

## KRULL AND GLOBAL DIMENSIONS OF SEMIPRIME NOETHERIAN $PI$ -RINGS

BY

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**ABSTRACT.** In this paper it is shown that if  $R$  is a semiprime Noetherian  $PI$ -ring of finite global dimension, then the Krull dimension of  $R$  is less than or equal to its global dimension. The proof depends upon two preliminary results on arbitrary Noetherian  $PI$ -rings, which are of independent interest: (i) any height two prime ideal of  $R$  contains infinitely many height one prime ideals; (ii) the localization of the polynomial ring  $R[x]$  at its set of monic elements is a Jacobson ring.

**Introduction.** Let  $R$  be a Noetherian ring and let  $K \dim R$  (respectively,  $\text{gl. dim } R$ ) denote the right Krull dimension (right global dimension) of  $R$ . If  $R$  is commutative, then one consequence of the Auslander-Buchsbaum-Serre theorem is that  $K \dim R \leq \text{gl. dim } R$ . While examples [10] show that this inequality need not hold in a ring which is only right Noetherian, it remains an open question whether one always has  $K \dim R \leq \text{gl. dim } R$  in a two-sided Noetherian ring. Some instances where an affirmative answer to this question has been obtained are (1) a filtered algebra whose associated graded ring is a regular commutative Noetherian ring [19], and (2) a ring which is module-finite over its center [12]. In this paper we will extend the dimension inequality to another class of "nearly commutative" Noetherian rings by proving, in Theorem 3.2, that Krull dimension is bounded by global dimension in any semiprime Noetherian  $PI$ -ring.

A proof for the special case of an affine prime  $PI$ -ring is given in [12, Proposition 4], but the method used there does not directly generalize. Rather, the starting point of our investigations is the broader class of prime Noetherian Jacobson  $PI$ -rings. In §1 we establish a useful property of the prime spectrum of these rings and show that they satisfy the dimension inequality. In §2 we reduce the general problem to the case previously considered by constructing, for any Noetherian  $PI$ -ring of finite global dimension, a faithfully flat extension  $R\langle x \rangle \supset R$  which not only is a Jacobson ring but also has the same Krull and global dimensions as  $R$ . Finally, in §3, the preceding results together with an elementary observation on rings of quotients allow us to prove the main theorem. A key underlying tool in all of these

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developments is Theorem 1.1, which states that any height two prime ideal in a right Noetherian  $PI$ -ring contains infinitely many height one prime ideals.

We conclude this introduction with a few remarks concerning our notation and terminology. Although it is the classical definition of Krull dimension which is most often used in  $PI$ -theory, in §2 of this paper we will be concerned with arbitrary Noetherian rings. To present a unified treatment, we will therefore use the more general Krull dimension of Rentschler and Gabriel (see [8]). Another concept which will be frequently used is that of “localization”. For us, this term will always mean classical localization [26, Chapter 2], and if  $S \subset R$  is a *right denominator set*, we let  $R_S$  denote the right ring of fractions. The unmodified word “Noetherian” will indicate a left and right Noetherian ring. Nearly all of the major theorems on  $PI$ -rings can be found in [22], while [21] is a good source for basic results in homological algebra.

**1. Jacobson  $PI$ -rings and regular prime ideals.** In this section we investigate the dimension properties of prime Noetherian Jacobson  $PI$ -rings. The results obtained here will not only play an important role in the proof of the main theorem in §3, but will also motivate the developments in §2. Proposition 1.2 and its corollary show that prime Jacobson rings have an abundance of regular prime ideals (see 1.3 for the definition), while Lemma 1.5 shows that the term “regular” can also be a given valid homological interpretation. The proof (Theorem 1.7) that the dimension inequality holds in the special case of a prime Noetherian Jacobson ring is an easy consequence of these more general considerations.

We begin with a result for which the Jacobson hypothesis is superfluous. Let  $\text{Spec}(R)$  denote the set of prime ideals in a ring  $R$  and if  $P \in \text{Spec}(R)$  define the *height of  $P$* ,  $\text{ht } P$ , as the length of a maximal chain of prime ideals descending from  $P$ . It was shown in [1, Theorem 5.1] that if  $R$  is a prime Noetherian  $PI$ -ring in which the intersection of the nonzero prime ideals is nonzero, then  $\text{ht } P \leq 1$  for all  $P \in \text{Spec}(R)$ . Although it readily follows that the set of height one prime ideals is infinite whenever  $\text{K dim } R \geq 2$ , the theorem does not imply the sharper analogue of [13, p. 107]: every height two prime ideal of  $R$  contains infinitely many height one primes. Adapting the methods of [1], we now give a proof of this fact.

**THEOREM 1.1.** *Let  $R$  be a right Noetherian  $PI$ -ring and let  $P$  be a prime ideal of  $R$ . If the height of  $P$  is strictly greater than one, then  $P$  contains infinitely many height one prime ideals of  $R$ .*

**PROOF.** Taking a prime ideal  $P' \subset P$  with  $\text{ht}(P/P') = 2$  and passing to the factor ring  $R/P'$ , we immediately reduce to the case where the ring  $R$  is prime and the height of  $P$  is exactly two.

