ON THE DIVISION OF DISTRIBUTIONS BY
ANALYTIC FUNCTIONS IN LOCALLY CONVEX SPACES

BY

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Abstract. Although the division of distributions by real polynomials and real
analytic functions (which are nonzero) is always possible in finite dimensional spaces
(from classical results of Hörmander and Lojasiewicz respectively), we show that this
is not always possible in infinite dimensional locally convex spaces. In particular, we
characterize those locally convex spaces where division is always possible.

I. Introduction. Let $S$ be a distribution and $g$ a $C^\infty$ function on an open subset $\Omega$
in a locally convex space. We say that $g$ divides $S$ if there exists a distribution $T$ on $\Omega$
so that $gT = S$.

This problem was first studied by L. Schwartz [8] who proved that the division by
a nonzero holomorphic function is always possible on a connected open subset of
$\mathbb{C}^n$. Later, L. Hörmander [6] and S. Lojasiewicz [7] respectively solved the division
convex spaces.

These works established that if $g$ does not vanish on any open subset of $\Omega$, the
division is always possible. Our purpose is to answer a natural question in Colombeau, Gay and Perrot [4] concerning division by real polynomials and real
analytic functions in infinite dimensional locally convex spaces.

We prove division is always possible by finite type real polynomials and finite
type real analytic functions which are nonzero (defined later). But the division by
general polynomials is not possible in general. We characterize the spaces where
division is always possible and will see that they form a rather limited class; in
particular, the unique infinite dimensional Fréchet space where division by nonzero
real polynomials or real analytic functions is always possible is $\mathbb{R}^N$, and the unique
Silva space is $\mathbb{R}^{(N)}$.

II. Notation. They are classical. We denote by $E$ a Hausdorff real locally convex
space and $\Omega$ an open subset of $E$. A mapping $f: \Omega \to \mathbb{R}$ is called a $C^\infty$
function if, for every convex balanced bounded subset $B$ of $E$, the restriction of $f$ to $\Omega \cap E_B$ (where
$E_B$ denotes, as usual, the vector space spanned by $B$ and normed with the gauge $\rho_B$
of $B$) is $C^\infty$ in the usual Fréchet sense of calculus in normed spaces. (This is the
We denote by $\mathcal{E}(\Omega)$ the vector space of the $C^\infty$ functions on $\Omega$. We consider on $\mathcal{E}(\Omega)$ the topology of the uniform convergence of functions and all their derivatives on the strictly compact subsets of $\Omega$ (i.e. compact in some $\Omega \cap E_B$). A basis of neighborhoods of zero in $\mathcal{E}(\Omega)$ is made of sets $V(K, L, \epsilon, \alpha) = \{ f \in \mathcal{E}(\Omega) \text{ such that } \sup_{x \in K; \eta \in L} |f^{(\alpha)}(x)h_1 \cdots h_i| \leq \epsilon \text{ if } 0 \leq i \leq \alpha \}$ where $K$ is a strictly compact subset of $\Omega$, $L$ a bounded subset of $E$, $\epsilon > 0$ and $\alpha \in \mathbb{N}$. We call here “distribution on $\Omega$” any element of the dual space $\mathcal{E}'(\Omega)$ of $\mathcal{E}(\Omega)$.

The function $g$ is called analytic on $\Omega$ if for each balanced convex bounded subset $B$ of $E$, the restriction of $f$ to $\Omega \cap E_B$ is locally the sum of a normally convergent series of $E_B$-continuous polynomials (see Colombeau [3]).

An analytic function $g$ on $\Omega$ is said to be of finite type if there exist two subspaces $E_1$ and $E_2$ of $E$ such that $E$ can be decomposed as the topological direct sum $E = E_1 \oplus E_2$ with $\dim E_1 < \infty$, and if for all $x \in \Omega$, $x = x_1 + x_2$ ($x_1 \in E_1$, $x_2 \in E_2$), we have $g(x_1) = g(x)$.

