HARDY SPACES AND JENSEN MEASURES

BY

TAKAHIKO NAKAZI

ABSTRACT. Suppose $A$ is a subalgebra of $L^\infty(m)$ on which $m$ is multiplicative. In this paper, we show that if $m$ is a Jensen measure and $A + A$ is dense in $L^2(m)$, then $A + A$ is weak-* dense in $L^\infty(m)$, that is, $A$ is a weak-* Dirichlet algebra. As a consequence of it, it follows that if $A + A$ is dense in $L^4(m)$, then $A$ is a weak-* Dirichlet algebra. (It is known that even if $A + A$ is dense in $L^4(m)$, $A$ is not a weak-* Dirichlet algebra.) As another consequence, it follows that if $B$ is a subalgebra of the classical Hardy space $H^\infty$ containing the constants and dense in $H^2$, then $B$ is weak-* dense in $H^\infty$.

1. Introduction. Let $(X, \mathcal{A}, m)$ be a nontrivial probability measure space and $A$ a subalgebra of $L^\infty = L^\infty(m)$ containing the constants. Suppose $m$ is multiplicative on $A$, that is, for $f, g \in A$ we have $\int_X fg \, dm = \int_X f \, dm \int_X g \, dm$. The abstract Hardy space $H^p = H^p(m), 0 < p \leq \infty$, associated with $A$ is defined as follows. For $0 < p < \infty$, $H^p$ is the $L^p = L^p(m)$-closure of $A$, while $H^\infty$ is defined to be the weak-* closure of $A$. If $A + A$ is weak-* dense in $L^\infty, A$ is called a weak-* Dirichlet algebra, which was introduced by T. P. Srinivasan and J. K. Wang [8]. The theory of weak-* Dirichlet algebras has emerged as the correct setting for many of the central results of abstract analytic function theory.

K. Hoffman and H. Rossi [5] gave an example such that even if $A + A$ is dense in $L^3, A$ is not a weak-* Dirichlet algebra. While G. Lumer [6] showed that if $A + A$ is dense in $L^p$ for all finite $p$, $H^p \cap L^\infty$ is a weak-* Dirichlet algebra. Recently, the author [7] proved that if $A + A$ is dense in $L^4$, then $H^4 \cap L^\infty$ is a weak-* Dirichlet algebra. Hence if $f \in A$, then [8]

$$\int_X \log |f| \, dm \geq \log \left| \int_X f \, dm \right|.$$ 

We say $m$ a Jensen measure when functions in $A$ satisfy the inequality above.

In this paper, we show that if $m$ is a Jensen measure and $A + A$ is dense in $L^2(m)$, then $A$ is a weak-* Dirichlet algebra. If $A + A$ is dense in $L^4$, then $m$ is a Jensen measure by the remark above and so $A$ is a weak-* Dirichlet algebra. Hence we answer affirmatively the question left open by Hoffman-Rossi [5], Lumer [6] and the author [7]. Moreover the result is applied to give another proof of a theorem of S. D. Fisher for backward shift invariant subalgebras [1].

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2. Jensen measure. Let $A$ be a subalgebra of $L^\infty$ containing the constants and $m$ a multiplicative measure on it. Set $A_0 = \{ f \in A : \int f \, dm = 0 \}$.

**Lemma 1.** Suppose $A + \overline{A}$ is dense in $L^2$. If $w \in L^\infty$ is a nonnegative function with $w^{-1} \in L^\infty$, then

$$\inf_{f \in A_0} \int |1 - f|^2 \, w \, dm = \left( \inf_{g \in A_0} \int |1 - g|^2 w^{-1} \, dm \right)^{-1}.$$

**Proof.** If $f, g \in A_0$, by Schwarz's inequality,

$$\int |1 - f|^2 \, w \, dm \geq \left( \int |1 - g|^2 w^{-1} \, dm \right)^{-1}.$$

There exists a unique $f_0$ in the closure of $A_0$ in $L^2(w \, dm)$ such that

$$\inf_{f \in A_0} \int |1 - f|^2 \, w \, dm = \int |1 - f_0|^2 \, w \, dm.$$

Then, by the minimum property of $\int |1 - f_0|^2 \, w \, dm$, $1 - f_0$ is orthogonal to $A_0$ in $L^2(w \, dm)$. Set $h_0 = (1 - f_0)w$. Since the infimum is positive, then

$$\int |1 - f_0|^2 \, w \, dm = \int (1 - f_0)(1 - f_0)w \, dm$$

$$= \int (1 - f_0)w \, dm = \int h_0 \, dm > 0.$$

By the hypothesis, $\overline{H^2}$ is the orthogonal complement of $A_0$ in $L^2$ and so $h_0 \in \overline{H^2}$. Hence $g_0 = h_0/\int h_0 \, dm$ belongs to $H^2$ and

$$\left( \int h_0 \, dm \right)^2 \int |1 - (1 - g_0)|^2 w^{-1} \, dm = \int |1 - f_0|^2 \, w \, dm = \int h_0 \, dm.$$

