

**THE ALBANESE MAPPING FOR
A PUNCTUAL HILBERT SCHEME: II.
SYMMETRIZED DIFFERENTIALS AND SINGULARITIES**

BY

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ABSTRACT. Let $f: X \rightarrow A$ be the canonical mapping from the irreducible and nonsingular surface X to its Albanese variety A , $X(n)$ the n -fold symmetric product of X , and H_X^n the punctual Hilbert scheme parameterizing 0-dimensional closed subschemes of length n on X . The latter is an irreducible and nonsingular variety of dimension $2n$, and the "Hilbert-Chow" morphism $\sigma_n: H_X^n \rightarrow X(n)$ is a birational map which desingularizes $X(n)$. Let $f_n: X(n) \rightarrow A$ denote the map induced by f by addition on A . This paper studies the singularities of the composite morphism

$$\varphi_n: H_X^n \xrightarrow{\sigma_n} X(n) \xrightarrow{f_n} A,$$

which is a natural generalization of the mapping $C(n) \rightarrow J$, where C is an irreducible and nonsingular curve and J is its Jacobian. Unlike the latter, however, φ_n need not be smooth for $n \gg 0$. We prove that φ_n is smooth for $n \gg 0$ only if $f: X \rightarrow A$ is smooth (Theorem 3), and over \mathbb{C} we prove the converse (Theorem 4). In case $X = A$ is an abelian surface, we show φ_n is smooth for n prime to the characteristic (Theorem 5), and give a counterexample to smoothness for all n (Theorem 6). We exhibit a connection (over \mathbb{C}) between singularities of φ_n and generalized Weierstrass points of X (Theorem 9).

Our method is as follows: We first show that the singularities of φ_n are the zeros of certain holomorphic 1-forms on H_X^n which are the "symmetrizations" of holomorphic 1-forms on X . We then study "symmetrized differentials" and their zeros on H_X^n (Theorems 1, 2). Our method works for curves C as well; we give an alternative proof of a result of Mattuck and Mayer [10, p. 226] which shows that the zeros of symmetrized differentials on $C(n)$ represent (for C complete nonsingular) the special divisors of degree n on C .

Introduction. Let $f: C \rightarrow J$ be the canonical map from the complete nonsingular curve C to its Jacobian J . Much of the classical theory of curves can be expressed in terms of the induced map $f_n: C(n) \rightarrow J$, where $C(n)$ is the n -fold symmetric product of C : The fibers of f_n are projective spaces which represent the complete linear systems of degree n on C ; the Riemann-Roch theorem (for positive divisors) computes the dimensions of these fibers for all $n > 0$. The special divisors of degree n are represented on $C(n)$ by the singularities of f_n ; in particular, the nonexistence of special divisors of degree $> 2g - 2$ ($g =$ genus of C) is equivalent to the smoothness of f_n for $n > 2g - 2$. The Weierstrass points of C are those points $P \in C$ for

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which the divisor gP is special, i.e., the point on $C(g)$ representing gP is a singularity of f_n . And so on. In this paper, we study a natural higher dimensional analogue of the mapping f_n . (Note: we work over an algebraically closed ground field k , of arbitrary characteristic except where noted.)

Let X be an irreducible and nonsingular projective variety of dimension $d > 1$, and $f: X \rightarrow A$ the canonical map from X to its Albanese variety A . The induced map $f_n: X(n) \rightarrow A$ is an analogue of $C(n) \rightarrow J$, but since $X(n)$ has singularities and $C(n)$ does not, the analogy is imperfect. However, when X is a surface, the punctual Hilbert scheme H_X^n provides a natural desingularization of $X(n)$; it is an irreducible and nonsingular complete variety of dimension $2n$ whose points represent the closed subschemes of X having Hilbert polynomial n . By associating to each such subscheme Z the 0-cycle $\sum_{i=1}^r n_i P_i$, where $\{P_1, \dots, P_r\} = \text{Supp } Z$ and n_i is the length of Z at P_i , we get a birational map

$$\sigma_n: H_X^n \rightarrow X(n)$$

(the ‘‘Hilbert-Chow’’ morphism), which is an isomorphism over the smooth (off-diagonal) locus of $X(n)$ [3].

We therefore study the composite morphism

$$\varphi_n: H_X^n \xrightarrow{\sigma_n} X(n) \xrightarrow{f_n} A;$$

since $H_C^n = C(n)$ [8], φ_n is a natural generalization of $f_n: C(n) \rightarrow J$. An earlier paper [6] studied the fibers of φ_n , and an analogue of ‘‘Riemann’s theorem’’ was proved: all the fibers of φ_n are irreducible and of dimension $2n - q$, for $n \gg 0$ ($q = \dim A$). This paper studies the singularities of φ_n . Guessing from the case of curves, we began the work in an attempt to answer the following

Question. Is φ_n smooth for $n \gg 0$. (And if not, where not?) We will see that the answer is, in general, *no*.

Our basic method for studying singularities of φ_n is this: We first observe that the singularities of φ_n are the zeros of certain holomorphic 1-forms on H_X^n , which arise by a process of ‘‘symmetrization’’ from holomorphic 1-forms on X . We therefore study ‘‘symmetrized differentials’’ with an eye toward finding their zeros; Theorems 1 (§3) and 2 (§4) are the main results in this direction. (We do not quote the theorems here since their precise statements require extensive preparations.) For curves C , the problem of determining the zeros of symmetrized differentials on $H_C^n = C(n)$ can be completely solved. The result ((4), §1) appears in a paper of Mattuck and Mayer [10] on the Riemann-Roch theorem for curves. Our methods work for curves as well as for surfaces, and we give an alternative proof of this result, which is the prototype of the kind of result we hoped to achieve for surfaces.

We now discuss the structure and contents of the paper in more detail. In §1 we define symmetrized differentials and establish their connection with the singularities of φ_n . §§2 and 3 are devoted to the formulation and proof of Theorem 1; §2 contains preliminaries on tangent bundles (especially the tangent bundle of H_X^n), and §3 contains the proof of Theorem 1, as well as some other basic results concerning symmetrized differentials. In §4 we use the ‘‘linear determinant’’ [8] to prove

Theorem 2; in tandem, Theorems 1 and 2 allow us in certain cases to find zeros of symmetrized differentials, essentially by direct computation.

In §5, we make four applications of Theorems 1 and 2 to obtain the principal results of the paper. Application 1 contains our alternative proof of (4) for curves. In Application 2, we answer the question concerning smoothness of φ_n for $n \gg 0$ raised above. We prove

THEOREM 3. *Let X be an irreducible and nonsingular surface for which the Albanese mapping $f: X \rightarrow A$ is not smooth. Then for $n = r(r+1)/2$, $r > 0$, the mapping $\varphi_n: H_X^n \rightarrow A$ is not smooth.*

Over the complex numbers, we have the converse as well:

THEOREM 4. *Let X be an irreducible and nonsingular surface/ \mathbb{C} , with $q = \dim A > 0$. Then the following are equivalent:*

- (i) $\varphi_n: H_X^n \rightarrow A$ is smooth for $n \gg 0$;
- (ii) the Albanese mapping $\varphi_1 = f: X \rightarrow A$ is smooth;
- (iii) every nonzero holomorphic 1-form $\omega = f^*(\omega_A)$ has no zeros on X ($0 \neq \omega_A \in \Gamma(\Omega_A^1)$);
- (iv) $\varphi_n: H_X^n \rightarrow A$ is smooth for all $n > 0$.

In particular, the conditions of Theorem 4 are satisfied when $X = A$ is an abelian surface/ \mathbb{C} . We study abelian surfaces in arbitrary characteristic in Application 3; we show that $\varphi_n: H_X^n \rightarrow A$ is smooth if n is prime to the characteristic (Theorem 5), but that whenever $n = m^2 \equiv 0 \pmod{\text{char } k}$, φ_n is not smooth (Theorem 6); indeed, it has singularities in every fiber.

In Application 4, we formulate and prove a generalization of (4) for curves to the case of surfaces (Theorem 8). This is then used in (the final) §6 to relate the differential behavior of φ_n to generalized Weierstrass points on X (Theorem 9), in analogy with the classical result for curves mentioned above.

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Conventions. In addition to those established above, we make the following conventions. (Recall that we work over an algebraically closed ground field k , of arbitrary characteristic except where noted.) By a k -scheme we mean a separated scheme of finite type over k ; a *variety* is a reduced k -scheme. A point of a k -scheme is a k -rational (or closed) point, unless otherwise stated. If $\sum_{i=1}^r n_i P_i$ is a 0-cycle on a variety Y , the associated point of $Y(n)$ is written $(\sum_{i=1}^r n_i P_i)$, and if Z is a 0-dimensional closed subscheme of length n on Y , the associated point of H_Y^n is written $[Z]$. We use C for a curve, X for a surface, and V for either, in discussions that apply to both cases. These will always be irreducible and nonsingular, but projective only when noted.

1. Symmetrized differentials and singularities of φ_n . Let V be an irreducible and nonsingular curve or surface, ω a holomorphic 1-form on V , and $\pi_i: V^n \rightarrow V$ the i th projection map. The 1-form $\omega^n = \sum_{i=1}^n \pi_i^*(\omega)$ on V^n is invariant under the action of

the symmetric group, so it induces on $V(n)$ a rational 1-form $\omega_{(n)}$; using the morphism $\sigma_n: H_V^n \rightarrow V(n)$, we pull back $\omega_{(n)}$ to obtain the 1-form $\omega_{[n]}$ (the *symmetrization* of ω_n) on H_V^n . *A priori* we know that $\omega_{(n)}$ is holomorphic on the locus $V(n)_0 = \{(P_1 + \dots + P_n) \mid P_i \neq P_j \text{ for } i \neq j\}$, over which $V^n \rightarrow V(n)$ is etale; thus $\omega_{[n]}$ is holomorphic on $\sigma_n^{-1}(V(n)_0)$, since σ_n is an isomorphism over $V(n)_0$. (Of course, when V is a curve, $H_V^n = V(n)$ and $\omega_{[n]} = \omega_{(n)}$.) We will show that $\omega_{[n]}$ is holomorphic everywhere on H_V^n (Proposition 3); in characteristic 0, this is a special case of [14, Theorem 3 or 11, p. 197]. It will follow that the map $\omega \mapsto \omega_{[n]}$ defines an injection $\Gamma(\Omega_V^1) \rightarrow \Gamma(\Omega_{H_V^n}^1)$ which is an isomorphism when V is projective (Proposition 4).

We now explain the connection between symmetrized differentials and singularities of the Albanese mapping $\varphi_n: H_V^n \rightarrow A$. We first recall the following criterion for smoothness at a point:

(1) A morphism $g: S \rightarrow W$ of nonsingular varieties is smooth at $s \in S$ if and only if the differential map $Dg: TS_s \rightarrow TW_w$ is surjective, where $w = g(s)$ and TS_s and TW_w are the tangent spaces to S at s and W at w , respectively [1, VII, 5.5].

This criterion may be applied to φ_n since its domain and range are nonsingular varieties. Let $[Z] \in H_V^n = H$ and $a = \varphi_n([Z]) \in A$. Dualized, (1) says that φ_n is smooth at $[Z]$ if and only if the map $(D\varphi_n)^*: TA_a^* \rightarrow TH_{[Z]}^*$ of cotangent spaces is injective. But a nonzero covector at a determines (and is determined by) a nonzero global holomorphic 1-form on A by translation; if the pullback of this 1-form to H has a zero at $[Z]$, it follows that $(D\varphi_n)^*$ is not injective. We conclude:

(2) $[Z] \in H_V^n$ is a singularity of φ_n if and only if for some nonzero global holomorphic 1-form ω_A on A , the pullback $\varphi_n^*(\omega_A)$ is zero at $[Z]$.

Therefore, to determine the singularities of φ_n (or prove that none exist) we have to determine the zeros of the differentials $\varphi_n^*(\omega_A)$ (or prove that none exist). The first (short) step is to observe that $\varphi_n^*(\omega_A)$ is a symmetrized differential; in fact,

(3) $\varphi_n^*(\omega_A) = \omega_{[n]}$, where $\omega = f^*(\omega_A)$ is the pullback of ω_A to V by the Albanese mapping $f: V \rightarrow A$.

PROOF. It is enough to prove that $f_n^*(\omega_A) = \omega_{(n)}$ on $V(n)$, or equivalently (separability of $V^n \rightarrow V(n)$), that the pullback of ω_A by the composite mapping $V^n \rightarrow V(n) \xrightarrow{f_n} A$ is $\omega^n = \sum_{i=1}^n \pi_i^*(\omega)$. But the latter map is the sum of maps $(\sum_{i=1}^n f \circ \pi_i): V^n \rightarrow A$, and since the pullback of ω_A by a sum of maps is the sum of the pullbacks [16, Proposition 17, p. 56], we have that

$$\left(\sum_{i=1}^n f \circ \pi_i \right)^* (\omega_A) = \sum_{i=1}^n \pi_i^*(f^*(\omega_A)) = \omega^n,$$

as desired. \square

REMARK. In general, the Albanese mapping $f: V \rightarrow A$ induces an injection $f^*: \Gamma(\Omega_A^1) \rightarrow \Gamma(\Omega_V^1)$ [17]. If V is complete and the base field is \mathbb{C} , then f^* is an isomorphism [18, §9].

The second (long) step is to determine the zeros of symmetrized differentials on H_V^n . Efforts to solve this problem have led to Theorems 1 (§3) and 2 (§4), which

solve the problem “in principle” if not always in practice. Briefly, these theorems enable us to compute the covector $\omega_{[n]}: TH_{[Z]} \rightarrow k$ induced at $[Z] \in H_V^n$ by the symmetrized differential $\omega_{[n]}$; if this covector is the zero map, the 1-form $\omega_{[n]}$ has a zero at $[Z]$. In practice, the computations can be carried out whenever a convenient basis of $TH_{[Z]}$ can be written down.

For example, when V is a curve C , the problem of determining the zeros of symmetrized differentials can be completely solved:

(4) $\omega_{(n)}$ is zero at $(\sum_{i=1}^n n_i P_i) \in C(n) \Leftrightarrow (\omega) > \sum_{i=1}^n n_i P_i$, where (ω) is the divisor of ω on C .

This is proved in [10, p. 226] using the theory of symmetric polynomials (Newton’s identities); it is a cornerstone of the proof of the Riemann-Roch theorem for curves given in the same paper. Theorems 1 and 2 can be applied in case $V = C$ to give an alternative proof of this result (Application 1, §5).

