AMPLENESS AND CONNECTEDNESS IN COMPLEX G/P

BY

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Abstract. This paper determines the “ampleness” of the tangent bundle of the complex homogeneous space, G/P, by calculating the maximal fibre dimension of the desingularization of a nilpotent subvariety of the Lie algebra of G.

1. Introduction. Let G be a connected semisimple complex Lie Group. Let P be a parabolic subgroup of G, and define the homogeneous space Z = G/P. Let φ: P(TZ) → PN be the map determined by the global sections of TZ, the tangent bundle of Z; see §3. Define the ampleness of TZ, amp(TZ), to be the maximum fibre dimension of φ, and the coampleness, ca(Z), to be dim Z — amp(TZ).

In this paper, I calculate ca(Z). The results are in Table 1 at the end of this section. I have also determined the number of irreducible components in a largest fibre of the ampleness map φ. This result is described in §5.

The following theorem of Sommese [21A, §3] is a generalization of the Barth-Larsen Lefschetz theorems.

Lefschetz theorem. Let f: X → Z be a regular immersion of the connected compact complex manifold X into Z. Let Y be a connected compact complex submanifold of Z. Let k be the ampleness of NY, the normal bundle of Y, i.e. k is the largest fibre dimension of the restriction of φ to P(N*Y). Sommese [21C, Corollary 1.4] shows that Z \ Y is k + cod Y convex in the sense of Andreotti-Grauert. Assume that 2 · dim Y ≥ k + dim Z and dim X ≤ dim Y + 1. Then \( \pi_i(X, f^{-1}(Y), x) = 0 \) for \( i < \dim X - \cod T - k \).

It is difficult to compute k exactly, but the inequality \( k \leq \amp TZ \) may be used.


Connectedness theorem. Let f: X → Z × Z be a regular map of the compact irreducible variety X to Z × Z and let Δ ⊂ Z × Z be the diagonal embedding of Z. Let \( l = \min\{rk(\frak{g}_i)\} \) where \( \frak{g} = \bigoplus \frak{g}_i \) is the decomposition of the Lie algebra of G into
simple ideals. Then

(i) \( \dim f(X) > 2 \dim(Z) - l \Rightarrow f^{-1}(\Delta) \neq \phi \).

(ii) \( \dim f(X) > 2 \dim(Z) - l \Rightarrow f^{-1}(\Delta) \) is connected.

From Table 1, we see that \( l \leq \text{ca}(Z) \). In [10A], Faltings explains that, in the above theorem, "\( l \)" may be replaced by the better bound "\( \text{ca}(Z) \)".

For \( Z \) other than projective space, it is not clear what are the higher homotopy results implied by the connectedness theorem; see Fulton and Lazarsfeld [12, §10.1].

I should mention, also, that part (i) of the connectedness theorem follows from work of Sommese [21, Proposition 1.1].

**Example 1.** \( G = \text{SL}(n + 1, \mathbb{C}) \), \( l = n = \text{rk}(G) \), and \( \text{ca}(G/P) = n \) for each parabolic \( P \). See Hansen [15, p. 3], for examples that the results (i), (ii) are sharp.

**Example 2.** \( G = \text{O}(2n, \mathbb{C}) \), \( l = n = \text{rk}(G) \), and \( \text{ca}(G/P) = 2n - 3 \) for each parabolic \( P \). Let \( Z \subset \mathbb{P}^{2n-1} \) be the \( 2n - 2 \) dimensional quadric given by the equation \( \Sigma_{i=1}^{2n} z_i^2 = 0 \). The above results imply that \( X \cap Y \neq \emptyset \) whenever \( X \) and \( Y \) are closed subvarieties of \( Z \) satisfying

\[
\dim(X) + \dim(Y) > 2(2n - 2) - (2n - 3) = 2n - 1.
\]

This result is sharp:

Let \( X \) and \( Y \) be the images, respectively, of the maps \( \mathbb{P}^{n-1} \to Z \) given by \( x \mapsto (x, ix) \) and \( y \mapsto (y, -iy) \). Clearly, \( X \cap Y = \emptyset \), while \( \dim(X) + \dim(Y) = 2n - 2 \).

**Example 3.** \( G = \text{O}(2n + 1, \mathbb{C}) \), \( l = n = \text{rk}(G) \), and

\[
\text{ca}(G/P) = \begin{cases} 
2n - 1 & \text{for one special } P, \\
2n - 2 & \text{for every other parabolic } P.
\end{cases}
\]

Let \( Z \) be the \( 2n - 1 \) dimensional quadric. The above results imply that

\[
X \cap Y \neq \emptyset \text{ whenever } X \text{ and } Y \text{ are closed subvarieties of } Z \text{ satisfying } \\
\dim(X) + \dim(Y) \geq 2n.
\]

In fact, as pointed out to me by Sommese, one has a stronger (and sharper) result by replacing \( 2n \) by \( 2n - 1 \) in (\( * \)). Sharpness follows from Example 2 since the \( 2n - 2 \) dimensional quadric is contained in \( Z \). The validity follows from the fact that \( H^*(Z, \mathbb{C}) \) is ring isomorphic to \( H^*(\mathbb{P}^{2n-1}, \mathbb{C}) \).