We notice that if $g$ is a polynomial on $E$, this definition coincides with the usual one for finite type continuous polynomials. We denote by $p_1$ and $p_2$ the two projections:

$$p_1 : E_1 \oplus E_2 \to E_1 \quad \text{and} \quad p_2 : E_1 \oplus E_2 \to E_2.$$ 

If $V$ is a convex balanced neighborhood of zero, we denote by $\rho_V$ or $\| \cdot \|_V$ the gauge of $V$, $E_V = E / \rho_V^*(0)$ and $s_V : E \to E / \rho_V^*(0)$ the canonical surjection map.

The notation $\tilde{V}$, $\tilde{B}, V(K, L, \epsilon, \alpha)$ will be used for the polars of $V$, $B, V(K, L, \epsilon, \alpha)$ respectively.

### III. Division by finite type analytic functions.

**Theorem 1.** Let $E$ be a Hausdorff real locally convex space, $\Omega$ an open subset of $E$ and $g$ a locally finite type analytic function on $\Omega$ which does not vanish identically on any open subset. Then, for every $S \in \mathcal{E}'(\Omega)$ there exists a $T \in \mathcal{E}'(\Omega)$ such that $S = gT$.

**Proof.** Let us consider the map

$$u : \mathcal{E}(\Omega) \to \mathcal{E}(\Omega), \quad f \mapsto gf.$$ 

We remark that this mapping is injective because $g$ does not vanish on any open subset of $\Omega$. Therefore, it suffices to prove that $u$ is open. Indeed, if $S \in \mathcal{E}'(\Omega)$ the linear form

$$gS(\Omega) \to \mathcal{E}(\Omega) \to \mathbb{R}, \quad gf \mapsto \langle S, f \rangle$$

would be continuous. Thus from the Hahn-Banach theorem it would be extended to $\mathcal{E}(\Omega)$ in a $T \in \mathcal{E}'(\Omega)$, and then $S = gT$.

For every $x_0 \in \Omega$, there exist two subspaces $E_1$ and $E_2$ of $E$ such that $E = E_1 \oplus E_2$ topologically, with $\dim E_1 < \infty$, and two convex balanced open neighborhoods of zero, $V_{x_0}$ and $W_{x_0}$ respectively in $E_1$ and $E_2$, with $x_0 + V_{x_0} + W_{x_0} \subset \Omega$, such that if
we set \( \Omega_1 = p_1(x_0) + V_{x_0} \) and \( \Omega_2 = p_2(x_0) + W_{x_0} \) we have \( g(\xi + y_1) = g(\xi + y_2) \) for all \( \xi \in \Omega_1 \) and \( y_1, y_2 \in \Omega_2 \). Let \( K_1 \) and \( K_2 \) be two strictly compact subsets of \( \Omega_1 \) and \( \Omega_2 \) respectively and \( B \) a convex balanced bounded subset of \( E_1 \). For every \( x \in \Omega \), we have

\[
\sup_{h_i \in B} \left| f^{(n)}(x)h_1 \cdots h_n \right| \leq 2^n \sup_{h_i \in B} \left| f^{(n)}(x)p_1(h_1) \cdots p_1(h_k)p_2(h_{k+1}) \cdots p_2(h_n) \right|.
\]

Let us fix \( k \in \{0 \cdots n\} \) and \( l_{k+1}, \ldots, l_n \in B \). If \( x \in \Omega_1 + \Omega_2 \), we set \( F(x) = f^{(n-k)}(x)p_2(l_{k+1}) \cdots p_2(l_n) \). If \( x = \xi + y \in K_1 + K_2 \) (\( \xi \in K_1, y \in K_2 \)) we set \( F_x : \Omega_1 \to \mathbb{R}, \ t \mapsto F_x(t) = F(t + y) \).