In case V is a surface, no such complete solution is obtained; we have several partial results (Applications 2–4 of §5) which yield some information about the singularities of φ_n , as indicated in the Introduction. Our first goal is the formulation and proof of Theorem 1, which occupies §3; the necessary preliminaries on tangent bundles are the subject of the next section.

2. Preliminaries on tangent bundles. Let Y be a k -scheme and let $k[\varepsilon]$ be the dual numbers ($\varepsilon^2 = 0$). If S is any k -scheme, we write $S[\varepsilon]$ for $S \otimes_k k[\varepsilon]$. Consider the contravariant functor Y_ε from k -schemes to sets, whose value on the k -scheme S is

$$Y_\varepsilon(S) = \text{Hom}(S[\varepsilon], Y),$$

and whose value on the morphism $j: S' \rightarrow S$ is the map

$$Y_\varepsilon(j): \text{Hom}(S[\varepsilon], Y) \rightarrow \text{Hom}(S'[\varepsilon], Y),$$

given by composition with $j \otimes 1: S'[\varepsilon] \rightarrow S[\varepsilon]$. If $S = \text{Spec } k$, then $Y_\varepsilon(S)$ is the set of tangent vectors on Y .

(5) The functor Y_ε is representable, and is represented by a k -scheme TY (the tangent bundle of Y), which is the vector bundle $\mathbf{V}(\Omega_{Y/k}^1)$ associated to the sheaf of (Kähler) differentials on Y .

TY is defined and studied in [4, IV, 16.5.12–13] (also [2, pp. 65–72]), though it is not explicitly pointed out there that TY represents Y_ε . To see this, we must construct a natural bijection

$$(6) \quad \text{Hom}(S[\varepsilon], Y) = \text{Hom}(S, TY)$$

for any k -scheme S ; by standard arguments, it suffices to do this when $Y = \text{Spec } B$ is affine and when Y_ε is restricted to the category of affine k -schemes. In this case, by definition, $TY = \mathbf{V}(\Omega_{Y/k}^1) = \text{Spec}(\text{Symm}_B(\Omega_{B/k}^1))$, where $\Omega_{B/k}^1$ is the module of Kähler differentials of B , trivial on k , and $\text{Symm}_B(\cdot)$ is the functor assigning to any B -module its symmetric algebra.

If $S = \text{Spec } R$, then a map $g: S \rightarrow TY$ has comorphism $g^*: \text{Symm}_B(\Omega_{B/k}^1) \rightarrow R$; in particular, R becomes a B -algebra by the map $g_0^*: B \rightarrow R$, which is the composition of g^* and the natural map $B \rightarrow \text{Symm}_B(\Omega_{B/k}^1)$ (the comorphism of the projection map $\rho: TY \rightarrow Y$). By the universal property of the symmetric algebra, g^*

corresponds uniquely to a B -module homomorphism $g^*: \Omega_{B/k}^1 \rightarrow R$, which in turn corresponds uniquely to a derivation $D_{g^*}: B \rightarrow R$, trivial on k , by the universal property of $\Omega_{B/k}^1$. Using this derivation, one constructs a homomorphism of k -algebras

$$B \rightarrow R \otimes_k k[\varepsilon] = R \oplus \varepsilon R, \quad b \mapsto g_0^*(b) + \varepsilon D_{g^*}(b),$$

which is the comorphism of a map $g': S[\varepsilon] \rightarrow Y$; one checks easily that $g \mapsto g'$ is a natural bijection, which gives (6) and establishes (5).

With g, g' as above, one sees easily that the following diagram commutes:

$$(7) \quad \begin{array}{ccc} S & \xrightarrow{g} & TY \\ \downarrow & \searrow g_0 & \downarrow \rho \\ S[\varepsilon] & \xrightarrow{g'} & Y \end{array}$$

In particular, if $S = \text{Spec } k$ and we fix g_0 to be the inclusion of the point $y \in Y$, we see that the fiber $\rho^{-1}(y)$ is (as expected) the tangent space TY_y .

If $Y = \text{Spec } B$ and $\Omega_{B/k}^1$ is free with basis $\omega_1, \dots, \omega_r$, then $\text{Sym}_B(\Omega_{B/k}^1)$ is the polynomial ring $B[\omega_1, \dots, \omega_r] = B \otimes_k k[\omega_1, \dots, \omega_r]$. Thus $TY = \text{Spec}(B[\omega_1, \dots, \omega_r])$ is the product of Y and a vector space, linear coordinates for which are given by the ω_i . Since the sheaf of holomorphic 1-forms on a nonsingular variety is locally free, the tangent bundle of a nonsingular variety is locally the product of the variety and a vector space. Moreover,

(8) to give a holomorphic 1-form on a nonsingular variety Y is the same as to give a holomorphic function on TY which is linear on each fiber TY_y .

PROOF. A holomorphic 1-form on Y determines a linear functional (covector) on TY_y for all $y \in Y$; the union (roughly speaking) of these functionals gives the desired function. More precisely, it is enough to consider the case where $Y = \text{Spec } B$ and $\Omega_{B/k}^1$ is free over B with basis $\omega_1, \dots, \omega_r$, so that $TY = \text{Spec } B[\omega_1, \dots, \omega_r]$. Since the ω_i give linear coordinates on each fiber TY_y , the holomorphic functions linear on each fiber are clearly given by the linear polynomials $\sum_{i=1}^r b_i \omega_i \in B[\omega_1, \dots, \omega_r]$, but these are also exactly the elements of $\Omega_{B/k}^1$, and (8) follows.

We now consider the case $Y = H_V^n = H$, where V is an irreducible and nonsingular curve or surface. (Actually, V could be any quasiprojective k -scheme.) By using (5) and the universal property of H , the functor of points of TH can be precisely described.

Let S be a k -scheme and $g: S \rightarrow TH$ a morphism. We have a commutative diagram (as in (7))

$$\begin{array}{ccc} S & \xrightarrow{g} & TH \\ \downarrow & \searrow g_0 & \downarrow \rho \\ S[\varepsilon] & \xrightarrow{g'} & H \end{array}$$

where g and g' are associated by (6). For fixed g_0 , we write L_{g_0} for the set of all liftings g (or g') making the diagram commute. By the universal property of H , the

map g_0 corresponds to a closed subscheme $W_{g_0} \subseteq V \times_k S = V_S$, flat and finite of degree n over S . Any $g' \in L_{g_0}$ corresponds to a first order (embedded) deformation of W_{g_0} , i.e., a closed subscheme $W_{g'} \subseteq V \times_k S[\epsilon]$ that is flat over $S[\epsilon]$ and satisfies $W_{g'} \times_{S[\epsilon]} S = W_{g_0}$. We now recall:

(9) The set of first order deformations $W_{g'}$ of W_{g_0} is in natural bijective correspondence with the module

$$\Gamma\left(\underline{\text{Hom}}_{O_{V_S}}(\mathcal{I}_{g_0}, O_{V_S}/\mathcal{I}_{g_0})\right),$$

where \mathcal{I}_{g_0} is the sheaf of ideals defining W_{g_0} [5, IV, Proposition 5.1].

In case $S = \text{Spec } k$, a map $g_0: S \rightarrow H$ corresponds to a 0-dimensional closed subscheme Z of V having length n , and g_0 is the inclusion of the point $[Z]$. Then L_{g_0} is the set of $k[\epsilon]$ -valued points (tangent vectors) of H localized at $[Z]$, so (9) shows

$$(10) \quad TH_{[Z]} \cong \Gamma\left(\underline{\text{Hom}}_{O_V}(\mathcal{I}, O_V/\mathcal{I})\right) \cong \prod_{i=1}^r \text{Hom}_{O_{V,P_i}}(\mathcal{I}_{P_i}, O_{V,P_i}/\mathcal{I}_{P_i}),$$

where \mathcal{I} is the sheaf of ideals defining Z , and $\text{Supp}(Z) = \{P_1, \dots, P_r\}$ [3, p. 517].

In particular, the tangent in space $TH_{[Z]}$ is a $\Gamma(V, O_V)$ module, hence any $h \in \Gamma(V, O_V)$ defines a linear map on $TH_{[Z]}$ by multiplication. We now show that for a fixed h , the linear maps induced on the $TH_{[Z]}$ for various $[Z]$ glue together in a natural way. Recall that a *bundle map* $TH \rightarrow TH$ is a morphism that commutes with the projection to H and is linear on each fiber $TH_{[Z]}$.

PROPOSITION 1. *Let V, H , and h be as above. Then there exists a bundle map $\hat{h}: TH \rightarrow TH$ such that for all $[Z] \in H$, the restriction of \hat{h} to $TH_{[Z]}$ is the linear map given (according to (10)) by multiplication by h .*

REMARK. In this proposition we implicitly assume that V is affine, since if V is projective, the holomorphic functions on V are constants and the conclusion is trivial.

PROOF. By (5), TH represents the functor H_ϵ , so we may construct \hat{h} as a morphism of functors.

Given a k -scheme S , we define a map

$$\hat{h}(S): \text{Hom}(S[\epsilon], H) \rightarrow \text{Hom}(S[\epsilon], H)$$

as follows: First break up $\text{Hom}(S[\epsilon], H)$ into the union of sets L_{g_0} as g_0 ranges over $\text{Hom}(S, H)$. Now (9) shows that L_{g_0} is in natural bijective correspondence with the $\Gamma(V, O_V)$ -module $\Gamma(\underline{\text{Hom}}_{O_{V_S}}(\mathcal{I}_{g_0}, O_{V_S}/\mathcal{I}_{g_0}))$, so that multiplication by h on L_{g_0} is defined and gives a map $L_{g_0} \rightarrow L_{g_0}$; the union of these maps over all $g_0 \in \text{Hom}(S, H)$ defines $\hat{h}(S)$.

To check this gives a map of functors we must verify that for any map $j: S' \rightarrow S$ of k -schemes, the induced diagram

$$\begin{array}{ccc} \text{Hom}(S[\epsilon], H) & \xrightarrow{\hat{h}(S)} & \text{Hom}(S[\epsilon], H) \\ \downarrow & & \downarrow \\ \text{Hom}(S'[\epsilon], H) & \xrightarrow{\hat{h}(S')} & \text{Hom}(S'[\epsilon], H) \end{array}$$

commutes, where the vertical arrows are defined by composition in the obvious way. This amounts to checking that for all $g_0 \in \text{Hom}(S, H)$, the diagram

$$\begin{array}{ccc}
 \Gamma\left(\underline{\text{Hom}}_{O_{V_S}}(\mathcal{G}_{g_0}, O_{V_S}/\mathcal{G}_{g_0})\right) & \xrightarrow{h} & \Gamma\left(\underline{\text{Hom}}_{O_{V_S}}(\mathcal{G}_{g_0}, O_{V_S}/\mathcal{G}_{g_0})\right) \\
 \downarrow & & \downarrow \\
 \Gamma\left(\underline{\text{Hom}}_{O_{V_S}}(\mathcal{G}_{g_0 \circ j}, O_{V_S}/\mathcal{G}_{g_0 \circ j})\right) & \xrightarrow{h} & \Gamma\left(\underline{\text{Hom}}_{O_{V_S}}(\mathcal{G}_{g_0 \circ j}, O_{V_S}/\mathcal{G}_{g_0 \circ j})\right) \\
 \parallel & & \parallel \\
 \mathcal{G}_{g_0} \otimes_{O_S} O_{S'} & & (O_{V_S}/\mathcal{G}_{g_0}) \otimes_{O_S} O_{S'}
 \end{array}$$

commutes, where the horizontal arrows are multiplication by h and the vertical arrows are the natural ones (where $\mathcal{G}_{g_0 \circ j}$, the ideal defining $(W_{g_0})_{S'} = W_{g_0 \circ j}$, equals $\mathcal{G}_{g_0} \otimes_{O_S} O_{S'}$ since W_{g_0} is flat over O_S), but here commutativity is clear. Thus the bundle map \hat{h} exists, and (evidently) on $TH_{[Z]}$, \hat{h} is the linear map given by multiplication by h as described above, so the proposition is proved. \square

3. Statement of proof of Theorem 1. We begin by proving Proposition 2, which is a preliminary version of Theorem 1, and which plays a key role in the proof of Proposition 3. Let V be an irreducible and nonsingular curve or surface (so that H_V^n is also irreducible and nonsingular), ω a holomorphic 1-form on V , and h a holomorphic function on V . We assume that $\omega_{[n]}$, the symmetrization of ω , is holomorphic on $H_V^n = H$; by (8), therefore, we may view $\omega_{[n]}$ as a holomorphic function $\omega_{[n]}: TH \rightarrow k$ which is linear on each fiber. Then the composite morphism

$$TH \xrightarrow{\hat{h}} TH \xrightarrow{\omega_{[n]}} k,$$

where \hat{h} is the bundle map given by Proposition 1, is also a holomorphic function linear on each fiber, and so defines another holomorphic 1-form on H , which we write $\omega_{[n]} \circ \hat{h}$.

PROPOSITION 2. *With V, h , and ω as above, we have that*

$$(h\omega)_{[n]} = \omega_{[n]} \circ \hat{h};$$

in particular, $(h\omega)_{[n]}$ is holomorphic on H_V^n .

For the proof of Proposition 2 and elsewhere we need the following

REMARKS. Let n_1, \dots, n_r be positive integers whose sum is n , and let $H_V^{n_i} = H^{n_i}$, $H_V^n = H^n$. Then there is a rational map

$$(11) \quad \lambda = \lambda_{n_1, \dots, n_r}: \prod_{i=1}^r H^{n_i} \rightarrow H^n$$

which is defined on the open set U of points $([Z_1], \dots, [Z_r])$ which satisfy $\text{Supp } Z_i \cap \text{Supp } Z_j = \emptyset$ for $i \neq j$, and whose value at such a point is

$$\lambda([Z_1], \dots, [Z_r]) = \left[\prod_{i=1}^r Z_i \right] \in H^n.$$

The map λ exists since the family of closed subschemes on U whose member over $([Z_1], \dots, [Z_r])$ is $\prod_{i=1}^r Z_i$ is easily seen to be flat and proper.

