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A TRACE FORMULA FOR COMPACT MANIFOLDS

BY

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ABSTRACT. An integral formula for the Euler characteristic is given, in which the data consists of a finite dimensional transitive vector space V of vector fields and a volume form Ω supported in a small neighborhood of the origin of V.

1. Introduction. Throughout M will be a smooth, closed, connected and oriented manifold of dimension m. Let $\mathfrak{G}_M$ be the Lie algebra of all its smooth vector fields. Since M is compact we can choose a finite dimensional vector subspace $V \subseteq \mathfrak{G}_M$ which spans every tangent space; so we get the short exact sequence of bundles

$$0 \to E \xrightarrow{i} M \times V \xrightarrow{e} T \to 0. \tag{1}$$

Here T is the tangent bundle of M, the *evaluation map* e is defined by

$$e(x, v) = (x, v(x)); \tag{2}$$

further we have $E = \ker(e)$ and i is the inclusion map. Since M is compact each $v \in \mathfrak{G}_M$ arises from a group homomorphism $t \mapsto v_t$ of $(\mathbf{R}, +)$ into the group $\mathrm{Diff}(M)$ of diffeomorphisms of M. Using this we define the *exponential map* ε: $M \times V \to M \times M$ by

$$\varepsilon(x, v) = (x, v_1(x)). \tag{3}$$

Since M is compact one can show that there exists a neighbourhood U of $0 \in V$ such that the exponential map is a fibration of $M \times U$ into $M \times M$. Let $\dim V = n$. We equip V with an orientation and choose a smooth form Ω of degree n supported inside U and such that

$$\int_V \Omega = 1. \tag{4}$$

We will equip E with the orientation induced from that of V and M.

Let π_1 and π_2 be the projections of $M \times M$ onto the first and second component respectively and let $\mathfrak{F}_1$ and $\mathfrak{F}_2$ denote the m dimensional foliations of $M \times M$ determined by their fibers. We have the codimension m foliation $\overline{\mathfrak{F}}_2 = \varepsilon^{-1}(\mathfrak{F}_2)$ of $M \times U$; further we shall choose on $M \times U$ an m dimensional tangent plane field $\overline{F}_1$ complementary to $\overline{\mathfrak{F}}_2$ and lying above $\mathfrak{F}_1$. A form θ on $M \times U$ is said to be of *type* (p, q) if it vanishes whenever $q + 1$ vectors are tangent to $\overline{\mathfrak{F}}_2$ or else $p + 1$ vectors

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are in $\bar{F}_1$. It is clear that each form θ on $M \times U$ can be written in a unique way as $\theta = \Sigma_{p,q}\theta_{p,q}$ where $\theta_{p,q}$ is a form of type (p, q). We will denote by π the projection $M \times V \to V$.

Given any smooth function f: $E \to \mathbf{R}$ we will define its *average value* by

$$f(M) = \int_E f \cdot i^*((\pi^*\Omega)_{m,n-m}). \tag{5}$$

It is particularly interesting to consider this average value for some geometrically interesting functions which are defined naturally on any bundle E of the above type. Here we will consider the *trace function* Tr: $E \to \mathbf{R}$ which is defined by

$$\mathrm{Tr}(x, v) = \mathrm{Tr}(v_1^* \mid \Lambda(x)), \tag{6}$$

where $\Lambda(x)$ is the graded algebra of forms at x and the expression on the right is the alternating sum of the traces $\Sigma_{p=0}^m(-1)^p \mathrm{Tr}(v_1^* \mid \Lambda^p(x))$. It turns out that in this case the average value is independent of the choice of V and Ω.

THEOREM. *For any smooth closed oriented manifold M*

$$\mathcal{X}(M) = \mathrm{Tr}(M). \tag{7}$$

Here $\mathcal{X}(M)$ is the Euler characteristic of M; thus the average value of Tr does not even depend on the orientation of M. In fact by using volume forms the same proof will yield the analogue of (7) for nonorientable closed manifolds. After some generalities about integration in §2 we will give the proof of the above theorem in §3. We remark that the same techniques were used in [2] to prove a different result. I do not know whether one can prove analogues of the formula (7) for some other geometrically interesting functions.