Then we have

\[
\sup_{h_i \in B} \left| F^{(k)}(x)p_1(h_1) \cdots p_1(h_k) \right| = \sup_{h_i \in B} \left| F_y^{(k)}(\xi)p_1(h_1) \cdots p_1(h_k) \right| \leq \sup_{h_i \in p_i(B)} \left| F_y^{(k)}(t)h_1 \cdots h_k \right|.
\]

The function \( F_y \) belongs to \( \mathcal{S}(\Omega_1) \), \( K_1 \) is a compact subset of \( \Omega_1 \), \( \Omega_1 \) is an open subset of \( E_1 \), \( \dim E_1 < \infty \) and \( g \) does not vanish identically on any open subset of \( \Omega_1 \). According to Lojasiewicz’ result \([7]\), there exist a compact subset \( K'_1 \) of \( \Omega_1 \), a convex balanced bounded subset \( B' \) of \( E_1 \), \( m_k \in \mathbb{N} \) and \( C_1 > 0 \) independent of \( x \), such that

\[
\sup_{h_i \in p_i(B)} \left| F_y^{(k)}(t)h_1 \cdots h_k \right| \leq C_1 \sup_{h_i' \in B'} \left| (gF_y)^{(p)}(t)h_1' \cdots h_p' \right|.
\]

But \( (gF_y)^{(p)}(t)h_1' \cdots h_p' = (gF)^{(p)}(t + y)h_1' \cdots h_p' \) since \( h_1' \in E_1 \) and \( (gF)(x) = g(x)f^{(n-k)}(x)p_2(l_{k+1}) \cdots p_2(l_n) \). If we remark that \( g^{(r)}(x)p_2(l_1') \cdots p_2(l_p') = 0 \) when \( r > 1 \) and \( l_1', \ldots, l_p' \in B \), applying the Leibnitz formula we have

\[
(gf)^{(n-k)}(x)p_2(l_{k+1}) \cdots p_2(l_n) = g(x)f^{(n-k)}(x)p_2(l_{k+1}) \cdots p_2(l_n).
\]

Therefore, for \( x \in K_1 + K_2 \)

\[
\sup_{h_i \in B} \left| f^{(n)}(x)p_1(h_1) \cdots p_1(h_k)p_2(l_{k+1}) \cdots p_2(l_n) \right| \leq C_1 \sup_{h_i' \in B', p \leq m} \left| (gf)^{(n-k+p)}(x)h_1' \cdots h_p'p_2(l_{k+1}) \cdots p_2(l_n) \right|.
\]
If \( m = \sup_{k=0}^{n} m_k \), \( B'' = B' + p_2(B) \) and \( K'' = K_1' + K_2' \), we obtain a first inequality

\[
\sup_{h_i \in B''} \left| f^{(n)}(x) h_1 \cdots h_n \right| \leq C_1 \sup_{h_i \in B''} \left| (f^g)^{(n-k+p)}(x) h_1 \cdots h_{n-k+p} \right|.
\]

Now let \( K \) be a strictly compact subset of \( \Omega \). In the same way, for every \( x \in K \), we can define convex balanced open neighborhoods of zero: \( V_x \) and \( W_x \) (of some \( E_1 \) and \( E_2 \) which are dependent on \( x \)). We can find therefore a finite set \( \mathcal{I} \) such that

\[
K \subseteq \bigcup_{i \in \mathcal{I}} x_i + \frac{1}{2} V_{x_i} + \frac{1}{2} W_{x_i}.
\]

If we set \( K_i = K \cap (x_i + \frac{1}{2} V_{x_i} + \frac{1}{2} W_{x_i}) \), we have the following inclusions:

\[
K_i \subseteq p_{i1}(K_i) + p_{i2}(K_i) \subseteq \left( p_{i1}(x_i) + V_{x_i} \right) + \left( p_{i2}(x_i) + W_{x_i} \right) \subseteq \Omega
\]

with \( p_{i1} \) and \( p_{i2} \) the projections on \( E_{i1} \) and \( E_{i2} \) (\( E = E_{i1} \oplus E_{i2} \) defined by \( x_i \)). The sets \( p_{i1}(K_i) \) and \( p_{i2}(K_i) \) being strictly compact in \( E_{i1} \) and \( E_{i2} \) respectively, we can then apply our previous result: if \( \alpha \in \mathbb{N} \) there exist a strictly compact \( K' \) of \( \Omega \), a bounded subset \( B' \) of \( E, m' \in \mathbb{N}, C > 0 \) such that, for every \( \epsilon > 0 \), \( gf \in V(K', B', \epsilon/C, m') \) implies \( f \in V(K, B, \epsilon, \alpha) \). Therefore \( u \) is open. \( \square \)

IV. Characterization of the spaces where division is always possible.

**Theorem 2.** Let \( E \) be a Hausdorff real locally convex space, and \( \Omega \) an open subset of \( E \). The division by a nonzero continuous polynomial is always possible if and only if for every convex balanced neighborhood of zero \( V \) and for every convex balanced bounded subset \( B \) of \( E \) we have \( \limsup \dim V(B) < \infty \).

