{k_{m-1}}, x_{k_m}) \pi(Te)(x_{k_m}) \right| \\
& \leq \int \left| \left( \frac{T^*}{\lambda} \right)^{n_1-1} (T^*e)(x_{k_1}) f_{k_2-k_1}(x_{k_1}, x_{k_2}) \right. \\
& \quad \left. \cdots f_{k_m-k_{m-1}}(x_{k_{m-1}}, x_{k_m}) \left( \frac{T}{\lambda} \right)^{n_2-1} (Te)(x_{k_m}) \right. \\
& \quad \left. - \left( \frac{T^*}{\lambda} \right)^{n_1-1} (T^*e)(x_{k_1}) f_{k_2-k_1}(x_{k_1}, x_{k_2}) \cdots f_{k_m-k_{m-1}}(x_{k_{m-1}}, x_{k_m}) \pi(Te)(x_{k_m}) \right| \\
& + \int \left| \left( \frac{T^*}{\lambda} \right)^{n_1-1} (T^*e)(x_{k_1}) f_{k_2-k_1}(x_{k_1}, x_{k_2}) \cdots f_{k_m-k_{m-1}}(x_{k_{m-1}}, x_{k_m}) \pi(Te)(x_{k_m}) \right. \\
& \quad \left. - \bar{\pi}(T^*e)(x_{k_1}) f_{k_2-k_1}(x_{k_1}, x_{k_2}) \cdots f_{k_m-k_{m-1}}(x_{k_{m-1}}, x_{k_m}) \pi(Te)(x_{k_m}) \right|.
\end{aligned}$$

The first integral in the last expression can be rewritten as

$$\int \left( \frac{T^*}{\lambda} \right)^{n_1-1} (T^*e)(x_{k_1}) \left[ \int f_{k_m-k_1}(x_{k_1}, x_{k_m}) \left( \left( \frac{T}{\lambda} \right)^{n_2-1} (Te)(x_{k_m}) - \pi(Te)(x_{k_m}) \right) dx_{k_m} \right] dx_{k_1}$$

$\rightarrow 0$  as  $n_1 \rightarrow \infty, n_2 \rightarrow \infty$ , since  $(T^*/\lambda)^{n_1-1}(T^*e) \rightarrow \bar{\pi}(T^*e)$  and  $(T/\lambda)^{n_2-1}(Te) \rightarrow \pi(Te)$ . Similarly, the second integral goes to zero.  $\langle (T^*)^{n_1-1}e, Te \rangle \lambda^{2-n} \rightarrow \langle \bar{\pi}T^*e, Te \rangle > 0$  by the discussion right after Lemma 2.2. Hence  $h_n(x_{k_1}, \dots, x_{k_m}) \rightarrow (2.2)$  in  $L_1$ . Then, the limit probability distribution which is the marginal distribution of  $X_{k_1}, \dots, X_{k_m}$  has density (2.2).

When (D3) is satisfied,

$$\langle (T^*)^{n_1-1}e, Te \rangle \lambda^{2-n} \rightarrow \langle T^*e, \phi \rangle \langle \phi^*, Te \rangle = \lambda^2 \int \phi^* \int \phi \neq 0.$$

Hence, by a similar method as in the normal case, the marginal density has form (2.3).  $\square$

Clearly these marginal density functions constitute a consistent family. Therefore the process mentioned in the beginning of this section is well defined. Also, it is easily seen that this process is strictly stationary. By using the fact that if r.v.'s  $Y$  and  $Z$  have joint density  $q(y, z)$ , then

$$P\{Z \in B \mid Y = y\} = \int_B \frac{q(y, z)}{\int q(y, z) dz} dz \quad \text{a.s.},$$

this process is Markovian. Hence we have

**THEOREM 2.2.** *The process defined by (2.2) or (2.3) is Markovian and strictly stationary.*

To prove (D3) and find out explicitly  $\lambda$ ,  $\phi$  and  $\phi^*$  is not easy. Usually we have to solve the eigenvalue problem of the integral equation  $\lambda\phi(x) = \int f(x, y)\phi(y) dy$ . However, we quote a sufficient condition for (D3) from Zabreyko [13], i.e.