2. Integration of forms. All our manifolds will be smooth, Hausdorff, paracompact, connected and oriented and all our diffeomorphisms orientation preserving. If F is a locally trivial vector bundle on a manifold N, and $x \in N$, then $\lambda^p(F)_x$ will denote the vector space formed by all skew-symmetric multilinear maps $T_x \times T_x \times \cdots \times T_x$ (p times) $\to F_x$, T_x being the tangent space to N at x. The direct sum of these vector spaces over all p is denoted by $\lambda(F)_x$. We equip $\lambda(F) = \cup_x \lambda(F)_x$ with the usual structure of a vector bundle over N and denote by $\Lambda(F)$ its compactly supported smooth sections; any such section is called a *form* on N with values in F.

Let $(x_1, x_2, \ldots, x_p)$ be a local oriented system of coordinates for the manifold P and $(x_{p+1}, x_{p+2}, \ldots, x_{p+n})$ a system for N over which F is trivial. Under the projection map φ: $P \times N \to N$ a vector bundle F over N pulls back to a vector bundle φ^*F over $P \times N$. For each form $w \in \Lambda(\varphi^*F)$ with support lying in the domain of the system $(x_1, x_2, \ldots, x_{p+n})$ of $P \times N$ we define a form $\int_\varphi w \in \Lambda(F)$ by

$$\left(\int_\varphi w\right)^\alpha_{i_1 i_2 \ldots} = \int_{\mathbf{R}^p} w^\alpha_{12\cdots p i_1 i_2 \ldots} dx_1 dx_2 \cdots dx_p, \qquad p < i_1 < i_2 < \ldots, \tag{8}$$

where $w^\alpha_{ijk\ldots}$ are the components of w with respect to a trivialization of φ^*F over the domain of the local system. We can verify that this definition does not depend on the trivialization and thus we are enabled to define a linear map $\int_\varphi$: $\Lambda(\varphi^*F) \to \Lambda(F)$.

Next one checks that if ψ is a diffeomorphism of $P \times N$ such that $\varphi\psi = \phi$ then

(9) $$\int_\varphi \psi^* w = \int_\varphi w.$$

Hence corresponding to each locally trivial fibration $L \xrightarrow{\varphi} N$ one has an *integration map* $\int_\varphi$: $\Lambda(\varphi^* F) \to \Lambda(F)$. We list below some simple properties of this map.

(a) *Change of variables formula.* Let (ψ_1, ψ_2): $\phi_1 \to \phi_2$ be a morphism of fibrations (i.e. $\phi_2\psi_1 = \psi_2\phi_1$) such that ψ_1 maps each fiber of ϕ_1 diffeomorphically onto a fiber of ϕ_2. Then

(10) $$\psi_2^* \int_{\varphi_2} w = \int_{\varphi_1} \psi_1^* w.$$

(b) *Fubini's formula.* Let the composition $\varphi_1\varphi_2$ of two fibrations be also a fibration. Then

(11) $$\int_{\varphi_1\varphi_2} = \int_{\varphi_1}\int_{\varphi_2}.$$

(c) *Constants can be taken outside the integral sign.* For example if we are given two bundles F, G over N, a fibration φ: $L \to N$ and forms $\theta \in \Lambda(\mathrm{Hom}(\varphi^* F, \varphi^* G))$, $w \in \Lambda(F)$ then

(12) $$\int_\varphi \theta(\varphi^* w) = \left(\int_\varphi \theta\right) w.$$

Here the product on the two sides is defined analogously to the ordinary exterior product: if Ω is a $\mathrm{Hom}(F, G)$ valued p-form and w is a F valued q-form then Ωw is a G valued $(p+q)$-form whose value on $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{p+q})$, $\varepsilon_i \in T_x$, is obtained by skew-symmetrizing $(\Omega(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_p))(w(\varepsilon_{p+1}, \varepsilon_{p+2}, \ldots, \varepsilon_{p+q}))$.

Some conventions. For a constant map φ: $N \to \{x\}$ instead of $\int_\varphi$ we write $\int_N$. For a trivial bundle $F \times N$ over N we write $\Lambda(N, F)$ in place of $\Lambda(F \times N)$; if there is no confusion possible one denotes $\Lambda(N, \mathbf{R})$ by $\Lambda(N)$ or even by Λ; the corresponding vector bundle being $\lambda(N)$ or even λ. If $w \in \Lambda(N, F)$ one often writes $\int_{x \in N} w(x)$ instead of $\int_N w$; this sometimes avoids the necessity of writing out the definition of w separately.