**Proof. First part:** Let us suppose there exist a convex balanced 0-neighborhood \( V \) in \( E \), and a convex balanced bounded subset \( B \) of \( E \) with \( \limsup \dim V(B) = \infty \). We may assume \( B \subseteq V \). By induction, there is a system \( \{e_n\}_{n \in \mathbb{N}} \), with \( e_n \in s_V(B) \), and \( v_n \in (E_V, p_V)' \) with \( v_n(e_m) = 0 \) if and only if \( n \neq m \). We may assume \( \|v_n\|_\nu = 1 \) and \( v_n(e_n) > 0 \). Let us set \( a_n = n^{-1} v_n(e_n) \) and

\[
P_V : E_V \to \mathbb{R}, \quad x \mapsto \sum_{n \in \mathbb{N}^*} a_n^2 v_n(x)^2.
\]

We have

\[
a_n \leq \frac{\|v_n\|_\nu \|e_n\|_\nu}{n} \leq \frac{1}{n} \quad \text{and} \quad |P_V(x)| \leq \sum_{n \in \mathbb{N}^*} n^{-n} \|x\|^2_\nu.
\]

\( P_V \) is thus a continuous polynomial on \( E \). Therefore, \( P = P_V \circ s_V \) is a continuous polynomial on \( E \). Since \( e_n \in s_V(B) \), there exists a \( j_n \in B \) such that \( s_V(j_n) = e_n \).

Hence, we have \( \|j_n\|_B \leq 1 \) and \( (n^{-1} j_n)_{n \in \mathbb{N}^*} \) converges to zero in \( (B, p_B) \).

By translation, we may suppose that \( 0 \in \Omega \); so there is an \( n_0 \in \mathbb{N}^* \) such that \( n^{-1} j_n \in \Omega \) if \( n \geq n_0 \). Let us define \( S \in \mathcal{E}'(\Omega) \) by

\[
\mathcal{E}(\Omega) \to \mathbb{R}, \quad f \mapsto \langle S, f \rangle = \sum_{n \geq n_0} n^{-2} f(n^{-1} j_n),
\]
and let us suppose there is a $T \in \mathcal{D}'(\Omega)$ with $\langle T, Pf \rangle = \langle S, f \rangle$, $\forall f \in \mathcal{D}(\Omega)$. Therefore there exist a strictly compact subset $k$ of $\Omega$, a bounded subset $L$ of $E$, $\varepsilon > 0$ and $\alpha \in \mathbb{N}$ with $T \in V(K, L, \varepsilon, \alpha)$. 

Let $\varphi$ be the function defined by

$$
\varphi \in \mathcal{D}(\mathbb{R}) \text{ and there is a constant } C \geq 0 \text{ such that }
\sup_{u \in \mathbb{R}} |\varphi^{(k)}(u)| \leq C.
$$

For every $n \in \mathbb{N}^*$ and $x \in E$ we set $f_n(x) = a_n^m \varphi[2(a_n^{-1}v_n \circ s_v(x) - 1)]$; thus $f_n \in \mathcal{D}(E)$ and for $k \leq \alpha$

$$
|f_n^{(k)}(x)h_1 \cdots h_k| \leq a_n^m 2^{k_1}a_n^{-k_1} |v_n \circ s_v(h_1) \cdots |v_n \circ s_v(h_k)| .
$$

Since $B$ is a bounded set, $\sup_{n \in \mathbb{N}^*} |v_n \circ s_v(h)| \leq b$ for some constant $b$. But since $(2^k b^k a_n^{-k})_{n \in \mathbb{N}^*}$ converges to zero, there is $N_0 \in \mathbb{N}^*$ such that $f_n \leq V(K, L, \varepsilon, \alpha)$ if $n \geq N_0$.