We first prove the theorem under the additional assumption that the prime ring  $R$  is integral over its center  $C$ . Letting  $\mathfrak{p} = P \cap C$  and localizing  $R$  at the central multiplicative subset  $C - \mathfrak{p}$ , we may further assume that  $C$  is local with unique maximal ideal  $\mathfrak{p}$ . For each  $c \in \mathfrak{p}$  let  $Q_c$  be a minimal prime ideal of  $cR$  which is contained in  $P$ . By Jategaonkar's principal ideal theorem [22, p. 231] each  $Q_c$  has height one, hence the containment  $Q_c \subset P$  is proper. If the set  $\{Q_c \mid c \in \mathfrak{p}\}$  is finite,

then we can conclude from [13, p. 55] that  $\mathfrak{p} \subset Q_c \cap C$  for some  $c \in \mathfrak{p}$ . This yields  $Q_c \cap C = \mathfrak{p} = P \cap C$ , which contradicts incomparability for Noetherian integral extensions [1, Theorem 3.1.ii].

In the general case let  $\hat{R}$  denote the *characteristic closure* of  $R$  as defined in [22, p. 208]. By [1, Theorem 2.4.iii]  $\hat{R}$  is a finite  $R$ -module, so by [18, Corollary 4.2] there exists a prime ideal  $\hat{P}$  in  $\hat{R}$  which lies over  $P$  and whose height is at least two. Inasmuch as  $\hat{R}$  is integral over its center [1, Theorem 2.4.i], the argument above gives an infinite set of height one prime ideals in  $\hat{R}$ , say  $\{\hat{Q}_i\}$ , each member of which is contained in  $\hat{P}$ . Now the pair  $R$  and  $\hat{R}$  also satisfy incomparability [18, Theorem 4.6], so  $Q_i = \hat{Q}_i \cap R$  is a height one prime ideal of  $R$  for each  $i$ . Further, since  $\hat{R}$  is Noetherian, only finitely many distinct prime ideals in  $\hat{R}$  can contract to a given prime ideal of  $R$ . Thus,  $\{Q_i\}$  is an infinite set of height one prime ideals contained in  $P$ .  $\square$

Theorem 1.1, it is worth remarking, not only implies the  $G$ -ring result in [1] but also leads to a more direct proof and a generalization of Lemma 5 in [23]. Another useful consequence is the following proposition.

**PROPOSITION 1.2.** *Let  $R$  be a prime Noetherian Jacobson  $PI$ -ring and let  $I$  be a nonzero ideal of  $R$ . Then for every nonnegative integer  $n$  less than or equal to the Krull dimension of  $R$ , there exists a prime ideal  $P$  in  $R$  such that  $\text{ht } P = n$  and  $P \not\supset I$ .*

**PROOF.** The proof is by induction on  $n$ . If  $n = 0$ , the result is trivial, while if  $n = 1$  it follows from our assumption that  $R$  is a Jacobson ring. So suppose that  $n > 1$  and let  $0 \subset Q_1 \subset \dots \subset Q_n$  be a saturated chain of prime ideals in  $R$  whose length is  $n$ . By Theorem 1.1 the ideal  $Q_2$  contains infinitely many height one prime ideals of  $R$ . Observe, however, that if  $\mathcal{P}$  is a family of height one prime ideals in a prime Noetherian ring, then  $\bigcap \{Q \mid Q \in \mathcal{P}\}$  is nonzero iff  $\mathcal{P}$  is finite. Thus, there exists a prime ideal  $Q'_1$  which is contained in  $Q_2$  but which does not contain  $I$ . If  $\bar{R} = R/Q'_1$ , then the way in which  $Q'_1$  was chosen shows that  $n - 1 \leq K \dim \bar{R}$ . Applying the induction hypothesis to the ring  $\bar{R}$ , we can find a  $\bar{P}_1 \in \text{Spec}(\bar{R})$  such that  $\text{ht } \bar{P}_1 = n - 1$  and  $\bar{P}_1 \not\supset \bar{I} = (I + Q'_1)/Q'_1$ . If  $P$  is any height  $n$  prime contained in the inverse image of  $\bar{P}_1$ , then  $P$  has the desired properties.  $\square$

Recall that a prime  $PI$ -ring is said to have  $PI$ -degree  $n$ , written  $\text{pid}(R) = n$ , if its central simple quotient ring has dimension  $n^2$  over its center. Following [1, §2] we call a prime ideal  $P$  in  $R$  *regular* if  $\text{pid}(R/P) = \text{pid}(R)$ , and we define the ideal  $V(R)$  by  $V(R) = \bigcap \{Q \in \text{Spec}(R) \mid \text{pid}(R/Q) < \text{pid}(R)\}$  ( $V(R) = R$ , if the set on the right is empty). As an easy corollary to Proposition 1.2, we have a partial generalization of Markov's theorem [1, Corollary 7.2] on chains of regular prime ideals in affine  $PI$ -rings.