Since $w, w^{-1} \in L^\infty$, $1 - g_0$ belongs to the closure of $A_0$ in $L^2(w^{-1} \, dm)$ and

$$\int h_0 \, dm = \left( \int |1 - (1 - g_0)|^2 w^{-1} \, dm \right)^{-1}.$$

This implies the lemma.

**Lemma 2 (Szegö's Theorem).** Suppose $A + \overline{A}$ is dense in $L^2$ and $m$ is a Jensen measure. If $w \in L^1$ is a nonnegative function, then

$$\inf_{f \in A_0} \int |1 - f|^2 \, w \, dm = \exp \int \log w \, dm.$$

**Proof.** By the inequality of arithmetic and geometric means and Jensen's inequality, for any $f, g \in A_0$,

$$\int |1 - f|^2 \, w \, dm \geq \exp \int \log w \, dm$$

and

$$\int |1 - g|^2 w^{-1} \, dm \geq \exp \int \log w^{-1} \, m$$
if \( m \) is a Jensen measure. If \( w, w^{-1} \in L^\infty \), by Lemma 1
\[
\inf_{f \in A_0} \int |1 - f|^2 w \, dm = \exp \int \log w \, dm.
\]
If \( w^{-1} \notin L^\infty \) with \( w \in L^\infty \), for any \( \varepsilon > 0 \),
\[
\exp \int \log(w + \varepsilon) \, dm = \inf \int |1 - f|^2 (w + \varepsilon) \, dm
\]
\[
\geq \inf \int |1 - f|^2 w \, dm \geq \exp \int \log w \, dm.
\]
and so letting \( \varepsilon \) tend to zero, the lemma follows. For any \( w \in L^1 \), let \( w_n = \min\{w, n\} \), then
\[
\exp \int \log w \, dm \geq \left( \inf \int |1 - g|^2 w^{-1} \, dm \right)^{-1}
\]
\[
\geq \left( \inf \int |1 - g|^2 w^{-1} \, dm \right)^{-1} = \exp \int \log w_n \, dm
\]
and so letting \( n \) tend to infinity, the lemma follows.

**Theorem 1.** \( A + \overline{A} \) is dense in \( L^2 \) and \( m \) is a Jensen measure if and only if \( A \) is a weak-* Dirichlet algebra.

**Proof.** If \( A \) is a weak-* Dirichlet algebra, then \( A + \overline{A} \) is dense in \( L^2 \) clearly and it is known [8] that \( m \) is a Jensen measure. If \( A + \overline{A} \) is dense in \( L^2 \) and \( m \) is a Jensen measure, then Szegö’s theorem is valid by Lemma 2. Srinivasan and Wang [8] imply that Szegö’s theorem is equivalent to that \( A + \overline{A} \) is weak-* dense in \( L^\infty \).

**Theorem 2.** Let \( A \) be a subalgebra of \( L^\infty \) containing the constants and \( m \) a multiplicative measure on it. If \( A + \overline{A} \) is dense in \( L^2 \), then the following (1) ~ (6) are equivalent.

1. \( A + \overline{A} \) is weak-* dense in \( L^\infty \), that is, \( A \) is a weak-* Dirichlet algebra.

2. \( A + \overline{A} \) is dense in \( L^4 \).

3. \( m \) is a Jensen measure.

4. If \( f \in H^1 \) is a real function, then \( f \) is a constant.

5. If \( f \in H^{1/2} \) is a nonnegative function, then \( f \) is a constant.

6. There is a constant \( \gamma_p \), defined for \( 0 < p < 1 \), such that
\[
\|f\|_p \leq \gamma_p \|f + \overline{g}\|_1, \quad f \in A, \ g \in A_0.
\]

**Proof.** (1) \( \iff \) (3) is equivalent to Theorem 1. (1) \( \iff \) (2) is clear by the remark in Introduction and (1) \( \iff \) (3). (1) \( \implies \) (6) is known (cf. [2, p. 107]). (6) \( \implies \) (4) is clear.