(D4)  $T$  and  $T^*$  are  $U_0$ -bounded.

**3. The Gaussian case.** Let us consider the bivariate Gaussian case, i.e.

$$f(x, y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-r^2}} \times \exp \left\{ -\frac{1}{2(1-r^2)} \left[ \left( \frac{x-m_1}{\sigma_1} \right)^2 - 2r \frac{x-m_1}{\sigma_1} \frac{y-m_2}{\sigma_2} + \left( \frac{y-m_2}{\sigma_2} \right)^2 \right] \right\}$$

where  $|r| < 1$ .

When  $m_1 = 0 = m_2$ , after some tedious calculation it is easy to see that  $\exp(-y^2/\sigma^2)$  is an eigenfunction where

$$\sigma^2 = -(\sigma_2^2 - \sigma_1^2) + \sqrt{(\sigma_1^2 + \sigma_2^2) - 4r^2\sigma_1^2\sigma_2^2}.$$

For the general case, let us try the function

$$\phi_0(y) = \exp(-(y - m_3)^2/\sigma^2).$$

with the integral equation

$$(3.1) \quad \lambda_0\phi_0(x) = \int f(x, y)\phi_0(y) dy,$$

where  $\lambda_0$  and  $m_3$  are to be determined later in the calculation.

The exponent of the integrand in (3.1) is

$$(3.2) \quad -\frac{1}{2(1-r^2)} \left[ \frac{(x-m_1)^2}{\sigma_1^2} - 2r \frac{x-m_1}{\sigma_1} \frac{y-m_2}{\sigma_2} + \frac{(y-m_2)^2}{\sigma_2^2} \right] - \frac{(y-m_3)^2}{\sigma^2} \\ = -\frac{1}{2(1-r^2)} \left[ \frac{(x-m_1)^2}{\sigma_1^2} - 2r \frac{x-m_1}{\sigma_1} \frac{y-m_2}{\sigma_2} + \frac{(y-m_2)^2}{\sigma_2^2} + \frac{b^+(y-m_3)^2}{2\sigma_1^2\sigma_2^2} \right]$$

where  $b^+ = \sigma_2^2 - \sigma_1^2 + \sqrt{(\sigma_1^2 + \sigma_2^2) - 4r^2\sigma_1^2\sigma_2^2}$ . Write the terms in the [ ] of (3.2) as

$$\left( \frac{b^+}{2\sigma_1^2\sigma_2^2} + \frac{1}{\sigma_2^2} \right) y^2 - 2 \left( \frac{b^+ m_3}{2\sigma_1^2\sigma_2^2} + \frac{m_2}{\sigma_2^2} + \frac{r(x-m_1)}{\sigma_1\sigma_2} \right) y \\ + \left( \frac{b^+ m_3^2}{2\sigma_1^2\sigma_2^2} + \frac{m_2^2}{\sigma_2^2} + \frac{2rm_2(x-m_1)}{\sigma_1\sigma_2} + \frac{(x-m_1)^2}{\sigma_1^2} \right).$$

Let us denote the sum of those terms in the last ( ) as  $A$  and complete the square of the above formula w.r.t.  $y$ , then we have

$$(3.3) \quad \left( \sqrt{\frac{c^+}{2\sigma_1^2\sigma_2^2}} y - \sqrt{\frac{2\sigma_1^2\sigma_2^2}{c^+}} \frac{b^+ m_3 + 2\sigma_1^2 m_2 + 2r\sigma_1\sigma_2(x - m_1)}{2\sigma_1^2\sigma_2^2} \right)^2 - \frac{2\sigma_1^2\sigma_2^2}{c^+} \left( \frac{b^+ m_3 + 2\sigma_1^2 m_2 + 2r\sigma_1\sigma_2(x - m_1)}{2\sigma_1^2\sigma_2^2} \right)^2 + A,$$

where  $c^+ = \sigma_1^2 + \sigma_2^2 + \sqrt{(\sigma_1^2 + \sigma_2^2)^2 - 4r^2\sigma_1^2\sigma_2^2}$ .