**COROLLARY 1.3.** *If  $R$  is a prime Noetherian Jacobson  $PI$ -ring, then for every integer  $n \leq K \dim R$  there exists a regular prime ideal  $P$  in  $R$  with  $\text{ht } P = n$ .*

**PROOF.** Note that  $V(R)$  cannot be zero and apply Proposition 1.2.  $\square$

We will use the preceding corollary to reduce the dimension problem to the case of an Azumaya algebra. The proof of Lemma 1.4, the statement of which is well

known, requires little more than the facts that an Azumaya algebra is module-finite, faithfully projective, and separable over its center.

LEMMA 1.4. *If  $R$  is a Noetherian Azumaya algebra of finite global dimension, then  $K \dim R = \text{gl.dim } R$ .  $\square$*

Let  $\text{pd}_R(U)$  denote the projective dimension of a right  $R$ -module  $U$ . It will follow from Theorem 3.2 that  $\text{ht } P \leq \text{pd}_R(R/P)$  is true more generally for any localizable prime ideal in a semiprime Noetherian  $PI$ -ring.

LEMMA 1.5. *Let  $R$  be a prime Noetherian  $PI$ -ring. If  $P$  is a regular prime ideal of  $R$ , then  $\text{ht } P \leq \text{pd}_R(R/P)$ .*

PROOF. By [25] the prime  $P$  is localizable (that is,  $\mathcal{C}(P) = \{c \in R \mid \bar{c} \text{ regular in } \bar{R} = R/P\}$  is a right denominator set) and the localization  $R_P$  has unique maximal ideal  $PR_P$ . It follows from [24, Theorem 1] and Kaplansky's theorem [22, p. 36] that  $\text{gl.dim } R_P = \text{pd}_{R_P}(R_P/PR_P)$ . The regularity of  $P$  and the Artin-Procesi theorem [22, p. 70], on the other hand, imply that  $R_P$  is an Azumaya algebra. Since we may clearly assume that  $\text{pd}_R(R/P)$  is finite, Lemma 1.4 and the exactness of localization [26, p. 57] give

$$\text{ht } P \leq K \dim R_P \leq \text{gl.dim } R_P \leq \text{pd}_{R_P}(R_P/PR_P) \leq \text{pd}_R(R/P). \quad \square$$

If the inequality given in Lemma 1.5 held for every prime ideal in a Noetherian  $PI$ -ring, the dimension theorem would be a trivial consequence. Before proceeding to the proof of Theorem 1.7, we give an example of a prime ideal (nonregular, of course) for which it fails.

EXAMPLE 1.6. Let  $D$  be any commutative Noetherian domain of finite global dimension, let  $\mathfrak{m}$  be a maximal ideal in  $D$ , and let  $R$  be the subring of the  $2 \times 2$  matrix ring  $M_2(D)$  given by

$$R = \begin{pmatrix} D & D \\ \mathfrak{m} & D \end{pmatrix} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, d \in D; c \in \mathfrak{m} \right\}.$$

$R$  is clearly a prime Noetherian  $PI$ -ring, and since it is the idealizer of the maximal right ideal  $\begin{pmatrix} D & D \\ \mathfrak{m} & \mathfrak{m} \end{pmatrix}$  in  $M_2(D)$ ,  $\text{gl.dim } R = \max\{1, \text{gl.dim } D\}$  by [17, Theorem 2.9]. If  $e_{ij}$ ,  $1 \leq i, j \leq 2$ , are the usual matrix units, put  $M_1 = e_{12}R$ ,  $M_2 = e_{22}R$  and  $M = M_1 \oplus M_2$ . As can be verified directly,  $M$  is a maximal ideal in  $R$  and  $\text{ht } M = \text{ht } \mathfrak{m}$ . Now the right annihilator of  $e_{12}$  is  $e_{11}R$ , and so it follows that  $M$  is a projective right ideal of  $R$ . With appropriate choices of  $D$  and  $\mathfrak{m}$  we can thus take  $\text{ht } M$  to be any positive integer, yet  $\text{pd}_R(R/M)$  is always 1.  $\square$

THEOREM 1.7. *Let  $R$  be a prime Noetherian Jacobson  $PI$ -ring of finite global dimension. Then:*

- (i)  $K \dim R \leq \text{gl.dim } R$ ;
- (ii)  $K \dim R = \text{gl.dim } R$  if and only if there exists a regular prime ideal  $P$  in  $R$  with  $\text{ht } P = \text{gl.dim } R$ .

PROOF. If  $Q$  is any prime ideal in  $R$ , then the height of  $Q$  is necessarily finite [22, p. 233]. By Corollary 1.3 we can find a regular prime ideal  $P$  with  $\text{ht } P = \text{ht } Q$ ,

and Lemma 1.5 shows that  $\text{ht } Q \leq \text{gl.dim } R$ . Assertion (i) now follows from [8, Theorem 8.12] and Posner's theorem [22, p. 53]. The proof of (ii) is left to the reader.  $\square$

We observe that 1.3 is *not* true if  $R$  is not Jacobson. Thus, in order to establish the result in general, we must find some reduction to Jacobson rings.

Examples of prime Noetherian  $PI$ -rings with  $K \dim R < \text{gl.dim } R$  have long been known [5]. Most of these rings are semilocal, however, so one can ask if equality always holds in the Jacobson case. A slight modification of Fields' construction, unfortunately, yields an example of an affine prime Noetherian  $PI$ -ring whose Krull dimension is 1 but whose global dimension is any positive integer  $n$ .