The computational part of the paper concerns a desingularization of the unipotent variety of \( G \). This map has been studied by Springer [22] and Steinberg [25]. Their results on fibre dimensions deal largely with the case \( P = \) a Borel subgroup of \( G \). R. Elkik [8] has also noticed the connection between this desingularization and the cotangent bundle of \( G/P \) (see Remark 3.3 for more details). In [26], Steinberg explains that, following my letter to him, he did the computations for determining \( \text{ca}(G/P) \). For the case \( P = B \) a Borel subgroup, he gives the succinct formula

\[
\text{ca}(G/B) = |\rho^*|, \text{ where } \rho \text{ is a highest root for the simple group } G \text{ and } \rho^* \text{ is its dual, or coroot. This formula is equivalent to Theorem 4.2.}
\]

The notion of \( k \)-ampleness is due to Andrew Sommese [20, §1], and I would like to thank him for suggesting that I consider doing these computations for \( G/P \). The preprint of Faltings provided further motivation to compare ampleness and rank in the simple groups.
I would like to thank James Carrell, Hanspeter Kraft, George Maxwell, Peter Slodowy and Robert Steinberg for helpful letters and conversations. A special thanks goes to Bomshik Chang for discussions on Weyl groups.

### Table 1

| $G$ (all roots long) | $\text{ca}(G/P) = |\rho|$ for every parabolic subgroup $P$. |
|----------------------|--------------------------------------------------|
| $A_n$                | $n$                                              |
| $D_n$                | $2n - 3$                                         |
| $E_6$                | $11$                                             |
| $E_7$                | $17$                                             |
| $E_8$                | $29$                                             |

| $G$ Coxeter-Dynkin diagram (The short roots are darkened) | $\text{ca}(G/B)$ | $|\rho|$ |
|-----------------------------------------------------------|------------------|--------|
| $B_n$                                                      | $2n - 2$         | $2n - 1$ |
| $C_n$                                                      | $n$              | $2n - 1$ |
| $G_2$                                                      | $3$              | $5$     |
| $F_4$                                                      | $8$              | $11$    |

$\text{ca}(G/P_\Theta) = \text{ca}(G/B) + \min\{d(\alpha) : \alpha \in \Theta\}$

$|\rho|$ is the height of the highest root

d(\alpha) = \text{no. of nodes from } \alpha \text{ to the long roots}

2. Background material. In this section, I describe the notation that is used in the paper. The references Borel [3, 4, 5], Carter [6], Humphreys [16], Samelson [18], Serre [19] and Steinberg [24] contain proofs and elaborations.

Let $G$ be a connected complex semisimple Lie group. The centralizer of a subset, $A$, of $G$ is $Z(A) = \{g \in G : ga = ag \ \forall a \in A\}$. The normalizer of $A$ is $N(A) = \{g \in G : gAg^{-1} \subset A\}$.

Let $g$ denote the Lie algebra of $G$, with Lie bracket $[\ ,\ ]$. For each $b \in g$, $\text{ad}(b) \in \text{End}(g)$ is defined by $\text{ad}(b)(c) = [b, c]$. The Killing form on $g$ is given by $(b, c) = \text{Trace}(\text{ad}(b) \circ \text{ad}(c))$. This pairing is nondegenerate and induces an identification, $\mathfrak{k}^*$, of $\mathfrak{g}^*$, the vector space dual of $\mathfrak{g}$, with $\mathfrak{g}$.

For $g \in G$, let $C(g)(x) = gxg^{-1}$, and denote by $\text{Ad}(g) : \mathfrak{g} \to \mathfrak{g}$ the differential of $C(g)$ at the identity element of $G$. Let $\exp : \mathfrak{g} \to \tilde{G}$ be the exponential map. For each $a \in \mathfrak{g}$, $t \mapsto \exp(ta)$ is the 1-parameter subgroup of $G$ whose tangent vector at $t = 0$ is $a$. One has

$$\text{Ad}(\exp(a)) = e^{\text{ad}(a)} = \sum_{m=0}^{\infty} (\text{ad}(a))^m / m!.$$
Fix a maximal torus $H$ in a Borel subgroup $B$ of $G$. Let $\mathfrak{h}$ be the Lie algebra of $H$. Let $R \subset \mathfrak{h}^*$ be the root system with respect to $H$, and $R^+$ (resp. $R^-$) the positive (resp. negative) roots with respect to $B$. We write $\alpha > 0$ (resp. $\alpha < 0$) for $\alpha \in R^+$ (resp. $\alpha \in R^-$). Let $\Sigma = \{\alpha_1, \ldots, \alpha_n\}$ denote the simple roots, i.e. a basis for $R$; $n = \dim(H) = \text{rank}(G)$.

Each $\alpha \in R$ may be expressed as $\alpha = \sum n_i \alpha_i$ with either all $n > 0$ (i.e. $\alpha \in R^+$) or all $n_i \leq 0$ (i.e. $\alpha \in R^-$). The height of $\alpha$ is $|\alpha| = \sum n_i$. An element $\rho$ of $R$ is a highest root when $\rho + \alpha \in R$ whenever $\alpha > 0$. For simple $G$ there is a unique such root. Also, $R$ is a reduced root system, i.e. for each $\alpha \in R$, the only multiples of $\alpha$ that again belong to $R$ are $\pm \alpha$.