Now let Z be a closed subscheme of length n with support $\{P_1, \dots, P_r\}$, and having length n_i at P_i . Then $Z = \prod_{i=1}^r Z_i$, where Z_i is supported at P_i ; thus $\lambda = \lambda_{n_1, \dots, n_r}$ is defined at $([Z_1], \dots, [Z_r]) \in \prod_{i=1}^r H^{n_i}$, and $\lambda([Z_1], \dots, [Z_r]) = [Z]$. Let \mathcal{G} be the sheaf of ideals in \mathcal{O}_V defining Z , and \mathcal{G}_i the ideals in the local rings \mathcal{O}_{P_i} defining Z_i for $1 \leq i \leq r$. Then it can be shown that the differential map $D\lambda$ at $([Z_1], \dots, [Z_r])$ is given by the following sequence of isomorphisms (recall (10)):

$$(12) \quad T\left(\prod_{i=1}^r H^{n_i}\right)_{([Z_1], \dots, [Z_r])} \cong \bigoplus_{i=1}^r TH_{[Z_i]}^{n_i} \cong \bigoplus_{i=1}^r \text{Hom}_{\mathcal{O}_{P_i}}(\mathcal{G}_i, \mathcal{O}_{P_i}/\mathcal{G}_i) \\ \cong \text{Hom}_{\mathcal{O}_V}(\mathcal{G}, \mathcal{O}_V/\mathcal{G}) \cong TH_{[Z]}^n.$$

It follows that λ is étale at $([Z_1], \dots, [Z_r])$, since it is smooth (by (1)) and quasifinite there.

PROOF OF PROPOSITION 2. Since $H_V^n = H^n$ is nonsingular and irreducible, it is enough to show that $(h\omega)_{[n]}$ and $\omega_{[n]} \circ \hat{h}$ are equal on a nonempty open set where both are holomorphic (for then they are equal as rational 1-forms, and since $\omega_{[n]} \circ \hat{h}$ is holomorphic everywhere, so too is $(h\omega)_{[n]}$, and the equality holds everywhere). We show $(h\omega)_{[n]} = \omega_{[n]} \circ \hat{h}$ on the open set $\sigma_n^{-1}(V(n)_0)$, where $(h\omega)_{[n]}$ is known to be holomorphic.

Fix a point $[Z] \in \sigma_n^{-1}(V(n)_0)$; then $\sigma_n([Z]) = (P_1 + \dots + P_n)$, with $P_i \neq P_j$ for $i \neq j$. Since the quotient map $V^n \rightarrow V(n)$ is étale over $(P_1 + \dots + P_n)$, the rational map

$$V^n \rightarrow V(n) \xrightarrow{\sigma_n^{-1}} H^n$$

is étale at (P_1, \dots, P_n) and is clearly equal to the rational map $\lambda = \lambda_{1, \dots, 1}$ (n 1's) as discussed above (11), where we identify V and H_V^1 . It is therefore enough to prove that the pullbacks $\lambda^*((h\omega)_{[n]})$ and $\lambda^*(\omega_{[n]} \circ \hat{h})$ give the same covector at $(P_1, \dots, P_n) \in V^n$.

Clearly we have that $\lambda^*((h\omega)_{[n]}) = (h\omega)^n = \sum_{i=1}^n h_i \omega_i$, where $h_i = \pi_i^*(h)$ and $\omega_i = \pi_i^*(\omega)$. Hence, given a tangent vector $v = (v_1, \dots, v_n) \in TV_{(P_1, \dots, P_n)}^n \cong \bigoplus_{i=1}^n TV_{P_i}$, we see that

$$\lambda^*((h\omega)_{[n]})(v) = \sum_{i=1}^n h_i(P_1, \dots, P_n) \omega_i(v_1, \dots, v_n) \\ = \sum_{i=1}^n h(P_i) \omega(v_i).$$

We must show that $\lambda^*(\omega_{[n]} \circ \hat{h})$ has the same value on the tangent vector v .

Let us compute:

$$\lambda^*(\omega_{[n]} \circ \hat{h})(v) = \omega_{[n]} \circ \hat{h}(D\lambda(v)) = \omega_{[n]}(D\lambda(\hat{h}'(v))) \\ = \lambda^*(\omega_{[n]})(\hat{h}'(v)) = \omega^n(\hat{h}'(v)),$$

where $\omega^n = \sum_{i=1}^n \omega_i$ and \hat{h}' is the linear map on $TV_{(P_1, \dots, P_n)}^n$ defined by the equation $D\lambda(\hat{h}'(v)) = \hat{h}(D\lambda(v))$ (which exists since $D\lambda: TV_{(P_1, \dots, P_n)}^n \rightarrow TH_{[Z]}^n$ is an isomorphism).

We now compute \hat{h}' . If \mathcal{G} is the sheaf of ideals defining $Z = \coprod_{i=1}^n P_i$, then the sequence of isomorphisms (12) giving $D\lambda$ in the present case is as follows ($M_{P_i} =$ maximal ideal at P_i):

$$\begin{aligned} D\lambda: TV_{(P_1, \dots, P_n)}^n &\cong \bigoplus_{i=1}^n TV_{P_i} \cong \bigoplus_{i=1}^n \text{Hom}_{O_{P_i}}(M_{P_i}, O_{P_i}/M_{P_i}) \\ &\cong \text{Hom}_{O_V}(\mathcal{G}, O_V/\mathcal{G}) \cong TH_{[Z]}^n. \end{aligned}$$

With $v = (v_1, \dots, v_n)$ as above, note that multiplication by h on $D\lambda(v) \in \text{Hom}_{O_V}(\mathcal{G}, O_V/\mathcal{G})$ corresponds to multiplication of each $v_i \in \text{Hom}_{O_{P_i}}(M_{P_i}, O_{P_i}/M_{P_i})$ by h , which is the same as multiplication of each v_i by the value $h(P_i)$, since $h \equiv h(P_i) \pmod{M_{P_i}}$. Consequently, \hat{h}' is the linear map on $TV_{(P_1, \dots, P_n)}^n$ given by

$$v = (v_1, \dots, v_n) \mapsto (h(P_1)v_1, \dots, h(P_n)v_n).$$

Using this to complete the above calculation, we get

$$\begin{aligned} \lambda^*(\omega_{[n]} \circ \hat{h}')(v) &= \omega^n(\hat{h}'(v)) \\ &= \sum_{i=1}^n \omega_i(h(P_1)v_1, \dots, h(P_n)v_n) \\ &= \sum_{i=1}^n h(P_i)\omega(v_i) = \lambda^*((h\omega)_{[n]})(v), \end{aligned}$$

which proves the proposition. \square

Now we drop the *assumption* that $\omega_{[n]}$ is holomorphic in order to *prove* it so:

PROPOSITION 3. *Let V be an irreducible and nonsingular curve or surface, and ω a holomorphic 1-form on V . Then $\omega_{[n]}$ is holomorphic on H_V^n .*

PROOF. Let $[Z] \in H_V^n$, and let $U = \text{Spec } R$ be an affine open subset of V containing the finite set $\text{Supp } Z$. (U exists since V is quasiprojective—every nonsingular curve or surface is quasiprojective [19, vol II.2.6].) Then H_U^n is open in H_V^n , and $[Z] \in H_U^n$; we complete the proof by showing that the restriction $\omega_{[n]}|_{H_U^n}$ is holomorphic on H_U^n .

But clearly

$$\omega_{[n]}|_{H_U^n} = (\omega|U)_{[n]}$$

moreover, $\omega|U$, being a holomorphic 1-form on $U = \text{Spec } R$ (i.e., an element of $\Omega_{R/k}^1$), may be written

$$\omega|U = \sum_{\alpha=1}^r h_\alpha dg_\alpha,$$

where $g_\alpha, h_\alpha \in R$ [15, Proposition 1, p. 159]. Thus

$$\omega_{[n]}|_{H_U^n} = (\omega|U)_{[n]} = \left(\sum_{\alpha=1}^r h_\alpha dg_\alpha \right)_{[n]} = \sum_{\alpha=1}^r (h_\alpha dg_\alpha)_{[n]},$$

where we have used the obvious linearity of $\omega \mapsto \omega_{[n]}$ to obtain the last equality. It suffices now to prove that each summand $(h_\alpha dg_\alpha)_{[n]}$ is holomorphic; if we knew that each $(dg_\alpha)_{[n]}$ were holomorphic, Proposition 2 would apply to yield the desired conclusion. It remains therefore to prove

(13) if g is a holomorphic function on U , then the symmetrized differential $(dg)_{[n]}$ is holomorphic on H_V^n . (Here U need not be affine.)

PROOF. First note that on U^n ,

$$(dg)^n = \sum_{i=1}^n \pi_i^*(dg) = d\left(\sum_{i=1}^n g_i\right),$$

where $g_i = \pi_i^*(g)$; since $\sum_{i=1}^n g_i$ is symmetric, it determines a holomorphic function $g_{(n)}$ on $V(n)$, and $(dg)_{(n)} = d(g_{(n)})$. Thus

$$(dg)_{[n]} = \sigma_n^*((dg)_{(n)}) = \sigma_n^*(d(g_{(n)})) = d(\sigma_n^*(g_{(n)}))$$

is holomorphic, since $\sigma_n^*(g_{(n)})$ is a holomorphic function on H_V^n ; thus (13) holds and Proposition 3 is proved. \square

Propositions 2 and 3 are summarized in

THEOREM 1. *Let V be an irreducible and nonsingular curve or surface, h a holomorphic function on V , and ω a holomorphic 1-form on V . Then the 1-forms $\omega_{[n]}$, $(h\omega)_{[n]}$, and $\omega_{[n]} \circ \hat{h}$ are holomorphic on H_V^n , and*

$$(h\omega)_{[n]} = \omega_{[n]} \circ \hat{h},$$

where \hat{h} is the bundle map given by Proposition 1.

PROOF. Since $\omega_{[n]}$ is now known to be holomorphic (Proposition 3), Proposition 2 applies. \square

COROLLARY 1. *If g and h are holomorphic functions on V , then*

$$(d(gh))_{[n]} = (dg)_{[n]} \circ \hat{h} + (dh)_{[n]} \circ \hat{g}$$

(which for $n = 1$ is the Leibniz rule on $V = H_V^1$).

PROOF. Since $d(gh) = hdg + gdh$ on V , the corollary follows immediately from the theorem and the linearity of the map $\omega \mapsto \omega_{[n]}$. \square

COROLLARY 2. *If \hat{h} is the zero map on $TH_{[Z]}$ ($H = H_V^n$), then $(h\omega)_{[n]}$ is zero at $[Z] \in H$ for all holomorphic 1-forms ω on V .*

REMARK. The hypothesis on \hat{h} holds, for example, whenever $h \in \mathcal{I}$, where \mathcal{I} is the sheaf of ideals defining Z , since (10)

$$TH_{[Z]} = \Gamma(\underline{\text{Hom}}_{\mathcal{O}_V}(\mathcal{I}, \mathcal{O}_V/\mathcal{I}))$$

is killed by multiplication by any elements of \mathcal{I} .

PROOF. Since $(h\omega)_{[n]} = \omega_{[n]} \circ \hat{h}$ by the theorem, the corollary is obvious. \square

Corollary 2 suggests one way of finding zeros of symmetrized differentials, which we exploit in Proposition 6 of Application 2. More generally, Theorem 1 reduces the problem of computing the induced covector $\omega_{[n]}: TH_{[Z]} \rightarrow k$ to the case $\omega_{[n]} = dg_{[n]}$

is the symmetrization of an exact 1-form on V . Indeed, replacing V by an affine open subset containing $\text{Supp } Z$, we may write $\omega = \sum_{\alpha} h_{\alpha} dg_{\alpha}$ (as in the proof of Proposition 3), where h_{α} and g_{α} are holomorphic on V . Then $\omega_{[n]} = (\sum_{\alpha} h_{\alpha} dg_{\alpha})_{[n]} = \sum_{\alpha} (dg_{\alpha})_{[n]} \circ \hat{h}_{\alpha}$, so the covector $\omega_{[n]}: TH_{[Z]} \rightarrow k$ can be computed by summing the covectors $TH_{[Z]} \xrightarrow{\hat{h}_{\alpha}} TH_{[Z]} \xrightarrow{(dg_{\alpha})_{[n]}} k$. In the next section, we show how the covector induced by $(dg_{\alpha})_{[n]}$ can be computed explicitly (Theorem 2). Meanwhile, for completeness, we prove

PROPOSITION 4. *Let V be a nonsingular and irreducible curve or surface. Then the map $\omega \mapsto \omega_{[n]}$ defines an injection $\Gamma(\Omega_V^1) \rightarrow \Gamma(\Omega_{H_V^n}^1)$, which is an isomorphism if V is projective.*

PROOF. Since $\omega_{[n]}$ is holomorphic by Proposition 3, the map is defined, is clearly linear, and is injective since

$$\begin{aligned} \omega_{[n]} = 0 \quad \text{on } H_V^n &\Rightarrow \omega_{(n)} = 0 \quad \text{on } V(n) \\ &\Rightarrow \omega^n = 0 \quad \text{on } V^n \\ &\Rightarrow \omega = 0 \quad \text{on } V. \end{aligned}$$

It remains to prove, assuming V is projective, that the map is surjective.

Let ω_H be a holomorphic 1-form on H_V^n , and let $\omega_{V(n)}$ be the rational 1-form on $V(n)$ corresponding to ω_H by the birational map $\sigma_n: H_V^n \rightarrow V(n)$. Let ω_{V^n} be the pullback of $\omega_{V(n)}$ to V^n . In case V is a curve, $V(n) = H_V^n$ and $\omega_{V(n)} = \omega_H$ is holomorphic on $V(n)$; hence ω_{V^n} is holomorphic on V^n . In case V is a surface, $\omega_{V(n)}$ is holomorphic off the branch locus of $V^n \rightarrow V(n)$, which has codimension > 1 for $n > 1$; hence ω_{V^n} is again holomorphic. In either case, ω_{V^n} is invariant under the action of the symmetric group, and since any (global) holomorphic 1-form on a product of projective varieties is a sum of 1-forms coming from the factors, we see that $\omega_{V^n} = \sum_{i=1}^n \pi_i^*(\omega)$ for some holomorphic 1-form ω on V . Using the separability of $V^n \rightarrow V(n)$, we conclude that $\omega_{V(n)} = \omega_{(n)}$; it follows that $\omega_H = \omega_{[n]}$, which proves the surjectivity. \square

4. The linear determinant and Theorem 2. As above, we let V be an irreducible and nonsingular curve or surface (although much of this section remains valid when V is an arbitrary quasiprojective variety). The goal of this section is the proof of Theorem 2, which gives a formula for the covector induced on $TH_{[Z]}$ ($H = H_V^n$) by a symmetrized differential $dg_{[n]}$, where dg is an exact 1-form holomorphic on a neighborhood of $\text{Supp } Z \subseteq V$. This formula can be computed explicitly whenever a "convenient" basis of $TH_{[Z]}$ can be written down.