EXAMPLE 1.8. Let  $F$  be any field, let  $F[x]$  be the ring of polynomials over  $F$ , and let  $(x)$  denote the principal ideal generated by  $x$ . Mimicking the notation in [5, §2], we put

$$S_n = M_{2^n}(F[x]), \quad I_n = xS_n,$$

$$R^1 = \begin{pmatrix} F[x] & F[x] \\ (x) & F[x] \end{pmatrix}, \quad R^{n+1} = \begin{pmatrix} R^n & R^n \\ I_n & R^n \end{pmatrix}.$$

As is easily checked,  $R^{n+1}$  is a prime Noetherian  $PI$ -ring, the center of  $R^{n+1}$  is just the set of scalar matrices over  $F[x]$ , and  $K \dim R^{n+1} = 1$ . Now let  $\mathfrak{m}$  be any maximal ideal of  $F[x]$ . If  $\mathfrak{m} = (x)$ , then one only need replace the ring of power series  $F[[x]]$  by  $F[x]_{(x)}$  in the arguments of [5, §3] to see that  $\text{gl.dim } R_{(x)}^{n+1} = n$ . If  $\mathfrak{m} \neq (x)$ , on the other hand,  $R_{\mathfrak{m}}^{n+1} \simeq M_{2^n}(F[x]_{\mathfrak{m}})$  and so has global dimension one. By [2, p. 124]

$$\text{gl.dim } R^{n+1} = \sup_{\mathfrak{m}} \{ \text{gl.dim } R_{\mathfrak{m}}^{n+1} \} = n. \quad \square$$

We also note that  $R_{(x)}^1$  is a hereditary prime Noetherian  $PI$ -ring whose only regular prime ideal is zero. Thus part (ii) of Theorem 1.7 need not be true if  $R$  is not a Jacobson ring.

**2. The monic localization of polynomial rings.** Since Quillen's proof of Serre's conjecture there has been considerable interest in the ring  $R\langle x \rangle$  obtained by localizing the polynomial extension of a commutative ring  $R$  at its set of monic elements (see, for example, [14, Chapters 4, 5; 15, 3]). In this section we shall show that the same localization exists if  $R$  is an arbitrary right Noetherian ring and establish a few of its basic properties. The main results are as follows: (1)  $K \dim R\langle x \rangle = K \dim R$ ; (2)  $\text{gl.dim } R\langle x \rangle = \text{gl.dim } R$ , if  $R$  is also left Noetherian and of finite global dimension; (3)  $R\langle x \rangle$  is Jacobson, if  $R$  is also a  $PI$ -ring. The inspiration for the last result was the recent paper of Brewer and Heinzer [3], who obtained this theorem for commutative Noetherian rings.

Our proof of property (3) will make heavy use of the fact that the variable  $x$  commutes with  $R$ , but the commuting hypothesis plays no role in the proofs of (1) and (2). The arguments being virtually the same, we will therefore establish these results in the more general setting of skew polynomial rings or Ore extensions. We

briefly recall the definition. If  $R$  is a ring,  $\sigma: R \rightarrow R$  is a ring automorphism and  $\delta: R \rightarrow R$  is a  $\sigma$ -derivation (that is,  $\delta(ab) = \delta(a)\sigma(b) + a\delta(b)$  for all  $a, b \in R$ ), then the Ore extension  $R[x; \sigma, \delta]$  is the free left  $R$ -module with basis the set  $\{x^i \mid i \in \mathbb{N}\}$  and with multiplication deduced from the rules  $x^i x^j = x^{i+j}$ ,  $xr = \sigma(r)x + \delta(r)$ . One views the elements of  $R[x; \sigma, \delta]$  as polynomials  $g = \sum_i r_i x^i$ , of course, and if  $n$  is the largest integer  $i$  for which  $r_i \neq 0$ , then  $n$  is the degree of  $g$  and  $r_n$  is its leading coefficient. To simplify both statements and proofs, we fix the following notation:  $R$  is a ring with identity element;  $S$  is an Ore extension of  $R$ ;  $\mathfrak{N} = \{f \in S \mid f \text{ monic}\}$  is the set of elements in  $S$  whose leading coefficient is 1.

In anticipation of Proposition 2.2, where it is shown that  $\mathfrak{N}$  is a right denominator set in  $S$  whenever  $R$  is a right Noetherian ring, we call a right  $S$ -module  $W$   $\mathfrak{N}$ -torsion if for every  $w \in W$  there exists an  $f \in \mathfrak{N}$  with  $wf = 0$ . Note that when  $f \in \mathfrak{N}$ ,  $S/fS$  is a finitely generated  $R$ -module by the right division algorithm (which holds, because  $\sigma$  is an automorphism). Our first proposition shows that this property comes close to characterizing the  $\mathfrak{N}$ -torsion modules.