One has the decomposition

$$g = \mathfrak{h} \oplus \{\mathfrak{g}^\alpha : \alpha \in R\}$$

where $\mathfrak{g}^\alpha = \{a \in \mathfrak{g} : \text{ad}(b)(a) = \alpha(b)a \forall b \in \mathfrak{h}\}$ is the 1-dimensional root space associated to $\alpha$. Moreover, $[\mathfrak{g}^\alpha, \mathfrak{g}^\beta] \subset \mathfrak{g}^{\alpha + \beta}$, so that $[\mathfrak{g}^\alpha, \mathfrak{g}^\beta] = 0$ when $\rho$ is a highest root and $\alpha > 0$. Also $(\mathfrak{g}^\alpha, \mathfrak{g}^\beta) = 0$ when $\alpha + \beta \neq 0$ while $\mathfrak{g}^\alpha$ and $\mathfrak{g}^{-\alpha}$ are perfectly paired by the Killing form.

The homomorphism

$$x_\alpha = \exp(\mathfrak{g}^\alpha) : \mathfrak{g}^\alpha \to X_\alpha \subset G$$

is an isomorphism onto $X_\alpha$, the root subgroup associated to $\alpha$. One has $X_\rho \subset Z(X_\alpha)$ when $\rho$ is a highest root and $\alpha > 0$.

For each subset $\Theta \subset \Sigma$, one has the standard parabolic subgroup

$$(2.1) \quad P = P_\Theta = H \cdot \prod \{X_\alpha : \alpha \in \langle \Theta \rangle\} \cdot \prod \{X_\alpha : \alpha > 0, \alpha \not\in \langle \Theta \rangle\}.$$  

The products are taken in any fixed order, and $\langle \Theta \rangle$ denotes the span of $\Theta$ in $R$. The third factor is the unipotent radical, $U_\rho$ of $P$, and $P \subset N(U_\rho)$. Every parabolic subgroup of $G$ is conjugate to some standard parabolic.

Similarly, the Lie algebra of $P_\Theta$ is

$$\mathfrak{p} = \mathfrak{h} \oplus \{\mathfrak{g}^\alpha : \alpha \in \langle \Theta \rangle\} \oplus \{\mathfrak{g}^\alpha : \alpha > 0, \alpha \not\in \langle \Theta \rangle\}.$$  

The third summand is the nilpotent radical $N_\rho$ of $P$ and is also the annihilator of $\mathfrak{p}$ in $\mathfrak{g}$ with respect to the Killing form.

$$(2.2) \quad \text{We denote } U = U_B = \prod \{X_\alpha : \alpha > 0\}. \text{ Then } X_\rho \subset Z(U) \text{ when } \rho \text{ is a highest root. The map}$$

$$H \times \mathbb{C}^N \to B \quad (N = \#(R^+)), \quad (t, c) \mapsto t \cdot \prod \{x_\alpha(c_\alpha) : \alpha > 0\}$$

is an isomorphism of varieties.

$$(2.3) \quad \text{The finite group } \mathfrak{W} = N(H)/H \text{ is the Weyl group of } G \text{ with respect to } H.$$  

For each $\omega \in \mathfrak{W}$, let $n_\omega$ be a fixed representative in $N(H)$. The group $\mathfrak{W}$ embeds as a group of linear transformations on $\mathfrak{h}^*$, leaving $R$ invariant. One has $n_\omega X_\alpha n^{-1}_\omega = X_{\alpha(\omega)}$ for each root subgroup $X_\alpha$ and $\omega \in \mathfrak{W}$.

For each $\alpha \in R$, let $\sigma_\alpha \in \mathfrak{W}$ be the reflection through $\alpha$. Let $\sigma_i = \sigma_{\alpha_i}$ for each of the simple roots $\alpha_i$. Then, $\mathfrak{W}$ is generated by the simple reflections $\{\sigma_1, \ldots, \sigma_n\}$. The length, $l(\omega)$, of an element $\omega \in \mathfrak{W}$ is defined as the least number of simple
reflections by which one may write \( \omega = \sigma_i \cdots \sigma_{j_1} \). One has also that \( l(\omega) = \# \{ \alpha > 0 : \omega(\alpha) < 0 \} \). The simple reflection \( \sigma_i \) permutes the elements of \( R^+ \setminus \{ \alpha_i \} \). It follows that

\[
l(\omega \sigma_i) = \begin{cases} l(\omega) + 1, & \omega(\alpha_i) > 0, \\ l(\omega) - 1, & \omega(\alpha_i) < 0. \end{cases}
\]

There is a unique element, \( \omega_0 \), of \( \mathbb{W} \) taking \( R^+ \) to \( R^- \). One has \( l(\omega_0) = \#(R^+) - l(\omega) \) for each \( \omega \in \mathbb{W} \).

Let \( (, ) \) denote a \( \mathbb{W} \)-invariant positive definite pairing on \( \mathfrak{h}^* \). When \( G \) is simple, at most two root lengths occur in \( R \) (long and short). If only one length occurs, it is called long.