From the following equivalence:

$$
f_n(x) \neq 0 \leftrightarrow a_n < v_n \circ s_v(x) < 2a_n
$$

and

$$
P(x) = 0 \leftrightarrow v_n \circ s_v(x) = 0 \quad \forall n \in \mathbb{N}^*,
$$

$f_n$ is zero in a neighborhood of the zero set of $P$. Therefore $f_n/P$ makes sense and is an element of $\mathcal{D}(\Omega)$. If $N \geq \sup(n_0, N_0)$,

$$
\langle T, f_n \rangle = \langle T, Pf_n/P \rangle = \langle S, f_n/P \rangle
= \sum_{n \geq n_0} n^{-2} (f_n/P)(n^{-1}j_n) \geq N^{-2} (f_n/P)(N^{-1}j_n).
$$

But $P(N^{-1}j_n) = a_N^m N^{-2}(v_n(e_N))^2$ and $f_n(N^{-1}j_n) = 2a_N^m$. Therefore if $N \geq \sup(n_0, N_0)$, $|\langle T, f_n \rangle| \geq 2$. We get a contradiction.

**Second part:** If $S \in \mathcal{D}'(\Omega)$, there is a strictly compact subset $K$ of $\Omega$, a bounded subset $B$ of $E$, an $\varepsilon > 0$ and an $\alpha \in \mathbb{N}$ such that $S \in V(K, B, \varepsilon, \alpha)$. We may assume $K \subset B$ and $B$ convex balanced. Therefore $S_{|B \cap E_B} \in \mathcal{D}'(\Omega \cap E_B)$. We suppose that for every convex balanced 0-neighborhood $V$ in $E$, $\dim s_v(E_B) < \infty$. Let $P$ be a nonzero continuous polynomial on $E$. The set $\Delta = \{x \in E / |P(x)| < |P(0)| + 1\}$ is a neighborhood of zero; therefore there exists a convex balanced 0-neighborhood $V$ such that $V \subset \Delta$. Let $E_B$ be endowed with the topology induced by that of $E$. Therefore $E_B = (E_B \cap \rho v(0)) \oplus F$ topologically with $F \approx s_v(E_B)$. If $x \in E_B \cap V$, there are $u \in E_B \cap \rho v(0)$ and $v \in F$ such that $x = u + v$. If $\lambda \in \mathbb{R}$, we have $\lambda u + v = (\lambda - 1)u + u + v$ and $\rho v(\lambda u + v) = |\lambda - 1| \rho v(u) + \rho v(u + v) < 1$. Therefore, given $u \in E_B \cap \rho v(0)$ and $v \in F$, the polynomial $\lambda \mapsto P(\lambda u + v)$ is bounded on $\mathbb{R}$, and so a constant. Thus we have $P(u + v) = P(v)$ for all $u \in E_B \cap \rho v(0)$ and so $P$ is a finite-type polynomial on $E_B$. According to Theorem 1, there
exists a $T_1 \in \mathcal{E}'(\Omega \cap E_B)$ such that $P_{|\Omega \cap E_B} T_1 = S_{|\Omega \cap E_B}$. Let $r$ be the restriction map from $\mathcal{E}(\Omega)$ to $\mathcal{E}(\Omega \cap E_B)$ and $'r$ be its transpose. $T = 'r(T_1)$ is a solution of $PT = S$.

Let $\mathcal{G}$ be the class of the Hausdorff real locally convex spaces such that for every convex balanced 0-neighborhood $V$, and every convex balanced bounded subset $B$ of $E$, we have $\dim s_V(E_B) < \infty$.

**Theorem 2'.** Let $E$ be a Hausdorff real locally convex space and $\Omega$ an open subset of $E$. The division by any continuous analytic function which does not vanish on any open subset of $\Omega$ is always possible in $\mathcal{E}'(\Omega)$ if and only if $E \in \mathcal{G}$.