Now let us go back to (3.1) and integrate out the term involving (3.3).

$$\int \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-r^2}} \exp \left[ -\frac{1}{2(1-r^2)} \left[ \sqrt{\frac{c^+}{2\sigma_1^2\sigma_2^2}} y - \sqrt{\frac{2\sigma_1^2\sigma_2^2}{c^+}} \frac{b^+ m_3 + 2\sigma_1^2 m_2 + 2r\sigma_1\sigma_2(x - m_1)}{2\sigma_1^2\sigma_2^2} \right]^2 \right] dy = (c^+ \pi)^{-1/2}.$$

Hence the R.H.S. of (3.1) becomes

$$(3.4) \quad (c^+ \pi)^{-1/2} \exp - \frac{1}{2(1-r^2)} \left[ A - \frac{2\sigma_1^2\sigma_2^2}{c^+} \left( \frac{b^+ m_3 + 2\sigma_1^2 m_2 + 2r\sigma_1\sigma_2(x - m_1)}{2\sigma_1^2\sigma_2^2} \right)^2 \right].$$

This term is to be equal to  $\lambda_0 \phi_0(x)$ . So the coefficients of  $x$  and  $x^2$  in the exponent of (3.4) have to match those in the exponent of  $\phi_0(x)$  respectively.

The terms in the [ ] of (3.4) can be rewritten as

$$\frac{b^+}{2\sigma_1^2\sigma_2^2} x^2 + \left( \frac{2rm_2}{\sigma_1\sigma_2} + \frac{8r^2\sigma_1^2\sigma_2^2 m_1 - 4r\sigma_1\sigma_2(b^+ m_3 + 2\sigma_1^2 m_2)}{2\sigma_1^2\sigma_2^2 c^+} - \frac{2m_1}{\sigma_1^2} \right) x + \frac{b^+}{2\sigma_1^2\sigma_2^2} m_3^2 + K_1,$$

where

$$K_1 = \frac{m_2^2}{\sigma_2^2} + \frac{m_1^2}{\sigma_1^2} - \frac{2rm_1 m_2}{\sigma_1\sigma_2} - \frac{1}{2\sigma_1^2\sigma_2^2 c^+} (b^+ m_3 + 2\sigma_1^2 m_2 - 2r\sigma_1\sigma_2 m_1)^2.$$

Clearly

$$\frac{b^+}{2\sigma_1^2\sigma_2^2} \cdot \frac{1}{2(1-r^2)} = \frac{1}{\sigma^2}.$$

Now let us match the coefficients of  $x$  to determine  $m_3$ . After some tedious calculation, we have

$$(3.5) \quad m_3 = \left( \frac{2r\sigma_1\sigma_2}{c^+} - 1 \right)^{-1} \left( \frac{2r\sigma_1\sigma_2}{c^+} m_2 - m_1 \right).$$

By plugging into the value of  $m_3$ ,  $K_1$  can be rewritten in a more symmetric way as

$$(3.6) \quad K_1 = \frac{m_1^2}{\sigma_1^2} + \frac{m_2^2}{\sigma_2^2} - \frac{2rm_1m_2}{\sigma_1\sigma_2} - \frac{2c^+}{\sigma_1^2\sigma_2^2} \left[ \frac{(r\sigma_1\sigma_2 - \sigma_2^2)m_1 + (r\sigma_1\sigma_2 - \sigma_1^2)m_2}{2r\sigma_1\sigma_2 - c^+} \right]^2.$$

Hence,  $\phi_0$  is an eigenfunction,

$$\phi_0(y) := \exp - \frac{(y - m_3)^2}{\sigma^2}$$

corresponding to eigenvalue

$$\lambda_0 = (c^+ \pi)^{-1/2} \exp - \frac{k_1}{2(1 - r^2)}.$$

Let us try to find other eigenvalues. Let  $H_k$  denote Hermite polynomial of degree  $k$ , then

$$\exp - z^2 - 2uz = \sum_0^{\infty} \frac{H_n(u)}{n!} z^n.$$

We consider the case  $r \neq 0$  first. Let  $\lambda_k = \lambda_0[(c^-/c^+)^{1/2} \text{sign } r]^k$ ,  $k = 0, 1, 2, 3, \dots$ , where

$$c^- = \sigma_1^2 + \sigma_2^2 - \sqrt{(\sigma_1^2 + \sigma_2^2)^2 - 4r^2\sigma_1^2\sigma_2^2}.$$