The proof of Theorem 2 is essentially a direct computation made possible by an explicit description (given in [8]) of the morphism $\sigma_n: H_V^n \rightarrow V(n)$ as a morphism of functors (of points). In the proof, we apply this description of σ_n to $k[\varepsilon]$ -valued points (tangent vectors) of H_V^n .

We now summarize what we need from [8]. Let T be a k -scheme, and $v: T \rightarrow H_V^n$ a T -valued point of H_V^n , which corresponds to a closed subscheme $W = W_v \subseteq V \times T$, flat and finite of degree n over T . In [8, II.2] there is constructed a map

$$(14) \quad \theta_{W/T}: T \rightarrow W_T(n),$$

called the *canonical section* of W/T , whose range is the n -fold symmetric product of W over T . If $W(t)$ is the fiber of W over $t \in T$, then $\theta_{W/T}(t) = (\sum_{i=1}^n n_i P_i)$, where $\text{Supp } W(t) = \{P_1, \dots, P_r\}$ and n_i is the length of $W(t)$ at P_i . From v , we obtain a T -valued point of $V(n)$ by the composite map

$$(15) \quad v_\sigma: T \xrightarrow{\theta_{W/T}} W_T(n) \rightarrow (V \times T)_T(n) \rightarrow V(n);$$

$$V(n) \parallel \times_k T$$

the resulting map $v \mapsto v_\sigma$ of T -valued points is functorial and corresponds to the morphism $\sigma_n: H_V^n \rightarrow V(n)$.

What we need to get our hands on is the comorphism of v_σ , in the case where $V = \text{Spec } R$ and $T = \text{Spec } A$ are affine. Then $W = W_\sigma = \text{Spec}(R \otimes A)/I_\sigma$, where $(R \otimes A)/I_\sigma = M_\sigma = M$ is a finite and flat A -algebra, locally free of dimension n . For a B -module N , we write $N_B^{(n)} \subseteq \otimes_B^n N$ for the B -module of symmetric tensors. The comorphism of the map v_σ (15) is then given by the map

$$(16) \quad v_\sigma^*: R_k^{(n)} \rightarrow (R \otimes_k A)_A^{(n)} \rightarrow M_A^{(n)} \xrightarrow{\theta_{W/T}^*} A,$$

where only the last arrow (the comorphism of the canonical section (14)) requires further explanation: it is the horizontal arrow in the commutative diagram

$$(17) \quad \begin{array}{ccc} M_A^{(n)} & \xrightarrow{\theta_{W/T}^*} & A \\ \downarrow & \nearrow \text{ld} & \\ (\text{End}_A M)_A^{(n)} & & \end{array}$$

where the vertical arrow is induced by the natural map $M \rightarrow \text{End}_A M$ (sending $m \in M$ to the A -linear map given by multiplication by m), and the diagonal arrow is the *linear determinant* (ld).

If M is a free module of dimension n over A , the linear determinant is defined as follows: let $(\text{End}_A M)_A^{(n)}$ act on the 1-dimensional A -module $\Lambda_A^n M$ in the obvious way, so that every element $\rho \in (\text{End}_A M)_A^{(n)}$ corresponds to multiplication by a scalar on $\Lambda_A^n M$. By definition, this scalar is $\text{ld}(\rho)$.

For example, let $\lambda \in \text{End}_A M$. Form the n -fold symmetric tensors

$$\lambda_{(n)} = \sum_{i=1}^n 1 \otimes \dots \otimes 1 \otimes \lambda \otimes 1 \otimes \dots \otimes 1 \quad (\lambda \text{ in } i\text{th place})$$

and

$$\lambda^{(n)} = \lambda \otimes \lambda \otimes \dots \otimes \lambda,$$

and choose a basis m_1, \dots, m_n for M over A . By definition of ld, we see that

$$\sum_{i=1}^n m_1 \wedge m_2 \wedge \dots \wedge \lambda(m_i) \wedge \dots \wedge m_n = \text{ld}(\lambda_{(n)}) \cdot (m_1 \wedge m_2 \wedge \dots \wedge m_n),$$

and

$$\lambda(m_1) \wedge \lambda(m_2) \wedge \dots \wedge \lambda(m_n) = \text{ld}(\lambda^{(n)}) \cdot (m_1 \wedge m_2 \wedge \dots \wedge m_n),$$

hence

$$(18) \quad \text{ld}(\lambda_{(n)}) = \text{tr } \lambda \quad \text{and} \quad \text{ld}(\lambda^{(n)}) = \det \lambda.$$

Remaining in the case $V = \text{Spec } R$, $T = \text{Spec } A$, etc., let g be a holomorphic function on V ($g \in R$), and let $g_{(n)}$ (as in the proof of (13)) denote its symmetrization on $V(n)$. As a symmetric tensor, $g_{(n)} = \sum_{i=1}^n 1 \otimes \cdots \otimes g \otimes \cdots \otimes 1 \in R_k^{(n)}$ (g in i th place). Viewing g as an element of R , $R \otimes A$, and $R \otimes A/I_v = M_v$ simultaneously, we write $g: M_v \rightarrow M_v$ for the A -linear map given by multiplication by g . If we assume M_v is free over A , then (16), (17), and (18) combine to prove

$$(19) \quad v^*(g_{(n)}) = \text{tr}(g: M_v \rightarrow M_v) \in A;$$

this fact is the heart of the proof of Theorem 2.

We now turn to the statement of Theorem 2. Let $V = \text{Spec } R$, $Z = \text{Spec } R/I$ be a 0-dimensional closed subscheme of length n on V , and g be a holomorphic function on V . The symmetrized differential $dg_{[n]}$ induces a covector on $TH_{[Z]}$, and we write $dg_{[n]}(v)$ for the value this covector has on $v \in TH_{[Z]}$. Our goal is to derive a formula for $dg_{[n]}(v)$; to this end, we express v as a map $v: T = \text{Spec } k[\epsilon] \rightarrow H_V^n$ localized on $[Z]$. As above, the map v corresponds to a closed subscheme $W_v \subseteq V \times T$ which is flat and finite of degree n over T ; thus, $W_v = \text{Spec}((R \otimes k[\epsilon])/I_v)$, where $(R \otimes k[\epsilon])/I_v = M_v$ is flat (and therefore free) of dimension n over $k[\epsilon]$, and $I_v \otimes_{k[\epsilon]} k = I$. As a convenient notation, given $a + \epsilon b \in k[\epsilon]$ ($a, b \in k$), we call b the ϵ -coefficient of $a + \epsilon b$, and write $C_\epsilon(a + \epsilon b) = b$. We may now state

THEOREM 2. *With $V, Z, v \in TH_{[Z]}$, g , etc., as above, we have that*

$$dg_{[n]}(v) = C_\epsilon(\text{tr}(g: M_v \rightarrow M_v)),$$

where $g: M_v \rightarrow M_v$ is the $k[\epsilon]$ -linear map given by multiplication by g on M_v .

Before proceeding with the proof, we need the following

REMARKS. Let U be a variety and $u \in U$. A tangent vector to U at u is usually defined to be an element $v \in \text{Der}_k(O_{U,u}, k) \approx \text{Hom}_k(M_u/M_u^2, k)$, and when so defined, the covector $d\rho$ is given by the equation

$$d\rho(v) = v(\rho),$$

for all $\rho \in O_{U,u}$ [2, p. 65]. Furthermore, when v is given as a map $v: T = \text{Spec } k[\epsilon] \rightarrow U$, the derivation $v: O_{U,u} \rightarrow k$ is recovered from the comorphism $v^*: O_{U,u} \rightarrow k[\epsilon] = k \oplus \epsilon k$ by the formula

$$v(\rho) = C_\epsilon(v^*(\rho)).$$

Thus, when a tangent vector is given as a map $v: T \rightarrow U$, we have that

$$(20) \quad d\rho(v) = C_\epsilon(v^*(\rho)), \quad \rho \in O_{U,u}.$$

PROOF OF THEOREM 2. As we saw in the proof of (13),

$$dg_{[n]} = \sigma_n^*(dg_{(n)}) = d(\sigma_n^*(g_{(n)}))$$

is an exact holomorphic 1-form on H_V^n . Thus

$$\begin{aligned} dg_{[n]}(v) &= d(\sigma_n^*(g_{(n)}))(v) \\ &= C_\epsilon(v^*(\sigma_n^*(g_{(n)}))) \quad (\text{by 20}) \\ &= C_\epsilon(v_\sigma^*(g_{(n)})) \quad (\text{since } v_\sigma = \sigma_n \circ v) \\ &= C_\epsilon(\text{tr}(g: M_v \rightarrow M_v)) \quad (\text{by (19)}), \end{aligned}$$

as desired. \square

5. Applications of Theorems 1 and 2. We now apply Theorems 1 and 2 to obtain the results outlined in the Introduction. Before proceeding with the applications, we need to make some technical preparations. Let V be an irreducible and nonsingular curve or surface, ω a holomorphic 1-form on V , and Z a 0-dimensional closed subscheme of V of length n , with support $\{P_1, \dots, P_r\}$; thus $Z = \coprod_{i=1}^r Z_i$, where Z_i has length n_i and is concentrated at P_i .

PROPOSITION 5. *The 1-form $\omega_{[n]}$ is zero at $[Z] \in H_V^n = H^n$ if and only if each $\omega_{[n_i]}$ is zero at $[Z_i] \in H^{n_i} = H^{n_i}$, for $1 \leq i \leq r$.*

PROOF. By the remarks that preceded the proof of Proposition 2, there is a rational map (11)

$$\lambda = \lambda_{n_1, \dots, n_r}: \prod_{i=1}^r H^{n_i} \rightarrow H^n,$$

defined and étale at $([Z_1], \dots, [Z_r])$, with $\lambda([Z_1], \dots, [Z_r]) = [Z]$. We claim that

$$\lambda^*(\omega_{[n]}) = \sum_{i=1}^r pr_i^*(\omega_{[n_i]}),$$

where pr_i is the i th projection; the proposition follows immediately from the claim, since λ is étale at $([Z_1], \dots, [Z_r])$.

To prove the claim, note that the diagram

$$\begin{array}{ccc} \prod_{i=1}^r H^{n_i} & \xrightarrow{\lambda} & H^n \\ \sigma_{n_1} \times \dots \times \sigma_{n_r} \downarrow & & \downarrow \sigma_n \\ \prod_{i=1}^r V(n_i) & \xrightarrow{\alpha} & V(n) \end{array}$$

commutes (on $\text{dom}(\lambda)$), where α is the map that adds 0-cycles. It is therefore enough to prove that $\alpha^*(\omega_{(n)}) = \sum_{i=1}^r pr_i^*(\omega_{(n_i)})$ on $\prod_{i=1}^r V(n_i)$; briefly, this is true since both these differentials pull back to $\omega^n = \sum_{i=1}^n \pi_i^*(\omega)$ on V^n , by the quotient map $V^n = \prod_{i=1}^n V^{n_i} \rightarrow \prod_{i=1}^n V(n_i)$ (see [14, Lemma 4]). Thus Proposition 5 is proved. \square

Proposition 5 reduces the problem of computing the covector $\omega_{[n]}: TH_{[Z]} \rightarrow k$ to the case $Z = \text{Spec}(O_P/I)$ is a 0-dimensional closed subscheme of length n concentrated at $P \in V$ ($H = H_V^n$ and $O_P = O$ is the local ring at P). In this case, $TH_{[Z]} = \text{Hom}_O(I, O/I)$, by (10); thus a tangent vector $v \in TH_{[Z]}$ is given by an

O -homomorphism $v: I \rightarrow O/I$. However, v can also be given as a map $v: T = \text{Spec } k[\varepsilon] \rightarrow H$ localized at $[Z]$; in order to compute using Theorem 2, we need to recall how these two descriptions of v are related. The latter corresponds to a closed subscheme $W_v \subseteq V \times T$, etc., which is defined by an ideal $I_v \subseteq O \otimes_k k[\varepsilon] = O[\varepsilon]$, such that $M_v = O[\varepsilon]/I_v$ is free of dimension n over $k[\varepsilon]$, and $I_v \otimes_{k[\varepsilon]} k = I$ (notation as in §4). If a tangent vector at $[Z]$ is given as an O -homomorphism $v: I \rightarrow O/I$, then the proof of (9) essentially shows that the corresponding ideal $I_v \subset O[\varepsilon] = O \oplus \varepsilon O$ is obtained as follows:

$$I_v = \{ \alpha + \varepsilon\beta \mid \alpha \in I, \beta \in v(\alpha) \},$$

where we view $v(\alpha) \in O/I$ as a coset of I in O . Consequently,

$$(21) \quad \alpha \equiv -\varepsilon\beta \pmod{I_v},$$

where $\alpha \in I$ and $\beta \in v(\alpha)$. We use this frequently below when making computations in M_v (such as computing $\text{tr}(g: M_v \rightarrow M_v)$ as required by Theorem 2).

APPLICATION 1. Let C be an irreducible and nonsingular curve. We apply Theorems 1 and 2 to obtain a proof of (4). We begin with the key computation.