**Proof.** We use the same proof as in Theorem 2, replacing polynomial by analytic function.

**Remarks.** (1) Theorems 2 and 2' are no more valid if we omit the continuous property. If $E \not\in \mathcal{G}$, let us consider on $E$ the weak topology $(E, \sigma)$. Since $E$ and $(E, \sigma)$ have the same bounded sets, the topological spaces of the $C^\infty$ functions on $E$ and on $(E, \sigma)$ are also the same. The division by a polynomial $P$ in $(E, \sigma)$ is equivalent to the division in $E$. If $P$ is the polynomial of the counterexample, the division by $P$ is not possible in $(E, \sigma)$ and $(E, \sigma) \not\in \mathcal{G}$. We deduce from Theorem 2 that the polynomial $P$ is not continuous on $(E, \sigma)$.

(2) If $(E, \| \cdot \|)$ is a Banach space and if $E \in \mathcal{G}$, it is obvious that $E$ is finite dimensional. So:

**Corollary 1.** Let $E$ be a real Banach space, $E$ is a finite dimensional space if and only if the division of any distribution by any nonzero continuous polynomial is always possible in $\mathcal{E}'(E)$.

(3) If $E$ is a Fréchet space and if $E \in \mathcal{G}$, we easily prove that, for every convex balanced 0-neighborhood $V$, $\dim s_V(E)$ is finite. The unique Fréchet spaces having this property are isomorphic to $\mathbb{R}^n$ or to $\mathbb{R}^\mathbb{N}$. (Let $E'$ be the dual of $E$, equipped with the equicontinuous bornology; since $\dim s_V(E) < \infty$, $(E')^0_V$ is finite dimensional. If $(e_i')_{i \in I}$ is an algebraic basis of $E'$, $E'$ is bornologically isomorphic to $\mathbb{R}^I$.) Let $E''$ be the bornological dual of $E'$ equipped with the topology of the uniform convergence on the equicontinuous subsets of $E'$. Therefore $E'' \cong \mathbb{R}^I$ topologically. Since $E$ is a subspace of $E''$ and $E''$ is a Schwartz space, thus $E$ is a Schwartz space. Since $E$ is a complete Schwartz space, $E = E''$ (see Hogbe-Nlend [5, p. 95]).

**Corollary 2.** If $E$ is an infinite dimensional Fréchet space, $E \in \mathcal{G}$ if and only if $E \cong \mathbb{R}^\mathbb{N}$.

(4) Let $F$ be a Fréchet space and $E$ its strong dual. If we suppose $E \in \mathcal{G}$, we prove that $F \in \mathcal{G}$: if $B$ is a convex balanced bounded subset of $F$ and $V$ a convex balanced neighborhood of zero, then $(F_B, \rho_V) \cong s^0_B(E_B^0)$. Since $E \in \mathcal{G}$, $\dim s^0_B(E_B^0) < \infty$ and $F_B/\rho_V^{-1}(0) \cap F_B$ is a finite dimensional space.

Since $F$ is a Fréchet space and $F \in \mathcal{G}$, according to (3), $F \cong \mathbb{R}^n$ or $F \cong \mathbb{R}^\mathbb{N}$. We deduce $E \cong \mathbb{R}^n$ or $E \cong \mathbb{R}^\mathbb{N}$. 

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Corollary 3. If $E$ is the strong dual of an infinite dimensional Fréchet space, $E \in \mathcal{S}$ if and only if $E \cong \mathbb{R}^{(\infty)}$.

(5) If $E$ is a nuclear bornological vector space, the Fourier transform is a bornological and topological isomorphism between $\mathcal{S}'(E)$ and $\mathcal{F}\mathcal{S}'(E)$ (see J. M. Ansemil and J. F. Colombeau [1]). Let $P$ be a nonzero real polynomial on $E$; we define the convolution operator $\Box$ by

$$0: \mathcal{F}\mathcal{S}'(E) \to \mathcal{S}'(E), \quad \mathcal{F}f \mapsto \mathcal{F}(Pf).$$

It is obvious that the surjectivity of the operator $\Box$ is equivalent to the surjectivity of the mapping

$$\mathcal{S}'(\Omega) \to \mathcal{S}'(\Omega), \quad f \mapsto Pf.$$

According to Theorem 2, we deduce that for some $\mathcal{F}\mathcal{S}'(E)$ all convolution operators are not surjective.

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References


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