<table>
<thead>
<tr>
<th>( G )</th>
<th>Squared length ratio (long : short)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_n, D_n, E_n )</td>
<td>all roots long</td>
</tr>
<tr>
<td>( B_n, C_n, F_4 )</td>
<td>2 : 1</td>
</tr>
<tr>
<td>( G_2 )</td>
<td>3 : 1</td>
</tr>
</tbody>
</table>

The roots of a given length form an orbit of the Weyl group in \( R \). The reflection \( \sigma_\alpha \) is given by the formula \( \sigma_\alpha(\beta) = \beta - \alpha^*(\beta)\alpha \) where \( \alpha^*(\beta) = 2(\alpha, \beta)/(\alpha, \alpha) \in \mathbb{Z} \).

**Lemma 2.4.** Assume that \( \mathfrak{g} \) is simple. Let \( \alpha, \beta \in R \) such that \( \alpha \neq \pm \beta \), \( (\alpha, \beta) \neq 0 \) and \( (\alpha, \alpha) \leq (\beta, \beta) \). Then

(i) \( \alpha^*(\beta) \cdot \beta^*(\alpha) = 1, 2 \text{ or } 3 \),

(ii) if \( (\alpha, \beta) > 0 \) then \( \alpha^*(\beta) = (\beta, \beta)/(\alpha, \alpha) \) and \( \beta^*(\alpha) = 1 \), and

(iii) if \( \alpha \) and \( \beta \) are simple then \( (\alpha, \beta) < 0 \) and \( \alpha^*(\beta) = -(\beta, \beta)/(\alpha, \alpha) \) and \( \beta^*(\alpha) = -1 \).

**Proof.** (i) \( \alpha^*(\beta) \cdot \beta^*(\alpha) = 4(\alpha, \beta)^2/(\alpha, \alpha)(\beta, \beta) \), so \( \alpha^*(\beta) \cdot \beta^*(\alpha) = 0, 1, 2, 3 \) or 4 by Schwarz's inequality, but the hypotheses preclude the values 0 and 4.

(ii) \( \alpha^*(\beta)/\beta^*(\alpha) = (\beta, \beta)/(\alpha, \alpha) \), so the result follows from (i).

(iii) Serre [19, V8, Lemme 3], for example, shows that \( (\alpha, \beta) < 0 \), and the result now follows from (ii).

**3. Ampleness of the tangent bundle.** With the notation of §2, \( Z = G/P \) is a compact complex homogeneous space with group \( G \). Sommese [20, §1] defines the \( k \)-ampleness of a vector bundle over a compact complex space. This notion is discussed, in particular, for the tangent bundle, \( TZ \), in Goldstein [14, §2] which notation is reviewed below.

The global sections \( S_0, \ldots, S_N \) of \( TZ \) determine the “ampleness map”

\[
\phi : T^*Z \to \mathfrak{g}^*
\]

where \( \mathfrak{g}^* \) is the vector space dual of the Lie algebra, \( \mathfrak{g} \), of \( G \). Explicitly,

\[
\phi(\alpha) = z^*\alpha \quad \text{where} \quad \alpha \in T^*_zZ, \quad z^* : G \to Z,
\]

\[ g \mapsto gz \quad \text{and} \quad z^{**} : T^*_zZ \to T^*_zG = \mathfrak{g}^*. \]
The bundle $TZ$ is $k$-ample when

$$\text{amp}(TZ) := \sup \{ \dim(\phi^{-1}(\alpha)) : \alpha \in \mathfrak{g} \setminus \{0\} \}$$

has value at most $k$. The coampleness is $\text{ca}(Z) := \dim(Z) - \text{amp}(TZ)$.

This section reduces the determination of $\text{ca}(G/P)$ to a calculation with Weyl groups (Proposition 3.6). First, the ampleness map is translated into Lie group data (Lemma 3.2).

**Notation.** Recall that $\mathfrak{p} \subset \mathfrak{g}$ is the Lie algebra of the parabolic subgroup $P$. Let $\mathfrak{p}^\perp = \{ \alpha \in \mathfrak{g}^*: \mathfrak{p} \subset \ker(\alpha) \}$.

Let $\Phi: G \times \mathfrak{g}^* \to \mathfrak{g}^*$ be the map

$$(g, \alpha) \mapsto (\text{Ad}(g))^{-1}(\alpha).$$

Let $\Phi: G \times \mathfrak{p}^\perp \to \mathfrak{g}^*$ be the restriction of $\Phi$ to $G \times \mathfrak{p}^\perp$.

Define an action of $P$ on $G \times \mathfrak{p}^\perp$ by

$$h \cdot (g, \alpha) = (gh^{-1}, (\text{Ad}(h))^{-1}(\alpha)).$$

The map $\Phi$ is constant on the orbits of $P$, and so induces a map

$$\phi: (G \times \mathfrak{p}^\perp)/P \to \mathfrak{g}^*.$$

The group $G$ acts on $G \times \mathfrak{p}^\perp$ by

$$h \cdot (g, \alpha) = (hg, \alpha).$$

This action commutes with the action of $P$ just described, thus defining a $G$-action on the space $(G \times \mathfrak{p}^\perp)/P$. The map $\phi$ is $G$-equivariant, where the action of $G$ on $\mathfrak{g}^*$ is the dual adjoint action.