Let $P \in C$, $O_p = O$, the local ring at P , t be a local parameter at P , $I = (t^n) \subset O$, and $Z = \text{Spec } O/I$; thus $[Z] = (nP) \in C(n) = H_C^n$. It is easy to give a k -basis of $TC(n)_{(nP)} = \text{Hom}_O(I, O/I)$; indeed, since $\{1, t, \dots, t^{n-1}\} \pmod{I}$ is a k -basis of O/I , the O -homomorphisms determined by $t^n \mapsto t^i$ ($0 \leq i \leq n-1$) will do. Hence, any $v \in TC(n)_{(nP)}$ is determined by a unique choice of n scalars $a_i \in k$, thus:

$$(22) \quad v(t^n) = (a_0 + a_1t + \dots + a_{n-1}t^{n-1}) \in O/I.$$

We now compute $(t^r dt)_{(n)}(v)$ for $0 \leq r \leq n-1$. By Theorem 1,

$$(t^r dt)_{(n)}(v) = dt_{(n)} \circ \widehat{t^r}(v) = dt_{(n)}(t^r \cdot v),$$

where $t^r \cdot v = v_r \in \text{Hom}_O(I, O/I)$ is given by

$$(23) \quad v_r(t^n) = (a_0t^r + a_1t^{r+1} + \dots + a_{n-r-1}t^{n-1}) \in O/I,$$

obtained by multiplying (22) through by t^r . By Theorem 2,

$$dt_{(n)}(v_r) = C_\varepsilon(\text{tr}(t: M_{v_r} \rightarrow M_{v_r}));$$

to compute the trace, we use the $k[\varepsilon]$ -basis of $M_{v_r} = O[\varepsilon]/I_{v_r}$ given by $\{1, t, \dots, t^{n-1}\}$ (Nakayama's lemma). We have that

$$t \cdot t^i = t^{i+1} \quad \text{for } 0 \leq i \leq n-2, \quad \text{and} \\ t \cdot t^{n-1} = t^n = -\varepsilon(a_0t^r + a_1t^{r+1} + \dots + a_{n-r-1}t^{n-1}),$$

where the last equality results from (21) and (23). The trace is $-\varepsilon \cdot a_{n-r-1}$, which has ε -coefficient $-a_{n-r-1}$; consequently, for v as in (22),

$$(24) \quad (t^r dt)_{(n)}(v) = dt_{(n)}(v_r) = -a_{n-r-1}, \quad 0 \leq r \leq n-1.$$

We may now prove (4); in view of Proposition 5, it is sufficient to prove the following statement:

$$(25) \quad \omega_{(n)} \text{ is zero at } (nP) \in C(n) \Leftrightarrow (\omega) > nP, \text{ where } (\omega) \text{ is the divisor of } \omega \text{ on } C.$$

PROOF. We use the notation established above. Locally at P we write $\omega = hdt = (b_0 + b_1t + \dots + b_{n-1}t^{n-1} + h_n)dt$, where $h_n \in (t^n)$; using the linearity of $\omega \mapsto \omega_{(n)}$ and Theorem 1 we obtain

$$\omega_{(n)} = \sum_{i=0}^{n-1} b_i(t^i dt)_{(n)} + dt_{(n)} \circ \hat{h}_n \quad \text{locally at } (nP).$$

Now choose $v \in TC(n)_{(nP)} = \text{Hom}_O(I, O/I)$ as in (22). Then

$$\begin{aligned} \omega_{(n)}(v) &= \sum_{i=0}^{n-1} b_i(t^i dt)_{(n)}(v) + dt_{(n)} \circ \hat{h}_n(v) \\ &= - \sum_{i=0}^{n-1} b_i \cdot a_{n-i-1} + 0, \end{aligned}$$

where the first term results from (24) and the second term is zero since multiplication by $h_n \in (t^n) = I$ kills the tangent space (as in Corollary 2 of Theorem 1). Therefore,

$$\begin{aligned} \omega_{(n)} \text{ is zero at } (nP) \in C(n) &\Leftrightarrow \omega_{(n)}(v) = 0 \quad \text{for all } v \in TC(n)_{(nP)} \\ &\Leftrightarrow - \sum_{i=0}^{n-1} b_i \cdot a_{n-i-1} = 0 \quad \text{for all choices of the } a \text{'s} \\ &\Leftrightarrow b_0 = b_1 = \dots = b_{n-1} = 0 \\ &\Leftrightarrow h \in (t^n) \\ &\Leftrightarrow (\omega) > nP \quad \text{on } C, \end{aligned}$$

which completes the proof. \square

We set the stage for our next application by proving that if C is a complete nonsingular curve of genus g and A is its Albanese (= Jacobian) variety, then

(26) for $n \geq 2g - 1$, the map $\varphi_n = f_n: C(n) \rightarrow A$ is smooth.

Indeed, by (2), it is enough to show that the pullback $\varphi_n^*(\omega_A)$ has no zeros on $C(n)$, where ω_A is an arbitrary nonzero holomorphic 1-form on A . But by (3), $\varphi_n^*(\omega_A) = \omega_{(n)}$ is the symmetrization of the nonzero 1-form $\omega = f^*(\omega_A)$ on C , and by (4), $\omega_{(n)}$ is zero at $(\sum_{i=1}^r n_i P_i) \in C(n) \Leftrightarrow (\omega) > \sum_{i=1}^r n_i P_i$. But (ω) has degree $2g - 2$, so for $n \geq 2g - 1$, $\omega_{(n)}$ can have no zeros, and (26) follows. \square

APPLICATION 2. We show that no statement analogous to (26) can hold for the map $\varphi_n: H_X^n \rightarrow A$ for surfaces X : in general, φ_n will not be smooth for $n \gg 0$. More precisely, we have

THEOREM 3. *Let X be an irreducible and nonsingular surface for which the Albanese mapping $f: X \rightarrow A$ is not smooth. Then for $n = r(r + 1)/2$, $r > 0$, the mapping $\varphi_n: H_X^n \rightarrow A$ is not smooth.*

REMARK. The hypothesis holds, for example, whenever $\dim A \geq 3$.

The proof of Theorem 3 is based on a calculation requiring only Theorem 1, which we state as Proposition 6. Let X be an irreducible and nonsingular surface, ω a holomorphic 1-form on X , and x, y local parameters at $P \in X$. Note that M_P^r has

colength $r(r + 1)/2$ in the local ring O_P (since the classes (mod M_P^r) of $1, x, y, x^2, \dots, x^{r-1}, x^{r-2}y, \dots, y^{r-1}$ form a basis of O_P/M_P^r over the ground field k), and let $Z_r = \text{Spec } O_P/M_P^r$, a 0-dimensional closed subscheme of length $r(r + 1)/2$ concentrated at P .

PROPOSITION 6. *If ω is zero at $P \in X$, then $\omega_{[n]}$ is zero at $[Z_r] \in H_X^n$ for all $r > 0$, $n = r(r + 1)/2$.*

PROOF. Since ω is zero at P , we may write $\omega = h_1 dx + h_2 dy$ locally at P , with $h_1, h_2 \in M_P$. By Theorem 1

$$\omega_{[n]} = dx_{[n]} \circ \hat{h}_1 + dy_{[n]} \circ \hat{h}_2$$

in a neighborhood of $[Z_r] \in H_X^n$; it therefore suffices to prove that \hat{h}_1 and \hat{h}_2 are each the zero map on $TH_{[Z_r]} = \text{Hom}_{O_P}(M_P^r, O_P/M_P^r)$, which follows from the

LEMMA. *For any $h \in M_P$ and $r > 0$, multiplication by h on $\text{Hom}_{O_P}(M_P^r, O_P/M_P^r)$ is the zero map.*

PROOF. In effect, we must prove that the image of any $v \in \text{Hom}_{O_P}(M_P^r, O_P/M_P^r)$ lies in M_P^{r-1}/M_P^r , or equivalently, that

$$\text{Hom}_{O_P}(M_P^r, M_P^{r-1}/M_P^r) = \text{Hom}_{O_P}(M_P^r, O_P/M_P^r).$$

Since the LHS is a subspace of the RHS, it suffices to prove that $\dim_k(\text{LHS}) = 2n$, which is the dimension of the RHS.

To do this, note that M_P^r has $r + 1$ generators $s_0 = x^r, s_1 = x^{r-1}y, \dots, s_r = y^r$, which give a k -basis of M_P^r/M_P^{r+1} , and likewise M_P^{r-1}/M_P^r has a k -basis of r elements given by $t_0 = x^{r-1}, t_1 = x^{r-2}y, \dots, t_{r-1} = y^{r-1}$; moreover, $r(r + 1) = w(r(r + 1)/2) = 2n$. One checks easily that given any choice of $2n$ scalars $a_{ij} \in k$ ($0 \leq i \leq r, 0 \leq j \leq r - 1$), we get a well-defined O_P -linear map $v: M_P^r \rightarrow M_P^{r-1}/M_P^r$ by setting $v(s_i) = \sum_{j=0}^{r-1} a_{ij} t_j, 0 \leq i \leq r$. Thus $\dim_k(\text{LHS}) = 2n$, which proves the lemma and the proposition. \square

PROOF OF THEOREM 3. By hypothesis, $f: X \rightarrow A$ is not smooth, say at the point $P \in X$; by (2) applied to $f = \varphi_1$, this means that for some nonzero 1-form ω_A on A , the pullback $f^*(\omega_A) = \omega$ has a zero at P . Let $Z_r = \text{Spec } O_P/M_P^r$ as above. Then (2), (3), and Proposition 6 combine to prove that

$$(27) [Z_r] \in H_X^n \text{ is a singularity of } \varphi_n, \text{ where } n = r(r + 1)/2, r > 0.$$

Letting $r = 1, 2, 3, \dots$, we obtain the desired conclusion. \square

REMARK. It is shown in [6, Theorem 3] that for $n \gg 0$, all the fibers of φ_n have the minimal dimension $2n - q$. It follows that φ_n is flat for $n \gg 0$, since any dominating map from a Cohen-Macaulay (in particular, nonsingular) variety to a nonsingular variety is flat wherever the fibers have the minimal dimension [1, V, 3.5]. Since the singularities of a flat map are singular points of their (scheme-theoretic) fibers [1, VII, 1.8], we see that the singularities of φ_n for $n \gg 0$ (for example, those given by (27)) are singular points of their fibers.

According to Theorem 3

(28) $\varphi_n: H_X^n \rightarrow A$ is smooth for $n \gg 0 \Rightarrow$ the Albanese mapping $f: X \rightarrow A$ is smooth, where X is an irreducible and nonsingular surface.

Our next goal is to prove the converse of (28) holds if the base field is $k = \mathbb{C}$ (Theorem 4). In characteristic $p > 0$, the converse is *false*; a counterexample will be given in Application 3 below. For the proof of Theorem 4 we will need

PROPOSITION 7. *Let X be an irreducible and nonsingular surface in characteristic 0, ω a holomorphic 1-form on X , and Z a 0-dimensional closed subscheme of length n on X . If $\omega_{[n]}$ is zero at $[Z] \in H_X^n$, then ω is zero on $\text{Supp } Z \subset X$.*

PROOF. In view of Proposition 5, we may assume that $\text{Supp } Z = \{P\} \subset X$.

We use the theory of induced differentials developed in [11 and 14]. Given any map $d: S \rightarrow X(n)$ having domain a nonsingular variety, the symmetrized differential $\omega_{(n)}$ induces a holomorphic differential denoted $d^*(\omega_{(n)})$ on S (which is defined even if the image of S lies in the singular locus of $X(n)$). The induced differential behaves functorially. If d is dominating, then the induced differential is just the usual pullback of $\omega_{(n)}$ as a rational differential.

Consider the diagonal map $d = d_n: X \rightarrow X(n)$ given by $d(Q) = (nQ)$ for all $Q \in X$; by factoring d through X^n in the obvious way and using the functorial behavior of the induced differential, we see that $d^*(\omega_{(n)}) = n \cdot \omega$ on X .

Now consider the diagram

$$\begin{array}{ccc} & & H_X^n \\ & \nearrow L & \downarrow \sigma_n \\ X & \xrightarrow{d} & X(n) \end{array}$$

where the dashed arrow has yet to be defined. If there exists a dashed arrow L defined in a neighborhood of P , which maps P to $[Z]$ and renders the diagram commutative (where defined), we can conclude that

$$n\omega = d^*(\omega_{(n)}) = L^*\sigma_n^*(\omega_{(n)}) = L^*(\omega_{[n]})$$

holds near P . But by hypothesis $\omega_{[n]}$ is zero at $[Z]$, hence its pullback $n \cdot \omega$ is zero at the preimage of $[Z]$, which is P . Since the characteristic is 0, it follows that ω is zero at P , as desired.

To complete the proof, therefore, it suffices to construct a lifting $L: U \rightarrow H$ as described above, defined on a neighborhood U of $P \in X$. By the universal property of H , L will correspond to a family of subschemes of X , flat and finite of length n over U , such that the subscheme over $Q \in U$ is supported only at Q (to ensure commutativity of the diagram), and such that the subscheme lying over P is Z .

First consider the case $X = \mathbb{A}^2 = \text{Spec } k[x, y]$ is the affine plane, $P = (0, 0)$, and Z is defined by the ideal $I = (g_1(x, y), \dots, g_r(x, y)) \subset k[x, y]$. Consider the closed subscheme $W \subset \mathbb{A}^2 \times \mathbb{A}^2$ defined by the ideal

$$I' = (g_1(x_1 - x_2, y_1 - y_2), \dots, g_r(x_1 - x_2, y_1 - y_2)),$$

where $(x_1, y_1; x_2, y_2)$ is a coordinate system on $\mathbb{A}^2 \times \mathbb{A}^2$. Map W to \mathbb{A}^2 by the second projection; we claim that in this way W is a flat and finite family of subschemes of length n over \mathbb{A}^2 .

Note that the fiber of $pr_2: W \rightarrow \mathbf{A}^2$ over (α, β) is the closed subscheme $W(\alpha, \beta)$ of the first factor \mathbf{A}^2 defined by the ideal

$$I_{(\alpha, \beta)} = (g_1(x_1 - \alpha, y_1 - \beta), \dots, g_r(x_1 - \alpha, y_1 - \beta));$$

since this is the pullback of $I = I_{(0,0)}$ by an isomorphism (translation) of \mathbf{A}^2 , we see that $W_{(\alpha, \beta)}$ is supported at (α, β) and has length n there. This proves that the support of W is the diagonal of $\mathbf{A}^2 \times \mathbf{A}^2$, hence $pr_2: W \rightarrow \mathbf{A}^2$ is finite, and since the length of $W_{(\alpha, \beta)}$ is always n , $pr_2: W \rightarrow \mathbf{A}^2$ is flat as well, by [13, Proposition 8, p. 44]. Finally, since $W_{(0,0)} = Z$, the lifting $L: \mathbf{A}^2 \rightarrow H_A^n$ exists as required in this case.

In the general case, since X is nonsingular, there exists a rational map $\beta: X \rightarrow \mathbf{A}^2$ which carries P to $(0, 0)$ and is étale in some neighborhood U of P . Let Z' be the closed subscheme of length n supported at $(0, 0) \in \mathbf{A}^2$ that corresponds to Z under the local analytic isomorphism $\hat{O}_{\mathbf{A}^2, (0,0)} \xrightarrow{\sim} \hat{O}_{X,P}$. Form the closed subscheme $W \subseteq \mathbf{A}^2 \times \mathbf{A}^2$ as above, starting with Z' at $(0, 0)$, and let W^* be the pullback:

$$\begin{array}{ccc} W^* & \xrightarrow{(\beta \times \beta)^*} & W \\ \downarrow & \square & \downarrow \\ U \times U & \xrightarrow{\beta \times \beta} & \mathbf{A}^2 \times \mathbf{A}^2 \end{array}$$

The support of W^* is the pullback of the diagonal in $\mathbf{A}^2 \times \mathbf{A}^2$; since $(\beta \times \beta)^*: \text{Supp } W^* \rightarrow \text{Supp } W$ is étale, we have that $\text{Supp } W^*$ is nonsingular, and therefore is the disjoint union of its connected (irreducible) components, one of which is the diagonal of $U \times U$. Moreover, W^* has no embedded components, since it is étale over W and W has no embedded components (being flat over \mathbf{A}^2). Thus W^* is the disjoint union of its minimal primary components, each of which is étale over W .