(4) \( \implies \) (3) Suppose \( w \in L^\infty \) is a nonnegative function with \( w^{-1} \in L^\infty \). Let \( f_0 \) (resp. \( g_0 \)) be the orthogonal projection of 1 into the closure of \( A_0 \) in \( L^2(w \, dm) \) (resp. \( L^2(w^{-1} \, dm) \)). Then by Lemma 1 and Schwarz’s lemma, \( |1 - f_0|^2 w = |1 - g_0|^2 w^{-1} \) and \( (1 - f_0)(1 - g_0) = k \geq 0 \). If \( f = 1 - g_0 \), then \( kw = |f|^2 \) and \( f \in H^2 \) and \( k \in H^1 \) because \( w, w^{-1} \in L^\infty \). By the hypothesis, \( k \) is a constant 1. Thus \( w = |f| \) for \( f, f^{-1} \in H^2 \cap L^\infty \). This implies that \( H^2 \cap L^\infty \) is a logmodular algebra and so \( m \) is a Jensen measure [4]. (1) \( \implies \) (5) is known [9]. (5) \( \implies \) (4) is clear.
(5) is Neuwirth-Newman's theorem and (6) is Kolmogoroff's theorem. Even if \( A + \hat{A} \) is not dense in \( L^2 \), (3) implies (5) (cf. [3, pp. 135 ~ 136]).

3. Subalgebras of \( H^\infty \) on the unit circle. Let \( T \) be the unit circle in the complex plane and \( d\theta/2\pi \) the normalized Lebesgue measure on \( T \). In this section, we consider \( A, L^\infty \) in case \( X = T \) and \( m = d\theta/2\pi \). If \( A \) is the set of all analytic polynomials on \( T \), then \( A \) is a weak-* Dirichlet algebra of \( L^\infty(T) = L^\infty(d\theta/2\pi) \). Then, for \( 0 < p < \infty \), \( H^p(T) = H^p(d\theta/2\pi) \) is the classical Hardy space.

**Theorem 4.** Let \( B \) be a subalgebra of \( H^\infty(T) \) containing the constants. If \( B \) is \( L^2 \)-dense in \( H^2(T) \), then \( B \) is weak-* dense in \( H^\infty(T) \).

**Proof.** Since \( L^2(T) = H^2(T) + e^{-i\theta}H^2(T) \) and \( d\theta/2\pi \) is a Jensen measure, and \( B \) is \( L^2 \)-dense in \( H^2(T) \), so it is a weak-* Dirichlet algebra of \( L^\infty(T) \) by Theorem 1, then \( H^2(T) \cap L^\infty(T) = L^\infty(T) \) is a weak-* closure of \( B \) by Theorem 2.4.1 of [8].

**Corollary (S. D. Fisher).** Let \( B \) be a nontrivial subalgebra of \( H^\infty(T) \) which (i) contains the constants, (ii) is weak-* closed, and (iii) contains \( e^{-i\theta}f \) whenever \( f \in B \) and \( f(0) = \int_0^{2\pi} f d\theta/2\pi = 0 \). Then \( B = H^\infty(T) \).

**Proof.** Let \( \mathfrak{M} \) be a closure of \( B \) in \( L^2(T) \), then the orthogonal complement \( \mathfrak{M}^\perp \) of \( \mathfrak{M} \) in \( H^2(T) \) is a shift invariant subspace, that is, \( e^{i\theta}\mathfrak{M}^\perp \subset \mathfrak{M}^\perp \). By the well-known theorem of Beurling, \( \mathfrak{M}^\perp = qH^2(T) \) for some inner function \( q \) with \( q(0) = 0 \) if \( \mathfrak{M}^\perp \neq \{0\} \). \( e^{-i\theta}q \) is orthogonal to \( qH^2(T) \) and so \( e^{i\theta}q \in \mathfrak{M} \cap L^\infty(T) \). Since \( \mathfrak{M} \cap L^\infty \) is an algebra, \( e^{-i\theta}q^2 \) is orthogonal to \( qH^2(T) \) and so \( \int qe^{-in\theta}d\theta/2\pi = 0 \) for \( n \geq 1 \). Thus \( q \) is a zero constant and so this implies \( \mathfrak{M}^\perp = \{0\} \) and \( \mathfrak{M} = H^2(T) \). Theorem 4 implies \( B = H^\infty(T) \).

**References**


**Division of Applied Mathematics, Research Institute of Applied Electricity, Hokkaido University, Sapporo 060, Japan**