**LEMMA 2.1.** *Let  $R$  be a ring and  $S$  an Ore extension of  $R$ . A right  $S$ -module  $W$  is  $\mathfrak{N}$ -torsion if and only if every finitely generated  $S$ -submodule of  $W$  is finitely generated as an  $R$ -module.*

**PROOF.** ( $\Rightarrow$ ) It is enough to show that every cyclic  $S$ -submodule  $wS \subset W$  is finitely generated over  $R$ . Since  $wS$  is isomorphic to a factor of  $S/fS$  for any  $f \in \mathfrak{N}$  which annihilates  $w$ , the latter assertion is clear.

( $\Leftarrow$ ) If  $wS$  is a finitely generated  $R$ -module, then it can be generated by the set  $\{w, wx, \dots, wx^n\}$  for some positive integer  $n$ .  $\square$

While the proof of the next result is implicit to that of Lemma 1.2 in [11], our argument is brief and is included for the sake of completeness.

**PROPOSITION 2.2.** *Let  $S$  be an Ore extension of  $R$ . If  $R$  is right Noetherian, then  $\mathfrak{N}$  is a right denominator set in  $S$ .*

**PROOF.** Given  $g \in S$  and  $f \in \mathfrak{N}$ , let  $J = \{h \in S \mid gh \in fS\}$  and note that the map  $s \rightarrow gs + fS$  induces an embedding of  $S/J$  into  $S/fS$ . Since  $S/fS$  is finitely generated over  $R$  and  $R$  is right Noetherian,  $S/J$  is  $\mathfrak{N}$ -torsion by Lemma 2.1. In particular  $\mathfrak{N} \cap J \neq \emptyset$ , so there exist  $f_1 \in \mathfrak{N}$  and  $g_1 \in S$  with  $gf_1 = fg_1$ . This establishes the right Ore condition for  $\mathfrak{N}$ , and the rest is obvious.  $\square$

Although the converse of Proposition 2.2 is false, the precise class of rings for which the conclusion holds is far from clear, even in the simple case  $S = R[x]$ . If  $A$  is an arbitrary ring and  $a, b \in A$ , then a necessary condition for the pair  $g = a$ ,  $f = x - b$  to satisfy the monic Ore condition is that the right ideal  $\sum_{i=0}^{\infty} b^i a A$  be finitely generated. It is not hard to construct examples of left and right Ore domains which do not satisfy the latter condition.

We turn now to the Krull and global dimensions of the localization  $S_{\mathfrak{N}}$ . We will need the following fact, the easy proof of which follows from the definitions.

**LEMMA 2.3.** *If  $R$  is a right Noetherian ring, then the extension  $S_{\mathfrak{N}}$  is left faithfully flat.*  $\square$

**THEOREM 2.4.** *If  $R$  is a right Noetherian ring, then  $K \dim S_{\mathfrak{N}} = K \dim R$ .*

**PROOF.** Let  $\mathfrak{L}(A)$  denote the lattice of right ideals in a ring  $A$ . We will prove the theorem by constructing strictly increasing functions  $\varepsilon: \mathfrak{L}(R) \rightarrow \mathfrak{L}(S_{\mathfrak{N}})$  and  $\mu: \mathfrak{L}(S_{\mathfrak{N}}) \rightarrow \mathfrak{L}(R)$  [4, p. 113].

In view of Lemma 2.3 we can let  $\varepsilon$  be the extension map,  $\varepsilon(I) = IS_{\mathfrak{N}}$ . To define  $\mu$ , we first consider the “leading coefficient map”  $\lambda: \mathfrak{L}(S) \rightarrow \mathfrak{L}(R)$ . In detail, if  $J$  is a right ideal of  $S$ , let  $\lambda_i(J)$  be the right ideal of  $R$  generated by the leading coefficients of the elements in  $J$  which have degree  $i$ . Because we have chosen a left normalization for the elements of  $S$ ,  $\lambda_i(J) \subset \lambda_{i+1}(J)$  for all  $i$  and  $\lambda(J) = \bigcup_{i=0}^{\infty} \lambda_i(J)$  is a right ideal of  $R$ . Note that the map  $\lambda$  is increasing, but not strictly so. Indeed, if  $J \subset K$  are right ideals of  $S$ , then  $\lambda(J) = \lambda(K)$  iff  $K/J$  is a finitely generated  $R$ -module. For if  $\lambda(J) = \lambda(K)$ , then  $\lambda(J) = \lambda_n(J) = \lambda_n(K) = \lambda(K)$  for some  $n \in \mathbb{N}$  since  $R$  is right Noetherian. Taking a finite set of generators  $\{g_1, \dots, g_m\}$  for the  $R$ -module  $K_{n-1} = \{g \in K \mid \deg g < n\}$  and arguing as in the proof of the Hilbert basis theorem, one sees that the set of images  $\{g_1 + J, \dots, g_m + J\}$  generates  $K/J$  as  $R$ -module. Conversely, if there exists a  $g \in K$  whose leading coefficient does not belong to  $\lambda(J)$ , then the  $R$ -submodule  $\sum_i (gx^i + J)R$  of  $K/J$  is not finitely generated. Thus, if we put  $\mu(L) = \lambda(L \cap S)$ , then Lemma 2.1 shows that  $\mu$  is strictly increasing.  $\square$

A rather infamous example of Goodearl [7, p. 317] shows that the assumption that  $\text{gl.dim } R < \infty$  cannot be omitted in the next proposition.