We are going to prove  $\lambda_k$  is an eigenvalue. Let  $u(y) = \sqrt{c^+} y / 2\sqrt{1 - r^2} \sigma_1\sigma_2$ ,  $z(x) = -r(x - m_4) / \sqrt{(1 - r^2)c^+}$ , and  $m_4 = m_1 - [b^+ m_3 + 2\sigma_1^2 m_2] / 2\sigma_1\sigma_2 r$ .

(3.7)

$$\begin{aligned} T(\phi_0(y)H_k(u(y)))(x) &= \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1 - r^2}} \\ &\cdot \exp - \frac{1}{2(1 - r^2)} \left( \left( \frac{x - m_1}{\sigma_1} \right)^2 - 2r \frac{x - m_1}{\sigma_1} \frac{y - m_2}{\sigma_2} + \left( \frac{y - m_2}{\sigma_2} \right)^2 \right) \\ &\cdot \exp - \frac{(y - m_3)^2}{\sigma^2} H_k(u(y)) dy \\ &= \lambda_0 \phi_0(x) \frac{\sqrt{c^+} \pi}{2\pi\sigma_1\sigma_2\sqrt{1 - r^2}} \int_{-\infty}^{\infty} \exp - \frac{1}{2(1 - r^2)} \\ &\cdot \left[ \sqrt{\frac{c^+}{2\sigma_1^2\sigma_2^2}} y \right. \\ &\quad \left. - \sqrt{\frac{2\sigma_1^2\sigma_2^2}{c^+} \frac{b^+ m_3 + 2\sigma_1^2 m_2 + 2r\sigma_1\sigma_2(x - m_1)}{2\sigma_1^2\sigma_2^2}} \right]^2 H_k(u(y)) dy \end{aligned}$$

(last equality comes from what we have done for  $k = 0$ )

$$\begin{aligned} &= \lambda_0 \phi_0(x) \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp[-(u+z(x))^2] H_k(u) du \\ &= \lambda_0 \phi_0(x) \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \sum_0^{\infty} \frac{H_n(u)}{n!} z^n(x) e^{-u^2} H_k(u) du \\ &= \lambda_0 \phi_0(x) z^n(x) 2^k = \lambda_0 \left( \sqrt{\frac{c^-}{c^+}} \operatorname{sign} r \right)^k \phi_0(x) Q_k(x) = \lambda_k \phi_0(x) Q_k(x) \end{aligned}$$

where

$$Q_k(x) = (-1)^k 2^k \left( \frac{\sqrt{c^+}}{2\sqrt{1-r^2}\sigma_1\sigma_2} \right)^k (x - m_4)^k.$$

Observe that the leading coefficient of  $H_k(u(x))$  is  $(-1)^k 2^k (\sqrt{c^+}/2\sqrt{1-r^2}\sigma_1\sigma_2)^k$  which is the same as that of  $Q_k(x)$ .

Clearly

$$\begin{aligned} \operatorname{span}\{\phi_0(x)H_k(u(x))\}_0^n &= \operatorname{span}\{\phi_0(x)Q_k(x)\}_0^n \\ &= \operatorname{span}\{\phi_0(x)x^k\}_0^n = M_n. \end{aligned}$$

Hence  $T: M_n \rightarrow M_n$  is 1-1 onto. But  $T - \lambda_k I: M_k \rightarrow M_{k-1}$ , since

$$(T - \lambda_k I)(\phi_0(\cdot)H_k(u(\cdot))) = \lambda_k(\phi_0 Q_k - \phi_0 H_k(u(\cdot))) \in M_{k-1}.$$

Hence  $\lambda_k$  is an eigenvalue of  $T$ . Also, it is not hard to see that there exists a corresponding eigenfunction  $\phi_k$  of the form  $\phi_0 P_k$  with  $P_k$  a polynomial of degree  $k$ . And  $P_k$  can be calculated inductively through a lengthy computation from the relation  $T(\phi_0(\cdot)H_k(u(\cdot))) = \lambda_k \phi_0 Q_k$ . Let  $N_k = \operatorname{span}\{\phi_0 P_k\}$ , then  $M_n = \bigvee_0^n N_k$ . Since  $\operatorname{span}\{e^{-(x/a)^2} x^k\}_0^\infty$  is  $L^2(\mathbf{R})$ ,  $\bigvee_0^\infty N_k = L^2(\mathbf{R})$ .