Let W_L be the primary component of W^* supported on the diagonal of $U \times U$, and map W_L to U by the second projection. As above it follows that $pr_2: W_L \rightarrow U$ is finite. To see $pr_2: W_L \rightarrow U$ is flat, we must check that the fibers all have the same length n , but this follows since these fibers are étale over the corresponding fibers of $pr_2: W \rightarrow \mathbf{A}^2$, all of which have length n . Finally, by construction, the fiber of W_L over P is Z , so the lifting L exists as required in the general case, and the proposition is proved. \square

THEOREM 4. *Let X be an irreducible and nonsingular surface/ \mathbf{C} , with $q = \dim A > 0$. Then the following are equivalent:*

- (i) $\varphi_n: H_X^n \rightarrow A$ is smooth for $n \gg 0$;
- (ii) the Albanese mapping $\varphi_1 = f: X \rightarrow A$ is smooth;
- (iii) every nonzero holomorphic 1-form $\omega = f^*(\omega_A)$ has no zeros on X ($0 \neq \omega_A \in \Gamma(\Omega_A^1)$);
- (iv) $\varphi_n: H_X^n \rightarrow A$ is smooth for all $n > 0$.

PROOF. (i) \Rightarrow (ii). This is (28).

(ii) \Leftrightarrow (iii). This follows from (2) applied to $\varphi_1 = f: X \rightarrow A$.

(iii) \Rightarrow (iv). Suppose (iii) holds, and that for some n , $\varphi_n: H_X^n \rightarrow A$ has a singularity at $[Z]$. Then, by (2), $\varphi_n^*(\omega_A)$ is zero at $[Z]$ for some nonzero holomorphic 1-form ω_A

on A , and by (3), $\varphi_n^*(\omega_A) = \omega_{[n]}$, where $\omega = f^*(\omega_A)$. Proposition 7 implies that ω is zero on $\text{Supp } Z$, but (iii) says that ω can have no zeros, contradiction.

(iv) \Rightarrow (i). Clear. \square

REMARK. If X is complete, then (as remarked in §1) $f^*: \Gamma(\Omega_A^1) \rightarrow \Gamma(\Omega_X^1)$ will be an isomorphism. Then (iii) reads: every nonzero holomorphic 1-form ω on X has no zeros on X .

The class of surfaces satisfying the conditions of the theorem is quite restricted:

PROPOSITION 8. *Let X be a complete nonsingular surface/ \mathbb{C} with irregularity $q > 0$, satisfying the conditions (i)–(iv) of Theorem 4. Then either X is an abelian surface or X has the invariants $q = 1, p_g = 0$.*

REMARK. Clearly every abelian surface/ \mathbb{C} will satisfy the conditions of the theorem. Examples of surfaces satisfying (i)–(iv) in the latter class include $X = \mathbb{P}^1 \times E$, where E is an elliptic curve, and $X = E(2)$ (use (4) to show (iii) holds for $E(2)$).

PROOF. (I am indebted to the referee for this argument.) By (ii), $f: X \rightarrow A$ is smooth, therefore $0 < q \leq 2$. If $q = 2$, then X is an étale cover of an abelian surface, and consequently is itself an abelian surface [12, p. 167]. If $q = 1$, then a nowhere zero 1-form ω on X gives an exact sequence

$$\begin{array}{ccccccc} 0 & \rightarrow & \mathcal{O}_X & \rightarrow & \Omega_X^1 & \rightarrow & K_X & \rightarrow & 0 \\ & & f & \mapsto & f \cdot \omega & & & & \\ & & & & h & \mapsto & \omega \wedge h & & \end{array}$$

of sheaves on X . From the long exact cohomology sequence, duality, and Hodge theory, one finds $h^{1,1} = 2 - 2p_g > 0 \Rightarrow p_g = 0$. \square

APPLICATION 3. Theorem 4 implies that $\varphi_n: H_A^n \rightarrow A$ is smooth for all $n > 0$ if A is an abelian surface/ \mathbb{C} . In this application we study φ_n for abelian surfaces defined over algebraically closed fields k of any characteristic. We give an easy proof that φ_n is smooth for n prime to the characteristic, and find a counterexample to smoothness for all n .

THEOREM 5. *Let A be an abelian surface. If $\text{char } k = p \nmid n$, then the map $\varphi_n: H_A^n \rightarrow A$ is smooth.*

PROOF. Let $[Z] \in H_A^n = H$ and let $\varphi_n([Z]) = a \in A$; by (1), we must prove that the differential map $D\varphi_n: TH_{[Z]} \rightarrow TA_a$ is surjective. To do this, we first observe that A acts on H_A^n in a natural way: translation by $b \in A$ on H_A^n carries $[Z]$ to $[Z_b]$, where $[Z_b]$ is defined by pullback as shown:

$$\begin{array}{ccc} Z_b & \xrightarrow{\quad} & Z \\ \downarrow & \square & \downarrow \\ A & \xrightarrow{\quad} & A \\ c & \xrightarrow{\quad} & c - b \end{array}$$

(For details concerning this action, see [6, Part 2].) In particular, if $\sigma_n([Z]) = (a_1 + \dots + a_n) \in A(n)$, then $\sigma_n([Z_b]) = ((a_1 + b) + \dots + (a_n + b))$ ($(a_i + b)$ is the sum in A); consequently

$$\varphi_n([Z_b]) = \varphi_n([Z]) + nb = a + nb.$$

Consider now the following commutative diagrams of maps and associated differential maps

$$\begin{array}{ccc}
 & H_A^n & TH_{[Z]} \\
 \tau \nearrow & \downarrow \varphi_n & \\
 A & \rightarrow & A \\
 b \mapsto & a + nb & \\
 & & \begin{array}{ccc}
 & TA_e & \rightarrow & TA_a \\
 D\tau \nearrow & & & \downarrow D\varphi_n
 \end{array}
 \end{array}$$

where $e \in A$ is the identity and τ is the map $b \mapsto [Z_b]$ (whose image is the orbit of $[Z]$). Since the isogeny of multiplication by n on A is étale whenever $p \nmid n$ (our hypothesis), we have that the horizontal arrow in the right-hand digram is surjective. But this forces the vertical arrow $D\varphi_n$ to be surjective as well, which proves the theorem. \square

Let $\text{char } k = p > 0$. We will show that for any (irregular) irreducible and nonsingular surface X and any $n = m^2 \equiv 0 \pmod{p}$, the map $\varphi_n: H_X^n \rightarrow A$ has singularities (Theorem 6). To produce these singularities, we need the next proposition, whose proof is an explicit computation based on Theorems 1 and 2.

Let $O_p = O$ be the local ring of $P \in X$, x, y local parameters at P , $I = (x^m, y^m) \subset O$, where $p \mid m$, and $Z = \text{Spec } O/I$. The colength of I is $n = m^2$, as may be seen by counting the monomials $x^i y^j$ ($0 \leq i, j \leq m - 1$), whose residues (mod I) form a k -basis of O/I . The point $[Z] \in H_X^n = H$ has a remarkable property, given by

PROPOSITION 9. *The symmetrization $\omega_{[n]}$ of any holomorphic 1-form ω on X is zero at $[Z] = [\text{Spec } O_p/(x^m, y^m)]$, where $n = m^2 \equiv 0 \pmod{p}$.*

PROOF. Since $\omega = h_1 dx + h_2 dy$ locally at P , its symmetrization $\omega_{[n]}$ is equal to $dx_{[n]} \circ \hat{h}_1 + dy_{[n]} \circ \hat{h}_2$ near $[Z]$ by Theorem 1, so it suffices to prove that $dx_{[n]}$ and $dy_{[n]}$ are zero at P ; by symmetry, it is enough to do this for $dx_{[n]}$. We will use Theorem 2 to show explicitly that $dx_{[n]}(v) = 0$ for all $v \in TH_{[Z]}$.

We first note that a basis of $TH_{[Z]} = \text{Hom}_O(I, O/I)$ is given by the O -homomorphisms determined by $x^m \mapsto x^{r_1} y^{s_1}$ and $y^m \mapsto x^{r_2} y^{s_2}$ ($0 \leq r_1, s_1, r_2, s_2 \leq m - 1$) (count dimensions); hence, any $v \in TH_{[Z]}$ is uniquely specified by a choice of $2n$ scalars $a_{r,s}, b_{r,s} \in k$, thus (Σ over $0 \leq r, s \leq m - 1$):

$$\begin{aligned}
 (29) \quad & v(x^m) = \sum a_{r,s} x^r y^s \in O/I, \\
 & v(y^m) = \sum b_{r,s} x^r y^s \in O/I.
 \end{aligned}$$

By Theorem 2, $dx_{[n]}(v) = C_\varepsilon(\text{tr}(x: M_v \rightarrow M_v))$; to compute the trace, we use the $k[\varepsilon]$ -basis $\{x^i y^j \mid 0 \leq i, j \leq m - 1 \pmod{I_v}\}$ of $M_v = O[\varepsilon]/I_v$ (Nakayama's lemma again). We have that

$$x \cdot (x^i y^j) = x^{i+1} y^j \quad \text{for } 0 \leq i \leq m - 2, 0 \leq j \leq m - 1,$$

which contributes nothing to the trace, and

$$x \cdot (x^{m-1} y^j) = x^m y^j = -\varepsilon \left(\sum a_{r,s} x^r y^s \right) \cdot y^j = -\varepsilon \left(\sum a_{r,s} x^r y^{s+j} \right),$$

where the next-to-last equality results from (21) and (29). Whenever $s + j \geq m$, we have that $y^{s+j} = -\varepsilon \left(\sum b_{r',s'} x^{r'} y^{s'} \right) \cdot y^{s+j-m}$ (again by (21) and (29)); since $\varepsilon^2 = 0$, the terms where $s + j \geq m$ drop out of the sum. From the remaining terms, we get a

contribution to the trace when $x^{m-1}y^j = x^r y^{s+j}$ ($\Rightarrow r = m - 1$ and $s = 0$), and this contribution is $-\epsilon a_{m-1,0}$. Since we get this contribution m times ($0 \leq j \leq m - 1$), the trace equals $-m\epsilon a_{m-1,0}$, which has ϵ -coefficient $-ma_{m-1,0}$. But this is zero, since we have assumed that $p \mid m$, so the proposition is proved. \square

THEOREM 6. *Let X be any irreducible and nonsingular irregular surface defined over an algebraically closed field k of characteristic $p > 0$. Then for any $n = m^2 \equiv 0 \pmod{p}$, the mapping $\varphi_n: H_X^n \rightarrow A$ has singularities.*

PROOF. Let P be any point of X , and let $Z = \text{Spec } O_P/(x^m, y^m)$ as above. Then (2), (3), and Proposition 9 imply immediately that

(30) $[Z] = [\text{Spec } O_P/(x^m, y^m)]$ is a singularity of the mapping $\varphi_n: H_X^n \rightarrow A$, where $n = m^2 \equiv 0 \pmod{p}$, whence the theorem. \square

Specializing to the case of an abelian surface $X = A$, we conclude that $\varphi_n: H_A^n \rightarrow A$ is not smooth for n as in the theorem; this gives the counterexample to smoothness of φ_n for all n promised above. It also gives a counterexample to the converse of (28) in characteristic p : the Albanese mapping $f: A \rightarrow A$ is smooth, but φ_n is not smooth for $n \gg 0$. But there is more to say. We first note that every fiber of φ_n has singularities of the type given by (30): given $a \in A$, choose $c \in A$ such that $nc = a$ (divisibility of A), and form the subscheme $Z = \text{Spec } O_c/(x_m, y_m)$ as above; then $[Z]$ is a singularity of φ_n lying over $\varphi_n([Z]) = nc = a$. This exemplifies the well-known failure of Sard's theorem in char p (every $a \in A$ is a "critical value" of φ_n).

Furthermore, since the fibers of φ_n all have the minimal dimension $2n - q$ [6, Theorem 4], our earlier argument (remark following proof of Theorem 3) shows that the singularities of φ_n are singular points of their fibers. By contrast, it is proved in [6, Theorem 6, Corollary 1] that if $p \nmid n$, then the fibers of $\varphi_n: H_A^n \rightarrow A$ are nonsingular varieties (this follows from Theorem 5) which desingularize the corresponding fibers of $f_n: A(n) \rightarrow A$; the case where $p \mid n$ is not settled by the method used there. We now see that this desingularization fails if $p \mid n$.

APPLICATION 4. Our goal here is to formulate and prove a natural generalization of (4) for curves to the case of surfaces which appears as Theorem 8 below. We first rephrase (25), the local version of (4), in the language of jets.

Let Y be a nonsingular variety of dimension γ , and $\Omega_{Y,y}^1$ the $O_{Y,y}$ -module of germs of holomorphic 1-forms at $y \in Y$. We say that two holomorphic 1-forms ω_1 and ω_2 on Y have the same m -jet at $y \Leftrightarrow$ the germ of $\omega_1 - \omega_2 \in M_y^{m+1} \cdot \Omega_{Y,y}^1$. In other words, if we write $\omega_1 = h_{1,1} dx_1 + \dots + h_{1,r} dx_r$ and $\omega_2 = h_{2,1} dx_1 + \dots + h_{2,r} dx_r$ using the local parameters x_1, \dots, x_r at y , then the power series expansions of $h_{1,i}$ and $h_{2,i}$ agree to order m for $1 \leq i \leq r$. In this language, we see that (25) is equivalent to the following statement:

(31) $\omega_{(n)}$ is zero at $(nP) \in C(n) \Leftrightarrow$ the $(n - 1)$ -jet of ω is zero at $P \in C$, where C is an irreducible and nonsingular curve and ω is a holomorphic 1-form on C .