**THEOREM 2.5.** *Let  $R$  be a left and right Noetherian ring of finite global dimension. If  $S = R[x; \sigma, \delta]$  is an Ore extension of  $R$  with  $\sigma\delta = \delta\sigma$ , then  $\text{gl.dim } S_{\mathfrak{N}} = \text{gl.dim } R$ .*

**PROOF.** Since  $\text{gl.dim } R < \infty$  and  $S_{\mathfrak{N}}$  is a faithfully flat left  $R$ -module, we have  $\text{gl.dim } R \leq \text{gl.dim } S_{\mathfrak{N}}$  by [21, p. 250]. To obtain the reverse inequality, we argue by contradiction. If  $\text{gl.dim } R = n$ , then a modified version of the Serre exact sequence [6, Remark (iii)] shows that  $\text{gl.dim } S \leq n + 1$ , whence the global dimension of  $S_{\mathfrak{N}}$  is at most  $n + 1$ . So suppose  $\text{gl.dim } S_{\mathfrak{N}} = n + 1$ . By Auslander’s theorem [21, p. 237] we can find a right ideal  $L \subset S_{\mathfrak{N}}$  with  $\text{pd}_{S_{\mathfrak{N}}}(S_{\mathfrak{N}}/L) = n + 1$ , and if  $K = L \cap S$ , then  $\text{pd}_S(S/K)$  is also  $n + 1$ . It follows from the generalization of [20, Theorem 3.8] given in [9, Satz IV.12] that there exists a right ideal  $K_0$  in  $S$  such that  $K \subset K_0$ ,  $\text{pd}_S(K_0/K) = n + 1$  and  $K_0/K$  is a finitely generated  $R$ -module. Now  $K_0/K$  is  $\mathfrak{N}$ -torsion by Lemma 2.1, yet  $S/K$  is  $\mathfrak{N}$ -torsion free by construction. We thus obtain  $n + 1 = \text{pd}_S(K_0/K) = 0$ , which is absurd.  $\square$

The remainder of this section is devoted to proving that  $S_{\mathfrak{N}}$  is a Jacobson ring if  $R$  is a Noetherian  $PI$ -ring and  $S$  is the ring of polynomials over  $R$ . To emphasize the fact that  $S$  is not an arbitrary Ore extension of  $R$ , we abandon our previous notation and borrow that of [15]. We thus write  $R\langle x \rangle$  for the localization  $R[x]_{\mathfrak{N}}$ .

If  $\beta = gf^{-1} \in R\langle x \rangle$ , put  $d(\beta) = \deg g - \deg f$ . Using the fact that  $\mathfrak{N}$  is a right Ore set in  $R[x]$ , it is easy to see that the value of  $d$  is independent of the representation of  $\beta$ , hence that the set  $B = \{\beta \in R\langle x \rangle \mid d(\beta) \leq 0\}$  is well defined. Somewhat more tedious manipulations with the Ore condition show that  $B$  is in fact a subring of  $R\langle x \rangle$ . The relation between  $B$  and  $R\langle x \rangle$  is given by the following proposition, which is lifted almost verbatim from [14, Proposition IV.1.4].

LEMMA 2.6. *Let  $R$  be a right Noetherian ring and let  $B$  and  $R\langle x \rangle$  be as above. Put  $s = x^{-1}$  and let  $R[s]$  denote the subring of  $R\langle x \rangle$  generated by  $R$  and  $s$ . Then:*

- (i) *if  $\mathfrak{S} = \{s^n \mid n \in \mathbf{N}\}$ ,  $\mathfrak{S}$  is a right denominator set in  $B$  and  $B_{\mathfrak{S}} \simeq R\langle x \rangle$ ;*
- (ii) *if  $\mathfrak{U} = \{1 + sh \mid h \in R[s]\}$ ,  $\mathfrak{U}$  is a right denominator set in  $R[s]$  and  $R[s]_{\mathfrak{U}} \simeq B$ .*

PROOF. (i)  $\mathfrak{S}$  is clearly a denominator set in  $B$  and if  $B[s^{-1}]$  is the smallest subring of  $R\langle x \rangle$  which contains both  $B$  and  $s^{-1}$ , then the inclusion  $B \hookrightarrow B[s^{-1}]$  induces an isomorphism  $B_{\mathfrak{S}} \simeq B[s^{-1}]$ . It thus suffices to show that  $B[s^{-1}] = R\langle x \rangle$ , and one only need change the notation  $h/f$  to  $hf^{-1}$  to see that the proof of [14, IV.1.4.i] applies.

(ii) Since  $s$  is a central element of  $R[s]$ , the ideal  $sR[s]$  satisfies the Artin-Rees property [16, p. 487]. By Smith's theorem [16, p. 493]  $\mathfrak{U}$  is a right denominator set. Inasmuch as  $R[s]$  is a subring of  $B$ , proving the second assertion reduces to showing that every  $u \in \mathfrak{U}$  is a unit in  $B$  and that every  $b \in B$  can be written in the form  $b = hu^{-1}$ , where  $h \in R[s]$  and  $u \in \mathfrak{U}$ . Lam's arguments again apply [14, IV.1.4.ii].