**Lemma 3.1.** There is a commutative diagram (3.1.1), where $\Gamma$ is a $G$-equivariant isomorphism of vector bundles over $Z$.

\[
\begin{array}{ccc}
G & \xleftarrow{\alpha} & G \times \mathfrak{p}^\perp \\
\pi \downarrow & & \phi \\
G/P & \xleftarrow{\bar{\rho}} & (G \times \mathfrak{p}^\perp)/P \\
\| & & \phi \\
Z & \xleftarrow{\rho} & T^*Z \\
\end{array}
\]

The maps $\alpha$, $\beta$, $\pi$, $\bar{\rho}$ and $\rho$ are the natural projections.

**Proof.** It remains to construct $\Gamma$. Define, first, a map

$$\hat{\Gamma}: G \times \mathfrak{p}^\perp \to T^*Z,$$

$$(g, \alpha) \mapsto g^{-1}(\pi^*(\alpha)).$$

Here, let $z_0$ denote the point $\pi(e)$ of $G/P$ represented by the coset $P$, 

$$\pi^*: T^*_0Z \to T^*_eG = \mathfrak{g}^*.$$
Note that $\pi^*$ is 1-1 and has image equal to $\mathfrak{u}^\perp$. It is elementary to verify that $\tilde{\Gamma}$ is $G$-equivariant, is constant on the orbits of $P$, that the induced map

$$\Gamma: (G \times \mathfrak{u}^\perp)/P \to T^*Z$$

is an isomorphism, and that $\phi \circ \Gamma = \tilde{\phi}$. □

**Lemma 3.2.** There is a commutative diagram (3.2.1).

\[
\begin{array}{ccc}
T^*Z & \overset{\Lambda_1}{\sim} & (G \times N_P)/P \\
\phi & \downarrow & \phi \\
\mathfrak{g}^* & \overset{\mathcal{K}}{\sim} & \mathcal{U} \\
\end{array}
\]

\[
(3.2.1)
\]

\[
\begin{array}{ccc}
(G \times U_P)/P \\
\Psi & \downarrow & \Psi \\
\end{array}
\]

The map $\Lambda_1$ is a vector bundle isomorphism, and $\Lambda_2$ is an isomorphism of varieties.

Here, $U_P$ and $N_P$ are the unipotent and nilpotent radicals, respectively, of $P$. The action of $\mathcal{E}$ on $U_P$ is given by

$$h^{-1} (g, x) = (gh^{-1}, h_x h^{-1}).$$

The action of $P$ on $N_P$ is given by

$$h^{-1} (g, v) = (gh^{-1}, \text{Ad}(h)(v)).$$

The map $\mathcal{K}$ is the Killing identification, and the exponential map, $\exp$, takes the nilpotent elements $\mathcal{U}$ of $\mathfrak{g}$ isomorphically onto the unipotent variety $V$ of $G$ (cf. Springer [22, §3]). We have $\Psi(g, x) = gxg^{-1}$, and $\tilde{\Psi}(g, v) = \text{Ad}(g)(v)$.

**Proof.** One verifies that $\mathcal{K}(\text{ad}(c)^*) = -\text{ad}(c)$ for each $c \in \mathfrak{g}$. Thus, $\mathcal{K}(\text{Ad}(g)^*) = \text{Ad}(g)^{-1}$. Also, $\mathcal{K}(\mathfrak{u}^\perp) = N_P$ and $\exp(N_P) = U_P$. Using these facts, together with Lemma 3.1, diagram (3.2.1) may be constructed. □

The goal in the remainder of this paper is to calculate the largest fibre dimension of the ampleness map $\phi$ (or $\tilde{\phi}$ or $\psi$).

**Remark 3.3.1.** The map $\psi$ has been studied by Springer [22] and Steinberg [25] when $P = B$. Let

$$W := \{(gP, u) \in G/P \times V : g^{-1}ug \in U_P\}.$$ 

The projection $\pi: W \to V$ is equivalent to the map $\psi$ of Lemma 3.2. Springer shows, in the case $P = B$, that $\pi$ is generically 1-1 and each fibre of $\pi$ is connected. Steinberg extends these results to general parabolics. In the case $P = B$, he shows that

$$\dim(\pi^{-1}(u)) = \frac{1}{2} \times (\dim(Z(u)) - \text{rk}(G))$$

and obtains some results on the number of irreducible components in each fibre of $\pi$.

**Remark 3.3.2.** The space $(G \times N_P)/P$ has also been studied by Elkik [8, §1], the identification with $T^*(G/P)$ being done in the language of schemes.
We next reduce the computation of $\text{ca}(G/P)$ to the case where $G$ is simple. Without loss of generality, assume that $G$ is simply connected, so $G = \times_{i=1}^{m} G_i$, where the $G_i$ are simple normal subgroups of $G$. From the description of standard parabolics, one has $P = \times_{i=1}^{m} P_i$, so that $G/P \cong \times_{i=1}^{m} (G/P_i)$. Let $Z_i = G/P_i$ and let $Z = G/P = \times_{i=1}^{m} Z_i$. Let

$$\phi_i: T^* Z_i \to g_i^*$$

be the ampleness maps. Then

$$\phi = \bigoplus_{i=1}^{m} \phi_i: \bigoplus_{i=1}^{m} (T^* Z_i) \to \bigoplus_{i=1}^{m} g_i^*$$

is the ampleness map of $Z$. It follows that

$$\text{ca}(Z) = \min\{\text{ca}(Z_i): i = 1, \ldots, m\}. \quad (3.4)$$

For the remainder of the paper, we assume that $G$ is simple.