Two things remain to be proved. I.e. there is no eigenvalue other than  $\{\lambda_k\}_0^\infty$  and every eigenfunction is in  $\bigcup_k N_k$ .

Observe the fact from (3.6) that  $K_1$  remains the same if we interchange  $m_1$  with  $m_2$  and  $\sigma_1$  with  $\sigma_2$ . I.e. when we consider  $T^*$  and go through the same procedures as we did for  $T$ , we shall get exactly the same  $\lambda_k$ . Let  $\phi_k^*(N_k^*)$  correspond to  $\phi_k(N_k)$  in the obvious way. Since  $\bigvee_0^\infty N_k = L^2(\mathbf{R})$ ,  $\bigvee_0^\infty N_k^* = L^2(\mathbf{R})$ .

**LEMMA 3.1.** *There is no eigenvalue of  $T$  other than  $\{\lambda_k\}_0^\infty$  and every eigenfunction is in  $\bigcup_0^\infty N_k$ .*

**PROOF.** Suppose  $\exists$  a eigenvalue  $\lambda \notin \{\lambda_k\}_0^\infty$ . Let  $v$  be a corresponding eigenfunction, then

$$\lambda \langle v, \phi_k^* \rangle = \langle Tv, \phi_k^* \rangle = \langle v, T\phi_k^* \rangle = \lambda_k \langle v, \phi_k^* \rangle$$

which implies  $\langle v, \phi_k^* \rangle = 0 \forall k$ . But  $\bigvee_0^\infty N_k^* = L^2(\mathbf{R})$ , then  $v = 0$ . Therefore the first assertion holds.

If  $u \neq 0$  is an eigenfunction, say, corresponding to  $\lambda_n$ , then  $u \notin \bigvee_{k \neq n} N_k$ . The proof goes as follows. Suppose  $u \in \bigvee_{k \neq n} N_k$ . Let  $\mathfrak{N}_n$  denote the eigenspace of  $T^*$  corresponding to  $\lambda_n$ , then as we did before  $\langle \phi_k, w \rangle = 0 \forall k \neq n, \forall w \in \mathfrak{N}_n$ . Hence  $\langle u, w \rangle = 0 \forall w \in \mathfrak{N}_n$ . But this contradicts the fact that  $\exists \tilde{w} \in \mathfrak{N}_n \ni \langle u, \tilde{w} \rangle \neq 0$  (Friedman [3]).

Since  $N_n$  is finite dimensional  $N_n \oplus \bigvee_{k \neq n} N_k = \bigvee_k N_k = L^2(\mathbf{R})$ ,  $u$  can be written as  $a + b$  with  $a \in N_n$  and  $b \in \bigvee_{k \neq n} N_k$ . But  $b = u - a$  is also an eigenfunction with eigenvalue  $\lambda_n$ , hence,  $u = a$  by the previous discussion. Therefore all the eigenfunctions are in  $\bigcup N_k$ .  $\square$

Clearly  $c^- < c^+$  ( $\because r \neq 0$ ), then by a previous lemma  $\lambda_0 > |\lambda_1| > |\lambda_2| > \dots > |\lambda_k| > \dots$  and every eigenspace is one dimensional. The same assertions hold for  $T^*$ .

For  $r = 0$ , the situation is much simpler.  $\text{Range}(T)$  is one dimensional,  $\lambda_0$  is the only nonzero eigenvalue and  $\bigvee_{k > 1} N_k$  is the eigenspace corresponding to 0. The same assertions hold for  $T^*$ . To sum up, we have the following theorem.