□

The next lemma extends the characterization of commutative Noetherian Jacobson rings given in [13, p. 108] to Noetherian *PI*-rings. We call a prime ideal  $P$  in  $R$  a *G-ideal* if  $\bigcap \{Q \in \text{Spec}(R) \mid Q \supseteq P\}$  properly contains  $P$ .

LEMMA 2.7. *Let  $R$  be a right Noetherian *PI*-ring. Then the following conditions are equivalent:*

- (i)  *$R$  is a Jacobson ring;*
- (ii) *every  $P \in \text{Spec}(R)$  with  $K \dim R/P = 1$  is contained in infinitely many maximal ideals;*
- (iii) *every  $G$ -ideal in  $R$  is maximal.*

PROOF. (i)  $\Rightarrow$  (ii). This is an easy consequence of Kaplansky's theorem on primitive *PI*-rings [22, p. 36].

(ii)  $\Rightarrow$  (iii). This follows from [1, Theorem 5.1] or Theorem 1.1.

(iii)  $\Rightarrow$  (i). Apply the following observation which is due to R. Irving: If  $\text{Spec}(R)$  satisfies the ascending chain condition, then every prime ideal in  $R$  is an intersection of *G*-ideals. □

THEOREM 2.8. *If  $R$  is a right Noetherian *PI*-ring, then  $R\langle x \rangle$  is a Jacobson ring.*

PROOF. We will show that  $R\langle x \rangle$  satisfies the second criterion of Lemma 2.7.

Let  $P$  be a prime ideal of  $R\langle x \rangle$  with  $K \dim R\langle x \rangle/P = 1$  and let  $N$  be any maximal ideal of  $R\langle x \rangle$  which contains  $P$ . If the ring  $B$  is defined as in Lemma 2.6, then part (i) of that proposition shows that  $Q = P \cap B$  and  $M = N \cap B$  are prime ideals of  $B$  with  $Q$  properly contained in  $M$ . We claim that  $M + sB \neq B$  (here, as in 2.6,  $s = x^{-1}$ ). Otherwise, we have  $1 = m + bs$  for some  $m \in M$ ,  $b \in B$ . By the definition of  $B$  we can write  $b = gf^{-1}$ , where  $g$  and  $f$  are elements of  $R[x]$ ,  $f$  is monic and  $\deg g \leq \deg f$ . Making the appropriate substitutions, we obtain

$$m = 1 - bs = 1 - gf^{-1}x^{-1} = (xf - g)(xf)^{-1}.$$

But  $\deg xf > \deg f \geq \deg g$ , so the polynomial  $xf - g$  is necessarily monic. The equation above thus implies that  $m$  is a unit in  $R\langle x \rangle$ , which contradicts our choice of  $N$ .

The preceding paragraph shows that the Krull dimension of  $B/Q$  is at least two, and since  $B$  is a right Noetherian ring by Lemma 2.6(ii),  $B/Q$  has infinitely many height one prime ideals by Theorem 1.1. Now  $s$  is a unit in  $R\langle x \rangle$ , so  $s \notin Q$  and only finitely many of the height one primes in  $B/Q$  can contain the image of  $s$ . It follows from Lemma 2.6(i) that  $\text{Spec}(R\langle x \rangle/P)$  is also infinite.  $\square$

We do not know whether  $R\langle x \rangle$  is Jacobson for an arbitrary right Noetherian ring, but we conclude this section with an example which shows that the statement of Lemma 2.7(ii), hence the proof of Theorem 2.8, can fail even when the ring  $R$  is fully bounded. Our construction depends on known properties of the set of prime ideals in a localization of a Noetherian ring; a detailed proof of these facts is given in [4, p. 123].

**EXAMPLE 2.9.** Let  $K$  be any field which has an automorphism  $\tau$  of infinite order and let  $R$  be the skew power series ring  $R = K[[t; \tau]]$ .  $R$  is a Noetherian domain with unique maximal right ideal  $tR$  and is obviously fully bounded. We shall show that  $tR\langle x \rangle$  is the only nonzero prime ideal in  $R\langle x \rangle$ , and for this it will be enough to show that if  $P$  is a nonzero prime ideal of  $R[x]$  which does not contain  $t$ , then  $P$  meets  $\mathfrak{N}$ . If  $t \notin P$ , then  $P \cap R = 0$  and so  $P$  survives in the localization  $R[x]_{\mathcal{C}} \simeq D[x]$ , where  $\mathcal{C} = \{t^n \mid n \geq 0\}$  and  $D = R_{\mathcal{C}}$  is the quotient ring of  $R$ . Since  $D$  is a division ring, the ideal  $PD[x]$  must contain a nonzero central element of  $D[x]$ . But, if  $k$  is the fixed ring of  $\tau$ , then  $\text{Cen}(R) = k = \text{Cen}(D)$  and so there exists  $0 \neq c \in \text{Cen}(R[x]) \cap PD[x]$ . It follows that  $ct^n \in P$  for some  $n \in \mathbb{N}$ , and our choice of  $P$  forces  $c$  to be an element of  $P$ . As  $k$  is a field, we may assume that  $c$  is monic.  $\square$

We remark that in spite of its “pathological” spectrum,  $R\langle x \rangle$  is still a Jacobson ring. Indeed, the polynomial  $tx - 1$  generates a maximal right ideal in  $R\langle x \rangle$ , so the ring is primitive.