We seek to generalize (31) to surfaces; the first step is another explicit computation using Theorems 1 and 2, which we state as Proposition 10. Let X be an irreducible and nonsingular surface, $O_P = O$ the local ring at $P \in X$, x, y local parameters at P , $I = (x^n, y) \subset O$, and $Z = \text{Spec } O/I$; thus Z has length n

$(\{1, x, \dots, x^{n-1}\} \pmod I)$ is a basis of O/I and is concentrated at P . Let $\omega = h_1 dx + h_2 dy$ be a holomorphic 1-form on X . Expand h_1 and h_2 in power series ($\Sigma = \Sigma_{0 \leq k, l}$):

$$h_1 = \sum b_{k,l} x^k y^l, \quad h_2 = \sum c_{k,l} x^k y^l.$$

PROPOSITION 10. *The symmetrized differential $\omega_{[n]}$ is zero at $[Z] \in H_X^n \Leftrightarrow b_{0,0} = b_{1,0} = \dots = b_{n-1,0} = 0$ and $n \cdot c_{0,0} = 0$ ($\Rightarrow c_{0,0} = 0$ if $\text{char } k = p \nmid n$).*

PROOF. We first note that a basis of $TH_{[Z]} = \text{Hom}_O(I, O/I)$ is given by the O -homomorphisms determined by $x^n \mapsto x^i$ and $y \mapsto x^j$, where $0 \leq i, j \leq n-1$ (count dimensions); hence, any element $v \in TH_{[Z]}$ is uniquely specified by a choice of $2n$ scalars $\alpha_r, \beta_r \in k$, thus:

$$(32) \quad \begin{aligned} v(x^n) &= \sum_{r=0}^{n-1} \alpha_r x^r \in O/I, \\ v(y) &= \sum_{r=0}^{n-1} \beta_r x^r \in O/I. \end{aligned}$$

We now compute $\omega_{[n]}(v)$ in terms of the α 's, β 's, b 's, and c 's; by requiring that the result be zero for all choices of the α 's and β 's (all $v \in TH_{[Z]}$), we will obtain the conditions stated in the proposition.

By Theorem 1,

$$\begin{aligned} \omega_{[n]}(v) &= dx_{[n]} \circ \hat{h}_1(v) + dy_{[n]} \circ \hat{h}_2(v) \\ &= dx_{[n]} \left(\left(\sum b_{k,l} x^k y^l \right) \cdot v \right) + dy_{[n]} \left(\left(\sum c_{k,l} x^k y^l \right) \cdot v \right) \\ &= dx_{[n]} \left(\left(\sum_{k=0}^{n-1} b_{k,0} x^k \right) \cdot v \right) + dy_{[n]} \left(\left(\sum_{k=0}^{n-1} c_{k,0} x^k \right) \cdot v \right), \end{aligned}$$

where the terms having x -degree $\geq n$ or y -degree ≥ 1 have dropped out since $x^n \equiv y \equiv 0 \pmod I$. Let $v_1 = \left(\sum_{k=0}^{n-1} b_{k,0} x^k \right) \cdot v$ and $v_2 = \left(\sum_{k=0}^{n-1} c_{k,0} x^k \right) \cdot v$; using (32) and multiplying out, we see that v_1 and v_2 are given as O -homomorphisms by

$$(33) \quad \begin{aligned} v_1(x^n) &= \sum_{s=0}^{n-1} \left(\sum_{r+k=s} b_{k,0} \alpha_r \right) \cdot x^s \in O/I, \\ v_1(y) &= \sum_{s=0}^{n-1} \left(\sum_{r+k=s} b_{k,0} \beta_r \right) \cdot x^s \in O/I, \\ v_2(x^n) &= \sum_{s=0}^{n-1} \left(\sum_{r+k=s} c_{k,0} \alpha_r \right) x^s \in O/I, \\ v_2(y) &= \sum_{s=0}^{n-1} \left(\sum_{r+k=s} c_{k,0} \beta_r \right) x^s \in O/I. \end{aligned}$$

We now compute $dx_{[n]}(v_1)$ and $dy_{[n]}(v_2)$.

$dx_{[n]}(v_1)$. By Theorem 2, $dx_{[n]}(v_1) = C_\varepsilon(\text{tr}(x: M_{v_1} \rightarrow M_{v_1}))$; to compute the trace, we use the $k[\varepsilon]$ -basis $\{x^i \mid 0 \leq i \leq n-1\} \pmod{I_{v_1}}$ of $M_{v_1} = O[\varepsilon]/I_{v_1}$. We have that

$$x \cdot x^i = x^{i+1} \quad \text{for } 0 \leq i \leq n-2,$$

which contributes nothing to the trace, and

$$x \cdot x^{n-1} = x^n = -\varepsilon \left(\sum_{s=0}^{n-1} \left(\sum_{r+k=s} b_{k,0} \alpha_r \right) \cdot x^s \right),$$

where the last equality results from (21) and (33). The sole contribution to the trace comes from the last expression when $s = n - 1$, and this contribution has ε -coefficient

$$dx_{[n]}(v_1) = - \sum_{r+k=n-1} b_{k,0} \alpha_r.$$

$dy_{[n]}(v_2)$. By Theorem 2, $dy_{[n]}(v_2) = C_\varepsilon(\text{tr}(y: M_{v_2} \rightarrow M_{v_2}))$; we use the $k[\varepsilon]$ -basis $\{x^i \mid 0 \leq i \leq n - 1\} \pmod{I_{v_2}}$ to compute the trace.

Before beginning, we remark that

$$y \equiv -\varepsilon \left(\sum_{s=0}^{n-1} \left(\sum_{r+k=s} c_{k,0} \beta_r \right) x^s \right) \pmod{I_{v_2}}$$

by (21) and (33). For each i , $0 \leq i \leq n - 1$, we have that

$$y \cdot x^i = -\varepsilon \left(\sum_{s=0}^{n-1} \left(\sum_{r+k=s} c_{k,0} \beta_r \right) \cdot x^{i+s} \right)$$

the terms for which $i + s \geq n$ drop out of this sum since $x^n \equiv -\varepsilon$ (expression in x), and $\varepsilon^2 = 0$. Among the remaining terms, we get a contribution to the trace whenever $x^i = x^{i+s}$ ($\Rightarrow s = 0$), and this contribution is $-\varepsilon c_{0,0} \beta_0$. Since we get this contribution n times ($0 \leq i \leq n - 1$), the trace is equal to $-n \in c_{(0,0)} \beta_0$, which has ε -coefficient

$$dy_{[n]}(v_2) = -nc_{0,0} \beta_0.$$

Adding these results, we see that

$$\omega_{[n]}(v) = - \left(\sum_{r+k=n-1} b_{k,0} \alpha_r \right) - nc_{0,0} \beta_0;$$

this expression is zero for all $v \in TH_{[Z]}$

\Leftrightarrow the expression is zero for all choices of the α_r and β_0

\Leftrightarrow the coefficients are all zero: $b_{k,0} = 0$ for $0 \leq k \leq n - 1$ and $n \cdot c_{0,0} = 0$,

which proves the proposition. \square

To generalize (31) to surfaces X , it is not enough to consider the behavior of a symmetrized differential at a single point $[Z] \in H_X^n$ lying over $(nP) \in X(n)$; we must rather consider its behavior on the locus $\sigma_n^{-1}((nP))$ of all the points $[Z]$ lying over (nP) . We recall the following (remarkable) fact concerning this locus:

(34) $\sigma_n^{-1}((nP))$ is irreducible (Brianchon) and of dimension $n - 1$ [7, Chapter 5, and Theorem 2.13].

As in Proposition 10, let x, y be local parameters at $P \in X$, and $\omega = h_1 dx + h_2 dy$ be a holomorphic 1-form on X , where h_1 and h_2 have the power series expansions ($\Sigma = \sum_{0 \leq i, j}$):

$$h_1 = \sum b_{i,j} x^i y^j, \quad h_2 = \sum c_{i,j} x^i y^j.$$

Let $I_{\alpha_1, \dots, \alpha_{n-1}} = (x^n, y - \alpha_1 \cdot x - \dots - \alpha_{n-1} \cdot x^{n-1})$, an ideal of colength n in O_P for each choice of the scalars α_i , and let $Z_{\alpha_1, \dots, \alpha_{n-1}} = \text{Spec } O_P / I_{\alpha_1, \dots, \alpha_{n-1}}$. The locus G (for "generic type") of all the points $[Z_{\alpha_1, \dots, \alpha_{n-1}}]$ has dimension $n - 1$; consequently,

(35) G is an open dense subset of $\sigma_n^{-1}((nP))$.

We may now prove

THEOREM 7. *The symmetrized differential $\omega_{[n]}$ is zero at every point of $\sigma_n^{-1}((nP)) \subset H_X^n \Leftrightarrow$*

$$\begin{aligned}
 (36) \quad & b_{i,j} = c_{i,j} = 0 \quad \text{for } 0 \leq i + j \leq n - 2, \\
 & b_{n-1,0} = 0, \\
 & b_{n-2,1} + c_{n-1,0} = 0, \\
 & b_{n-3,2} + c_{n-2,1} = 0, \\
 & \vdots \\
 & b_{0,n-1} + c_{1,n-2} = 0, \\
 & c_{0,n-1} = 0.
 \end{aligned}$$

PROOF. By (35), it is enough to prove: $\omega_{[n]}$ is zero on $G \Leftrightarrow$ the conditions (36) hold. To do this, we first find the conditions imposed on the coefficients $b_{i,j}, c_{i,j}$ when $\omega_{[n]}$ is required to be zero at a particular point $[Z_{\alpha_1, \dots, \alpha_{n-1}}] \in G$ (these are polynomial relations in the b 's, c 's, and α 's). We then show that requiring these relations to hold for all choices of the α 's is equivalent to imposing the conditions (36) on ω .

For the first step, fix a point $[Z_{\alpha_1, \dots, \alpha_{n-1}}] \in G$, and make the change of coordinates

$$x = x, \quad y' = y - \alpha_1 x - \dots - \alpha_{n-1} x^{n-1}.$$

Rewriting ω in x, y' coordinates, we get

$$\begin{aligned}
 \omega &= h_1 dx + h_2 dy \\
 &= h_1 dx + h_2 \left(dy' + \left(\sum_{t=1}^{n-1} t \alpha_t x^{t-1} \right) dx \right) \\
 &= \left(h_1 + h_2 \left(\sum_{t=1}^{n-1} t \alpha_t x^{t-1} \right) \right) dx + h_2 dy';
 \end{aligned}$$

expressing h_1 and h_2 as power series in x and y' , this becomes $(\Sigma = \Sigma_{0 \leq i,j})$

$$\begin{aligned}
 (37) \quad \omega &= \left[\sum b_{i,j} x^i \left(y' + \sum_{t=1}^{n-1} \alpha_t x^t \right)^j \right. \\
 &\quad \left. + \left(\sum c_{i,j} x^i \left(y' + \sum_{t=1}^{n-1} \alpha_t x^t \right)^j \right) \cdot \left(\sum_{t=1}^{n-1} t \alpha_t x^{t-1} \right) \right] dx \\
 &\quad + \left(\sum c_{i,j} x^i \left(y' + \sum_{t=1}^{n-1} \alpha_t x^t \right)^j \right) dy'.
 \end{aligned}$$

Let β_k be the coefficient of x^k in the power series multiplying dx in (37). Since the ideal of $Z_{\alpha_1, \dots, \alpha_{n-1}}$ in x, y' coordinates is $(x^n, y') \subset O_P$, Proposition 10 implies that

(38) $\omega_{[n]}$ is zero at $[Z_{\alpha_1, \dots, \alpha_{n-1}}] \Leftrightarrow \beta_0 = \beta_1 = \dots = \beta_{n-1} = 0$ and $n \cdot c_{0,0} = 0$; although not made explicit, these are the polynomial relations we had in mind.

We now show by induction on k that requiring β_k ($0 \leq k \leq n-1$) to be zero for all choices of $\alpha_1, \dots, \alpha_{n-1}$ is equivalent to the conditions (36). First note that the "pure x " part of the dx -coefficient in (37) is given by

$$\begin{aligned} \sum_{0 \leq i, j} b_{i,j} x^i \left(\sum_{t=1}^{n-1} \alpha_t x^t \right)^j + \left(\sum_{0 \leq i, j} c_{i,j} x^i \left(\sum_{t=1}^{n-1} \alpha_t x^t \right)^j \right) \cdot \left(\sum_{t=1}^{n-1} t \alpha_t x^{t-1} \right) \\ = \sum_{0 \leq i, j} x^i \left(\sum_{t=1}^{n-1} \alpha_t x^t \right)^j \cdot \left(b_{i,j} + c_{i,j} \left(\sum_{t=1}^{n-1} t \alpha_t x^{t-1} \right) \right), \end{aligned}$$

after factoring. By the multinomial theorem, this becomes

$$(39) \quad \sum_{0 \leq i, j} \left(\sum_{s_1 + \dots + s_{n-1} = j} \binom{j}{s_1, \dots, s_{n-1}} \alpha_1^{s_1} \dots \alpha_{n-1}^{s_{n-1}} x^{(i+s_1+2s_2+\dots+(n-1)s_{n-1})} \right) \cdot \left(b_{i,j} + c_{i,j} \left(\sum_{t=1}^{n-1} t \alpha_t x^{t-1} \right) \right).$$

For a given choice of i and j in the above sum, we call $i + j$ the *weight* of the associated terms. It is then clear that β_k , the coefficient of x^k in (39), can only involve terms of weight $\leq k$. This, together with our induction hypotheses, will simplify the computations.

Step $k = 0$. By inspection of (39), we see that

$$\beta_0 = b_{0,0} + \alpha_1 \cdot c_{0,0};$$

this equals zero for all choices of $\alpha_1 \Leftrightarrow b_{0,0} = c_{0,0} = 0$, which are the conditions of weight 0 in (36).