**Proposition 3.5.** Let $G$ be a (simple) complex Lie group. Let $\rho$ be the highest root of $G$ with respect to some ordering, and let $x_\rho$ be any nonidentity element of $X_\rho$, the root subgroup of $G$ associated to $\rho$. Let $P$ be a parabolic subgroup of $G$, with unipotent radical $U_P$. Then

$$\text{ca}(G/P) = \text{cod}_G\{g \in G: g^{-1} x_\rho g \in U_P\}.$$ 

**Proof.** We use the notation of Lemma 3.2. The projectivization of the ampleness map

$$P(\psi): (G \times P(N_P))/P \to P(\mathcal{R})$$

is proper and $G$-equivariant. The space $P(\mathcal{R})$ possesses a unique closed $G$-orbit $\Theta$, which is the orbit of a highest root-vector line.\(^2\) Thus, any point of $P(\mathcal{R})$ may be specialized, within its orbit, to a point of $\Theta$ and we have that the maximum $\psi$-fibre dimension (over $\mathcal{R} \setminus \{0\}$) occurs at $v_\rho \in \mathcal{R}$, where $v_\rho$ is a highest root vector.

Viewing the ampleness map now as

$$\psi: (G \times U_P)/P \to V,$$

the maximum fibre dimension of $\psi$ is

$$\dim(\psi^{-1}(x_\rho)) = \dim\{\{(g, u) \in G \times U_P: g u g^{-1} = x_\rho\}\} - \dim(P)$$

$$= \dim\{\{g \in G: g^{-1} x_\rho g \in U_P\}\} - \dim(P)$$

where $x_\rho = \exp(v_\rho)$. Thus,

$$\text{ca}(G/P) = \dim(G/P) - \dim(\psi^{-1}(x_\rho))$$

$$= \dim(G) - \dim\{g \in G: g^{-1} x_\rho g \in U_P\}. \quad \square$$

\(^2\)This fact, as related to me by H. Kraft and P. Slodowy, is well known: The adjoint action of $G$ on $\mathfrak{g}$ is irreducible. So, there is a unique line in $\mathfrak{g}$ which is invariant under the Borel group viz. the highest root-vector line (see e.g. [24, §3, 4a]). Using the fact that any two Borel subgroups of $G$ are conjugate, one concludes that the orbit of this line in $P(\mathcal{R})$ is the unique closed orbit.
The last result of this section expresses $\text{ca}(G/P)$ in terms of the length function of the Weyl group of $G$.

**Proposition 3.6.** Let $P$ be the standard parabolic subgroup of the simple group $G$ associated to the subset $\Theta \subset \Sigma$. Then

\[
\text{ca}(G/P) = \min \{ l(\omega) : \omega \in \mathfrak{h}^*, \omega(\rho) < 0 \text{ and } \omega(\rho) \notin \langle \Theta \rangle \}.
\]

Moreover, the irreducible components of a largest fibre of the ampleness map are in 1-1 correspondence with those $\omega$'s which minimize (3.6.1).

**Proof.** By the Bruhat decomposition, each $g \in G$ may be expressed uniquely in the form

\[
g = u_n^0 b \quad \text{with } u \in U, \omega \in \mathfrak{h}^* \text{ and } b \in B.
\]

By Proposition 3.5, we have that

\[
\text{ca}(G/P) = \text{cod}_G \{ g \in G : g^{-1} x_\rho g \in U_P \}.
\]

Now,

\[
g^{-1} x_\rho g = b^{-1} n_w^{-1} u^{-1} x_\rho u n_w b \in U_P \iff n_w^{-1} x_\rho n_w \in U_P
\]

\[
\iff \omega^{-1}(\rho) \in N_P \iff \omega^{-1}(\rho) > 0 \text{ and } \omega^{-1}(\rho) \notin \langle \Theta \rangle.
\]

The first equivalence follows from $P \subset N(U_P)$ and $x_\rho \in Z(U)$. The last two follow from 2.2, 2.3 and 2.1.

We conclude that the maximum fibre dimension for the ampleness map is

\[
M = \max \{ \dim(U_n^\omega B) : \omega^{-1}(\rho) > 0, \omega(\rho) \notin \langle \Theta \rangle \} - \dim(P)
\]

\[
= \dim(B) - \dim(P) + \max \{ l(\omega) : \omega(\rho) > 0, \omega(\rho) \notin \langle \Theta \rangle \}.
\]

Here we have used the formula

\[
\dim(U_n^\omega B) = \dim(B) + l(\omega)
\]

and then made the substitution $\omega$ for $\omega^{-1}$. Hence

\[
\text{ca}(G/P) = \dim(G/P) - M
\]

\[
= \#(R^+) - \max \{ l(\omega) : \omega(\rho) > 0, \omega(\rho) \notin \langle \Theta \rangle \}
\]

\[
= \min \{ l(\omega_0) : \omega(\rho) > 0, \omega(\rho) \notin \langle \Theta \rangle \}.
\]

(Recall that $\dim(G/B) = \#(R^+)$ and that $l(\omega_0) = \#(R^+) - l(\omega)$ where $\omega_0$ is the Weyl group element taking $R^+$ to $R^-$. The proposition now follows by making the substitution $\omega \to \omega_0$ and noting that $\omega_0(\rho) = -\rho$.)