**THEOREM 3.1.** *Let  $T$  be an integral operator with bivariate Gaussian density*

$$\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-r^2}} \exp\left\{-\frac{1}{2(1-r^2)}\left[\left(\frac{x-m_1}{\sigma_1}\right)^2 - 2r\frac{x-m_1}{\sigma_1}\frac{y-m_2}{\sigma_2} + \left(\frac{y-m_2}{\sigma_2}\right)^2\right]\right\},$$

where  $|r| < 1$ , as its kernel. Then, all the eigenvalues are of the form

$$\lambda_k = \lambda_0 \left( \sqrt{\frac{c^-}{c^+}} \text{sign } r \right)^k,$$

where  $k = 0, 1, 2, \dots$ ,  $c^\pm = \sigma_1^2 + \sigma_2^2 \pm \sqrt{(\sigma_1^2 + \sigma_2^2)^2 - 4r^2\sigma_1^2\sigma_2^2}$ ,

$$\lambda_0 = (c^+ \pi)^{-1/2} \exp - \frac{1}{2(1-r^2)} \left\{ \frac{m_1^2}{\sigma_1^2} + \frac{m_2^2}{\sigma_2^2} - \frac{2rm_1m_2}{\sigma_1\sigma_2} - \frac{2c^+}{\sigma_1^2\sigma_2^2} \cdot \left[ \frac{(r\sigma_1\sigma_2 - \sigma_2^2)m_1 + (r\sigma_1\sigma_2 - \sigma_1^2)m_2}{2r\sigma_1\sigma_2 - c^+} \right] \right\}.$$

If  $r \neq 0$ , then  $\lambda_0 > |\lambda_1| > |\lambda_2| > \dots > |\lambda_k| > \dots$ , and every eigenspace is one dimensional which is determined by the corresponding eigenfunction  $P_k\phi_0$ , where  $\phi_0 = \exp - (x - m_3)^2/\sigma^2$ ,

$$\sigma^2 = -(\sigma_2^2 - \sigma_1^2) + \sqrt{(\sigma_1^2 + \sigma_2^2)^2 - 4r^2\sigma_1^2\sigma_2^2},$$

$$m_3 = \left( \frac{2r\sigma_1\sigma_2}{c^+} - 1 \right)^{-1} \left( \frac{2r\sigma_1\sigma_2}{c^+} m_2 - m_1 \right),$$

and  $P_k$  is a polynomial of degree  $k$  which can be determined from (3.7). For  $r = 0$ ,  $\lambda_0$  is the only nonzero eigenvalue with eigenfunctions multiples of  $\phi_0$ . And the eigenspace corresponding to 0 is

$$\text{span}\{\phi_0 x^k\}_1^\infty = \text{span}\left\{\phi_0 H_k\left(\frac{\sqrt{c^+}}{2\sqrt{1-r^2}\sigma_1\sigma_2}x\right)\right\}_1^\infty$$

where  $H_k$  is Hermite polynomial of degree  $k$ . The same assertions hold for  $T^*$ .

**THEOREM 3.2.** Condition (D3) is satisfied and the marginal density is

$$\lambda_0^{k_1-k_m}\left(\int\phi_0^*\phi_0\right)^{-1}\phi_0^*(x_{k_1})f_{k_2-k_1}(x_{k_1},x_{k_2})\cdots f_{k_m-k_{m-1}}(x_{k_{m-1}},x_{k_m})\phi_0(x_{k_m})$$

where  $\phi_0^*(x) = \exp - (x - \hat{m}_3)^2 / \hat{\sigma}^2$  with

$$\hat{\sigma}^2 = -(\sigma_1^2 - \sigma_2^2) + \sqrt{(\sigma_1^2 + \sigma_2^2)^2 - 4r^2\sigma_1^2\sigma_2^2},$$

$$\hat{m}_3 = \left(\frac{2r\sigma_1\sigma_2}{c^+} - 1\right)^{-1}\left(\frac{2r\sigma_1\sigma_2}{c^+}m_1 - m_2\right).$$