Induction Step. Suppose that we have shown that $\beta_0 = \beta_1 = \dots = \beta_{k-1} = 0$ ($0 < k < n$), for all choices of the α 's, is equivalent to the conditions

$$(40) \quad \begin{aligned} b_{i,j} = c_{i,j} = 0 \quad \text{for } 0 \leq i + j \leq k - 2, \\ b_{k-1,0} = 0, \\ b_{k-2,1} + c_{k-1,0} = 0, \\ b_{k-3,2} + c_{k-2,1} = 0, \\ \vdots \\ b_{0,k-1} + c_{1,k-2} = 0, \\ c_{0,k-1} = 0. \end{aligned}$$

To complete the proof, we will show that if we assume $\beta_0 = \beta_1 = \dots = \beta_{k-1} = 0$ for all choices of the α 's (\Leftrightarrow (40)), then assuming in addition that $\beta_k = 0$ for all α 's is equivalent to the conditions having the same pattern as (40) with k in place of

$k - 1$. Since β_k depends only on the terms of weight $\leq k$, and since $b_{i,j} = c_{i,j} = 0$ for $i + j \leq k - 2$ by our assumption, only terms of weight k or $k - 1$ will appear in β_k . With this in mind, we read from (39) that

$$\begin{aligned}
 (41) \quad \beta_k &= \sum_{i+j=k} \left(\sum_{\substack{s_1 + \dots + s_{n-1} = j \\ i + s_1 + 2s_2 + \dots + (n-1)s_{n-1} = k}} \binom{j}{s_1, \dots, s_{n-1}} \alpha_1^{s_1} \dots \alpha_{n-1}^{s_{n-1}} (b_{i,j} + \alpha_1 c_{i,j}) \right) \\
 &+ \sum_{i+j=k-1} \left(\sum_{\substack{s_1 + \dots + s_{n-1} = j \\ i + s_1 + 2s_2 + \dots + (n-1)s_{n-1} = k}} \binom{j}{s_1, \dots, s_{n-1}} \alpha_1^{s_1} \dots \alpha_{n-1}^{s_{n-1}} (b_{i,j} + \alpha_1 c_{i,j}) \right) \\
 &+ \sum_{i+j=k-1} \left(\sum_{\substack{s_1 + \dots + s_{n-1} = j \\ i + s_1 + 2s_2 + \dots + (n-1)s_{n-1} = k-1}} \binom{j}{s_1, \dots, s_{n-1}} \alpha_1^{s_1} \dots \alpha_{n-1}^{s_{n-1}} (2\alpha_2 c_{i,j}) \right).
 \end{aligned}$$

By considering each summation in (41) separately (e.g., in the first sum, the conditions $i + j = k$, $s_1 + \dots + s_{n-1} = j$, and $i + s_1 + 2s_2 + \dots + (n - 1)s_{n-1} = k \Rightarrow s_1 = j, s_2 = \dots = s_{n-1} = 0$), we see that the above sum equals

$$\begin{aligned}
 \sum_{i+j=k} (\alpha_1^i b_{i,j} + \alpha_1^{i+1} c_{i,j}) &+ \sum_{\substack{i+j=k-1 \\ j \geq 1}} (j\alpha_1^{j-1} \alpha_2 b_{i,j} + j\alpha_1^j \alpha_2 c_{i,j}) \\
 &+ \sum_{i+j=k-1} 2\alpha_1^i \alpha_2 c_{i,j}.
 \end{aligned}$$

Regrouping this sum according to the various monomials in the α 's, we obtain

$$\begin{aligned}
 (42) \quad \beta_k &= b_{k,0} + \sum_{j=1}^k \alpha_1^j (b_{k-j,j} + c_{k-j+1,j-1}) + \alpha_1^{k+1} \cdot c_{0,k} \\
 &+ \sum_{j=0}^{k-2} ((j+1)b_{k-j-2,j+1} + (j+2)c_{k-j-1,j}) \cdot \alpha_1^j \cdot \alpha_2 \\
 &+ (k+1)c_{0,k-1} \alpha_1^{k-1} \alpha_2,
 \end{aligned}$$

where the last term is zero by (40). Since the distinct monomials in the α 's are algebraically independent, requiring β_k to be zero for all choices of the α 's is equivalent to requiring that the coefficients of the monomials all vanish, i.e.,

$$\begin{aligned}
 b_{k,0} &= 0, \\
 b_{k-1,1} + c_{k,0} &= 0, \\
 b_{k-2,2} + c_{k-1,1} &= 0, \\
 &\vdots \\
 b_{0,k} + c_{1,k-1} &= 0, \\
 c_{0,k} &= 0,
 \end{aligned}$$

which are the conditions of weight k we desired, and (at weight $k - 1$),

$$(j + 1)b_{k-j-2,j+1} + (j + 2)c_{k-j-1,j} = 0$$

for $0 \leq j \leq k - 2$. But from (40) we see that

$$b_{k-j-2,j+1} + c_{k-j-1,j} = 0$$

for $0 \leq j \leq k - 2$, requiring these equations to hold simultaneously forces $b_{k-j-2,j+1} = c_{k-j-1,j} = 0$ for $0 \leq j \leq k - 2$, since the determinant

$$\begin{vmatrix} j + 1 & j + 2 \\ 1 & 1 \end{vmatrix} = -1 \neq 0$$

(in all characteristics). In view of (40), we now have that $b_{i,j} = c_{i,j} = 0$ for $0 \leq i + j \leq k - 1$, as well as the desired conditions of weight k , so the induction step is complete and the theorem is proved. \square

REMARK. Since the expression (42) for β_k only involves α_1 and α_2 , we see *a posteriori* that $\alpha_3, \dots, \alpha_{n-1}$ are superfluous for the proof. Put another way, we have that $\omega_{[n]}$ is zero on all of $\sigma_n^{-1}((nP)) \Leftrightarrow \omega_{[n]}$ is zero at every point $[Z_{\alpha_1, \alpha_2, 0, \dots, 0}] \Leftrightarrow$ the conditions (36) hold.

The generalization of (31) to surfaces which we have been seeking is given by the following

THEOREM 8. *Let X be an irreducible and nonsingular surface and let ω be a holomorphic 1-form on X . If the $(n - 1)$ -jet of ω is zero at $P \in X$, then $\omega_{[n]}$ is zero at every point of $\sigma_n^{-1}((nP)) \subset X^n$. Moreover, if we assume in addition that ω is closed at P and that $\text{char}(k) = p \nmid n$, then the converse holds as well.*

PROOF. The first assertion follows immediately from Theorem 7:

$$\begin{aligned} \text{the } (n - 1)\text{-jet of } \omega \text{ is zero at } P &\Rightarrow \text{conditions (36) hold} \\ &\Rightarrow \omega_{[n]} \text{ is zero on } \sigma_n^{-1}((nP)). \end{aligned}$$

(ALTERNATIVE PROOF. Suppose that the $(n - 1)$ -jet of $\omega = h_1 dx + h_2 dy$ is zero at $P \in X$. Then $h_1, h_2 \in M_P^n$, and since $M_P^n \subset I$ for all ideals I of colength n in O_P [7, Lemma 1.1], we have that \hat{h}_1 and \hat{h}_2 are each the zero map on $TH_{[Z]}$ for all $[Z] \in \sigma_n^{-1}((nP))$. The assertion now follows immediately from Corollary 2 of Theorem 1.)

We now prove the second assertion. In view of Theorem 7, we must prove that under the additional hypotheses given above, the conditions (36) imply the conditions $b_{ij} = c_{ij} = 0$ for $0 \leq i + j \leq n - 1$ (which say that the $(n - 1)$ -jet of ω at P is zero). Since ω is closed ($\partial h_1 / \partial y = \partial h_2 / \partial x$), the b 's and c 's of weight $n - 1$ satisfy the relations

$$(j + 1)b_{n-j-2,j+1} - (n - j - 1)c_{n-j-1,j} = 0$$

for $0 \leq j \leq n - 2$; confronting these with the weight $n - 1$ relations

$$b_{n-j-2,j+1} + c_{n-j-1,j} = 0$$

for $0 \leq j \leq n - 2$, from (36), we conclude that $b_{n-j-2,j+1} = c_{n-j-1,j} = 0$ for $0 \leq j \leq n - 2$ (which proves what we want), since the determinant

$$\begin{vmatrix} j + 1 & -(n - j - 1) \\ 1 & 1 \end{vmatrix} = n \neq 0$$

(since we have assumed that $p \nmid n$). \square

6. Connection with generalized Weierstrass points. As noted in the Introduction, a Weierstrass point of a complete nonsingular curve C is a point $P \in C$ for which the point (gP) is a singularity of the Jacobian mapping $f_g: C(g) \rightarrow J$ (i.e. gP is a special divisor), where g is the genus of C . To bring this paper to a close, we show that (for one definition of generalized Weierstrass points) a similar interpretation is possible for the generalized Weierstrass points on a surface X , in terms of singularities of the Albanese mapping $\phi_n: H_X^n \rightarrow A$. Here we assume that X is an irreducible and nonsingular projective surface/ \mathbb{C} . The following definition of the Weierstrass points on X (which generalizes the classical definition for curves) is a special case of a general definition given in [9], which applies to varieties of any dimension and any bundles on those varieties. Let $J_p^m(\Omega_X^1)$ denote the O_p -module of m -jets of holomorphic 1-forms at $P \in X$.

DEFINITION. $P \in X$ is a Weierstrass point of order m (for the cotangent bundle Ω_X^1) \Leftrightarrow the natural map

$$j_p^m: \Gamma(\Omega_X^1) \rightarrow J_p^m(\Omega_X^1)$$

(which takes a global 1-form to its m -jet at P) is not of maximal rank.

In other words, P is a Weierstrass point of order m if more than the “expected number” of independent holomorphic 1-forms on X have zero m -jets at P . More precisely, if we let $K_p^m \subset \Gamma(\Omega_X^1)$ be the kernel of j_p^m , then

$$(43) P \in X \text{ is a Weierstrass point of order } m \Leftrightarrow$$

$$\dim_k(K_p^m) > \max\{q - (m + 1)(m + 2), 0\},$$

where $q = \dim_k(\Gamma(\Omega_X^1)) = \dim A$ is the irregularity of X , and (as is easily checked) $(m + 1)(m + 2) = \dim_k(J_p^m(\Omega_X^1))$.

Since X is projective/ \mathbb{C} , every global 1-form on X is closed, and (as previously noted) the Albanese mapping $f: X \rightarrow A$ induces by pullback an isomorphism $f^*: \Gamma(\Omega_A^1) \rightarrow \Gamma(\Omega_X^1)$ [18, §9]. Hence, by Theorem 8,

$$(44) \omega \in K_p^m \Leftrightarrow \omega_{[m+1]} \text{ is zero at every point } [Z] \in \sigma_{m+1}^{-1}((m + 1)P) \subset H_X^{m+1}.$$

But, for nonzero ω , the zeros of $\omega_{[m+1]}$ are singularities of φ_{m+1} , so (replacing $m + 1$ by n) (44) suggests a theorem relating singularities of $\varphi_n: H_X^n \rightarrow A$ and Weierstrass points of order $n - 1$ on X ; it is now simply a matter of stating this theorem precisely.

To do this, let $0 \neq \omega \in \Gamma(\Omega_X^1)$, and let $\omega_A \in \Gamma(\Omega_A^1)$ be such that $\omega = f^*(\omega_A)$. Suppose that $\omega_{[n]} (= \varphi_n^*(\omega_A))$, by (3) is zero at every point $[Z] \in \sigma_n^{-1}((nP))$ for some $P \in X$, and let $a \in A$ be the image of $\sigma_n^{-1}((nP))$ under φ_n . Then, for each $[Z] \in \varphi_n^{-1}((nP))$,

$$D\varphi_n(TH_{[Z]}) \subset \ker \omega_A \subsetneq TA_a,$$

where by $\ker \omega_A$ we mean the kernel of the covector induced by ω_A at a . Therefore,

$$\left\langle \bigcup_{[Z] \in \sigma_n^{-1}((nP))} D\varphi_n(TH_{[Z]}) \right\rangle \subset \ker \omega_A \subsetneq TA_a,$$

where $\langle \rangle$ denotes linear span. Hence, the hypothesis that $\omega_{[n]}$ is zero everywhere on $\sigma_n^{-1}((nP))$ implies not merely that each $[Z] \in \sigma_n^{-1}((nP))$ is a singularity of φ_n , but something stronger—namely, that the images of all the tangent spaces $TH_{[Z]}$ lie in the same proper subspace of codimension 1 in TA_a . It is now easy to check that

(45) the codimension μ of $\langle \bigcup_{[Z] \in \sigma_n^{-1}((nP))} D\varphi_n(TH_{[Z]}) \rangle$ in TA_a is equal to the dimension of the subspace of all $\omega \in \Gamma(\Omega_X^1)$ for which $\omega_{[n]}$ is zero everywhere on $\sigma_n^{-1}((nP))$.

We can now state

THEOREM 9. *Let X be an irreducible and nonsingular projective surface of irregularity $q > 0/\mathbb{C}$. Then $P \in X$ is a Weierstrass point of order $n - 1 \Leftrightarrow$*

$$\mu > \max\{q - n(n + 1), 0\},$$

where μ is as in (45).

PROOF. By (43), P is Weierstrass of order $n - 1 \Leftrightarrow$

$$\dim(K_P^{n-1}) > \max\{q - n(n + 1), 0\},$$

but by (44) and (45), $\dim_\kappa(K_P^{n-1}) = \mu$; the theorem follows. \square

EXAMPLE. Let C be a complete nonsingular curve of genus 2 and let A be its Jacobian. Let P be a Weierstrass point on C , which means that a basis ω_1, ω_2 of $\Gamma(\Omega_C^1)$ may be chosen such that $\text{ord}_P(\omega_1) = 0, \text{ord}_P(\omega_2) = 2$. According to the definition above, P is therefore a generalized Weierstrass point of order 1 on C , since $j_P^1: \Gamma(\Omega_C^1) \rightarrow J_P^1(\Omega_C^1)$ is not of maximal rank. (P is not a generalized Weierstrass point of any other order on C .)

Now consider the direct product $X = C \times C$, a surface of irregularity four. A basis of $\Gamma(\Omega_X^1)$ is obtained by pulling back the 1-forms ω_1 and ω_2 by each of the two projection maps. Using this basis, one easily checks that $(P, P) \in C \times C$ is a generalized Weierstrass point of order 1 (and no other order) on X .

Finally, consider the symmetric product $X' = C(2)$, a surface of irregularity two. As is well known, the Jacobian map $X' = C(2) \rightarrow A$ represents X' as the blow-up of A at a point $a \in A$; the exceptional locus L represents the unique g_2^1 carried by C , which is the canonical system. The points Q of L (one of which is $(2P)$) are generalized Weierstrass points of order 0 on X' , which, since $\dim \Gamma(\Omega_{X'}^1) = \dim J_Q^0(\Omega_{X'}^1) = 2$, is a fancy way of saying that each point of L is a zero of some global 1-form on X' . (This last follows from (2), since $C(2) \rightarrow A$ is not smooth at any point of L . Theorem 9 is trivially verified in this case.) Using the basis of $\Gamma(\Omega_{X'}^1)$ given by the symmetrizations of ω_1 and ω_2 , and the analysis of symmetrized differentials given in [10], it can be shown that $(2P)$ is not a generalized Weierstrass point of any higher order on X' . Further analysis of the generalized Weierstrass points on $C(2)$ or $C(n)$ for curves of higher genus should be possible using [10], but that is beyond the scope of this (now concluded) paper.

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