3. Proof of Theorem. We will use all notation of §1. We start by defining a linear map s: $\Lambda(M) \to \Lambda(M)$ by

(13) $$(sw)(x) = \int_{v \in V} (v_1^* w)(x) \cdot \Omega(v), \quad \forall x \in M.$$

(In other words for each $w \in \Lambda(M)$ and for each $x \in M$, $(sw)(x) = \int_V R_w(x)$ where the form $R_w(x) \in \Lambda(V, \lambda_x)$ is defined by $(R_w(x))(v) = \Omega(v) \cdot (v_1^* w)(x)$.) Let d: $\Lambda(M) \to \Lambda(M)$ be the exterior derivative. By differentiating the above formula under the integral sign, which is clearly permissible, one sees that s is a chain map, i.e. $sd = ds$. Further *s induces the identity map in de Rham cohomology*; to see this we define p: $\Lambda(M) \to \Lambda(M)$ by

(14) $$(pw)(x) = \int_{(t, v) \in I \times V} (\iota_v v_t^* w)(x) \cdot dt \cdot \Omega(v).$$

Here $I = [0, 1]$ and dt is the 1-form on $I \times V$ induced by the standard 1-form on I. Further $\iota_v\colon \Lambda \to \Lambda$ is the interior product with v. We now use (4) and the fact that $d\iota_v + \iota_v d$ is the Lie derivative with respect to $v \in \mathfrak{G}_M$ to check that p is a chain homotopy between the identity map and s.

We note that the projection $M \times V \to M$ equals $\pi_1\varepsilon$. So we see that $sw = \int_{\pi_1\varepsilon} R_w \forall w \in \Lambda(M)$; here R_w is a $(\pi_1\varepsilon)^*\lambda$ valued form on $M \times V$ defined by $R_w(x, v) = (\pi^*\Omega)(x, v) \cdot (v_1^* w)(x)$. Alternatively this formula can be written as

$$sw = \int_{\pi_1\varepsilon} L(\varepsilon^*\pi_2^* w), \quad \forall w \in \Lambda(M); \tag{15}$$

here L is a $\mathrm{Hom}(\varepsilon^*\pi_2^*\lambda, \varepsilon^*\pi_1^*\lambda)$ valued form on $M \times V$ defined by

$$L(x, v) = (\pi^*\Omega)(x, v) \cdot v_1^*. \tag{16}$$

We now use Fubini's theorem (11) and formula (12) to see for all $w \in \Lambda(M)$ that

$$sw = \int_{\pi_1} K(\pi_2^* w) \tag{17}$$

where K is a $\mathrm{Hom}(\pi_2^*\lambda, \pi_1^*\lambda)$ valued form on $M \times M$ given by

$$K = \int_\varepsilon L. \tag{18}$$

Let A be any vector bundle over M. Then a linear map $s\colon C^\infty(A) \to C^\infty(A)$ is said to be *an integral operator with smooth kernel* K if K is a $\mathrm{Hom}(\pi_2^* A, \pi_1^* A)$ valued smooth form on $M \times M$ such that formula (17) holds for all $w \in C^\infty(A)$. Forms on $M \times M$ can be bigraded by using the complementary foliations $\mathfrak{F}_1$ and $\mathfrak{F}_2$ (see §1); it is clear that formula (17) will continue to hold even if K is replaced by $K_{m,0}$. We have checked that the map s defined by (13) is an integral operator with smooth kernel which commutes with d and induces the identity map in de Rham cohomology. For such maps we have the following trace formula of [**1**]:

$$\mathfrak{X}(M) = \int_M \mathrm{tr}(\delta^* K_{m,0}). \tag{19}$$

Here $\delta\colon M \to M \times M$ is the diagonal map. So $\delta^* K_{m,0}$ is a $\mathrm{Hom}(\lambda, \lambda)$ valued form on M. We think of the tr on the right side of (19) as a section of $(\mathrm{Hom}(\lambda, \lambda))^*$ given by taking the alternating trace sum at each point. We note now that under $\int_\varepsilon$ a form of type (p, q) on $M \times V$ goes into a form of type $(p, q - n + m)$. On the other hand $\varepsilon i = \delta\tau$, where $\tau\colon E \to M$ denotes the bundle projection. Hence we can use (10); this gives $\mathfrak{X}(M) = \int_M \mathrm{tr} \int_\tau i^* L_{m,n-m}$. Next by taking constants outside the integral (as in (12)) and by recalling the definition (16) of L we see that this equals $\int_M \int_\tau \mathrm{Tr} \cdot i^*(\pi^*\Omega)_{m,n-m}$; now we have used Tr to denote the function $E \to \mathbf{R}$ defined in §1. Finally using Fubini's theorem we get the required trace formula (7).

References

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