**4. Relaxed regularity** (EQUAL, LINEAR). Consider the regularity (EQUAL, LINEAR) but now with an acceptance function  $A(x, y)$  (Grenander [7]). In other words, when we connect  $n$  generators together, hence  $2n$  random variables, via the regularity, these  $2n$  random variables have a density function

$$(4.1) \quad z^{-1} \prod_{k=1}^n f(\beta_k^1, \beta_k^2) \prod_{k=1}^{n-1} A(\beta_k^2, \beta_{k+1}^1).$$

As in §1, let us assume that

(A1)'  $f(x, y)$  and  $A(x, y) \in L^2(X \times X)$ ,

(A2)'  $\int f(x, y) dy$  and  $\int f(x, y) dx \in L^2(X)$ .

Let  $T$  and  $S$  denote the following two transformations:

$$T(\phi)(x) = \int f(x, y)\phi(y) dy,$$

$$S(\phi)(x) = \int A(x, y)\phi(y) dy.$$

To make (4.1) meaningful we have to assume that

(D1)'  $ST$  is not nilpotent.

With (D1)' it is easily seen that

$$z = \langle (S^*T^*)^{n-1}e, Te \rangle \neq 0.$$

Instead of considering each bond value separately as a r.v. as we did in §2, we regard each generator, i.e. a pair of random variables, as a whole piece. As before, the process  $\{Z_k\}$  of generators induced by the regularity with acceptance function is

specified by its marginal densities. As in §2, we can show that the marginal density  $h_n$  of  $Z_k, \dots, Z_{k_m}$  within  $\mathcal{C}_n$  is

$$\langle (S^*T^*)^{n-1}e, Te \rangle^{-1} (S^*T^*)^{n-1}(e) \left( x_{k_1}^1 \right) f(x_{k_1}^1, x_{k_1}^2) g_{k_2-k_1}(x_{k_1}^2, x_{k_2}^1) f(x_{k_2}^1, x_{k_2}^2) \\ \cdots f(x_{k_{m-1}}^1, x_{k_{m-1}}^2) g_{k_m-k_{m-1}}(x_{k_{m-1}}^2, x_{k_m}^1) f(x_{k_m}^1, x_{k_m}^2) (ST)^{n_2}(e) (x_{k_m}^2),$$

where

$$g_1(x, y) = A(x, y),$$

$$g_{p+1}(x, y) = \int A(x, z_1^1) f(z_1^1, z_1^2) \cdots f(z_p^1, z_p^2) A(z_p^2, y) dz_1^1 dz_1^2 \cdots dz_p^1 dz_p^2.$$

Suppose

(A3)'  $\lambda' = \text{radius } \sigma(ST) > 0$  and is the only eigenvalue with magnitude equal to  $\lambda'$ .

As before, let  $M_{\lambda'} (M_{\lambda'}^*)$  denote the eigenspace of  $ST (T^*S^*)$  with corresponding eigenvalue  $\lambda' (\lambda')$  and  $\pi (\bar{\pi})$  denote the projection onto  $M_{\lambda'} (M_{\lambda'}^*)$ .

If  $ST$  is normal, then

$$\langle (S^*T^*)^n e, Te \rangle \rightarrow \langle T^*e, \pi(ST)e \rangle.$$

By Lemma 2.2,  $\pi(ST)e \geq 0$  and  $\pi(ST)e \neq 0$  a.e. Suppose that  $\langle T^*e, \pi(ST)e \rangle = 0$ . Equivalently,

$$0 = \int \pi(ST)e(y) \left( \int f(x, y) dx \right) dy \\ = \int \left( \int f(x, y) \pi(ST)e(y) dy \right) dx \\ \Rightarrow \int f(x, y) \pi(ST)e(y) dy = 0 \quad \text{a.e.} \\ \Rightarrow \pi(ST)e(y) \in \mathfrak{N}(T) \Rightarrow ST(\pi(ST)e) = 0$$

which is a contradiction.

For the normal case, we have to assume

(D3)'  $M_{\lambda'}$  is one dimensional.

Clearly, we may choose  $\phi \in M_{\lambda'}$  and  $\phi^* \in M_{\lambda'}^*$  with  $\phi \geq 0$  a.e.  $\phi^* \geq 0$  a.e. and  $\langle \phi, \phi^* \rangle = 1$ .