**4. Computing** $\text{ca}(G/P)$. The calculation is divided into three parts (recall that $G$ is simple):

(i) Determine $\text{ca}(G/B)$ from the expression of the highest root $\rho = \sum n_i \alpha_i$ in terms of the simple roots $\Sigma = \{ \alpha_1, \ldots, \alpha_n \}$, as in Theorem 4.2.

(ii) For each standard maximal parabolic $P_i := P_{\Sigma \setminus \{ \alpha_i \}}$, determine $\text{ca}(G/P_i)$.

Let $d(i)$ be the least number of steps, in the Coxeter-Dynkin diagram of $G$, from $\alpha_i$ to the long roots. Then, by Theorem 4.7

\[
\text{ca}(G/P_i) = d(i) + \text{ca}(G/B).
\]
(iii) Now, let $P = P_\Theta$ be any standard parabolic. By formula (3.6.1), we have
$$ca(G/P) = \min \{ l(\omega) : \omega \rho < 0 \text{ and } \omega \rho \text{ involves some } \alpha_i, \text{ with } \alpha_i \notin \Theta \}$$
$$= \min \{ ca(G/P) : \alpha_i \notin \Theta \}$$
$$= \min \{ d(i) : \alpha_i \notin \Theta \} + ca(G/B).$$

The results are summarized in Table 1 in §1.

From formula (3.6.1), we see that, in particular,
$$ca(G/B) = \min \{ l(\omega) : \omega(\rho) < 0 \}.$$ (4.1.1)

To each simple root $\alpha_i$ associate the integer
$$\nu_i = \begin{cases} 1, & \alpha_i \text{ long,} \\ 2, & \alpha_i \text{ short, } G \neq G_2, \\ 3, & \alpha_i \text{ short, } G = G_2. \end{cases}$$ (4.1.2)

Suppose that $\beta$ is a long root of height at least 2, and that $(\alpha_i, \beta) > 0$ for some $i$. Then
$$|\sigma_i(\beta)| = |\beta| - \nu_i$$ since $\alpha_i^*(\beta) = \nu_i$, as in Lemma 2.4.

**Theorem 4.2.** Let $\rho = \sum n_i \alpha_i$ be the expression of the highest root $\rho$ in terms of simple roots. Then
$$ca(G/B) = \sum n_i/\nu_i.$$ (The $\nu_i$ are defined in (4.1.2).)

**Proof.** It is well known that for each $\beta > 0$ there is some simple root $\alpha_i$ with $(\alpha_i, \beta) > 0$. This, together with (4.1.3) and formula (4.1.1), proves the theorem. \(\square\)

Table 1 contains the results of calculating $ca(G/B)$ for each of the simple Lie groups.

We turn next to calculating $ca(G/P_i)$ where $P_i$ is the maximal parabolic $P_{\Sigma \setminus \{\alpha_i\}}$.

**Lemma 4.3.** Let $\omega \in \mathfrak{h}$ minimize $l(\omega)$ in formula (4.1.1). Then
(i) $\omega(\rho) = -\alpha_i$ for some (long) simple root $\alpha_i$, and
(ii) $\omega^{-1}(\alpha_j) > 0$ for every other simple root $\alpha_j$.

Let $\alpha_k$ be any long simple root. Then there exists a unique $\omega_k$ minimizing $l(\omega)$ in formula (4.1.1) and satisfying $\omega_k(\rho) = -\alpha_k$.

**Proof.** Suppose that $\omega^{-1}(\alpha_i) < 0$ and $\omega(\rho) \neq -\alpha_j$. Then $l(\sigma_j \omega) < l(\omega).$ But $\sigma_j$ permutes $R^+ \setminus \{\alpha_i\}$, so that $\sigma_j \omega(\rho) < 0$. This contradicts the minimality of $\omega$, and proves (i) and (ii).

The subdiagram of the Coxeter-Dynkin diagram for $G$ consisting of the long roots is connected, so we may assume that $(\alpha_i, \alpha_k) \neq 0$. Then, as in Lemma 2.4,
$$\alpha_i^*(\alpha_k) = \alpha_k^*(\alpha_i) = -1,$$
so
$$\sigma_i \sigma_k(\alpha_i) = \sigma_i(\alpha_i + \alpha_k) = -\alpha_i + \alpha_k + \alpha_i = \alpha_k.$
To see that \( \omega_k = \sigma \sigma_k \omega \) is the required Weyl group element, it remains to see that \( l(\omega_k) = l(\omega) \). This follows from properties (i) and (ii) of \( \omega \):

\[
\omega^{-1}(\alpha_i) > 0 \Rightarrow l(\sigma_i \omega) = l(\omega) + 1 \quad \text{and} \\
\omega^{-1}(\sigma_i) = \omega^{-1}(\alpha_i + \alpha_k) = -\rho + \omega^{-1}(\alpha_k) < 0 \\
= l(\sigma_i \sigma_k \omega) = l(\sigma_k \omega) - 1 = l(\omega).
\]