**3. The main theorem.** In this brief section we complete the proof of the dimension inequality for semiprime Noetherian  $PI$ -rings. If the ring is prime, then the theorem is an immediate consequence of the results in §§1 and 2. The techniques used in the proof of Theorem 1.7 cannot be applied directly to the semiprime case, however. Although one can prove a version of the localization theorem [25] for semiprime  $PI$ -rings, Corollary 1.3 need not hold for semiprime Jacobson rings, as the simple example  $M_2(F) \times F[x, y]$  shows. One is thus forced to seek an indirect argument, and our basic strategy is to apply Proposition 1.2 in the appropriate prime factor ring, then lift the result back to  $R$ . The efficacy of this method depends on Lemma 3.1, which can be viewed as an easy generalization of the well-known fact that a minimal prime ideal in a semiprime Goldie ring is localizable. We let  $r(K)$  denote the right annihilator of a subset  $K \subset R$ , and leave the straightforward proof of the lemma to the reader.

**LEMMA 3.1.** *Let  $\pi: R \rightarrow \bar{R}$  be a surjective ring homomorphism with kernel  $K$ . Suppose that  $\bar{\mathcal{S}}$  is a right denominator set in  $\bar{R}$  and let  $\mathcal{S} = \pi^{-1}(\bar{\mathcal{S}})$ . If  $r(K) \cap \mathcal{S} \neq \emptyset$ , then  $\mathcal{S}$  is a right denominator set in  $R$  and  $R_{\mathcal{S}} \simeq \bar{R}_{\bar{\mathcal{S}}}$ .  $\square$*

**THEOREM 3.2.** *If  $R$  is a semiprime Noetherian  $PI$ -ring of finite global dimension, then  $K \dim R \leq \text{gl. dim } R$ .*

PROOF. Since  $R\langle x \rangle$  is also a semiprime Noetherian  $PI$ -ring, Theorems 2.4, 2.5 and 2.8 show that we may assume that  $R$  is a Jacobson ring. As in the proof of Theorem 1.7, we shall show that  $\text{ht } Q \leq \text{gl.dim } R$  for every  $Q \in \text{Spec}(R)$ .

Choose a minimal prime  $Q_0 \subset Q$  with  $\text{ht}(Q/Q_0) = \text{ht } Q$ , put  $\bar{R} = R/Q_0$  and let  $r \rightarrow \bar{r}$  denote the canonical projection. In view of Theorem 1.7 we only need consider the case where  $Q_0 \neq 0$ , and because  $R$  is semiprime, this means that  $r(Q_0)$  has nonzero image in  $\bar{R}$ . Hence, if  $V(\bar{R})$  is defined as in the paragraph preceding Corollary 1.3,  $\bar{I} = r(Q_0) \cap V(\bar{R})$  is also nonzero. Applying Proposition 1.2 to the prime Noetherian Jacobson  $PI$ -ring  $\bar{R}$ , we can find a  $\bar{P} \in \text{Spec}(\bar{R})$  such that  $\text{ht } \bar{P} = \text{ht } \bar{Q}$  but  $\bar{P} \not\supset \bar{I}$ . Since  $\bar{P} \not\supset V(\bar{R})$ ,  $\bar{P}$  is a regular prime ideal of  $\bar{R}$  and since  $\bar{P} \not\supset r(Q_0)$ ,  $r(Q_0) \cap \mathcal{C}(\bar{P}) \neq \emptyset$  ( $\bar{P}$  denotes the inverse image of  $\bar{P}$ ). From [25] and Lemma 3.1 we deduce that  $P$  is a localizable prime ideal in  $R$ . But  $R_P \simeq \bar{R}_{\bar{P}}$  is an Azumaya algebra, and the argument of Lemma 1.5 shows that  $\text{ht } P \leq \text{gl.dim } R$ . Inasmuch as  $\text{ht } Q \leq \text{ht } P$  by construction, the theorem is proved.  $\square$

As the example which follows will show, the proof of Theorem 3.2 does not adapt to the case where the nil radical  $N(R)$  is nonzero. While one can prove that  $K \dim R \leq \text{gl.dim } R$  under some very special hypotheses (if, for example,  $N(R)$  is a prime ideal), the major difficulty here is the old question of whether  $\text{gl.dim } R \geq \text{gl.dim } R/N(R)$  always holds in a right Noetherian ring.

EXAMPLE 3.3. Let  $A$  be any prime Noetherian  $PI$ -ring of finite global dimension and let  $R$  be the subring of  $M_3(A)$  given by

$$R = \left\{ \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & a \end{pmatrix} \mid a, b, c, d, e, \in A \right\}.$$

It follows from [24, Theorem 1] that the global dimension of  $R$  is at most  $\text{gl.dim } A + 2$ , but if  $P_0 = (e_{11} + e_{33})R + e_{23}R$ , then  $P_0$  is a minimal prime ideal of  $R$  with  $r(P_0) = 0$ .  $\square$

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