Similarly to the discussion in §2,

$$\langle (S^*T^*)^n e, Te \rangle = \langle T^*e, (ST)^n e \rangle \rightarrow \langle T^*e, \phi \rangle \langle (ST)e, \phi^* \rangle.$$

As before, we know that  $\langle ST(e), \phi^* \rangle \neq 0$ . Suppose  $\langle T^*e, \phi \rangle = 0$ ,

$$\langle T^*e, \phi \rangle = \int \phi(y) \int f(x, y) dx dy = \int \left( \int f(x, y) \phi(y) dy \right) dx = 0$$

which implies  $T(\phi) = 0$ . But then  $ST(\phi) = 0$ ; a contradiction is obtained.

Similarly to Theorem 2.1 we have

**THEOREM 4.1.** *Suppose that (A1)', (A2)', and (A3)' are satisfied. If  $ST$  is normal, then the marginal density of  $Z_{k_1}, \dots, Z_{k_m}$  is*

$$\lambda^{k_1 - k_m - 1} \langle T^* e, \pi(ST)e \rangle^{-1} S^* \pi T^*(e) \left( x_{k_1}^1 \right) f \left( x_{k_1}^1, x_{k_1}^2 \right) g_{k_2 - k_1} \left( x_{k_1}^2, x_{k_2}^1 \right) f \left( x_{k_2}^1, x_{k_2}^2 \right) \cdots f \left( x_{k_{m-1}}^1, x_{k_{m-1}}^2 \right) g_{k_m - k_{m-1}} \left( x_{k_{m-1}}^2, x_{k_m}^1 \right) f \left( x_{k_m}^1, x_{k_m}^2 \right) \pi(ST)(e) \left( x_{k_m}^2 \right).$$

If (D3)' is true, then the marginal density is

$$\lambda^{k_1 - k_m - 1} S^*(\phi^*) \left( x_{k_1}^1 \right) f \left( x_{k_1}^1, x_{k_1}^2 \right) g_{k_2 - k_1} \left( x_{k_1}^2, x_{k_2}^1 \right) f \left( x_{k_2}^1, x_{k_2}^2 \right) \cdots f \left( x_{k_{m-1}}^1, x_{k_{m-1}}^2 \right) g_{k_m - k_{m-1}} \left( x_{k_{m-1}}^2, x_{k_m}^1 \right) f \left( x_{k_m}^1, x_{k_m}^2 \right) \phi \left( x_{k_m}^2 \right).$$

Similarly to Theorem 2.2, we have

**THEOREM 4.2.** *The process is strictly stationary and Markovian.*

**REMARKS.** (1) Heuristically (D1) means that there is a positive probability to concatenate  $n$  generators together.

(2) Given any two random generators, if we regard

$$\int f(x_3, x_1) f(x_3, x_2) dx_3 \left( \int f(x_1, x_3) f(x_2, x_3) dx_3 \right)$$

as the joint "density" of two out-bond values (in-bond values) of these two random generators after identifying their in-bond values (out-bond values), then  $T$  is normal means that these two joint "densities" are equal a.e. This gives us some sort of probabilistic equivalence between in-bond value and out-bond value.

(3) (D2), (D3), (D4) and (A3) are motivated by the following consideration. When the space  $X$  has only finitely many points,  $f$  can be regarded as a matrix. If  $f$  is aperiodic and irreducible, then clearly all the assumptions are satisfied. If  $X$  is a compact interval in  $\mathbf{R}$  and  $f$  is continuous and positive (or the  $n$ th iteration of  $f$  is positive), then again all the assumptions are true. One might "pretend" that  $f$  is a "transition probability function". To prove some limiting properties in the above-mentioned two special cases, it comes down to the existence of some sort of spectral limiting theorem which is the consequence of these assumptions. Since in the general case we do not have simple conditions as those in the special cases, we settle for these less intuitive assumptions. It will be very interesting to see some direct probabilistic interpretations of these assumptions.

(4) The above three remarks are applicable to the relaxed regularity discussed in §4.

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