The uniqueness of \( \omega_k \) is a standard result on parabolic subgroups of \( \mathbb{G} \), since \( \omega_k \) is the unique element for which \( \min\{l(\omega) : \omega(\rho) = -\alpha_k\} \) is attained, e.g. see Carter \[6, §2.5\]. \( \square \)

For the maximal parabolics, formula (3.6.1) reads

(4.4.1) \( \text{ca}(G/P) = \min\{l(\omega) : \omega(\rho) < 0 \text{ and } \omega(\rho) \text{ involves } \alpha_i\} \).

For each simple root \( \alpha_i \), let

(4.4.2) \( k(i) := \min\{l(\omega) : \omega(\alpha_j) \text{ involves } \alpha_i \text{ for some long simple root } \alpha_j\} \).

**Remark 4.5.** \( k(i) = 0 \iff \alpha_i \) is long, and \( k(i) = 1 \iff \alpha_i \) is short and is adjacent to a long root in the Coxeter-Dynkin diagram for \( G \).

For each simple root \( \alpha_i \), let \( d(i) \) = the number of nodes, in the Coxeter-Dynkin diagram for \( G \), from \( \alpha_i \) to the long roots.

**Lemma 4.6.** For each \( i \), we have \( k(i) = d(i) \).

**Proof.** The only diagrams having more than one short root are those for \( C_n \) and \( F_4 \). By Remark 4.5, it remains only to verify the lemma for these two cases. Samelson \[18, pp. 79–86\] contains the descriptions that I will be using of the root systems. The short roots are underlined. In each case, \( \Sigma = \{ \alpha_1, \ldots, \alpha_n\} \) is the implicit labeling of the simple roots, and \( \sigma_i \) is the reflection through \( \alpha_i \).

I. \( G = C_n \).

\[
R^+ = \{ 2\alpha_i, \alpha_i \pm \alpha_j, i < j \}, \\
\Sigma = \{ \alpha_1 - \alpha_2, \ldots, \alpha_{n-1} - \alpha_n, 2\alpha_n \}.
\]

Figure (4.6.1) shows the only paths (without backtracking) between the long positive roots:

(4.6.1) \( 2\alpha_n^o \rightarrow 2\alpha_{n-1}^o \rightarrow \cdots \rightarrow 2\alpha_2^o \rightarrow 2\alpha_1^o \).

It follows that \( k(i) = n - i = d(i) \).

II. \( G = F_4 \).

\[
R^+ = \{ \alpha_i, \alpha_i \pm \alpha_j, i < j, 1 \times (a_1 \pm a_2 \pm a_3 \pm a_4) \}, \\
\Sigma = \{ 1 \times (a_1 - a_2, -a_3 - a_4), a_4, a_3 - a_4, a_2 - a_3 \}, \\
\sigma_1 \sigma_2(\alpha_3) = \sigma_1(\alpha_3 + a_4) = \frac{1}{2} \times (a_1 - a_2 + a_3 + a_4).
\]

Thus, \( k(1) = 2 = d(1) \). \( \square \)
Theorem 4.7. Let $P_i = P_{\Sigma \setminus \{(\alpha_i)\}}$ be a maximal parabolic subgroup of the simple group $G$. Then

$$ca(G/P_i) = d(i) + ca(G/B).$$

Proof. By Lemma 4.6, it suffices to prove the result with $k(i)$ in place of $d(i)$. Let

$$\omega = \sigma_{i_m} \cdots \sigma_{i_1} \sigma_{i_0} \rho,$$

be a reduced expression for some $\omega$ minimizing $l(\omega)$ in formula (4.1.1). By considering the sequence,

$$\beta_r = \sigma_{i_r} \cdots \sigma_{i_1} \sigma_{i_0} (\rho), \quad r = 0, \ldots, m,$$

one sees that $\omega$ may be written as $\omega = \omega_2 \omega_1$ where $\omega_1$ minimizes $l(\omega)$ in formula (4.1.1), $\omega_2$ minimizes $l(\omega)$ in formula (4.4.2), and $l(\omega) = l(\omega_1) + l(\omega_2)$. \box

5. Let $L\Sigma$ denote the long roots in the basis $\Sigma$ of the root system $R$ of $G$, a simple complex Lie group. Let $n(P)$ be the number of irreducible components in a largest fibre of the ampleness map $\psi$ (cf. Remark 3.3.1). By Proposition 3.6 and Lemma 4.3, we have that $n(B) = \#(L\Sigma)$. (In fact, when $R$ contains only long roots, then $n(P_\Theta) = \#(L\setminus \Theta)$.) For the maximal parabolics, $n(P_i) = 1$. To see this, one need only verify the uniqueness of minimizing elements for formula (4.4.2). More generally,

$$n(P_\Theta) = \max\{1, \#(L\Sigma \setminus \Theta)\}.$$

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